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Abstract: This paper deals with a new event-based stabilization strategy for a class of linear hyperbolic systems of conservation laws. It includes an internal dynamics which serves as a filter mechanism for the event-triggered condition previously introduced in Espitia et al. (2016). The well-posedness as well as the global exponential stability of the resulting closed-loop system is studied. Some numerical simulations are performed to validate the theoretical results.

Keywords: Event-based control, hyperbolic systems, Lyapunov techniques, triggering conditions, piecewise continuous functions.

1. INTRODUCTION

In recent years, event-based control has gained a lot of attention not only because of its efficient way of using communications and computational resources by updating control inputs aperiodically (only when needed) but also because of its rigorous way to implement digitally continuous time controllers. For finite dimensional networked control systems, event-triggered strategies for stabilization have become an active research area, for which seminal contributions can be found in Åström and Bernhardsson (1999); Årzén (1999) or more recent ones in Heemels et al. (2012); Marchand et al. (2013); Postoyan et al. (2015) and the references therein. Typically, the framework of event-based control includes a feedback control law which is designed to stabilize the system along with a triggering strategy which determines the time instants when the control needs to be updated. The triggering strategy guarantees that a Lyapunov function decreases strictly. The most common triggering strategy uses a static rule obtained by an Input-to-State Stability (ISS) property as in Tabuada (2007). An extension to this strategy is done in Girard (2015) where an internal dynamics is introduced into the triggering rule, reducing the number of control updates in comparison to the static policy. Other approaches, among others, rely directly on the time derivative of the Lyapunov function (Marchand et al. (2013); Seuret et al. (2014)).

The design of event-based control strategies for infinite dimensional systems (namely those governed by partial differential equations (PDEs)), is rarely treated in the literature. For parabolic PDEs, event-based strategies are considered in Selivanov and Fridman (2015). For a class of hyperbolic systems of conservation laws, the closest framework to event-based control is the work on switched hyperbolic systems as in Lamare et al. (2015) which is highly inspiring, especially when dealing with the well-posedness of the closed-loop solution and with the filter mechanism in form of a dynamic variable enabling to reduce the number of switches. A recent work however has introduced two event-based boundary controllers for linear hyperbolic systems of conservations laws: inspired by two of the main strategies developed for finite dimensional systems, an extension by means of Lyapunov techniques for stability has been done in Espitia et al. (2016) for linear hyperbolic systems of conservation laws. It is worth recalling that stability analysis and continuous stabilization of such systems by means of boundary control have been considered for a long time in literature. For instance, backstepping design (Krstic and Smyshlyaev (2008)) and Lyapunov techniques (Coron, J-M et al. (2007)) are the most commonly used. In fact, some complex physical networks can be modeled by means of Hyperbolic PDEs. To mention few applications which stand out: hydraulic (Bastin et al. (2008)), road traffic (Coclite et al. (2005)), gas pipeline networks (Gugat et al. (2011)). They all motivate the use of boundary control. Furthermore, they all motivate the event-based boundary control which is actually a realistic approach for the actuator in those systems. In order to make the motivation a bit more clear, for instance in open channels modeled by the Saint-Venant equations, the actuation on the boundary might be expensive due to the actuator inertia when regulating the water level and the water flow rate by using gates opening as a control

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actions. Event-based control would suggest to modulate efficiently the gates opening, only when needed. The main contribution of this work relies on the extension of one of the event-based strategies proposed in Espitia et al. (2016). We introduce an internal dynamic to the triggering algorithm in order to reduce the number of control updates while guaranteeing both the well-posedness of the closed-loop solution and the global exponential stability as well as the absence of the so-called Zeno phenomena.

This paper is organized as follows. Section 2 contains some results provided in Espitia et al. (2016). The main result of this paper is then presented in Subsection 2.3. Section 3 provides a numerical example to illustrate the main results and to compare the two control strategies for the control of a system describing traffic flow on a roundabout. Finally, conclusions are given in Section 4.

Preliminary definitions and notation. The set of all functions $\phi : [0,1] \to \mathbb{R}^n$ such that $\int_0^1 |\phi(x)|^2 < \infty$ is denoted by $L^2([0,1],\mathbb{R}^n)$ that is equipped with the norm $\|\cdot\|_{L^2([0,1],\mathbb{R}^n)}$. The restriction of a function $y : I \to J$ on an open interval $(x_1,x_2) \subset I$ is denoted by $y|_{(x_1,x_2)}$. Given an interval $I \subseteq \mathbb{R}$ and a set $J \subseteq \mathbb{R}^n$ for some $n \geq 1$, a *piecewise left-continuous function* (resp. a *piecewise right-continuous function*) $y : I \to J$ is a function continuous on each closed interval subset of I except maybe on a finite number of points $x_0 < x_1 < \ldots < x_p$ such that for all $l \in \{0,..,p-1\}$ there exists y_l continuous on $[x_l, x_{l+1}]$ and $y_{l|(x_l,x_{l+1})} = y_{|(x_l,x_{l+1})}$. Moreover, at the points $x_0,\cdots,x_p$ the function is continuous from the left (resp. from the right). The set of all piecewise left-continuous functions (resp. piecewise right-continuous functions) is denoted by $\mathcal{C}_{lpw}(I,J)$ (resp. $\mathcal{C}_{rpw}(I,J)$). In addition, we have the following inclusions $\mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$, $\mathcal{C}_{rpw}([0,1],\mathbb{R}^n) \subset L^2([0,1],\mathbb{R}^n)$.

Linear Hyperbolic Systems Let us consider the linear system of conservation laws given in Riemann coordinates:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \qquad x \in [0,1], t \in \mathbb{R}^+ \tag{1}$$

along with the following boundary condition

$$y(t,0) = Hy(t,1) + Bu(t), \qquad t \in \mathbb{R}^+ \tag{2}$$

where $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^n$, Λ is a diagonal matrix in $\mathbb{R}^{n\times n}$ such that $\Lambda = \text{diag}(\lambda_1,\cdots,\lambda_n)$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$, $H \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $u : \mathbb{R}^+ \to \mathbb{R}^m$.
In addition, we consider the initial condition given by

$$y(0,x) = y^0(x), \qquad x \in [0,1] \tag{3}$$

where $y^0 \in \mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$. We assume that the linear hyperbolic system is only observed at right boundary $x = 1$ at any time. Therefore we define the output function as follows:

$$z(t) = y(t,1) \tag{4}$$

2. EVENT-BASED STABILIZATION

2.1 Preliminaries on stability

We define the notion of stability considered in the paper.

Definition 1. *The linear hyperbolic system* (1)-(3),(4) *with controller $u = \varphi(z)$ is globally exponentially stable (GES) if there exist $\nu > 0$ and $C > 0$ such that, for every $y^0 \in \mathcal{C}_{lpw}([0,1];\mathbb{R}^n)$, the solution satisfies, for all t in $\mathbb{R}^+$,*

$$\|y(t,\cdot)\|_{L^2([0,1];\mathbb{R}^n)} \leq Ce^{-\nu t}\|y^0\|_{L^2([0,1];\mathbb{R}^n)} \tag{5}$$

A particular case studied in literature (see e.g. de Halleux et al. (2003)) is when φ is given by $u = \varphi_c(z)$ as $u(t) = Kz(t)$. This corresponds to continuous time control for which it holds

$$y(t,0) = Gz(t) \qquad t \in \mathbb{R}^+ \tag{6}$$

with $G = H + BK$. The following inequality is stated in Coron, J.-M. et al. (2008) as a sufficient condition, usually called *dissipative boundary condition*, which guarantees that the system (1)-(3) with boundary condition (6) is GES. In this paper, such a sufficient condition is assumed to be satisfied.

Assumption 1. *The following inequality holds:*

$$\rho_1(G) = Inf\{\|\Delta G \Delta^{-1}\|; \Delta \in \mathcal{D}_{n,+}\} < 1 \tag{7}$$

where $\|\cdot\|$ denotes the usual 2-norm of matrices in $\mathbb{R}^{n\times n}$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal matrices whose elements on the diagonal are strictly positive.

Proposition 1. *[Diagne et al. (2012)] Under Assumption 1, there exist $\mu > 0$, and a diagonal positive definite matrix $Q \in \mathbb{R}^{n\times n}$ (with $Q = \Lambda^{-1}\Delta^2$) such that the following matrix inequality holds*

$$G^T Q \Lambda G < e^{-2\mu} Q \Lambda. \tag{8}$$

Moreover, the linear hyperbolic system (1)-(3),(4),(6) *is GES and* (5) *holds for some $C > 0$ and $\nu = \mu\underline{\lambda}$ where $\underline{\lambda} = \min_{1\leq i\leq n}\{\lambda_i\}$.*

Under the assumption of Proposition 1, inspired by (Diagne et al., 2012, Theorem 1), let us recall that the function defined, for all $y(\cdot) \in L^2([0,1],\mathbb{R}^n)$, by

$$V(y) = \int_0^1 y(x)^T Q y(x) e^{-2\mu x} dx \tag{9}$$

is a Lyapunov function for system (1)-(3),(4),(6).

2.2 ISS static event-based stabilization

We introduce in this subsection the main results of one event-based control scheme for linear hyperbolic systems of conservation laws introduced in Espitia et al. (2016). In that framework, ISS property with respect to a deviation between the continuous controller and the event-based controller, combined with a strict Lyapunov condition using (9), has been studied.

Definition 2. *[Definition of φ_s] Let $\varsigma, \kappa, \eta, \mu > 0$, K in $\mathbb{R}^{m\times n}$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$. Let us define φ_s the operator which maps z to u as follows:*

Let z be in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$ and let $\tilde{V}$ be given, at $t = \frac{1}{\underline{\lambda}}$, by

$$\tilde{V}(\tfrac{1}{\underline{\lambda}}) = \sum_{i=1}^n Q_{ii} \int_0^1 \left(H_i z(t - \tfrac{x}{\lambda_i})\right)^2 e^{-2\mu x} dx \tag{10}$$

and, for all $t > \frac{1}{\underline{\lambda}}$, by

$$\tilde{V}(t) = \sum_{i=1}^n Q_{ii} \int_0^1 \left(H_i z(t - \tfrac{x}{\lambda_i}) + B_i u(t - \tfrac{x}{\lambda_i})\right)^2 e^{-2\mu x} dx \tag{11}$$

and let $\varepsilon(t) = \varsigma \tilde{V}(\frac{1}{\underline{\lambda}})e^{-\eta t}$ for all $t \geq \frac{1}{\underline{\lambda}}$. If $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$, let the increasing sequence of time instants (t_k^u) be defined iteratively by $t_0^u = 0$, $t_1^u = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,

$$\begin{aligned} t_{k+1}^u = \inf\{t \in \mathbb{R}^+ | t > t_k^u \wedge \|BK(-z(t) + z(t_k^u))\|^2 \\ \geq \kappa \tilde{V}(t) + \varepsilon(t)\} \end{aligned} \tag{12}$$

If $\tilde V(\frac{1}{\underline{\lambda}})=0$, the time instants are $t_0^u=0$, $t_1^u=\frac{1}{\underline{\lambda}}$ and $t_2^u=\infty$.

Finally, let the control function, $z\mapsto \varphi_s(z)(t)=u(t)$, be defined by:

$$\begin{array}{lll} u(t) = 0 & \forall t\in[t_0^u,t_1^u) & \\ u(t) = Kz(t_k^u) & \forall t\in[t_k^u,t_{k+1}^u), & k\geq 1\end{array} \tag{13}$$

Remark 1. The boundary condition (2) with controller $u=\varphi_s(z)$ as defined in Definition 2 can be rewritten as:

$$y(t,0)=Gz(t)+d(t) \qquad t\in\mathbb{R}^+ \tag{14}$$

where

$$d(t)=BK(-z(t)+z(t_k^u)) \qquad t\in[t_k^u,t_{k+1}^u) \tag{15}$$

which can be seen as a deviation between the continuous controller $u=Kz$ and the event based controller of Definition 2. ∘

Proposition 2. *[Espitia et al. (2016)] Let y be a solution to (1)-(3). It holds that, for all $t\geq\frac{1}{\underline{\lambda}}$, $V(y(t,\cdot))=\tilde V(t)$, where $\tilde V(t)$ is given by (11).*

Theorem 1. *[Espitia et al. (2016)] Let K be in $\mathbb{R}^{n\times n}$ such that Assumption 1 holds for $G=H+BK$. Let $\mu>0$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$ and $\nu=\mu\underline{\lambda}$ be as in Proposition 1. Let σ be in $(0,1)$, $\alpha>0$ such that $(1+\alpha)G^TQ\Lambda G\leq e^{-2\mu}Q\Lambda$. Let ρ be the largest eigenvalue of $(1+\frac{1}{\alpha})Q\Lambda$, $\kappa=\frac{2\nu\sigma}{\rho}$, $\eta>2\nu(1-\sigma)$ and ε and φ_s be given in Definition 2. Let V be given by (9). Then the system (1)-(3),(4) with the controller $u=\varphi_s(z)$ has a unique solution and is globally exponentially stable.*

2.3 ISS dynamic event-based stabilization

In this section we introduce a second event-based control strategy relying on the previous one. It is inspired by Girard (2015) (for finite dimensional systems) where an internal dynamic variable is added to the event triggering condition in order to reduce the number of triggering times while guaranteeing the exponential stability. We recall that in ISS static event-based stabilization, events are triggered so that $\|d\|^2-\kappa\tilde V$ is always less than ε (see (12)). In this new approach, we will rather impose that the *weighted average value* of $\|d\|^2-\kappa\tilde V-\varepsilon$ is less than 0. Then, an internal dynamic will be presented under the form $m(t)=e^{-\eta t}\int_{\frac{1}{\underline{\lambda}}}^t e^{\eta s}\big(-\kappa\tilde V(s)-\varepsilon(s)+\|d(s)\|^2\big)ds$ for all $t\geq\frac{1}{\underline{\lambda}}$.

Definition 3. *[Definition of φ_d] Let σ be in $(0,1)$, $\tilde V(t)$, $\varepsilon(t)$ given as in Definition 2 for all $t\geq\frac{1}{\underline{\lambda}}$, and ρ and κ as in Theorem 1. Let us define φ_d the operator which maps z to u as follows:*

Let z be in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$. If $\tilde V(\frac{1}{\underline{\lambda}})>0$, let the increasing sequence of time instants (t_k^u) be defined iteratively by $t_0^u=0$, $t_1^u=\frac{1}{\underline{\lambda}}$, and for all $k\geq 1$,

$$t_{k+1}^u=\inf\{t\in\mathbb{R}^+|t>t_k^u\wedge m(t)\geq 0\} \tag{16}$$

where m satisfies the differential equation,

$$\begin{aligned}\dot m(t) &= -\eta m(t)+(-\kappa\tilde V(t)-\varepsilon(t)\\ &\quad +\|BK(-z(t)+z(t_k^u))\|^2)\\ m(\tfrac{1}{\underline{\lambda}}) &= 0\end{aligned} \tag{17}$$

for all $t\in[t_k^u,t_{k+1}^u)$ for a given $\eta>2\nu(1-\sigma)$.

If $\tilde V(\frac{1}{\underline{\lambda}})=0$, the time instants are $t_0^u=0$, $t_1^u=\frac{1}{\underline{\lambda}}$ and $t_2^u=\infty$.

Finally, let the control function, $z\mapsto\varphi_d(z)(t)=u(t)$, be defined by:

$$\begin{array}{lll} u(t) = 0 & \forall t\in[t_0^u,t_1^u) & \\ u(t) = Kz(t_k^u) & \forall t\in[t_k^u,t_{k+1}^u), & k\geq 1\end{array} \tag{18}$$

Note that $m(t_k^u)=0$ for all $k\geq 1$.

Proposition 3. *For any y^0 in $\mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$, there exists an unique solution to the closed-loop system (1)-(3),(4) and controller $u=\varphi_d(z)$.*

Proof. We recall a sufficient condition for the existence and uniqueness of solutions under a feedback control.

Lemma 1. *[Espitia et al. (2016)] Let φ be an operator from $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$ to $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^m)$ satisfying the following causality property: for all s in $\mathbb{R}^+$, for all z, $z^*\in\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$ $(\forall t\in[0,s], z(t)=z^*(t)) \implies (\forall t\in[0,s], u(t)=u^*(t))$ where $u=\varphi(z)$ and $u^*=\varphi(z^*)$. Let $y^0\in\mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$. Then, there exists a unique solution to the closed-loop system (1)-(3) with controller $u=\varphi(z)$ where z is defined by (4). Moreover, for all t in $\mathbb{R}^+$, $y(t,\cdot)\in\mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$ and for all $x\in[0,1]$ $y(\cdot,x)\in\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$.*

We will show that φ_d defined in Definition 3 satisfies hypothesis of Lemma 1. Once it is done, the result of Proposition 3 yields with φ_d.

Let us then prove that $u=\varphi_d(z)$ belongs to $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^m)$ provided z is in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$. Consider J a closed interval subset of $\mathbb{R}^+$. Since z is in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$, z has a finite number of discontinuities on J. We denote $t_1^z,\cdots,t_M^z\in J$ as the increasing sequence of these discontinuity time instants to which we add the extremities t_0^z and t_{M+1}^z of the interval J. The goal is to prove that u has a finite number of discontinuities on the time interval $[t_i^z,t_{i+1}^z]$, with $i\in\{0,...,M\}$. If $\tilde V(\frac{1}{\underline{\lambda}})=0$, there is only at most one discontinuity which is $t_1^u=\frac{1}{\underline{\lambda}}$. Let us see the case $\tilde V(\frac{1}{\underline{\lambda}})>0$. We define $w^i(t)$ as the continuation of $BKz(t)$ on the interval $[t_i^z,t_{i+1}^z]$ with the left limit of $BKz(t)$ in t_{i+1}^z, that is

$$w^i(t)=BKz(t), \quad \text{if} \quad t\in[t_i^z,t_{i+1}^z) \tag{19}$$

$$w^i(t_{i+1}^z)=\lim_{t\to(t_{i+1}^z)^-}BKz(t) \tag{20}$$

The definition of $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$ insures that the left limit of $BKz(t)$ exists and that $w^i(t)$ is continuous on the closed interval $[t_i^z,t_{i+1}^z]$. Then, it is uniformly continuous. It means that for all $\zeta>0$, there exists $\tau>0$ such that

$$\forall t,t'\in[t_i^z,t_{i+1}^z]:|t-t'|<\tau\Rightarrow\|w^i(t)-w^i(t')\|^2<\zeta$$

We denote $\overline{\tau}$ the value of τ when $\zeta=\varepsilon(t_{i+1}^z)$. We assume that there are at least two consecutive discontinuity instants in (t_i^z,t_{i+1}^z) and let t_k^u be the first one. Considering (16) and (17) in Definition 3 and using the continuity of m, ε and w^i, it holds at time $t=t_{k+1}^u$:

$$m(t_{k+1}^u)\geq 0 \tag{21}$$

Let us prove by contradiction that $|t_k^u-t_{k+1}^u|\geq\overline{\tau}$. To do that, let us assume that $|t_k^u-t_{k+1}^u|<\overline{\tau}$. Then, by uniform continuity, we have $\|w^i(t_k^u) - w^i(s)\|^2 < \varepsilon(t_{i+1}^z)$ for all $s \in [t_k^u, t_{k+1}^u]$. Since ε is a decreasing function, it holds also that $\|w^i(t_k^u) - w^i(s)\|^2 < \varepsilon(s)$. Due to the non-negativity of $\tilde{V}$, we have

$$\|w^i(t_k^u) - w^i(s)\|^2 < \varepsilon(s) + \kappa\tilde{V}(s) \tag{22}$$

Multiplying both sides of (22) by $e^{\eta s}$ and integrating on $[t_k^u, t_{k+1}^u]$, it yields,

$$\int_{t_k^u}^{t_{k+1}^u} e^{\eta s}\|w^i(t_k^u) - w^i(s)\|^2 ds$$
$$< \int_{t_k^u}^{t_{k+1}^u} e^{\eta s}\varepsilon(s)ds + \int_{t_k^u}^{t_{k+1}^u} e^{\eta s}\kappa\tilde{V}(s)ds$$

Multiplying both sides by $e^{-\eta t_{k+1}^u}$ and re-organizing the previous inequality, one gets,

$$e^{-\eta t_{k+1}^u}\int_{t_k^u}^{t_{k+1}^u} e^{\eta s}(\|w^i(t_k^u) - w^i(s)\|^2 - \kappa\tilde{V}(s) - \varepsilon(s))ds$$
$$< 0 \tag{23}$$

Using (17) and $m(t_k^u) = 0$, for all $t \geq t_k^u$, we have that $m(t) = e^{-\eta t}\int_{t_k^u}^t e^{\eta s}\Big(- \kappa\tilde{V}(s) - \varepsilon(s) + \|w^i(t_k^u) - w^i(s)\|^2\Big)ds$ Then, (23) is equivalent to $m(t_{k+1}^u) < 0$. which contradicts (21). Hence, $|t_k^u - t_{k+1}^u| \geq \overline{\tau}$. Therefore, $\overline{\tau}$ gives a lower bound for the duration between two input updates, depending only on the interval (t_i^z, t_{i+1}^z). Finally, an upper bound for the maximal number of input updates on (t_i^z, t_{i+1}^z) is given by:

$$\overline{s}_i = \left\lfloor \tfrac{t_{i+1}^z - t_i^z}{\overline{\tau}} \right\rfloor$$

The number of discontinuities of u on J is bounded by $\overline{S} = \sum_{i=1}^M \overline{s}_i + M + 2$ which is finite. To conclude, from (18) in Definition 3, u is piecewise constant, which yields $u \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$.

Due to the limitation of space, we refer to Espitia et al. (2016) to see the type of arguments that need to be used for the proof that operator φ_d satisfies the causality property.

We conclude then that Lemma 1 holds. This ends the proof of Proposition 3. •

Let us now state our main result of the paper.

Theorem 2. *Let K be in $\mathbb{R}^{n\times n}$ such that Assumption 1 holds for $G = H + BK$. Let $\mu > 0$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$ and $\nu = \mu\underline{\lambda}$ be as in Proposition 1. Let σ be in $(0,1)$, η and ε and φ_d be given in Definition 3. Let V be given by (9) and d given by (15). Then the system (1)-(3),(4) with the controller $u = \varphi_d(z)$ has a unique solution and is globally exponentially stable.*

Proof. The existence and uniqueness of a solution to system (1)-(3),(4), with $u = \varphi_d(z)$ is given by Proposition 3. Let us show that the system is GES but before proceeding, it is important to recall the following lemma which will be necessary to show that the system is GES and whose proof is given in the Appendix in Espitia et al. (2016).

Lemma 2. *Let y be a solution to (1)-(3) and let $V(y)$ be given by (9). Then, $t \mapsto V(y(t,\cdot))$ is continuous and right differentiable on $\mathbb{R}^+$ and its right time-derivative (denoted by D^+V) is given by:*

$$D^+V = y^T(\cdot,0)Q\Lambda y(\cdot,0) - y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1)$$
$$- 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx \tag{24}$$

Having stated this, assume first that $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$. Thanks to the boundary condition (2) with $u = \varphi_d(z)$, we obtain from its equivalent form (14) that (24) can be rewritten as follows:

$$D^+V = (Gz)^TQ\Lambda Gz + 2(Gz)^TQ\Lambda d + d^TQ\Lambda d$$
$$-z^Te^{-2\mu}Q\Lambda z - 2\mu\int_0^1 y^T(e^{-2\mu x}\Lambda Q)ydx \tag{25}$$

Using *Young's inequality* and the fact that $(1+\alpha)G^TQ\Lambda G \leq e^{-2\mu}Q\Lambda$, then from (25) it follows:

$$D^+V \leq -2\mu\int_0^1 y^T\Lambda Qye^{-2\mu x}dx + (1 + \tfrac{1}{\alpha})d^TQ\Lambda d$$

Since Q is diagonal positive definite, it holds $\Lambda Q \geq \underline{\lambda}Q$. Thus, taking $\nu = \mu\underline{\lambda}$, it yields, $D^+V \leq -2\nu V + (1 + \frac{1}{\alpha})d^TQ\Lambda d$ which can be rewritten as follows:

$$D^+V \leq -2\nu V + \rho\|d\|^2 \tag{26}$$

To show the global exponential stability of the closed-loop system, we consider the following Lyapunov function candidate W, for the augmented dynamical system, defined, for all $y(\cdot) \in \mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$ and $m \in \mathbb{R}^-$, $\varepsilon \in \mathbb{R}^+$, by

$$W(y,m,\varepsilon) = V(y) + \tfrac{\rho}{\eta - 2\nu(1-\sigma)}\varepsilon - \rho m \tag{27}$$

Computing the right time-derivative of (27), it yields,

$$D^+W = D^+V - \eta\tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon - \rho(-\eta m - \kappa\tilde{V} - \varepsilon + \|d\|^2) \tag{28}$$

Then, replacing (26) in (28), using $\kappa = \frac{2\sigma\nu}{\rho}$ and applying Proposition 2, we obtain for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+W(t) \leq -2\nu(1-\sigma)V(t)$$
$$+\rho\eta m(t) + \rho\varepsilon(t) - \eta\tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon(t)$$

which can be rewritten as follows:

$$D^+W(t) \leq -2\nu(1-\sigma)(W(t) - \tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon(t) + \rho m(t))$$
$$+\rho\eta m(t) + \rho\varepsilon(t) - \eta\tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon(t)$$

Simplifying the previous inequality, one gets

$$D^+W(t) \leq -2\nu(1-\sigma)W(t) + \rho(-2\nu(1-\sigma) + \eta)m(t)$$

From the definition of φ_d, events are triggered in order to guarantee for all $t \geq \frac{1}{\underline{\lambda}}$, that $m(t) \leq 0$. We obtain accordingly, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+W(t) \leq -2\nu(1-\sigma)W(t)$$

Now, using the *Comparison principle*, for all $t \geq \frac{1}{\underline{\lambda}}$, we have

$$V(y(t,\cdot)) \leq W(y(t,\cdot),m,\varepsilon)$$
$$\leq e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}W(y(\tfrac{1}{\underline{\lambda}},\cdot),m,\varepsilon) \tag{29}$$

The previous inequality holds even if $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$ since in this case $W(y(\frac{1}{\underline{\lambda}},\cdot),m,\varepsilon) = 0$ for all $t \geq \frac{1}{\underline{\lambda}}$. Knowing that

$m(\frac{1}{\lambda}) = 0$ and $\varepsilon(\frac{1}{\lambda}) = \varsigma V(y(\frac{1}{\lambda},\cdot))e^{-\eta\frac{1}{\lambda}}$, inequality (29) can be rewritten as follows,

$$V(y(t,\cdot)) \leq e^{-2\nu(1-\sigma)(t-\frac{1}{\lambda})}\Big(V(y(\tfrac{1}{\lambda},\cdot)) + \tfrac{\rho\varsigma}{\eta-2\nu(1-\sigma)}V(y(\tfrac{1}{\lambda},\cdot))e^{-\eta\frac{1}{\lambda}}\Big) \quad (30)$$

In addition, $V(y(\frac{1}{\lambda},\cdot))$ is given as follows (see Espitia et al. (2016) for further details):

$$V(y(\tfrac{1}{\lambda},\cdot)) \leq e^{2\theta\frac{\overline{\lambda}}{\lambda}}e^{2(\theta+\mu)}V(y^0) \quad (31)$$

Therefore, replacing (31) in (30) we get for all $t \geq \frac{1}{\lambda}$,

$$V(y(t,\cdot)) \leq e^{\left(2\theta\frac{\overline{\lambda}}{\lambda}+2(\theta+\mu)\right)} \times \left[1+\frac{\rho\varsigma e^{-\eta\frac{1}{\lambda}}}{\eta-2\nu(1-\sigma)}\right]e^{-2\nu(1-\sigma)t}V(y^0)$$

This ends the proof of Theorem 2. •

The following proposition states that the first triggering time after $t = \frac{1}{\lambda}$ occurs with φ_s than with φ_d. (Its proof is omitted due to space limitation).

Proposition 4. *Let $t^u_{2,s}$ be given by the rule (12) and let $t^u_{2,d}$ be given by the rule (16). It holds that after $t = \frac{1}{\lambda}$, $t^u_{2,s} \leq t^u_{2,d}$.*

3. NUMERICAL SIMULATIONS

We illustrate our results by considering the following example of a linear system of 2×2 hyperbolic conservation laws describing traffic flow on a simple roundabout (see Espitia et al. (2016)):

$$\partial_t y + \Lambda\partial_x y = 0 \quad (32)$$

with $y = [y_1 \quad y_2]^T$ and $\Lambda = \mathrm{diag}(1 \quad \sqrt{2})$, the boundary condition given by $y(t,0) = Hy(t,1) + Bu(t)$ where $H = \left(\begin{smallmatrix}0 & 0.7\\ 0.9 & 0\end{smallmatrix}\right)$, $B = I_2$ and $u(t) = Ky(t,1)$ with $K = \left(\begin{smallmatrix}0 & k_1\\ k_2 & 0\end{smallmatrix}\right)$. The initial condition is $y(0,x) = \left[\begin{smallmatrix}4x(x-1) & \sin(8\pi x)\end{smallmatrix}\right]^T$ for all $x \in [0,1]$.

3.1 Continuous stabilization: controller $u = \varphi_c(z)$

Here, $u(t) = \varphi_c(z)(t) = Kz(t)$ is the continuous controller acting from $t \geq \frac{1}{\lambda} = 1$. K has been designed such that $\rho_1(G) < 1$ with $G = H + BK$. With $K = \left(\begin{smallmatrix}0 & 0.3\\ -0.9 & 0\end{smallmatrix}\right)$ and $\Delta_G = \left(\begin{smallmatrix}0.9134 & 0\\ 0 & 1.2580\end{smallmatrix}\right)$, $\|\Delta_G G\Delta_G^{-1}\| = 0.7262 < 1$. It implies that the closed-loop system is GES. Condition (8) in Proposition 1 was checked with scalars $\mu = 0.1$, $\nu = 0.1$ and the symmetric matrix $Q = \left(\begin{smallmatrix}0.8346 & 0\\ 0 & 1.1191\end{smallmatrix}\right)$.

3.2 ISS static event-based stabilization: controller $u = \varphi_s(z)$

The boundary condition is now $y(t,0) = Hy(t,1) + Bu(t)$ where $u(t) = \varphi_s(z)(t)$. The parameters for the triggering algorithm are $\alpha = 0.5$, $\sigma = 0.9$. Therefore, $\rho = 4.7481$, $\kappa = 0.0379$ and $[(1+\alpha)G^TQ\Lambda G - e^{-2\mu}Q\Lambda] = \left(\begin{smallmatrix}-0.6833 & 0\\ 0 & -0.0439\end{smallmatrix}\right)$ is a symmetric negative definite matrix. Consequently,

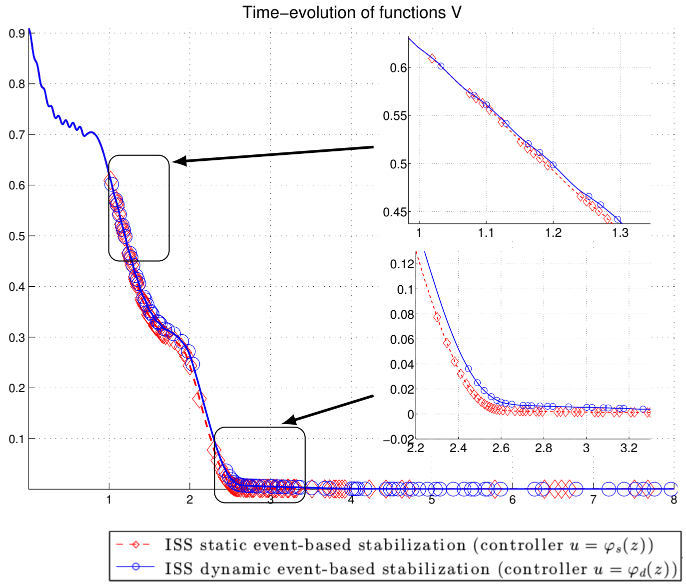


Fig. 1. Time-evolution of functions V.

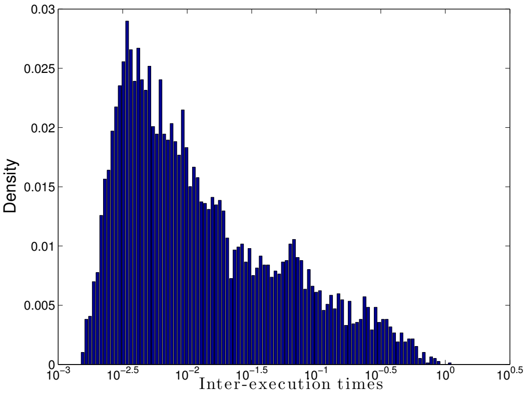
Theorem 1 holds. The function ε used in the triggering condition (12) is chosen to be $\varepsilon(t) = \varsigma V(1)e^{-\eta t}$, $t \in \mathbb{R}^+$ with $\eta = 1$, $V(1) = 0.6390$ and ς is such that $\varsigma V(1) = 1 \times 10^{-2}$. The number of events under this event-based approach was 109, counting them from $t \geq \frac{1}{\lambda} = 1$.

3.3 ISS dynamic event-based stabilization: controller $u = \varphi_d(z)$

The boundary condition is now $y(t,0) = Hy(t,1) + Bu(t)$ where $u(t) = \varphi_d(z)(t)$. The number of events under this event-based approach was 86, counting them from $t \geq 1$. Figure 1 shows functions V when stabilizing with φ_s and φ_d. It can be noticed that under the two event-based controllers φ_s and φ_d, global asymptotic stability is achieved with quite different observed rates despite similar theoretical guarantees. Besides this, the first triggering time occurs with φ_s. This is consistent with Proposition 4. In addition, for both event-based approaches, we ran simulations for several initial conditions given by $y^0_{a,b}(x) = [ax(1-x) \quad \frac{b}{2}\sin((2a)\pi x)]^T$, $a = 1,...,5$ and $b = 1,...,10$ on a frame of 8 s. We have computed the duration intervals between two control updates (inter-execution times). The mean value, standard deviation and the coefficient of variation of inter-execution times for both approaches are reported in Table 1 and the density of such inter-execution times is given in Figure 2. From this figure and Table 1, it can be observed that stabilization with φ_d results in larger inter-execution times than with φ_s which was expected because events generated according to φ_d-event-triggered rule, is a weighted average of those generated according to φ_s-event-triggered rule. The mean value of triggering times with φ_s was 158.3 events whereas, with φ_d, it was 109.1 events. It can be seen that using φ_d results in larger inter-execution times in average than φ_s. In addition, φ_d

Table 1. Mean value, standard deviation and variability of inter-execution times for φ_s and φ_d.

	Mean value	Standard deviation	Coefficient of variation
ISS static event-based	0.0432	0.0925	2.1427
ISS dynamic event-based	0.0640	0.0538	0.8411



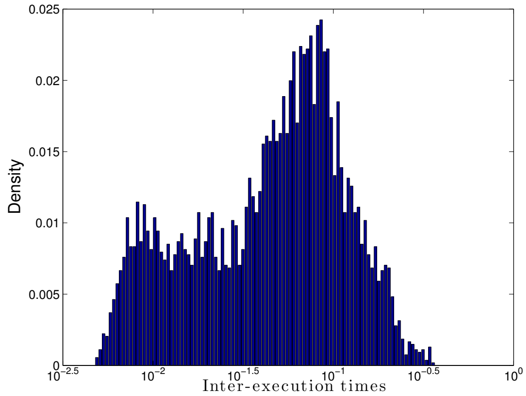


Fig. 2. Density of the inter-execution times with controller $u = \varphi_s(z)$ (left) and with controller $u = \varphi_d(z)$ (right).

reduces the variability of the inter execution times and with φ_s it is needed to sample faster than with φ_d.

4. CONCLUSION

In this paper, a new event-based boundary controller has been proposed. The analysis of global exponential stability is based on Lyapunov techniques. We have proved that under the new event-based stabilization strategy, the solution to the closed-loop system exists and is unique. This work leaves some open questions for future works. The event-based stabilization approaches may be applied to a linear hyperbolic system of balance laws.

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