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Large population and size scale limit of a stochastic particle model describing an age and space-structured population

P. Michel^{*†}

March 22, 2016

Abstract

The aim of this work is to study this stochastic individual-based model, structured with respect to age (progression within the cell cycle) and space (radial distance from the oocyte). We prove the existence of solutions and the convergence in large population and size scale limit to a partial differential equation.

Key-words: stochastic processes, partial differential equation, convergence.

AMS class. No: 35L60, 35L65, 60G55

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1 Introduction

The development of ovarian follicles is a unique instance of a morphogenesis process resulting from the interactions between somatic cells (granulosa cells) and germ cell (oocyte). In mammals, the initiation of follicular development from the pool of resting follicles is characterized by an increase in the oocyte size concomitant with the surrounding granulosa cells proliferating (see [19]). In [5], the authors have introduced a multi-scale stochastic model, taking in accounts for the molecular dialogue existing between the oocyte and granulosa cells, of the primordial follicle development. We propose, here, to study mathematically this model. The main line of this model can be summarized as follows :

- i) there are a population of small cells of diameter ϵ proliferate around large cell of radius r_O (see fig. 1),
- ii) there is a dialog (depending on the distance) between the large cell and the small cells (see fig. 1) which links growth and proliferation.

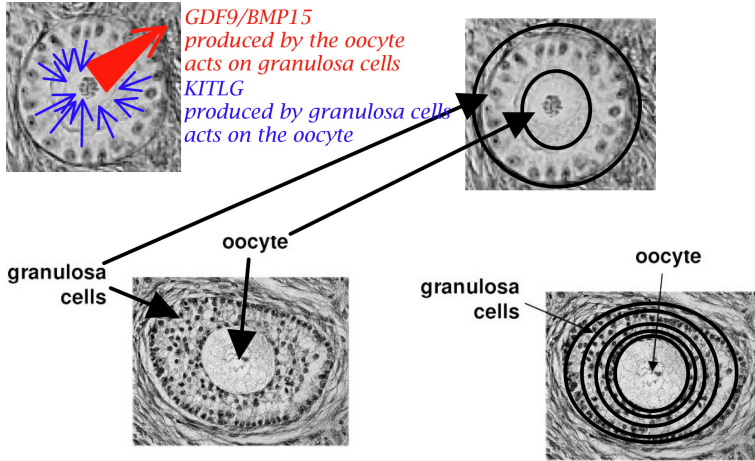


Figure 1: Development of preantral follicles. Each follicle is comprised of a germ cell (the oocyte) and granulosa cells. The oocyte produces GDF9/BMP15 which make granulas cells proliferate around the oocyte and granulosa cells (small ones) produce KITLG that make the oocyte grows. Therefore, the development of each follicle (in its basal follicular development), is coordinated by tight interactions existing between the oocytes and their surrounding granulosa cells.

The age of a cell is simply a positive real number and $Age = \mathbb{R}_+$. To simplify the computation, we choose an equivalent formulation of this model : the large cell keeps a constant radius ($= 1$) but the local volume of space around the oocyte evolves with respect to time. The physical space is

$$Space = \mathbb{R}^3/B(0, 1), \quad \text{where } B(0, 1) = \{(r, \theta, \phi) \in Space : r < 1\}. \quad (1)$$

Let $\epsilon > 0$ the diameter of a granulosa cell, then i^{th} layer is given by

$$\mathcal{L}_i^\epsilon = \{(r, \theta, \phi) \in Space : r \in [1 + (i - 1)\epsilon, 1 + i\epsilon]\}, \quad (2)$$

and for $N = \mathbb{E}_{nt}(1/\epsilon) \in \mathbb{N}^*$, a subdivision of these layers (see fig. 2) is given by

$$\mathcal{L}_{i,j,k}^\epsilon = \{(r, \theta, \phi) \in Space : r \in [1 + (i - 1)\epsilon, 1 + i\epsilon], \theta \in [\pi \frac{(j - 1)}{N}, \pi \frac{j}{N}[, \phi \in [-\pi + 2 \frac{k - 1}{N} \pi, -\pi + 2 \frac{k}{N} \pi]\}. \quad (3)$$

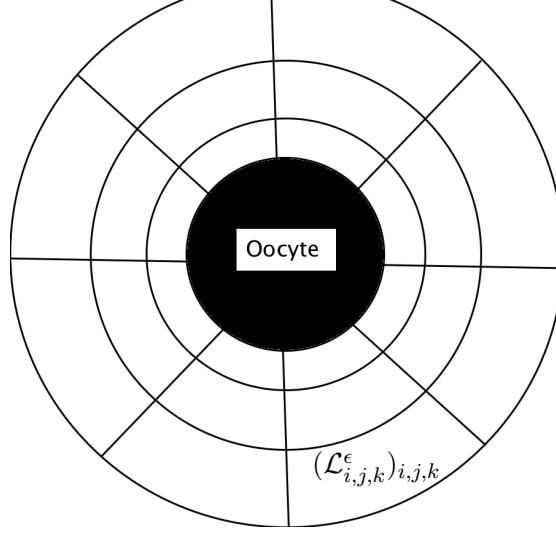


Figure 2: Space layer decomposition. The physical space surround the ball (oocyte) in its center. We subdivide this space in layers $(\mathcal{L}_i^\epsilon)_i$ and mesh these layers uniformly $(\mathcal{L}_{i,j,k}^\epsilon)_{i,j,k}$ of volumes $(Vol_{i,j,k}^\epsilon)_{i,j,k}$.

For $\epsilon > 0$, the volume of each \mathcal{L}_i^ϵ (resp. $\mathcal{L}_{i,j,k}^\epsilon$) (which evolves with the radius of this oocyte r_O) are given by

$$Vol_i^\epsilon = \frac{4\pi}{3}[(r_O + i\epsilon)^3 - (r_O + (i-1)\epsilon)^3] = \frac{4\pi}{3}[\epsilon^3(3i^2 - 3i + 1) + 3\epsilon^2 r_O(2i - 1) + 3\epsilon r_O^2], \quad (4)$$

and

$$Vol_{i,j,k}^\epsilon = Vol_i^\epsilon / N^2 \sim \epsilon^2 \frac{4\pi}{3}[\epsilon^3(3i^2 - 3i + 1) + 3\epsilon^2 r_O(2i - 1) + 3\epsilon r_O^2]. \quad (5)$$

For each time, a cell is defined by its age and its position, therefore by a dirac mass on $Age \times Space$. The whole population is then characterized by a punctual measure on this set, i.e., in $\mathcal{M}_P(Space \times Age)$. Let

$$Z_0^{M,\epsilon}(da, dp) = \frac{1}{M} \sum_{k=1}^{N_0^{M,\epsilon}} \delta_{(a_k^{M,\epsilon}, x_k^{M,\epsilon})} \in \mathcal{M}_P(Space \times Age), \quad (6)$$

where M is a normalization parameter and $(a_k, x_k)_k \subset Age \times Space$, the initial population sequence, such that

$$\sup_{M,\epsilon} (N_0^{M,\epsilon} / M) < \infty,$$

and $r_O^{M,\epsilon}(0) = r_0 > 0$. Let $Q(ds, Compt(dn), d\Theta, \Pi_{s-}(p, dp'))$ be a Poisson point measure on $\mathbb{R}_+ \times \varepsilon = \mathbb{R}_+ \times \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}^3$ with intensity

$$q(ds, Compt(dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) = ds \otimes Compt(dn) \otimes d\Theta \otimes \Pi_{s-}^\epsilon(p, dp'), \quad dp' = r^2 \sin(\theta) dr d\theta d\phi,$$

and independent of $Z_0^{M,\epsilon}$ (see [4, 25, 17, 15, 3]). Let us denote $X_k^{M,\epsilon}(t)$ and $A_k^{M,\epsilon}(t)$ the position and age of the k^{th} individual at time t (ranked in the lexicographic order on $\mathbb{R}^3 \times \mathbb{R}_+$, see [4, 25, 17, 15, 3] for details). Then the oocyte radius follows the equation

$$r_O^{M,\epsilon}(t) = r_O^{M,\epsilon}(0) + \sum_{i,j,k} \kappa((i-1)\epsilon) \int_0^t (r_O^{M,\epsilon}(s-))^\alpha \langle \Psi_{i,j,k}^\epsilon, Z_{s-}^{M,\epsilon} \rangle ds, \quad (7)$$

where $\alpha < 0$ and $\kappa \in C_b^0(\mathbb{R}_+, \mathbb{R}_+)$ and $\Psi_{i,j,k}^\epsilon$ a regular approximation of the characteristic function $\chi_{\mathcal{L}_{i,j,k}^\epsilon}$ (see (28) in the annex : section 5.1). The population at time t , denoted by $Z_t^{M,\epsilon}$, is the set of all individuals alive

at time t and follows the master equation

$$\begin{aligned} Z_t^{M,\epsilon}(da, dp) &= \frac{1}{M} \sum_{k=1}^{N_0^{M,\epsilon}} \delta_{(A_k^{M,\epsilon}(0)+t, X_k^{M,\epsilon}(0))} \\ &\quad + \frac{1}{M} \int_0^t \int_{\mathcal{E}} 1_{n < N_{s-}^{M,\epsilon}} \left[(2\delta_{(t-s, X_n^{M,\epsilon}(s-))} - \delta_{(A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))}) 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right. \\ &\quad \left. + (\delta_{(A_n^{M,\epsilon}(s-)+t-s, p')} - \delta_{(A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))}) 1_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^\epsilon(X_n^{M,\epsilon}(s-))} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')), \end{aligned} \quad (8)$$

where

$$B_{n,s-}^{M,\epsilon} = B(A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-)), \text{ with } B(a, p) = 1 - e^{-a/\lambda(\|p\|_2)}, \quad \lambda \in C^0(\mathbb{R}_+, \mathbb{R}_+), \quad (9)$$

$$P_{s-}^\epsilon(p) = C^\epsilon \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p) R(\langle \Psi_{i,j,k}^\epsilon, Z_{s-}^{M,\epsilon} \rangle \frac{Vol_G}{Vol_{i,j,k}^\epsilon}), \text{ with } R(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}}, \quad (10)$$

and

$$\Pi_{s-}^\epsilon(p, dp') = \frac{\Psi^\epsilon(|p' - p|) G_{s-}^\epsilon(p') dp'}{\iint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq}, \text{ with } G_{s-}^\epsilon(p') = 1 - \frac{P_{s-}^\epsilon(p')}{C^\epsilon}, \quad (11)$$

where $C^\epsilon = C/\epsilon^2 > 0$.

The main objective of the paper is to prove the convergence of the stochastic processes $(Z_t^{M,\epsilon}(da, dp))_{\epsilon, M}$ as the population increases to infinity and the size of a cell converges to zero (the existence of solutions to the system (7)-(17) is quite classical (see [4, 25, 17, 15, 3]) and so be proved in annex 5). Difficulties of this work appear in the manipulation of stochastic and deterministic tools. Indeed, stochastic processes, their convergence or/and tightness (compactness) are powerful mathematical tools but are not well suited for the punctual (or regular) convergence, therefore for a fixed initial population size (in an equivalent way, fixing M) we cannot pass to the limit as ϵ converges to zero. Thus, its not completely clear that $(Z_t^{M,\epsilon}(da, dp))_{\epsilon, M}$ converge for all sequences of $(\epsilon_k, M_k) \rightarrow (0, \infty)$. We show that the limit (étroite-weak limit [15, 4] for similar studies) of $(Z_t^{M,\epsilon}(da, dp))_{\epsilon, M}$ (for a fixed ϵ and $M \rightarrow \infty$) is well defined (unique) and is a strong solution to the following partial differential equation

$$\text{Intermediate eq.} \begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \rho^\epsilon - \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \rho^\epsilon(t, a, p') dp' + P_\infty^\epsilon(p) \rho^\epsilon(t, a, p) = 0, \\ r_O^\epsilon(t) = r_O^\epsilon(0) + \sum_{i,j,k} \kappa((i-1)\epsilon) \int_0^t (r_O^\epsilon(s))^\alpha \iiint \Psi_{i,j,k}^\epsilon(p) \rho^\epsilon(t, a, p) dadpds \\ \rho^\epsilon(t, a, p)|_{a=0} = 2 \int B(a, p) \rho^\epsilon(t, a, p) dadp, \quad \rho^\epsilon(0, a, p') = \rho_0^\epsilon(a, p), \quad r_O^\epsilon(0) = r_0. \end{cases} \quad (12)$$

with $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p) da$,

$$P_\infty^\epsilon(p') = \frac{C}{\epsilon^2} \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p') R\left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(t, p) dp Vol_G}{Vol_{i,j,k}^\epsilon}\right), \text{ with } R(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}}, \quad (13)$$

$$\Pi_\infty^\epsilon(p', dp) = \frac{\Psi^\epsilon(|p' - p|) G_{s-}^\epsilon(p) dp}{\iiint \Psi^\epsilon(|q - p'|) G_{s-}^\epsilon(q) dq}, \text{ with } G_{s-}^\epsilon(\cdot) = 1 - \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(\cdot) R\left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(t, p) dp Vol_G}{Vol_{i,j,k}^\epsilon}\right). \quad (14)$$

Then, we prove that, the limit as ϵ converge to zero, is a weak solution (using Sobolev spaces and weak-* convergence [2]) to the following nonlinear partial differential equation (Transport [20, 22], Keller Segel type [12, 13])

$$Final\ eq. \begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\rho + div(C\rho R\nabla(\log(1-R))) = 0, \\ \rho|_{a=0} = 2 \int B(a,p)\rho(t,a,p)da, \quad \rho_{t=0} = \rho_0, \\ r_O(t) = r_O(0) + \int_0^t (r_O^\epsilon(s))^\alpha \iiint \kappa(r)\rho^\epsilon(t,a,p)dadpds, \quad r_O(0) = r_0. \end{cases} \quad (15)$$

with $C > 0$, $R = R(\int_{\mathbb{R}_+} \rho(t,a,p)da \frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(t)+3r_O(t)^2})$ and

$$r'_O(t) = (r_O(t))^\alpha \iiint \kappa(r)\rho(t,a,p)r^2 \sin(\theta)dadr d\theta d\phi, \quad r_O(0) = r_0. \quad (16)$$

Therefore, the main difficulty is to mix both approach to prove the convergence. In section 2, we give main theorems of convergence and proof is done in section 3 (we put in annex (section 5) the most technicals lemmas).

2 Main results

Before proving the convergence of the stochastic process $Z_t^{M,\epsilon}$, we first show that we have existence for all M and ϵ of $Z_t^{M,\epsilon}$ (proposition 2.1). Then, we adopt the following approach (see fig. 2.2) : we show, in theorem 2.2 I, that we can extract a subsequence $M_k \rightarrow \infty$ such that $(Z_t^{M_k,\epsilon})_{M_k,\epsilon}$ converge to a solution ρ^ϵ to an intermediate partial differential equation (12), then we prove, in theorem 2.2 II, that we can extract a subsequence $\epsilon_k \rightarrow \infty$ such that ρ^{ϵ_k} converge to ρ solution to (15). Finally, in theorem 2.2 III, we show that there exists a subsequence $(\epsilon_k, M_k) \rightarrow (0, \infty)$ such that $Z_t^{M,\epsilon}$ converges to ρ (see fig. 2.2).

Assumptions

I- Geometric : Assume that for all p , we can define the matrix

$$M_\Psi := \frac{\iiint_q \Psi^\epsilon(|q-p|)(q-p)^t(q-p)dq}{\epsilon^2 \iiint \Psi^\epsilon(|q-p|)dq},$$

which satisfies

$$\begin{cases} M_\Psi \text{ is definite positive} \\ \forall p \forall \epsilon, \quad 0 < \inf_{\epsilon,p} \min\{\lambda \in Sp(M_\Psi)\} \leq \sup_{\epsilon,p} \max\{\lambda \in Sp(M_\Psi)\} < \infty \\ CM_\Psi \rightarrow^{C^0(Space)} Ch(p). \end{cases} \quad (17)$$

Remark 1 The construction we give in section 5.1 satisfies (17) and by symmetry of Ψ^ϵ we have directly that $Ch(p) = CId$ for all $p \in Space$. Moreover, we notice that

$$Vol_G \iiint \Psi_{i,j,k}^\epsilon dp / Vol_{i,j,k}^\epsilon \rightarrow^{\epsilon \rightarrow 0} \frac{Vol_G(3r^2 + 6r + 3)}{3r^2 + 6rr_O(t) + 3r_O(t)^2}$$

II- Uniform bounds on $Z_0^{M,\epsilon}$ and $\rho_0^\epsilon(a,p)$: assume there exists $m \geq 1$ and $w > 0$ s.t.

$$\sup_{(M,\epsilon) \in U_w} \mathbb{E} \left(\left(\iiint (1 + a^m + r^m) Z_0^{M,\epsilon}(da, dp) \right)^2 + \left(\iiint (1 + a^{2m} + r^{2m}) Z_0^{M,\epsilon}(da, dp) \right) \right) < \infty, \quad (18)$$

with

$$U_w := \{M\epsilon > w\}, \quad (19)$$

$$\sup_\epsilon \int (1 + \|p\|) \left[\int \rho_0^\epsilon(a,p)da + \left(\int \rho_0^\epsilon(a,p)da \right)^2 + \left(\int \left| \frac{\partial}{\partial a} \rho_0^\epsilon(a,p) \right| da \right)^2 \right] dp < \infty, \quad (20)$$

II- Convergence of $Z_0^{M,\epsilon}$ and $\rho_0^\epsilon(a, p)$:

$$Z_0^{M,\epsilon}(da, dp) \xrightarrow{M \rightarrow \infty} \rho_0^\epsilon(a, p)dadp, \quad \text{with } \rho_0^\epsilon(a, p) \in C_0^1(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3) \quad (21)$$

$$\rho_0^\epsilon(a, p) \xrightarrow{\epsilon \rightarrow 0} \rho_0(a, p), \quad \text{with } \rho_0(a, p) \in C_0^1(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3) \quad (22)$$

Proposition 2.1 (Existence) Under assumptions (6)-(11) and (18) there exists a solution $Z_t^{M,\epsilon} \in D(\mathbb{R}_+, \mathcal{M}_P)$ and $r_O \in C^0 \cap C_m^1(\mathbb{R}_+)$ to (7)-(8) for all $M, \epsilon > 0$. Moreover, assuming that (18) is satisfied then we have that

$$\sup_{M,\epsilon} \mathbb{E} \left(\sup_{t \in [0, T]} \left(\iiint \iiint (1+a^m+r^m) Z_t^{M,\epsilon}(da, dp) \right)^2 + \left(\iiint \iiint (1+a^{2m}+r^{2m}) Z_t^{M,\epsilon}(da, dp) \right) \right) < \infty, \quad \forall T > 0. \quad (23)$$

The proof is given in section 5.2.

Theorem 2.2 We have that.

I) Assume that (21) and (18) are satisfied. Then, for all $\epsilon > 0$, $(Z_t^{M,\epsilon}(da, dp))_M$ is tight on $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(\text{Age} \times \text{Space}), \text{weak}))$. Its limit values $\lim_{M_k \rightarrow \infty} Z_t^{M_k, \epsilon}(da, dp) = \rho^\epsilon(t, a, p)dadp$, are continuous measure-valued process satisfying (12)-(14).

II) Assume that (22), (20) and (17) are satisfied then $(\rho^\epsilon, r_O^\epsilon)$ solution to (12) weakly converge to (ρ, r_O) weak solution to (15)-(16).

III) Assume that (21), (22), (18) and (17) are satisfied. Then, for all $C > 0$, there exists a subsequence $(\epsilon_k, M_k) \subset U_C$ s.t. $M_k \epsilon_k \rightarrow \infty$ and

$$\lim_{k \rightarrow \infty} \langle Z_t^{M_k, \epsilon_k}(da, dp), \psi \rangle = \langle \rho(t, a, p)dadp, \psi \rangle, \quad \forall \psi \in C_0^1,$$

weak solution to (15)-(16).

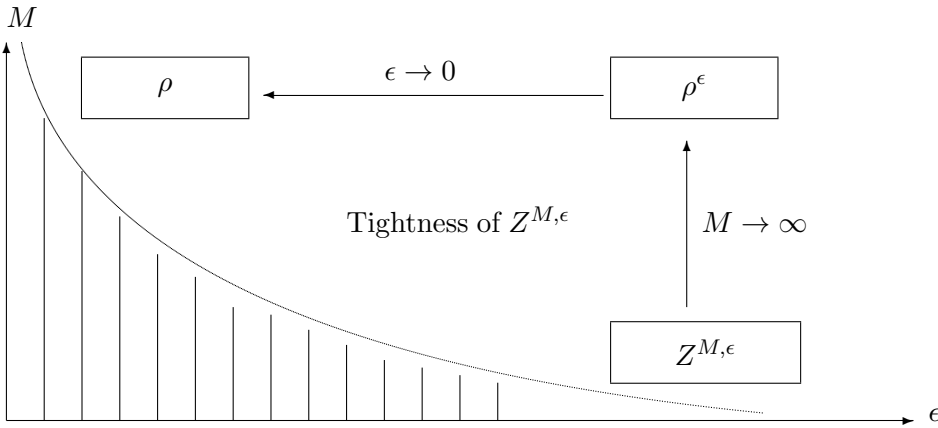


Figure 3: Convergence proof.

3 Proof of the main theorem

The proof of the main theorem follows the scheme given in figure 2.2. The first and third part are stochastic processes results while the second point is a partial differential equation result.

3.1 Proof of theorem 2.2 Part I : Convergence of the stochastic process $Z_t^{M,\epsilon}$ as $M \rightarrow \infty$

To prove tightness of the sequence $Z_t^{M,\epsilon}(da, dp)$ (as probability measure on $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(Age \times Space), vague))$), we use a Rolley criterium (see [24, 8, 9]) which establishes that it suffices to prove that for all f of a dense subspace of $(C_0(Age \times Space, \mathbb{R}), \|\cdot\|_\infty)$ (here $C_0^1(Age \times Space, \mathbb{R})$) the sequence $\langle f_t, Z_t^{M,\epsilon} \rangle$ is tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. In the section 5.3.2, we prove that

$$\langle f_t, Z_t^{M,\epsilon} \rangle = \underbrace{\mathcal{M}_t^{M,\epsilon}(f)}_{Martingale} + \underbrace{\mathcal{V}_t^{M,\epsilon}(f)}_{Finite\ Variation} .$$

Therefore, using a criterium of Aldous-Rebolledo [1, 9, 23], it suffices to prove that :

- for all $t \in \mathcal{T}$ (dense in \mathbb{R}_+), $\langle \mathcal{M}_t^{M,\epsilon}(f) \rangle$ and $\mathcal{V}_t^{M,\epsilon}(f)$ are tight on \mathbb{R}
- for all $T > 0$, $u > 0$, $\eta > 0$, there exists $\delta > 0$ and $N_0^{M,\epsilon} \in \mathbb{N}$ s.t.

$$\sup_{M \geq N_0^{M,\epsilon}} \mathbb{P} \left(|\langle \mathcal{M}_{T_M}^{M,\epsilon}(f) \rangle - \langle \mathcal{M}_{S_M}^{M,\epsilon}(f) \rangle| \geq \eta, \quad T_M < S_M + \delta \right) \leq u,$$

$$\sup_{M \geq N_0^{M,\epsilon}} \mathbb{P} \left(|\mathcal{V}_{T_M}^{M,\epsilon}(f) - \mathcal{V}_{S_M}^{M,\epsilon}(f)| \geq \eta, \quad T_{M,\epsilon} < S_M + \delta \right) \leq u,$$

for any sequences of stopping times (S_M, T_M) of the natural filtration \mathcal{F}_M , so that $S_M \leq T_M \leq T$. Both points are a direct consequence of the bounds (42)-(44) (see [26] for more details). Using Prohorov theorem, we can extract a subsequence $Z_t^{M_k,\epsilon}(da, dp)$ which vague converges to $Z_t^\epsilon(da, dp)$ and by construction

$$\sup_{t \in \mathbb{R}_+, f \in C^1(Age \times Space)} |\langle f_t, Z_t^{M_k,\epsilon} \rangle - \langle f_t, Z_t^{M_k,\epsilon} \rangle| \leq Cst \frac{\|f\|_{W^{1,\infty}}}{M_k},$$

the limit process is a.e. continuous. Finally, using a result of convergence [16], to prove the weak convergence, it suffices to add tightness of $\langle 1, Z_t^{M,\epsilon} \rangle$ (which came directly from bound (38)). Now using (42), we have that the martingale part of the process satisfies

$$\mathbb{E}(|\mathcal{M}_t^{M_k,\epsilon}(f)|)^2 \leq \mathbb{E}(|\mathcal{M}_t^{M_k,\epsilon}(f)|^2) = \mathbb{E}(\langle \mathcal{M}_t^{M_k,\epsilon}(f) \rangle) \leq Cst \frac{\|f\|_{W^{1,\infty}}}{M_k} \rightarrow_{k \rightarrow \infty} 0.$$

By passing to the limit in (13)-(14), we have that

$$P_{s-}^\epsilon \rightarrow_{M \rightarrow \infty} P_\infty^\epsilon(s-) = \frac{C}{\epsilon^2} \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(\cdot) R \left(\frac{\langle \Psi_{i,j,k}^\epsilon, Z_{s-}^\epsilon(da, dp) \rangle Vol_G}{Vol_{i,j,k}^\epsilon} \right), \text{ with } R(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}},$$

and

$$\Pi_{s-}^\epsilon(\cdot, dq) \rightarrow_{M \rightarrow \infty} \Pi_\infty = \frac{\Psi^\epsilon(|p' - p|) G_{s-}^\epsilon(p') dp'}{\iint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq}, \text{ with } G_{s-}^\epsilon(p') = 1 - \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(\cdot) R \left(\frac{\langle \Psi_{i,j,k}^\epsilon, Z_t^\epsilon(da, dp) \rangle Vol_G}{Vol_{i,j,k}^\epsilon} \right).$$

Therefore, we find that for all $f \in W^{1,\infty}$ (see lemma 5.2 and the section 5.3.2),

$$\begin{aligned} 0 &= \langle f_t, Z_t^{M,\epsilon} \rangle - \langle f_0, Z_0^{M,\epsilon} \rangle - \int_0^t \left\langle \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, a, p), Z_u^{M,\epsilon}(da, dp) \right\rangle du \\ &\quad - \int_0^t \left\langle (2f(s, 0, p) - f(s, a, p)) B(a, p) + \int_{Space} (f(s, p, p') - f(s, a, p)) P_\infty^\epsilon(s-)(p) \Pi_\infty(p, dp'), Z_s^{M,\epsilon} \right\rangle ds, \end{aligned}$$

is satisfied. The limit is a weak solution to the partial differential equation (12), which is unique (see the proof of II). \square

3.2 Proof of theorem 2.2 Part II : Convergence of ρ_t^ϵ as $\epsilon \rightarrow 0$

Using a fixed point theorem (contraction in Banach space), we prove (in lemma 5.8) the existence of solution to the master equation (12). To prove the convergence as $\epsilon \rightarrow 0$, we first prove, in lemmas 5.9-5.12, that under assumption (20), for any $T > 0$, $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p) da$ (resp. ρ^ϵ) belongs to a compact set of $L^2([0, T] \times Space)$ (resp. $L^2([0, T] \times Age \times Space)$) and $\nabla M_\epsilon^1(t, p)$ belongs to a weak-* compact set of $L^2([0, T] \times Space)$.

Assume (22) - (17), let $f \in C_c^1(Age \times Space)$, multiply (12) by f and integrate with respect to a : we find that

$$\begin{aligned} \frac{\partial}{\partial t} \int_p f(a, p) \rho^\epsilon(t, a, p) da dp - \iint B(a, p) f(a, p) \rho^\epsilon(t, a, p) da dp \\ = \iint \int_{p, p'} (f(a, p') - f(a, p)) \Pi_{s-}^\epsilon(p, dp') P_{s-}^\epsilon(p) \rho^\epsilon(t, a, p) da dp. \end{aligned}$$

Rewriting the second member as follows, we have that

$$\begin{aligned} \iint \int_{a, p, p'} (f(a, p') - f(a, p)) \Pi_{s-}^\epsilon(p, dp') P_{s-}^\epsilon(p) \rho^\epsilon(t, a, p) dp da \\ = \iint \int_{p, p'} C^\epsilon (f(a, p') - f(a, p)) \frac{\Psi^\epsilon(|p' - p|) G_{s-}^\epsilon(p') dp'}{\iint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq} \frac{P_{s-}^\epsilon(p)}{C^\epsilon} \rho^\epsilon(t, a, p) da dp = \\ = \iint \int_p {}^t \nabla f(a, p) \frac{C^\epsilon \iint \Psi^\epsilon(|q - p|) (q - p) G_{s-}^\epsilon(q) dq}{\iint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq} \frac{P_{s-}^\epsilon(p)}{C^\epsilon} \rho^\epsilon(t, a, p) da dp + o(1) \\ = \iint \int_p {}^t \nabla f(a, p) \frac{C^\epsilon \iint \Psi^\epsilon(|q - p|) (q - p) {}^t(q - p) dq}{\iint \Psi^\epsilon(|q - p|) dq} \frac{P_{s-}^\epsilon(p)}{C^\epsilon} \rho^\epsilon(t, a, p) \frac{\nabla G_{s-}^\epsilon(p)}{G_{s-}^\epsilon(p)} da dp + o(1) \\ = \iint \int_p \nabla f(a, p) \rho^\epsilon(t, a, p) Ch(p) R^t \nabla (\log(1 - R)) dp da + o(1). \end{aligned}$$

By passing to the limit $\nabla f(a, p) \rho^\epsilon(t, a, p) Ch(p)$ in L^2 and $\nabla (\log(1 - R))$ in weak-* L^2 (see lemmas 5.10-5.11 and [2]), we have that ρ is a solution to

$$\frac{\partial}{\partial t} \rho(t, a, p) + \frac{\partial}{\partial a} \rho(t, a, p) + \text{div}(\rho(t, a, p) Ch(p) R \nabla (\log(1 - R))) = 0, \quad \rho(t, 0, p) = \int B(a, p) \rho(t, a, p) da,$$

with $R = R(\int_{\mathbb{R}_+} \rho(t, a, p) da \frac{\text{Vol}_G(3r^2 + 6r + 3)}{3r^2 + 6rr_O(s) + 3r_O(s)^2})$. \square

3.3 Proof of theorem 2.2 Part III : Convergence of $Z_t^{M, \epsilon}$ as $M \rightarrow \infty$ and $\epsilon \rightarrow 0$

Let $T > 0$. Changing the time scale $t \mapsto t\epsilon$. We first notice that for $(M, \epsilon) \in U_1$, bounds given in the proof of theorem 2.2 Part I are independent of ϵ, M . Indeed, the time scale appeared in lemma 5.5 that gives

$$\sup_{M, \epsilon} \mathbb{E} \left(\sup_{t \in [0, T\epsilon]} \iint \iint r^m Z_t^{M, \epsilon}(da, dp) \right) < \infty, \quad \forall T > 0.$$

Then, changing the time scale, we find that all bounds given in Proof of theorem 2.2 Part I and in technical part 5.3.2 are in $O(T/\epsilon M)$ therefore uniform in U_1 (and so the tightness of $Z_t^{M,\epsilon}$ is assured in U_1). To obtain the convergence we need that $\epsilon M \rightarrow \infty$. Now, we construct a (ϵ_k, M_k) so that the limit is the one we are expecting (i.e., ρ).

Since C_0^1 is a separable set, there exists $(\psi_k)_k$ dense subset of C_0^1 . Let any sequence $(\epsilon_k)_k$ converging to 0. Using theorem 2.2, there exists M^1 s.t.

$$\sup_{t \leq T} |\langle Z_t^{M^1, \epsilon_1}(da, dp), \psi_1 \rangle - \langle \rho_{\epsilon_1}(t, a, p)dadp, \psi_1 \rangle| \leq 1.$$

For the same reason, we can find M^2 s.t.

$$\sup_{t \leq T} |\langle Z_t^{M^2, \epsilon_2}(da, dp), \psi_i \rangle - \langle \rho_{\epsilon_2}(t, a, p)dadp, \psi_j \rangle| \leq 1/4, \quad j = 1, 2.$$

and so on, there exists M^n s.t.

$$\sup_{t \leq T} |\langle Z_t^{M^n, \epsilon_n}(da, dp), \psi_j \rangle - \langle \rho_{\epsilon_n}(t, a, p)dadp, \psi_j \rangle| \leq 1/2^n, \quad j = 1..n.$$

Therefore, using the theorem 2.2 part I, we have that

$$\lim_{n \rightarrow \infty} \sup_{t \leq T} |\langle Z_t^{M^n, \epsilon_n}(da, dp), \psi \rangle - \langle \rho(t, a, p)dadp, \psi \rangle| = 0, \quad \forall \psi \in C_0^1.$$

□

4 Conclusion

In this paper, we prove the convergence of a stochastic process which represent the evolution of a cell population as its population size goes to infinity and its cell size converges to zero. A way to understand the result is to imagine that we observe the evolution of the cell population at a certain distance (depending of the number of cells). If the distance is fixed (i.e. a fixed cell size), then when the number of cells goes to infinity, the whole space is filled by cells (we are too close) and we can only observe $Z_t^{M,\epsilon} = Cst$ everywhere. If the distance is too far with respect to the size population, i.e. cells are too small, we observe a concentrated mass on the boundary of the oocyte (we are too far). To observe the cell population evolution we have to be, neither too close or too far, which explains the balance between ϵ (cell size, or observation distance to the follicle) and M (cell population size). The proof is robust to the change of the birth rate B and cell displacement rate (function R), as long as there are smooth and bounded. The next step is to study the dynamics of the final equation (15) and make the link with the partial differential equation of the follicle evolution given in [19, 6, 7].

5 Annex

5.1 Regular approximation of $\chi_{\mathcal{L}_{i,j,k}^\epsilon}$

We introduce a regular approximation of the characteristic function $\chi_{[0,\epsilon] \times [0,\pi/N] \times [0,2\pi/N]}$ (see fig. 4). Let $\eta \in]0, 1[$ and

$$\Psi^{\epsilon,\eta} : (r, \theta, \phi) \in \mathbb{R}^3 \mapsto \Psi_r^{\epsilon,\eta}(r) \Psi_\theta^{\epsilon,\eta}(\theta) \Psi_\phi^{\epsilon,\eta}(\phi), \quad (24)$$

a C^∞ positive function such that $\Psi^{\epsilon,\eta} \leq 1$, such that

$$(\Psi_r^\epsilon)|_{[\eta\epsilon, \epsilon(1-\eta)]} = 1, \quad \Psi_r^\epsilon|_{[-\eta\epsilon, \epsilon(1+\eta)]^c} = 0, \quad (25)$$

$$(\Psi_\theta^\epsilon)|_{[\frac{\eta}{N}, \frac{\pi-\eta}{N}]} = 1, \quad (\Psi_\theta^\epsilon)|_{[\frac{\pi+\eta}{N}, \pi-\frac{\eta}{N}]} = 1, \quad \Psi_\theta^\epsilon(\theta + \pi) = \Psi_\theta^\epsilon(\theta), \quad \forall \theta \quad (26)$$

$$(\Psi_\phi^\epsilon)|_{[-\pi+\frac{\eta}{N}, -\pi+\frac{2\pi-\eta}{N}]} = 1, \quad (\Psi_\phi^{\epsilon,\eta})|_{[-\pi+\frac{2\pi+\eta}{N}, \pi-\frac{\eta}{N}]} = 1, \quad \Psi_\phi^\epsilon(\phi + 2\pi) = \Psi_\phi^\epsilon(\phi), \quad \forall \phi \quad (27)$$

and satisfying the unity partition formulae

$$\sum_{i,j,k} \Psi_{i,j,k}^\epsilon(r, \theta, \phi) = 1, \quad \forall (r, \theta, \phi) \in [1 + \eta\epsilon, \infty[\times [0, \pi] \times [-\pi, \pi].$$

with $\Psi_{i,j,k}^\epsilon$ a regular approximation of the characteristic function $\chi_{\mathcal{L}_{i,j,k}^\epsilon}$:

$$\Psi_{i,j,k}^\epsilon(r, \theta, \phi) = \Psi^\epsilon\left((r-1) - i\epsilon, \theta - \frac{j}{N}, \phi - \frac{k}{N}\right). \quad (28)$$

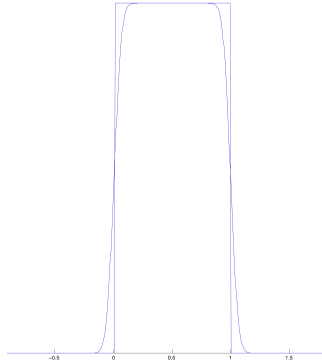


Figure 4: Regularization of characteristic function $\chi_{[0,1]}$. Using the convolution and Gaussian function it is easy to compute a regular approximation of $\chi_{[0,1]}$ which satisfies (25)-(27).

5.2 Proof of proposition 2.1 : Existence of the stochastic process $Z_t^{M,\epsilon}$

This process can be construct step by step (see [3, 15, 4, 25, 17]), the only point is to prove global bounds

$$0 < E\left(\frac{N_0^{M,\epsilon}}{M}\right) \leq E\left(\sup_{s \leq t} N_s^{M,\epsilon}\right) \leq E\left(\frac{N_0^{M,\epsilon}}{M}\right) e^{t/M} < \infty, \quad (29)$$

$$0 < r_O(0) \leq E(\sup_{s \leq t} r_O(s)) \leq r_O(0) e^{\frac{C}{2} E(\frac{N_0^{M,\epsilon}}{M}) e^{t/M} t} < \infty, \quad (30)$$

where $N_s^{M,\epsilon} = \langle Z_s^{M,\epsilon}, 1 \rangle$. Using (34), in the particular case $f = 1$, we have that (using (9), we have $B_{n,s-}^{M,\epsilon} \leq 1 < \infty$)

$$N_t^{M,\epsilon} = \iint_{\mathbb{R}_+ \times \text{Space}} 1 Z_t^{M,\epsilon}(da, dp) = \frac{1}{M} N_0^{M,\epsilon} + \frac{1}{M} \int_0^t \int_{\epsilon} 1_{n < N_{s-}^{M,\epsilon}} \left[11_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')).$$

Therefore, for $t \leq \tau_N^{M,\epsilon}$ (the stopping time $\tau_N^{M,\epsilon}$ is defined in (33)) and $N_t^{\bar{M},\epsilon} = E(\sup_{s \leq \min(t, \tau_N^{M,\epsilon})} N_s^{M,\epsilon})$, we have that

$$E\left(\frac{N_0^{M,\epsilon}}{M}\right) \leq N_t^{\bar{M},\epsilon} \leq E\left(\frac{N_0^{M,\epsilon}}{M}\right) + \int_0^t \frac{1}{M} N_s^{\bar{M},\epsilon} ds,$$

which implies (using Gronwall inequality on the right side) that

$$E\left(\frac{N_0^{M,\epsilon}}{M}\right) \leq \bar{N}_t \leq E\left(\frac{N_0^{M,\epsilon}}{M}\right) e^{\frac{t}{\bar{M}}}$$

and so, we find that

$$P\left(\inf_{M \geq N} \tau_N^{M,\epsilon} \geq t\right) = P\left(\tau_N^{M,\epsilon} \geq t\right) = P\left(\sup_{s \leq \min(t, \tau_N^{M,\epsilon})} N_s^{M,\epsilon} \geq N\right) \leq E\left(\frac{N_0^{M,\epsilon}}{M}\right) e^{\frac{t}{\bar{M}}} / N, \quad P\left(\lim_{N \rightarrow \infty} \tau_N^{M,\epsilon} = \infty\right) = 1.$$

Using Fatou lemma, we find that

$$E\left(\lim_{N \rightarrow \infty} \sup_{s \leq \min(t, \tau_N^{M,\epsilon})} N_s^{M,\epsilon}\right) = E\left(\lim_{N \rightarrow \infty} \inf_{s \leq \min(t, \tau_N^{M,\epsilon})} \sup N_s^{M,\epsilon}\right) \leq \lim_{N \rightarrow \infty} \inf N_t^{\bar{M},\epsilon} \leq E\left(\frac{N_0^{M,\epsilon}}{M}\right) e^{\frac{t}{\bar{M}}},$$

and finally $E(\sup_{s \leq t} N_s^{M,\epsilon}) \leq E\left(\frac{N_0^{M,\epsilon}}{M}\right) e^{\frac{t}{\bar{M}}}$. Similarly, we have (30). Let $(T_k)_k$ the sequences of successive jumps of the $Z_s^{M,\epsilon}$ process, then, $\lim_k T_k = \infty$, *a.e.* Indeed, let $U_M \subset \Omega$, such that $\lim_k T_k(\omega) < M$ for $\omega \in U_M$, then necessarily $\lim_{k \rightarrow \infty} N_{T_k}^{M,\epsilon}(\omega) < \infty$ (otherwise $\tau_N^{M,\epsilon} \leq M/2$ for N large enough, which is with null probability) and so, for all $\omega \in U_M$, we can construct the time sequence $(T_k(\omega))_k$ as a subsequence of a Poisson point process of intensity $(1 + C/\epsilon^2)N_\infty$ (using (9) and (10) we have that $1 + C/\epsilon^2 < \infty$) where $N_\infty = \lim_{k \rightarrow \infty} N_{T_k}(\omega)$ which is *a.e.* unbounded. Moreover, we have for all $\epsilon > 0$, F and f belongs to $W^{1,\infty}$, the infinitesimal generator of the Markovian process $(Z_t^{M,\epsilon})_{t \geq 0}$ issued from $Z_0^{M,\epsilon}$ is given by

$$\begin{aligned} \mathcal{L}F_f(Z_0^{M,\epsilon}) &= \frac{\partial}{\partial t} \mathbb{E}(F(\langle f, Z_t^{M,\epsilon} \rangle))|_{t=0} = F'(\langle f, Z_0^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} \frac{\partial}{\partial a} f(a, p) Z_0^{M,\epsilon}(da, dp) \\ &+ \left(\iint_{\mathbb{R}_+ \times \text{Space}} \left[(F(\langle f, Z_0^{M,\epsilon} \rangle) + \frac{2}{M} f(0, p) - \frac{1}{M} f(a, p)) - F(\langle f, Z_0^{M,\epsilon} \rangle) B(0, a) \right. \right. \\ &\quad \left. \left. + (F(\langle f, Z_0^{M,\epsilon} \rangle) + \frac{1}{M} f(a, p') - \frac{1}{M} f(a, p)) - F(\langle f, Z_0^{M,\epsilon} \rangle) P_0^\epsilon \Pi_0^\epsilon(p, dp') \right] Z_0^{M,\epsilon}(da, dp) \right). \quad (31) \end{aligned}$$

The infinitesimal generator of the Markovian process $(r_O^{M,\epsilon}(t))_{t \geq 0}$ issued from r_0 is given by

$$\mathcal{L}F(r_O^{M,\epsilon}(0)) = \frac{\partial}{\partial t} \mathbb{E}(F(r_O^{M,\epsilon}(t)))|_{t=0} = F'(r_0) \sum_{i,j,k} \kappa((i-1)\epsilon) r_0^\alpha \langle \Psi_{i,j,k}^\epsilon, Z_0^{M,\epsilon} \rangle. \quad (32)$$

Indeed, let $t \leq \tau_N^{M,\epsilon}$ with $\tau_N^{M,\epsilon} = \inf\{s \geq 0, N_s^{M,\epsilon} \geq N\}$, then using lemma 5.2, we have that

$$\begin{aligned} & \mathbb{E}\left(\int_0^{\min(t, \tau_N^{M,\epsilon})} F'(\langle f, Z_u^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} \frac{\partial}{\partial a} f(u, a, p) Z_u^{M,\epsilon}(da, dp) du\right) = \\ & \quad \mathbb{E}\left(F\left(\iint_{\mathbb{R}_+ \times \text{Space}} f(a, p) Z_t^{M,\epsilon}(da, dp)\right)\right) - \mathbb{E}\left(F\left(\iint_{\mathbb{R}_+ \times \text{Space}} f(a, p) Z_0(da, dp)\right)\right) \\ & - \left[\mathbb{E}\left(\int_0^{\min(t, \tau_N^{M,\epsilon})} \int_\epsilon^{1_{n < N_{s-}^{M,\epsilon}}} \left[(F(\langle f, Z_s \rangle) + \frac{2}{M} f(0, X_n^{M,\epsilon}(s-)) - \frac{1}{M} f(A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) - F(\langle f_s, Z_s^{M,\epsilon} \rangle) \right. \right. \right. \\ & \quad \left. \left. \left. 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} + (F(\langle f, Z_s \rangle) + \frac{1}{M} f(A_n^{M,\epsilon}(s-), p') - \frac{1}{M} f(A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) - F(\langle f, Z_s \rangle) \right) \right. \right. \\ & \quad \left. \left. \left. 1_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^\epsilon(X_n(s-))} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')) \right) \right], \end{aligned}$$

which can be rewritten (for simplicity) as

$$\mathbb{E}\left(F\left(\iint_{\mathbb{R}_+ \times \text{Space}} f(a, p) Z_{\min(t, \tau_N^{M,\epsilon})}(da, dp)\right)\right) = \mathbb{E}\left(F\left(\iint_{\mathbb{R}_+ \times \text{Space}} f(a, p) Z_0(da, dp)\right)\right) + \mathbb{E}\left(\Psi(\min(t, \tau_N^{M,\epsilon}), Z)\right),$$

with

$$\begin{aligned} \Psi(t, Z) &= \int_0^t F'(\langle f, Z_u^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} \frac{\partial}{\partial a} f(a, p) Z_u^{M,\epsilon}(da, dp) du \\ &+ \left(\int_0^t \iint_{\mathbb{R}_+ \times \text{Space}} 1_{n < N_{s-}^{M,\epsilon}} \left[(F(\langle f, Z_s \rangle) + \frac{2}{M} f(s, 0, p) - \frac{1}{M} f(s, a, p)) - F(\langle f, Z_s \rangle) B(p, a) \right. \right. \\ & \quad \left. \left. + (F(\langle f, Z_s \rangle) + \frac{1}{M} f(s, a, p') - \frac{1}{M} f(s, a, p)) - F(\langle f, Z_s \rangle) P_{s-}^\epsilon(p) \Pi_{s-}^\epsilon(p, dp') \right] Z_s^{M,\epsilon}(da, dp), \right) \end{aligned}$$

with $B(p, a) = 1 - e^{-a/\lambda(\|p\|_2)}$. We have the following bound

$$|\Psi(\min(t, \tau_N^{M,\epsilon}), Z)| \leq TN \|F\|_{W^{1,\infty}} (1 + \|f\|_{W^{1,\infty}} (1 + \sup_p \int |p' - p| P^\epsilon(p) \Pi_{s-}(p, dp'))).$$

and since $Z_t^{M,\epsilon}$ is càd and $\tau_N^{M,\epsilon} > 0$, we have

$$\begin{aligned} \frac{\partial}{\partial t} \Psi(\min(t, \tau_N^{M,\epsilon}), Z)|_{t=0} &= F'(\langle f_0, Z_0^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} \frac{\partial}{\partial a} f(a, p) Z_0^{M,\epsilon}(da, dp) \\ &+ \left(\iint_{\mathbb{R}_+ \times \text{Space}} \left[(F(\langle f, Z_0 \rangle) + \frac{2}{M} f(0, 0, p) - \frac{1}{M} f(0, a, p)) - F(\langle f, Z_0^{M,\epsilon} \rangle) B(0, a) \right. \right. \\ & \quad \left. \left. + (F(\langle f, Z_0 \rangle) + \frac{1}{M} f(0, a, p') - \frac{1}{M} f(0, a, p)) - F(\langle f, Z_0^{M,\epsilon} \rangle) P_0^\epsilon(p) \Pi_0^\epsilon(p, dp') \right] Z_0^{M,\epsilon}(da, dp), \right) \end{aligned}$$

which is dominated by $T \frac{N_0^{M,\epsilon}}{M} \|F\|_{W^{1,\infty}} (1 + \|f\|_{W^{1,\infty}} (1 + C/\epsilon)) < \infty$. Therefore by derivation under domination, we find (31). \square

5.3 Technical lemmes : Stochastic

In this part, we give the doob decomposition of the stochastic process $Z_s^{M,\epsilon}$ (issued from Ito calculus). Then, we give uniform bounds which are used to prove the tightness of $Z_s^{M,\epsilon}$.

5.3.1 Ito calculus and first lemmas

For convenience, we introduced stopping time $(\tau_N^{M,\epsilon})$

$$\tau_N^{M,\epsilon} = \inf\{s \geq 0, \quad \langle 1, Z_s^{M,\epsilon} \wedge r_O^{M,\epsilon}(s) \rangle \geq N\}. \quad (33)$$

We first give computational lemmas on the population evolution.

Lemma 5.1 *For all $f \in C^0(\text{Space} \times \text{Age})$, $t > 0$ and $r_O, Z_t^{M,\epsilon}$ solution to (7)-(8) we have*

$$\begin{aligned} \iint_{\mathbb{R}_+ \times \text{Space}} f(t, a, p) Z_t^{M,\epsilon}(da, dp) &= \frac{1}{M} \sum_{k=1}^{N_0^{M,\epsilon}} f(t, A_n^{M,\epsilon}(0) + t, X_n^{M,\epsilon}(0)) \\ &+ \frac{1}{M} \int_0^t \int_{\epsilon} 1_{n < N_{s-}^{M,\epsilon}} \left[(2f(t, t-s, X_n^{M,\epsilon}(s-)) - f(t, A_n^{M,\epsilon}(s-) + t-s, X_n^{M,\epsilon}(s-))) 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right. \\ &\quad \left. + (f(t, A_n^{M,\epsilon}(s-) + t-s, p') - f(t, A_n^{M,\epsilon}(s-) + t-s, X_n^{M,\epsilon}(s-))) 1_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^{\epsilon}(X_n(s-))} \right] \\ &\quad Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')) \quad (34) \end{aligned}$$

Proof Direct computation (see [4] for more details).

Lemma 5.2 *Let $F \in C^1(\mathbb{R}, \mathbb{R})$ and $f \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+ \times \text{Space}, \mathbb{R})$ s.t. $\forall p \in \text{Space}, \quad (s, a) \in \mathbb{R}_+^2 \mapsto f(s, a, p) \in C_b^{1,1}(\mathbb{R}_+ \times \text{Space}, \mathbb{R})$ with uniform (in space) bounds of the partial derivation of f then, for all $t > 0$,*

$$\begin{aligned} \int_0^t F'(\langle f_u, Z_u^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, a, p) Z_u^{M,\epsilon}(da, dp) du &= F(\langle f_t, Z_t^{M,\epsilon} \rangle) - F(\langle f_0, Z_0^{M,\epsilon} \rangle) \\ - \left[\int_0^t \int_{\epsilon} 1_{n < N_{s-}^{M,\epsilon}} \left[(F(\langle f_s, Z_s \rangle) + \frac{2}{M} f(s, 0, X_n^{M,\epsilon}(s-)) - \frac{1}{M} f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) - F(\langle f_s, Z_s^{M,\epsilon} \rangle) 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right. \right. \\ &\quad \left. \left. + (F(\langle f_s, Z_s \rangle) + \frac{1}{M} f(s, A_n^{M,\epsilon}(s-), p') - \frac{1}{M} f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) - F(\langle f_s, Z_s \rangle) \right. \right. \\ &\quad \left. \left. 1_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^{\epsilon}(X_n(s-))} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')) \right] \quad (35) \end{aligned}$$

$$\text{with } \langle f_s, Z_s^{M,\epsilon} \rangle = \iint_{\mathbb{R}_+ \times \text{Space}} f(s, a, p) Z_s^{M,\epsilon}(da, dp).$$

Proof Using (34) of lemma 5.1, in the particular case $(\frac{\partial}{\partial u} + \frac{\partial}{\partial a})f(u, a, p)$, and integrating in time, we have that $Z_t^{M,\epsilon}$ satisfies (for all f given in assumptions)

$$\begin{aligned} \int_0^t \iint_{\mathbb{R}_+ \times \text{Space}} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, a, p) Z_u^{M,\epsilon}(da, dp) du &= \frac{1}{M} \int_0^t \sum_{k=1}^{N_0^{M,\epsilon}} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, A_n^{M,\epsilon}(0) + u, X_n^{M,\epsilon}(0)) du \\ &\quad + \frac{1}{M} \int_0^t \int_0^u \int_{\epsilon} 1_{n < N_{s-}^{M,\epsilon}} \left[\left(2 \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, u-s, X_n^{M,\epsilon}(s-)) \right. \right. \\ &\quad \left. \left. - \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, A_n^{M,\epsilon}(s-) + u-s, X_n^{M,\epsilon}(s-)) \right) 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right. \\ &\quad \left. + \left(\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, A_n^{M,\epsilon}(s-) + u-s, p') - \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, A_n^{M,\epsilon}(s-) + u-s, X_n^{M,\epsilon}(s-)) \right) \right. \\ &\quad \left. 1_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^{\epsilon}(X_n(s-))} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')) du. \quad (36) \end{aligned}$$

Now, using Fubini theorem $\int_{u=0}^t \int_{s=0}^u = \int_{s=0}^t \int_{u=s}^t$ and $((\frac{\partial}{\partial u} + \frac{\partial}{\partial a})f)(u, \cdot + u, \cdot) = \frac{d}{du}(f(u, \cdot + u, \cdot))$, we have that

$$\begin{aligned} \int_0^t \iint_{\mathbb{R}_+ \times \text{Space}} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, a, p) Z_u^{M,\epsilon}(da, dp) du &= \frac{1}{M} \sum_{k=1}^{N_0^{M,\epsilon}} [f(t, A_n^{M,\epsilon}(0)+t, X_n^{M,\epsilon}(0)) - f(0, A_n^{M,\epsilon}(0), X_n^{M,\epsilon}(0))] \\ &\quad + \frac{1}{M} \int_0^t \int_\epsilon^t \mathbf{1}_{n < N_{s-}^{M,\epsilon}} \left[(2f(t, t-s, X_n^{M,\epsilon}(s-)) - f(s, 0, X_n^{M,\epsilon}(s-))) \right. \\ &\quad \left. - (f(t, A_n^{M,\epsilon}(s-) + t-s, X_n^{M,\epsilon}(s-))) - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-)) \right] \mathbf{1}_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \\ &\quad + (f(t, A_n^{M,\epsilon}(s-) + t-s, p') - f(s, A_n^{M,\epsilon}(s-), J)) - (f(t, A_n^{M,\epsilon}(s-) + t-s, X_n^{M,\epsilon}(s-)) - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) \\ &\quad \left. \mathbf{1}_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^\epsilon(X_n(s-))} \right] du Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')). \end{aligned}$$

Using formula (34), we find finally that

$$\begin{aligned} \int_0^t \iint_{\mathbb{R}_+ \times \text{Space}} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, a, p) Z_u^{M,\epsilon}(da, dp) du &= \langle f_t, Z_t^{M,\epsilon} \rangle - \langle f_0, Z_0^{M,\epsilon} \rangle \\ &\quad - \frac{1}{M} \left[\int_0^t \int_\epsilon^t \mathbf{1}_{n < N_{s-}^{M,\epsilon}} \left((2f(s, 0, X_n^{M,\epsilon}(s-)) - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) \mathbf{1}_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right. \right. \\ &\quad \left. \left. + (f(s, A_n^{M,\epsilon}(s-), p') - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) \mathbf{1}_{0 \leq \Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^\epsilon(X_n(s-))} \right) \right. \\ &\quad \left. Q(ds, (dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')) \right]. \end{aligned}$$

Now, using the Itô formula with jump processes, we find that (35) is satisfied (see [4, 25, 17, 15, 3] for more details). \square

5.3.2 Doob decomposition of $Z_t^{M,\epsilon}$

Let $m \geq 1$ and assume that

$$\sup_{M,\epsilon} \mathbb{E} \left(\langle 1 + a^{2m} + r^{2m}, Z_0^{M,\epsilon}(da, dp) \rangle + \langle 1 + a^m + r^m, Z_0^{M,\epsilon}(da, dp) \rangle^2 \right) < \infty, \quad (37)$$

then we have that for all $T > 0$ (see lemmas 5.3-5.7),

$$\sup_{M,\epsilon} \mathbb{E} \left(\sup_{t \in [0, T\epsilon]} \langle 1 + a^{2m} + r^{2m}, Z_t^{M,\epsilon}(da, dp) \rangle + \sup_{t \in [0, T\epsilon]} \langle 1 + a^m + r^m, Z_t^{M,\epsilon}(da, dp) \rangle^2 \right) < \infty. \quad (38)$$

Moreover, for all f ,

$$\langle f_t, Z_t^{M,\epsilon} \rangle = \mathcal{M}_t^{M,\epsilon}(f) + \mathcal{V}_t^{M,\epsilon}(f),$$

where

$$\begin{aligned} \mathcal{M}_t^{M,\epsilon}(f) &= \langle f_t, Z_t^{M,\epsilon} \rangle - \langle f_0, Z_0^{M,\epsilon} \rangle - \int_0^t \left\langle \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial a} \right) f(u, a, p), Z_u^{M,\epsilon}(da, dp) \right\rangle du \\ &\quad - \int_0^t \langle (2f(s, 0, p) - f(s, a, p)) B(a, p) + \int_{\text{Space}} (f(s, p, p') - f(s, a, p)) P_{s-}^\epsilon(p) \Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds, \quad (39) \end{aligned}$$

is an L^2 martingale càdlàg nul at $t = 0$ of quadratic previsible increasing process

$$\begin{aligned} \langle \mathcal{M}_t^{M,\epsilon}(f) \rangle &= \\ &= \frac{1}{M} \int_0^t \langle (2f(s, 0, p) - f(s, a, p))^2 B(a, p) + \int_{\text{Space}} (f(s, a, p') - f(s, a, p))^2 P_{s-}^\epsilon \Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds, \quad (40) \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}_t^{M,\epsilon}(f) &= \langle f_0, Z_0^{M,\epsilon} \rangle + \int_0^t \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a})f(u, a, p), Z_u^{M,\epsilon}(da, dp) \rangle du \\ &\quad + \int_0^t \langle (2f(s, 0, p) - f(s, a, p))B(a, p) + \int_{Space} (f(s, p, p') - f(s, a, p))P_{s-}^\epsilon(p)\Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds, \end{aligned} \quad (41)$$

which satisfy in average (in $d\mathcal{P}(\omega)$) an uniform $W_{loc}^{1,\infty}(\mathbb{R}_+)$ bound. More precisely, for all stopping time T, S

$$\mathbb{E}\left(\sup_{t \leq T} |\langle \mathcal{M}_t^{M,\epsilon}(f) \rangle|\right) \leq \left(9\|f\|_{L^\infty}^2 + \|f\|_{W^{1,\infty}}^2(1+C)\right)\mathbb{E}\left(\frac{|T|}{M} \sup_{s \leq T} \langle 1, Z_s^{M,\epsilon} \rangle\right), \quad (42)$$

$$\mathbb{E}\left(|\langle \mathcal{M}_T^{M,\epsilon}(f) \rangle - \langle \mathcal{M}_S^{M,\epsilon}(f) \rangle|\right) \leq \left(9\|f\|_{L^\infty}^2 + \|f\|_{W^{1,\infty}}^2(1+C)\right)\mathbb{E}\left(\frac{|T-S|}{M} \sup_{s \leq \max(T,S)} \langle 1, Z_s^{M,\epsilon} \rangle\right), \quad (43)$$

$$\mathbb{E}\left(\sup_{t \leq T} |\langle \mathcal{V}_t^{M,\epsilon}(f) \rangle|\right) \leq \mathbb{E}\left(|\langle f_0, Z_0^{M,\epsilon} \rangle|\right) + \left(3\|f\|_{L^\infty} + \|f\|_{W^{1,\infty}}(1+C)\right)(1 + \mathbb{E}\left(|T| \sup_{s \leq T} \langle 1, Z_s^{M,\epsilon} \rangle\right)), \quad (44)$$

and

$$\mathbb{E}\left(|\langle \mathcal{V}_T^{M,\epsilon}(f) \rangle - \langle \mathcal{V}_S^{M,\epsilon}(f) \rangle|\right) \leq \left(3\|f\|_{L^\infty} + \|f\|_{W^{1,\infty}}(1+C)\right)\mathbb{E}\left(\frac{|T-S|}{M} (1 + \sup_{s \leq \max(T,S)} \langle 1, Z_s^{M,\epsilon} \rangle)\right). \quad (45)$$

Proof To prove bound (38), we use technical lemmas 5.3-5.7 (proves are similar to [26, 17]). Now, for all $f \in C^0(Space \times Age)$, $t > 0$ and r_O , $Z_t^{M,\epsilon}$ solution to (7)-(8) we have (see lemma 5.2)

$$\begin{aligned} 0 &= \langle f_t, Z_t^{M,\epsilon} \rangle - \langle f_0, Z_0^{M,\epsilon} \rangle - \int_0^t \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a})f(u, a, p), Z_u^{M,\epsilon}(da, dp) \rangle du \\ &\quad - \frac{1}{M} \left[\int_0^t \int_\epsilon 1_{n < N_{s-}^{M,\epsilon}} \left[(2f(s, 0, X_n^{M,\epsilon}(s-)) - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) 1_{0 \leq \Theta < B(A_n^M(s-), X_n^M(s-))} \right. \right. \\ &\quad \quad \left. \left. + (f(s, A_n^{M,\epsilon}(s-), p') - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) \right. \right. \\ &\quad \quad \left. \left. 1_{B(A_n^M(s-), X_n^M(s-)) \leq \Theta < B(A_n^M(s-), X_n^M(s-)) + P_{s-}^\epsilon(X_n^M(s-))} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \right] \end{aligned} \quad (46)$$

Using (34), we have that

$$\begin{aligned} \mathcal{M}_t^{M,\epsilon}(f) &= -\frac{1}{M} \int_0^t \int_\epsilon 1_{n < N_{s-}^{M,\epsilon}} \left[(2f(s, 0, X_n^{M,\epsilon}(s-)) - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) 1_{0 \leq \Theta < B(A_n^M(s-), X_n^M(s-))} \right. \\ &\quad \left. + (f(s, A_n^{M,\epsilon}(s-), p') - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) \right. \\ &\quad \left. 1_{B(A_n^M(s-), X_n^M(s-)) \leq \Theta < B(A_n^M(s-), X_n^M(s-)) + P_{s-}^\epsilon(X_n^M(s-))} \right] \tilde{Q}(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \end{aligned} \quad (47)$$

with $\tilde{Q} = Q - ds \otimes \text{Compt}(dn) \otimes d\Theta \otimes \Pi_{s-}^{M,\epsilon}(p, dp')$ be the compensated Poisson process of Q introduced in the introduction. Therefore $\mathcal{M}_t^{M,\epsilon}(f)$ is a local martingale associated to the stopping times sequence $(\tau_N^{M,\epsilon})_N$ introduced in (33). Using Ito formula, we have that

$$\begin{aligned} \mathcal{M}_t^{M,\epsilon}(f)^2 &= - \int_0^t \int_\epsilon 1_{n < N_{s-}^{M,\epsilon}} \\ &\quad \left[(\langle f_s, Z_s^{M,\epsilon} \rangle + \frac{2}{M} f(s, 0, X_n^{M,\epsilon}(s-)) - \frac{1}{M} f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-)))^2 - \langle f_s, Z_s^{M,\epsilon} \rangle^2 \right] 1_{0 \leq \Theta < B(A_n^M(s-), X_n^M(s-))} \\ &\quad + (\langle f_s, Z_s^{M,\epsilon} \rangle + \frac{1}{M} f(s, A_n^{M,\epsilon}(s-), p') - \frac{1}{M} f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-)))^2 - \langle f_s, Z_s^{M,\epsilon} \rangle^2 \\ &\quad \left. 1_{B(A_n^M(s-), X_n^M(s-)) \leq \Theta < B(A_n^M(s-), X_n^M(s-)) + P_{s-}^\epsilon(X_n^M(s-))} \right] \tilde{Q}(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \end{aligned} \quad (48)$$

Using lemma 5.2, we find that

$$\begin{aligned} & \langle f_t, Z_t^{M,\epsilon} \rangle^2 - \langle f_0, Z_0^{M,\epsilon} \rangle^2 - 2 \int_0^t \langle f_u, Z_u^{M,\epsilon} \rangle \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a}) f(u, a, p), Z_u^{M,\epsilon}(da, dp) \rangle du \\ & - 2 \int_0^t \langle f_s, Z_s^{M,\epsilon} \rangle \langle (2f(s, 0, p) - f(s, a, p)) B(a, p) + \int_{Space} (f(s, a, p') - f(s, a, p)) P_{s-}^\epsilon(p) \Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds \\ & - \frac{1}{M} \int_0^t \langle (2f(s, 0, p) - f(s, a, p))^2 B(a, p) + \int_{Space} (f(s, a, p') - f(s, a, p))^2 P_{s-}^\epsilon(p) \Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds = -\mathcal{M}_t^{M,\epsilon}(f)^2, \end{aligned}$$

which means that

$$\begin{aligned} \mathcal{M}_t^{M,\epsilon}(f)^2 &= Local_{Martingale} \\ &+ \frac{1}{M} \int_0^t \langle (2f(s, 0, p) - f(s, a, p))^2 B(a, p) + \int_{Space} (f(s, a, p') - f(s, a, p))^2 P_{s-}^\epsilon(p) \Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds \end{aligned}$$

where

$$\begin{aligned} Local_{Martingale} &= -\langle f_t, Z_t^{M,\epsilon} \rangle^2 + \langle f_0, Z_0^{M,\epsilon} \rangle^2 + 2 \int_0^t \langle f_u, Z_u^{M,\epsilon} \rangle \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a}) f(u, a, p), Z_u^{M,\epsilon}(da, dp) \rangle du \\ &+ 2 \int_0^t \langle f_s, Z_s^{M,\epsilon} \rangle \langle (2f(s, 0, p) - f(s, a, p)) B(a, p) + \int_{Space} (f(s, a, p') - f(s, a, p)) P_{s-}^\epsilon(p) \Pi_{s-}^\epsilon(p, dp'), Z_s^{M,\epsilon} \rangle ds. \end{aligned}$$

Using uniqueness of Doob decomposition we find that (40) is satisfied and

$$\mathbb{E}(\langle \mathcal{M}_t^{M,\epsilon}(f) \rangle) \leq \frac{T}{M} 3(1+C) \|f\|_{L^\infty}^2 \mathbb{E} \left(\sup_{t \in [0, T]} \langle 1, Z_t^{M,\epsilon}(da, dp) \rangle + \sup_{t \in [0, T]} \langle 1, Z_t^{M,\epsilon}(da, dp) \rangle^2 \right) < \infty.$$

Therefore $\mathcal{M}_t^{M,\epsilon}(f)$ is a L^2 martingale of angle brackets process (40). We have directly (using Rolle's theorem) that $\mathcal{M}_t^{M,\epsilon}(f)$ and $\mathcal{V}_t^{M,\epsilon}(f)$ satisfy (42)-(44). \square

5.3.3 Uniform bounds on $Z_t^{M,\epsilon}$

Lemma 5.3 *Let $q \geq 1$ and assume that*

$$\sup_{M,\epsilon} \mathbb{E} \left(\left(\iiint \mathbb{1} Z_0^{M,\epsilon}(da, dp) \right)^q \right) < \infty,$$

then we have that

$$C_0^T = \sup_{M,\epsilon} \mathbb{E} \left(\left(\sup_{t \in [0, T]} \iiint \mathbb{1} Z_t^{M,\epsilon}(da, dp) \right)^q \right) < \infty, \quad \forall T > 0.$$

Proof Using equation (35) for $F : x \mapsto x^q$ and $f = 1$, we find that

$$\begin{aligned} 0 &= (\langle 1, Z_t^{M,\epsilon} \rangle)^q - (\langle 1, Z_0^{M,\epsilon} \rangle)^q \\ &- \int_0^t \int_\epsilon^{N_s^M} \left[\left((\langle 1, Z_s^{M,\epsilon} \rangle + \frac{1}{M})^q - \langle 1, Z_s^{M,\epsilon} \rangle^q \right) \mathbb{1}_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \right]. \end{aligned}$$

Since $(1+y)^q - y^q \leq q2^{q-1}(1+y^{q-1})$, we have that

$$\langle 1, Z_t^{M,\epsilon} \rangle^q \leq \langle 1, Z_0^{M,\epsilon} \rangle^q + q2^{q-1} \int_0^t \int_\epsilon^{N_s^M} \left[(1 + \langle 1, Z_s^{M,\epsilon} \rangle^{q-1}) \mathbb{1}_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \right],$$

$$\sup_{u \leq \min(T_N^M, t)} \langle 1, Z_u^{M, \epsilon} \rangle^q \leq \langle 1, Z_0^{M, \epsilon} \rangle^q + q2^{q-1} \int_0^t \int_{\mathcal{E}} \mathbf{1}_{n < N_{s-}^M} \left[(1 + \langle 1, Z_s^{M, \epsilon} \rangle^{q-1}) \mathbf{1}_{0 \leq \Theta < B_{n, s-}^{M, \epsilon}} Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \right],$$

and $\mathbb{E}(\sup_{u \leq \min(T_N^M, t)} \langle 1, Z_u^{M, \epsilon} \rangle^q) \leq \mathbb{E}(\langle 1, Z_0^{M, \epsilon} \rangle^q) + q2^{q-1} \int_0^t \int_{Age \times Space} \mathbb{E}((1 + \langle 1, Z_s^{M, \epsilon} \rangle^{q-1}) Z_s^{M, \epsilon}) ds$. Using the stopping time $\tau_N^{M, \epsilon}$ with $\bar{N}_t = E(\sup_{s \leq \min(t, \tau_N^{M, \epsilon})} N_s)$ and noticing that $x^q + x \leq 2(1 + x)$ for all $x \geq 0$ and $q \geq 1$, we have that

$$\mathbb{E}\left(\sup_{u \leq \min(T_N^M, t)} \langle 1, Z_u^{M, \epsilon} \rangle^q\right) \leq [\mathbb{E}(\langle 1, Z_0^{M, \epsilon} \rangle^q) + tq2^q] + q2^q \int_0^t \mathbb{E}\left(\sup_{u \leq \min(T_N^M, s)} \langle 1, Z_u^{M, \epsilon} \rangle^q\right) ds.$$

Now, using by Gronwall lemma, we find that

$$\mathbb{E}\left(\sup_{u \leq \min(\tau_N^{M, \epsilon}, t)} \langle 1, Z_u^{M, \epsilon} \rangle^q\right) \leq [\mathbb{E}(\langle 1, Z_0^{M, \epsilon} \rangle^q) + tq2^q] e^{q2^q t}.$$

Thus, $\lim_{N \rightarrow \infty} \tau_N^{M, \epsilon} = \infty$ for all (M, ϵ) and for all $t \leq T$

$$C_0^t = \sup_{M, \epsilon} \mathbb{E}(\sup_{u \leq t} \langle 1, Z_u^{M, \epsilon} \rangle^q) \leq [\sup_{M, \epsilon} \mathbb{E}(\langle 1, Z_0^{M, \epsilon} \rangle^q) + tq2^q] e^{q2^q t}. \quad (49)$$

□

Lemma 5.4 *Let $m \geq 1$ and assume that (where $dp = r^2 \sin(\theta) dr d\theta d\phi$)*

$$\sup_{M, \epsilon} \mathbb{E}\left(\iiint\iiint a^m Z_0^{M, \epsilon}(da, dp)\right) < \infty, \quad (50)$$

then we have that

$$C_m^T = \sup_{M, \epsilon} \mathbb{E}\left(\sup_{t \in [0, T]} \iiint\iiint a^m Z_t^{M, \epsilon}(da, dp)\right) < \infty, \quad \forall T > 0. \quad (51)$$

Proof We have that $Z_u^{M, \epsilon}$ satisfies (35), for $F : x \mapsto x$ and $f(a) = a^m$, we find that

$$\begin{aligned} \langle a^m, Z_t^{M, \epsilon} \rangle &= \langle a^m, Z_0^{M, \epsilon} \rangle + m \int_0^t \iint_{\mathbb{R}_+ \times Space} a^{m-1} Z_u^{M, \epsilon}(da, dp) du \\ &\quad - \frac{1}{M} \left[\int_0^t \int_{\mathcal{E}} \mathbf{1}_{n < N_{s-}^M} A_n^{M, \epsilon}(s-) \mathbf{1}_{0 \leq \Theta < B_{n, s-}^{M, \epsilon}} Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \right], \end{aligned}$$

$$\langle a^m, Z_t^{M, \epsilon} \rangle \leq \langle a^m, Z_0^{M, \epsilon} \rangle + m \int_0^t \iint_{\mathbb{R}_+ \times Space} a^{m-1} Z_u^{M, \epsilon}(da, dp) du,$$

and finally we find that

$$\mathbb{E}\left(\langle a^m, Z_t^{M, \epsilon} \rangle\right) \leq \mathbb{E}\left(\langle a^m, Z_0^{M, \epsilon} \rangle\right) + m \int_0^t \mathbb{E}\left(\langle a^{m-1} Z_u^{M, \epsilon}(da, dp) \rangle\right) du,$$

and using that $a^{m-1} \leq a^m + 1$ and (49), we have that

$$\mathbb{E}\left(\sup_{u \leq \min(\tau_N^{M, \epsilon}, t)} \langle a^m, Z_u^{M, \epsilon} \rangle\right) \leq \mathbb{E}\left(\langle a^m, Z_0^{M, \epsilon} \rangle\right) + mt \left[\sup_{M, \epsilon} \mathbb{E}(\langle 1, Z_0^{M, \epsilon} \rangle) + 2t \right] e^{2t} + m \int_0^t \mathbb{E}\left(\langle a^m, Z_u^{M, \epsilon}(da, dp) \rangle\right) du.$$

Using Gronwall lemma we find

$$\mathbb{E}\left(\sup_{u \leq \min(\tau_N^{M,\epsilon}, t)} \langle a^m, Z_u^{M,\epsilon} \rangle\right) \leq \left(\mathbb{E}\left(\langle a^m, Z_0^{M,\epsilon} \rangle\right) + mt[\sup_{M,\epsilon} \mathbb{E}(\langle 1, Z_0^{M,\epsilon} \rangle) + 2t]e^{2t}\right)e^{mt}.$$

Therefore, under assumption (50), we prove that (51) is satisfied. \square

Lemma 5.5 *Let $m \geq 1$ and assume that (where $dp = r^2 \sin(\theta) dr d\theta d\phi$)*

$$\sup_{M,\epsilon} \mathbb{E}\left(\iiint r^m Z_0^{M,\epsilon}(da, dp)\right) < \infty, \quad \sup_{M,\epsilon} \mathbb{E}\left(\iiint 1 Z_0^{M,\epsilon}(da, dp)\right) < \infty, \quad (52)$$

then we have that

$$D_m^T = \sup_{M,\epsilon} \mathbb{E}\left(\sup_{t \in [0, T\epsilon]} \iiint r^m Z_t^{M,\epsilon}(da, dp)\right) < \infty, \quad \forall T > 0. \quad (53)$$

Proof We have that $Z_u^{M,\epsilon}$ satisfies (35), for $F : x \mapsto x$ and $f(a) = r^m$, we find that

$$\begin{aligned} \sup_{u \leq \min(\tau_N^{M,\epsilon}, t)} \langle r^m, Z_u^{M,\epsilon} \rangle &= \langle r^m, Z_0^{M,\epsilon} \rangle + \frac{1}{M} \left[\int_0^t \int_{\epsilon} 1_{n < N_{s-}^M} \left[(X_n^{M,\epsilon}(s-))^m 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right. \right. \\ &\quad \left. \left. + ((p')^m - (X_n^{M,\epsilon}(s-))^m) 1_{B_{n,s-}^{M,\epsilon} \leq \Theta < B_{n,s-}^{M,\epsilon} + P_{s-}^\epsilon} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')) \right], \end{aligned}$$

and finally we find that

$$\begin{aligned} \mathbb{E}\left(\sup_{u \leq \min(\tau_N^{M,\epsilon}, t)} \langle r^m, Z_u^{M,\epsilon} \rangle\right) &\leq \mathbb{E}\left(\langle r^m, Z_0^{M,\epsilon} \rangle\right) \\ &\quad + \int_0^t \mathbb{E}\left(\langle r^m, Z_u^{M,\epsilon} \rangle\right) + \|P_{s-}^\epsilon\| \epsilon m 2^m (2\mathbb{E}\left(\langle 1, Z_u^{M,\epsilon} \rangle\right) + \mathbb{E}\left(\langle r^m, Z_u^{M,\epsilon} \rangle\right)) du, \end{aligned}$$

$$\mathbb{E}\left(\sup_{u \leq \min(\tau_N^{M,\epsilon}, t)} \langle r^m, Z_u^{M,\epsilon} \rangle\right) \leq \mathbb{E}\left(\langle r^m, Z_0^{M,\epsilon} \rangle\right) + \int_0^t \mathbb{E}\left(\langle r^m, Z_u^{M,\epsilon} \rangle\right) (1 + \|P_{s-}^\epsilon\| \epsilon m 2^m) du + \|P_{s-}^\epsilon\| \epsilon m 2^{m+1} t C_0^T.$$

Using Gronwall lemma we find that for all $t < T$:

$$\mathbb{E}\left(\sup_{u \leq \min(\tau_N^{M,\epsilon}, t\epsilon)} \langle r^m, Z_u^{M,\epsilon} \rangle\right) \leq \left(\mathbb{E}\left(\langle r^m, Z_0^{M,\epsilon} \rangle\right) + \|P_{s-}^\epsilon\| \epsilon^2 m 2^{m+1} t C_0^T\right) e^{(1 + \|P_{s-}^\epsilon\| \epsilon^2 m 2^m)t},$$

with and (53) holds. \square

Lemma 5.6 *Let $m \geq 1$ and assume that*

$$\sup_{M,\epsilon} \mathbb{E}\left(\left(\iiint a^m Z_0(da, dp)\right)^2\right) < \infty, \quad \sup_{M,\epsilon} \mathbb{E}\left(\iiint a^{2m} Z_0(da, dp)\right), \quad (54)$$

then we have that

$$C_{m,2}^T = \sup_{M,\epsilon} \mathbb{E}\left(\left(\sup_{t \in [0, T]} \iiint a^m Z_t^{M,\epsilon}(da, dp)\right)^2\right) < \infty, \quad \forall T > 0. \quad (55)$$

Proof We have that (see lemma 5.2), for all $t \leq T$,

$$\begin{aligned} (\langle a^m, Z_t^{M,\epsilon} \rangle)^2 &= (\langle a^m, Z_0^{M,\epsilon} \rangle)^2 + 2m \int_0^t (\langle a^m, Z_u^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} a^{m-1} Z_u^{M,\epsilon}(da, dp) du \\ &+ \int_0^t \int_{\epsilon} 1_{n < N_{s-}^M} \left[-\frac{2}{M} A_n^{M,\epsilon}(s-) (\langle a^m, Z_s^{M,\epsilon} \rangle) + \frac{1}{M^2} A_n^{M,\epsilon}(s-) 2m 1_{0 \leq \Theta < B_{n,s-}^{M,\epsilon}} \right] Q(ds, (dn), d\Theta, \Pi_{s-}^\epsilon(p, dp')), \end{aligned}$$

therefore, we find that,

$$\begin{aligned} \mathbb{E} \left(\sup_{u \leq \min(\tau_N^{M,\epsilon}, t)} (\langle a^m, Z_u^{M,\epsilon} \rangle)^2 \right) &\leq \mathbb{E} \left((\langle a^m, Z_0^{M,\epsilon} \rangle)^2 \right) \\ &+ 2m \int_0^t \mathbb{E} \left((\langle a^m, Z_u^{M,\epsilon} \rangle) \iint_{\mathbb{R}_+ \times \text{Space}} a^{m-1} Z_u^{M,\epsilon}(da, dp) \right) du + \int_0^t \frac{1}{M} \mathbb{E} \left(\langle a^{2m}, Z_u^{M,\epsilon} \rangle \right) du, \end{aligned}$$

and using the lemma 5.4 and the Gronwall lemma (and noticing that $a^{m-1} \leq a^m + 1$), (55) holds. \square

Lemma 5.7 *Let $m \geq 1$ and assume that*

$$\sup_{M,\epsilon} \mathbb{E} \left(\left(\iiint r^m Z_0(da, dp) \right)^2 \right) < \infty, \quad \sup_{M,\epsilon} \mathbb{E} \left(\left(\iiint r^{2m} Z_0(da, dp) \right) \right),$$

and (19), then we have that

$$D_{m,2}^T = \sup_{M,\epsilon} \mathbb{E} \left(\left(\sup_{t \in [0, T\epsilon]} \iiint r^m Z_t^{M,\epsilon}(da, dp) \right)^2 \right) < \infty, \quad \forall T > 0.$$

The proof of this lemma is similar to the previous ones.

5.4 Technical lemmas : PDE

In this part, we prove the existence, regularity and compactness of solution to the intermediate equation.

5.4.1 Proof of existence and uniqueness of solution to (12)

Lemma 5.8 (*Existence/Uniqueness*) *Let $Z_0^\epsilon(a, p) = \rho_0^\epsilon(a, p)dadp$ satisfying (22) with $\rho_0^\epsilon \in C^1$ and*

$$\rho_0^\epsilon(a = 0, p) = 2 \int B(a, p) \rho_0^\epsilon(a, p) da, \quad (56)$$

then $Z_t^\epsilon(a, p) = \lim_{M_k \rightarrow \infty} Z_t^{M_k, \epsilon}(da, dp)$ is equal to $\rho^\epsilon(t, a, p)dadp$ with $\rho^\epsilon \in C^1$ solution to (12).

Proof Let $\mathcal{T} : g \mapsto f$ solution to

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) f - K[g](t, p) = -K_T(t, p) f(t, a, p), \\ f(t, 0, p) = 2 \int B(a, p) f(t, a, p) da, \quad f(t = 0, \cdot, \cdot) = \rho_0^\epsilon(\cdot, \cdot), \end{cases} \quad (57)$$

with $K[f](t, p) = \iiint \tilde{P}_\infty^\epsilon(p') \frac{\tilde{\Pi}_\infty^\epsilon(p', dp)}{dp} f(t, a, p') dp'$, $K_T(t, p) = \tilde{P}_\infty^\epsilon(p)$,

$$\tilde{P}_\infty^\epsilon(p') = \frac{C}{\epsilon^2} \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p') R \left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) g da dp \text{Vol}_G}{\text{Vol}_{i,j,k}^\epsilon} \right),$$

and

$$\frac{\tilde{\Pi}_\infty^\epsilon(p', dp)}{dp} = \frac{\Psi^\epsilon(|p - p'|)\tilde{G}_{s-}^\epsilon(p)}{\iint \Psi^\epsilon(|q - p'|)\tilde{G}_{s-}^\epsilon(q)dq}, \text{ with } \tilde{G}_{s-}^\epsilon(\cdot) = 1 - \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(\cdot)R\left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p)gdadpVol_G}{Vol_{i,j,k}^\epsilon}\right).$$

Using Gronwall lemma and $\|B\|_\infty \leq 1$, we have that

$$\iiint f(t, a, p)dadp \leq \iiint \rho_0^\epsilon(a, p)dadpe^{2t}.$$

By computation, and using that $|R'| \leq \frac{1}{\sigma}$, we find that

$$|K[g] - K[\tilde{g}]|(t, p) \leq \frac{3Vol_G}{\sigma(1 - R(\iiint \rho_0^\epsilon(a, p)dadpe^{2t}))Vol_{i,j,k}^\epsilon} \iiint |g - \tilde{g}|dadp.$$

Therefore, using the Characteristics (in (a, t)) of the transport equation (57) (see [20]), we have existence and uniqueness of the solution. Moreover, for $\phi : t \mapsto \int_t^\infty 2e^{-\frac{C}{\epsilon^2}(s-t)}ds$, and $f = \mathcal{T}(g)$, $\tilde{f} = \mathcal{T}(\tilde{g})$, we have that (multiplying (57) by ϕ and integrating in (a, p))

$$\frac{d}{dt} \iiint |f - \tilde{f}|\phi(t)dadp \leq \frac{3Vol_G}{\sigma(1 - R(\iiint \rho_0^\epsilon(a, p)dadpe^{2t}))Vol_{i,j,k}^\epsilon} \iiint |g - \tilde{g}|\phi(t)dadp,$$

and so

$$\iiint |f - \tilde{f}|\phi(t)dadp \leq \int_0^t \frac{3Vol_G}{\sigma(1 - R(\iiint \rho_0^\epsilon(a, p)dadpe^{2s}))Vol_{i,j,k}^\epsilon} \iiint |g - \tilde{g}|\phi(s)dadpds.$$

Therefore $\mathcal{T} : g \in E \mapsto f \in E$ with $E = C([0, T], L^1(\mathbb{R}_+ \times \mathbb{R}^3))$, is a contracting mapping for $T > 0$ small enough and there exists a unique solution to (12). Now, we construct a solution on $[T, 2T]$... and finally on \mathbb{R}_+ . Since Ψ_ϵ and R are C^∞ , the regularity of ρ^ϵ , solution to the transport equation (12) with the boundary condition 56, is given by those of ρ_0^ϵ (under the assumption (56), see [20, 18, 11, 27]). Using uniqueness, we find that $Z_t^\epsilon(a, p) = \rho^\epsilon(t, a, p)dadp$. \square

5.4.2 Compactness of $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p)da$

Lemma 5.9 *Let $T > 0$, ρ^ϵ solution to (12) and assume that (20) is satisfied then we can extract a convergent subsequence of $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p)da$ in $L^2([0, T] \times Space)$.*

Proof Since $M_\epsilon^1(0, p) \in L^2(Space)$ by assumption (20), we have, by integrating equation (12) that $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p)da$ follows the equation

$$\begin{cases} \frac{\partial}{\partial t} M_\epsilon^1(t, p) - \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} M_\epsilon^1(t, p') dp' + P_\infty^\epsilon(p) M_\epsilon^1(t, p) = 2 \int B(a, p) \rho^\epsilon(t, a, p) da, \\ M_\epsilon^1(t = 0, p) = \int \rho_0^\epsilon(a, p) da. \end{cases} \quad (58)$$

First integrating (58) with respect to p and Gronwall lemma, we have that

$$\int M_\epsilon^1(0, p)dp \leq \int M_\epsilon^1(t, p)dp \leq \int M_\epsilon^1(0, p)dpe^{2T}, \quad \forall t \in [0, T],$$

therefore ρ^ϵ is uniformly bounded in $L^\infty([0, T], L^1(Age \times Space))$. Secondly, multiplying (58) by M_ϵ^1 and integrating with respect to p we find that

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp &= \iint \left[\frac{P_\infty^\epsilon(p') M_\epsilon^1(t, p')}{\iiint \Psi^\epsilon(|q - p'|) G_{s-}^\epsilon(q) dq} G_{s-}^\epsilon(p) (M_\epsilon^1(t, p) - M_\epsilon^1(t, p')) \Psi^\epsilon(|p' - p|) dp dp' \right. \\ &\quad \left. + \iint B(a, p) \rho^\epsilon(t, a, p) da M_\epsilon^1(t, p) dp, \right. \end{aligned}$$

that can be rewritten as,

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp &= \frac{1}{2} \iint \left[\frac{P_\infty^\epsilon(p') M_\epsilon^1(t, p')}{\iiint \Psi^\epsilon(|q - p'|) G_{s-}^\epsilon(q) dq} G_{s-}^\epsilon(p) - \frac{P_\infty^\epsilon(p) M_\epsilon^1(t, p)}{\iiint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq} G_{s-}^\epsilon(p') \right] \\ &\quad (M_\epsilon^1(t, p) - M_\epsilon^1(t, p')) \Psi^\epsilon(|p' - p|) dp dp' + \iint B(a, p) \rho^\epsilon(t, a, p) da M_\epsilon^1(t, p) dp. \end{aligned}$$

Let $p_+ = (p + p')/2$ and $p_- = (p - p')/2$, then we have $p = p_+ + p_-$, $p' = p_+ - p_-$ and for all A, B C^1 -functions we have that

$$\begin{aligned} A(p')B(p) - A(p)B(p') &= A(p_+ - p_-)B(p_+ + p_-) - B(p_+ - p_-)A(p_+ + p_-) \\ &= (A(p_+) - \nabla A(p_+)p_- + o(p_-))(B(p_+) + \nabla B(p_+)p_- + o(p_-)) \\ &\quad - (A(p_+) + \nabla A(p_+)p_- + o(p_-))(B(p_+) - \nabla B(p_+)p_- + o(p_-)) \\ &= 2(A(p_+)\nabla B(p_+)p_- - B(p_+)\nabla A(p_+)p_-) + o(p_-). \end{aligned}$$

Noticing that $M_\epsilon^1(t, p) - M_\epsilon^1(t, p') = M_\epsilon^1(t, p_+ + p_-) - M_\epsilon^1(t, p_+ - p_-) = 2\nabla M_\epsilon^1(t, p_+)p_- + o(p_-)$ and changing the variables in the integral, we find that

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp &= 4 \iint \left[\left(\frac{P_\infty^\epsilon M_\epsilon^1(t, \cdot)}{\iiint \Psi^\epsilon(|q - \cdot|) G_{s-}^\epsilon(q) dq} (p_+) \nabla G_{s-}^\epsilon(p_+) 2p_- \right. \right. \\ &\quad \left. \left. - G_{s-}^\epsilon(p_+) \nabla \frac{P_\infty^\epsilon M_\epsilon^1(t, \cdot)}{\iiint \Psi^\epsilon(|q - \cdot|) G_{s-}^\epsilon(q) dq} (p_+) 2p_- \right) \right. \\ &\quad \left. (2\nabla M_\epsilon^1(t, p_+)p_-) \Psi^\epsilon(2|p_-|) dp_+ dp_- + \iint B(a, p) \rho^\epsilon(t, a, p) da M_\epsilon^1(t, p) dp + o(1). \right. \end{aligned}$$

Using that

$$P_\infty^\epsilon(p') = \frac{C}{\epsilon^2} \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p') R \left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(s, p) dp Vol_G}{Vol_{i,j,k}^\epsilon} \right), \text{ with } R(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}},$$

$$\Pi_\infty^\epsilon(p', dp) = \frac{\Psi^\epsilon(|p' - p|) G_{s-}^\epsilon(p) dp}{\iiint \Psi^\epsilon(|q - p'|) G_{s-}^\epsilon(q) dq}, \text{ with } G_{s-}^\epsilon(\cdot) = 1 - \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(\cdot) R \left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(s, p) dp Vol_G}{Vol_{i,j,k}^\epsilon} \right),$$

we find that

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp &= 4C \iint \left[M_\epsilon^1(t, \cdot) \frac{R}{1 - R} (M_\epsilon^1(t, \cdot)) (1 - R)' \nabla M_\epsilon^1(t, p_+) 2 \frac{p_-}{\epsilon} \right. \\ &\quad \left. - M_\epsilon^1(t, p_+) (1 - R) \left(\frac{R}{1 - R} \right)' (M_\epsilon^1(t, p_+)) \nabla M_\epsilon^1(t, p_+) 2 \frac{p_-}{\epsilon} - R(M_\epsilon^1(t, p_+)) \nabla M_\epsilon^1(t, p_+) 2 \frac{p_-}{\epsilon} \right] \\ &\quad (2\nabla M_\epsilon^1(t, p_+) \frac{p_-}{\epsilon}) \Psi^\epsilon(2|p_-|) dp_+ dp_- + \iint B(a, p) \rho^\epsilon(t, a, p) da M_\epsilon^1(t, p) dp + o(1). \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp = \\ & - 4C \int \left[1 + \frac{2M_\epsilon^1(t, p_+)}{\sigma} \right] R(M_\epsilon^1(t, p_+)) (\nabla M_\epsilon^1(t, p_+)) \int 2 \frac{p_-}{\epsilon} \Psi^\epsilon(2|p_-|) 2 \frac{p_-}{\epsilon} dp_- {}^t \nabla M_\epsilon^1(t, p_+) \Psi^\epsilon(2|p_-|) dp_+ \\ & \quad + \iint B(a, p) \rho^\epsilon(t, a, p) da M_\epsilon^1(t, p) dp + o(1). \end{aligned}$$

Therefore, we have the following bound

$$\frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp + 4CR(0)\lambda_m \iint \left[1 + \frac{2M_\epsilon^1(t, p_+)}{\sigma} \right] \|\nabla M_\epsilon^1(t, \cdot)\|^2(t, p_+) dp_- \leq \int (M_\epsilon^1(t, p))^2 dp,$$

with $\lambda_m > 0$ the infimum (with respect of ϵ) of the minimum of the eigenvalues of the definite positive matrix $\int 2 \frac{p_-}{\epsilon} \Psi^\epsilon(2|p_-|) 2 \frac{p_-}{\epsilon} dp_-$. Finally, using Gronwall inequality we have that for all $T > 0$

$$\int (M_\epsilon^1(t, p))^2 dp \leq \int (M_\epsilon^1(0, p))^2 dp e^{2T}, \quad \forall t \leq T,$$

and integrating the last inequality, we found that

$$\int_{[0, T]} \iint \left[1 + \frac{2M_\epsilon^1(t, p_+)}{\sigma} \right] \|\nabla M_\epsilon^1(t, \cdot)\|^2(t, p_+) dp_- \leq \frac{\int (M_\epsilon^1(0, p))^2 dp}{4CR(0)\lambda_m} e^{2T} (T + 1).$$

We notice that, for all $T > 0$, the same computation leads to (multiplying (58) by $(pM_\epsilon^1)^2$ and integrating with respect to p)

$$\int p (M_\epsilon^1(t, p))^2 dp \leq \left[\int p (M_\epsilon^1(0, p))^2 dp + T \|M_\epsilon^1\|_{C([0, T], H^1(\text{Space}))} \lambda_M \right] e^{2T}, \quad \forall t \leq T,$$

with $\lambda_M > 0$ the supremum (with respect of ϵ) of the maximum of the eigenvalues of the definite positive matrix $\int 2 \frac{p_-}{\epsilon} \Psi^\epsilon(2|p_-|) 2 \frac{p_-}{\epsilon} dp_-$. Thus, we have that $(M_\epsilon^1)_\epsilon$ is uniformly bounded in $L^2([0, T], H^1(\text{Space}) \times L^2_{pdp}(\text{Space}))$. Moreover, we have directly that $|\frac{\partial}{\partial t} \frac{1}{2} \int (M_\epsilon^1(t, p))^2 dp|$ is uniformly bounded in $L^2([0, T])$. Therefore, by the (Lions-Aubin) result (see [14]), we can extract a convergent subsequence of $(M_\epsilon^1)_\epsilon$ in $L^2([0, T] \times (\text{Space}))$. \square

Lemma 5.10 *Let $T > 0$. Assume that $(\nabla M_\epsilon^1(t, \cdot))_\epsilon$ is $L^\infty([0, T]; L^2(\text{Space}))$ and M_ϵ^1 converges $C^0([0, T]; L^2(\text{Space}))$ to M^1 as ϵ to 0 then*

$$\sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p') R \left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(t, \cdot) dadp Vol_G}{Vol_{i,j,k}^\epsilon} \right) \rightarrow_{C^0([0, T]; L^2(\text{Space}))} R \left(\frac{Vol_G(3r^2 + 6r + 3)}{3r^2 + 6rr_O(s) + 3r_O(s)^2} M^1(t, p) \right).$$

Proof Indeed, we have that

$$\begin{aligned} & \int \left| \sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p') R \left(\frac{\iiint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(t, \cdot) dadp Vol_G}{Vol_{i,j,k}^\epsilon} \right) - R \left(\frac{Vol_G(3r^2 + 6r + 3)}{3r^2 + 6rr_O(s) + 3r_O(s)^2} M_\epsilon^1(t, p) \right) \right|^2 dp \\ & \leq \|R'\|_\infty^2 \int \left(\int_{u \leq 1} |M_\epsilon^1(p + \epsilon u) - M_\epsilon^1(p)| du \right)^2 dp \leq \epsilon^2 \int \left(\int_{u \leq 1} \int_{s \in [0, 1]} |\nabla M_\epsilon^1(p + \epsilon su)| duds \right)^2 dp \\ & \leq \epsilon^2 \int_{u \leq 1} \int_{s \in [0, 1]} \left(\int |\nabla M_\epsilon^1(p + \epsilon su)|^2 dp \right) duds, \end{aligned}$$

and

$$\begin{aligned} \int \left(R \left(\frac{\text{Vol}_G(3r^2 + 6r + 3)}{3r^2 + 6rr_O(s) + 3r_O(s)^2} M^1(t, p) \right) - R \left(\frac{\text{Vol}_G(3r^2 + 6r + 3)}{3r^2 + 6rr_O(s) + 3r_O(s)^2} M_\epsilon^1(t, p) \right) \right)^2 dp \\ \leq cst \|R'\|_\infty^2 \int (M^1(t, p) - M_\epsilon^1(t, p))^2 dp. \end{aligned}$$

Under the uniform boundness of $\nabla M_\epsilon^1(t, \cdot)_\epsilon$ and the $C^0([0, T]; L^2(\text{Space}))$ convergence of M_ϵ^1 we have that $\sum_{i,j,k} \Psi_{i,j,k}^\epsilon(p') R \left(\frac{\iint \Psi_{i,j,k}^\epsilon(p) M_\epsilon^1(t, \cdot) dadp \text{Vol}_G}{\text{Vol}_{i,j,k}^\epsilon} \right)$ converges (as $\epsilon \rightarrow 0$) to $R(M^1(t, p))$ in $C^0([0, T]; L^2(\text{Space}))$. \square

Lemma 5.11 *Let $T > 0$, ρ^ϵ solution to (12) and assume that (20) is satisfied then we can extract a convergent subsequence of $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p) da$ which limit is a weak solution to*

$$\frac{\partial}{\partial t} M^1(t, p) + \text{div}(M^1(t, p) Ch(p) R \nabla \left(\log(1 - R) \right)) = H,$$

$$M^1(0, p) = \int \rho_0^\epsilon(a, p) da,$$

and $H \in C([0, T], H^1(\text{Space}))$ and R defined in (15).

Proof Let $f \in C_c^1(\mathbb{R}_+ \times \text{Space})$, multiplying (12) by f and integrating with respect to a we find that :

$$\frac{\partial}{\partial t} \int_p f(p) M_\epsilon^1(t, p) dp - \iint B(a, p) f(p) \rho^\epsilon(t, a, p) dadp = \iiint_{p,p'} (f(p') - f(p)) \Pi_{s-}^\epsilon(p, dp') P_{s-}^\epsilon(p) M_\epsilon^1(t, p) dp.$$

Rewriting the second member as follows, we have that

$$\begin{aligned} \iiint_{p,p'} (f(p') - f(p)) \Pi_{s-}^\epsilon(p, dp') P_{s-}^\epsilon(p) M_\epsilon^1(t, p) dp \\ = \iiint_{p,p'} C^\epsilon (f(p') - f(p)) \frac{\Psi^\epsilon(|p' - p|) G_{s-}^\epsilon(p') dp'}{\iint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq} \frac{P_{s-}^\epsilon(p)}{C^\epsilon} M_\epsilon^1(t, p) dp = \\ = \iiint_p {}^t \nabla f(p) \frac{C^\epsilon \iint_q \Psi^\epsilon(|q - p|) (q - p) G_{s-}^\epsilon(q) dq}{\iint \Psi^\epsilon(|q - p|) G_{s-}^\epsilon(q) dq} \frac{P_{s-}^\epsilon(p)}{C^\epsilon} M_\epsilon^1(t, p) dp + o(1) \\ = \iiint_p {}^t \nabla f(p) \frac{C^\epsilon \iint_q \Psi^\epsilon(|q - p|) (q - p) {}^t(q - p) dq}{\iint \Psi^\epsilon(|q - p|) dq} \frac{P_{s-}^\epsilon(p)}{C^\epsilon} M_\epsilon^1(t, p) \frac{\nabla G_{s-}^\epsilon}{G_{s-}^\epsilon}(p) dp + o(1) \\ = \iiint_p \nabla f(p) M_\epsilon^1(t, p) Ch(p) R_\epsilon^t \nabla \left(\log(1 - R_\epsilon) \right) dp + o(1). \end{aligned}$$

Since, $M_\epsilon^1(t, p)$ is bounded $C([0, T], H^1(\text{Space}))$, we have that $\log(1 - R_\epsilon) \xrightarrow{\text{Distribution}} \log(1 - R)$ and by Banach Aoluglu in L^2 (and identifying the limit) $\nabla \log(1 - R_\epsilon) \xrightarrow{*-\text{weak } L^2} \nabla \log(1 - R)$. Now, by lemmes 5.9 and 5.10, we have that $\nabla f(p) M_\epsilon^1(t, p) Ch(p) R_\epsilon \rightarrow L^2 \nabla f(p) M^1(t, p) Ch(p) R$ and so (there exists $H \in C([0, T], L^2)$, the limit of $\iint B(a, p) f(p) \rho^\epsilon(t, a, p) dadp$ as $\epsilon \rightarrow 0$)

$$\frac{\partial}{\partial t} \int_p f(p) M^1(t, p) dp - \iint f(p) H(t, p) dp = \iiint_p \nabla f(p) M^1(t, p) Ch(p) R^t \nabla \left(\log(1 - R) \right) dp.$$

Therefore the limit is $C([0, T], H^1)$ and is a weak solution to

$$\frac{\partial}{\partial t} M^1(t, p) + \text{div}(M^1(t, p) Ch(p) R \nabla \left(\log(1 - R) \right)) = H,$$

with $M^1(0, p) = \int \rho_0(a, p) da$. \square

5.4.3 Proof of existence and uniqueness of solution to (15)bis

Lemma 5.12 *Let $T > 0$, ρ^ϵ solution to (12) and assume that (20) is satisfied then we can extract a convergent subsequence of $(\rho^\epsilon)_\epsilon$ in $L^2([0, T] \times \text{Space} \times \text{Age})$.*

Proof Differentiate (12) with respect to a leads to

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) \frac{\partial}{\partial a} \rho^\epsilon - \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \frac{\partial}{\partial a} \rho^\epsilon(t, a, p') dp' + P_\infty^\epsilon(p) \frac{\partial}{\partial a} \rho^\epsilon(t, a, p) = 0, \\ \rho^\epsilon|_{a=0} = 2 \int B(a, p) \rho^\epsilon(t, a, p) da, \quad \rho^\epsilon|_{t=0} = \rho_0^\epsilon. \end{cases}$$

Therefore, we have that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \int \left|\frac{\partial}{\partial a} \rho^\epsilon\right| dp \leq 0,$$

and for all t, a and ϵ , we find that

$$\frac{\partial}{\partial t} \iint \left|\frac{\partial}{\partial a} \rho^\epsilon\right| dp da \leq \int \left|\frac{\partial}{\partial a} \rho^\epsilon\right|(a=0) dp$$

Now, using the intermediate equation, we find that

$$\begin{aligned} \frac{\partial}{\partial a} \rho^\epsilon(a=0) &= -2 \int B(a, p) \frac{\partial}{\partial t} \rho^\epsilon(t, a, p) da \\ &\quad + 2 \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \int B(a, p') \rho^\epsilon(t, a, p') dadp' - 2P_\infty^\epsilon(p) \int B(a, p) \rho^\epsilon(t, a, p) da, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial a} \rho^\epsilon(a=0) &= -2 \int B(a, p) \left[-\frac{\partial}{\partial a} \rho^\epsilon + \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \rho^\epsilon(t, a, p') dp' - P_\infty^\epsilon(p) \rho^\epsilon(t, a, p)\right] da \\ &\quad + 2 \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \int B(a, p') \rho^\epsilon(t, a, p') dadp' - 2P_\infty^\epsilon(p) \int B(a, p) \rho^\epsilon(t, a, p) da, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial a} \rho^\epsilon(a=0) &= -2 \int \rho^\epsilon \frac{\partial}{\partial a} B(a, p) da - 2 \int B(a, p) \left[\iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \rho^\epsilon(t, a, p') dp' - P_\infty^\epsilon(p) \rho^\epsilon(t, a, p)\right] da \\ &\quad + 2 \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \int B(a, p') \rho^\epsilon(t, a, p') dadp' - P_\infty^\epsilon(p) \int B(a, p) \rho^\epsilon(t, a, p) da, \text{ and finally,} \end{aligned}$$

$$\frac{\partial}{\partial a} \rho^\epsilon(a=0) = -2 \int \rho^\epsilon \frac{\partial}{\partial a} B(a, p) da - 2 \iiint P_\infty^\epsilon(p') \frac{\Pi_\infty^\epsilon(p', dp)}{dp} \int (B(a, p) - B(a, p')) \rho^\epsilon(t, a, p') dadp'.$$

Therefore, using the same computation as lemma 5.9, we have that

$$\left\| \frac{\partial}{\partial a} \rho^\epsilon(a=0) \right\|_{L^2(\text{Space})} \leq 8 \|B\|_{W^{1, \infty}} (1 + \lambda_M^2) \sup_\epsilon \|M_\epsilon^1\|_{C([0, T], H^1(\text{Space}))}$$

with $\lambda_M > 0$ the sup (with respect of ϵ) of the maximum of the eigenvalues of the definite positive matrix $\int 2 \frac{p_-}{\epsilon} \Psi^\epsilon(2|p_-|) 2 \frac{t p_-}{\epsilon} dp_-$. And so, using the same computation as lemma 5.9, we have that

$$\rho^\epsilon \in C([0, T], H^1(\text{Age} \times \text{Space})).$$

Therefore, by the (Lions-Aubin) result (see [14]), we can extract a convergent subsequence of $(\rho^\epsilon)_\epsilon$ in $L^2([0, T] \times \text{Space})$. \square

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