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# Time-splitting approximation of the Cauchy problem for a stochastic conservation law

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## Abstract

In this paper, we present a time discretization of a first-order hyperbolic equation of nonlinear type set in  $\mathbb{R}^d$  and perturbed by a multiplicative noise. Using an operator splitting method, we are able to show the existence of an approximate solution. Thanks to recent techniques of well-posedness theory on this kind of stochastic equations, we show the convergence of such an approximate solution towards the unique stochastic entropy solution of the problem, as the time step of the splitting scheme converges to zero.

*Keywords:* Stochastic PDE, multiplicative noise, Itô integral, time-splitting method, Young measures, entropy solution

*2014 MSC:* 35L60, 60H15, 35L60

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## 1. Introduction

We are interested in the Cauchy problem for a nonlinear hyperbolic scalar conservation law with a multiplicative stochastic perturbation of type:

$$\begin{cases} du + \operatorname{div} \vec{f}(u) dt = h(u) dW & \text{in } ]0, T[ \times \mathbb{R}^d \times \Omega, \\ u(\omega, 0, x) = u_0(x), & \omega \in \Omega, x \in \mathbb{R}^d, \end{cases} \quad (1)$$

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where  $\text{div}$  is the divergence operator with respect to the space variable (which belongs to  $\mathbb{R}^d$ ),  $d \geq 1$ ,  $T > 0$  and  $W = \{W_t, \mathcal{F}_t; 0 \leq t \leq T\}$  is a standard adapted one-dimensional continuous Brownian motion defined on the classical Wiener space  $(\Omega, \mathcal{F}, P)$ . By denoting  $Q = ]0, T[ \times \mathbb{R}^d$ , this equation has to be understood in the following way: P-a.s. in  $\Omega$  and  $\forall \varphi \in \mathcal{D}(Q)$

$$\int_Q u \partial_t \varphi + \vec{f}(u) \cdot \nabla_x \varphi dx dt = \int_Q \int_0^t h(u) dW(s) \partial_t \varphi dx dt.$$

Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. The mathematical stake consists in introducing a selective criterion in order to identify the physical solution. In the present work we consider a stochastic version of the entropy condition proposed by S.N. Kruzhkov in the 70s, the one used in [2] and presented in Section 2.

We assume the following hypotheses:

- 10 H<sub>1</sub>:  $\vec{f} : \mathbb{R} \rightarrow \mathbb{R}^d$  is a Lipschitz-continuous function with  $\vec{f}(0) = \mathbf{0}$ .
- H<sub>2</sub>:  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz-continuous function with  $h(0) = 0$ .
- H<sub>3</sub>:  $u_0 \in L^2(\mathbb{R}^d)$ .
- H<sub>4</sub>: There exists  $M > 0$  such that  $\text{supp} h \subset [-M, M]$ .
- H<sub>5</sub>:  $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ <sup>1</sup>.

15 **Remark 1.**

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<sup>1</sup>where  $BV(\mathbb{R}^d)$  denotes the set of integrable functions with bounded variation on  $\mathbb{R}^d$ .

- 20
 .  $H_1, H_2$  and  $H_3$  are claimed conditions from the theoretical point of view to ensure the well-posedness in the sense of [2]. Let us first mention that  $H_1$  can be weakened by assuming that  $\vec{\mathbf{f}}$  is a locally Lipschitz continuous function. Indeed, since the solution  $u$  is bounded by a constant  $M_1$  depending only on  $M$  and  $\|u_0\|_\infty$ , the result holds by a truncation argument of  $\vec{\mathbf{f}}$  outside  $[-M_1, M_1]$ . Secondly, since  $\operatorname{div}[\vec{\mathbf{f}}(0)] = 0$ , one can assume by convenience that  $\vec{\mathbf{f}}(0) = \mathbf{0}$ .
  
- 25
 .  $H_4$  and  $H_5$  are specific conditions from the numerical analysis point of view. These are technical assumptions to control the estimates in the forthcoming lemmas, in particular to apply Lemma 3.3. Note that  $H_4$  is a necessary condition to keep the solution  $u$  bounded.

### 1.1. Former results

30
 Only few papers have been devoted to the theoretical study of hyperbolic scalar conservation laws with a multiplicative stochastic forcing: the development of a well-posedness theory has been done in [4]-[9] by the way of strong entropy solution, in [6] by the use of kinetic formulation, and in [2]-[3] with the notion of stochastic entropy solution. For a thorough exposition of all these papers, we refer the reader to the introduction of [1]. Concerning the numerical

35
 analysis of such stochastic problems, there is also, to our knowledge, few papers. Let us cite the work of Holden-Risebro [10] where a time-discretization of the

equation is proposed by the use of an operator-splitting method. They proposed a result of convergence to prove the existence of pathwise weak solutions to the Cauchy problem for (1) set in  $\mathbb{R}$ . In the recent paper of Bauzet [1], a generaliza-  
40 tion of the work of Holden-Risebro [10] is proposed in a bounded domain  $D$  of  $\mathbb{R}^d$ . The author proved that the pathwise weak solution obtained in [10] is the unique entropy weak solution of the stochastic conservation law and that the whole sequence of approximation given by the time-splitting scheme converges strongly. Kröker-Rohde [11] are interested in a recent work in a method of  
45 handling the finite volume schemes for the approximate solution of the Cauchy problem for (1) and investigate on a space-discretization of the equation. For a class of strongly monotone numerical fluxes they established the pathwise convergence of a semi-discrete finite volume solution towards a stochastic entropy solution. Since the authors use a stochastic version of the compensated  
50 compactness approach, the study is restricted to the one-dimensional case.

### *1.2. Goal of the study and main result*

In a recent published paper [1], a generalization of the time-splitting method introduced much earlier by Holden and Risebro [10] is proposed to approximate solution of the stochastic conservation law (1) set in a bounded domain  $D$  of  
55  $\mathbb{R}^d$  and with homogeneous Dirichlet boundary condition. Precisely, the author

showed that the pathwise weak limit obtained in the former study of Holden and Risebro is the unique stochastic entropy solution of (1) and that the whole sequence of approximate solutions converges strongly with respect to all its variables. Our aim in the present paper is to prove that the tools developed in [1] are sufficiently strong to be extended to an unbounded domain and allow us to complete the work of Bauzet-Vallet-Wittbold [2] by a numerical analysis using their well-posedness theory for stochastic entropy solution. The main result of the present paper which deals with the convergence of our numerical scheme is stated in the following theorem.

**Theorem 1.1.** *Assume that hypotheses  $H_1$  to  $H_5$  hold. Set  $N \in \mathbb{N}^*$  and let  $\Delta = \frac{T}{N}$  be the time step of the time-splitting operator scheme. Then, the associated approximate solution denoted  $u^\Delta$  in the sequel and defined p.13 converges in  $L^p_{loc}(\Omega \times Q)$  for any finite  $p$  towards the unique stochastic entropy solution of the stochastic conservation law (1).*

The paper is organized as follows. Section 2 is devoted to the presentation of the theoretical background : we recall the definition of a solution for (1) proposed in [2] and their well-posedness result. In Section 3, we present firstly the time-splitting scheme used to approximate the solution of our problem. Several preliminaries results satisfied by the time-splitting approximate solution  $u^\Delta$  are

75 then stated. The remainder of Section 3 is devoted to show the convergence of  $u^\Delta$  towards the unique stochastic entropy solution of (1). To illustrate our proposal, we present in Section 4 numerical experiments around the stochastic burgers' equation in the one-dimensional case.

### 1.3. Notations

- 80 .  $E$  denotes the integral over  $\Omega$  with respect to the probability measure  $P$ .
- . Consider  $BV(\mathbb{R}^d)$  the set of integrable functions with bounded variation on  $\mathbb{R}^d$  endowed with the norm  $\|v\|_{BV(\mathbb{R}^d)} = \|v\|_{L^1(\mathbb{R}^d)} + TV_x(v)$ , where  $TV_x(v)$  denotes the total variation of  $v$  on  $\mathbb{R}^d$  (see Evans-Gariepy [7]).
  - . Denote by  $N_w^2(0, T, L^2(\mathbb{R}^d))$  the set of the predictable processes of
 

85  $L^2(]0, T[ \times \Omega, L^2(\mathbb{R}^d))$  (Da Prato-Zabczyk[5] p.94).
  - . Consider  $\mathcal{E}$  the set of any  $C^{2,1}(\mathbb{R})$  nonnegative convex approximation of the absolute-value function such that  $\eta(0) = 0$  and that there exists  $\delta > 0$  such that  $\eta'(x) = 1$  (resp.  $-1$ ) if  $x > \delta$  (resp.  $x < \delta$ ).
  - .  $F^\eta$  denotes the entropy flux defined for any  $a, b \in \mathbb{R}$  and for any smooth
 

90 entropy  $\eta \in \mathcal{E}$  by  $F^\eta(a, b) = \int_b^a \eta'(\sigma - b) f'(\sigma) d\sigma$ . Note in particular that  $F^\eta$  is a Lipschitz-continuous function.

## 2. Existence and uniqueness result

Let us recall the definitions and the result introduced in the paper of Bauzet-Vallet-Wittbold [2]. These results are obtained under hypotheses  $H_1$  to  $H_3$ .

95 **Definition 2.1. (Stochastic entropy solution)**

A function  $u$  of  $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d)) \cap L^\infty(0, T; L^2(\Omega, L^2(\mathbb{R}^d)))$  is an entropy solution of the stochastic scalar conservation law (1) with the initial condition  $u_0 \in L^2(\mathbb{R}^d)$ , if  $P$ -a.s in  $\Omega$ , for any  $\eta \in \mathcal{E}$  and for any  $(\lambda, \varphi) \in \mathbb{R} \times \mathcal{D}^+([0, T] \times \mathbb{R}^d)$

$$0 \leq \int_{\mathbb{R}^d} \eta(u_0 - \lambda) \varphi(0, x) dx + \int_Q \eta(u - \lambda) \partial_t \varphi dx dt + \int_Q F^\eta(u, \lambda) \cdot \nabla_x \varphi dx dt \\ + \int_0^T \int_{\mathbb{R}^d} \eta'(u - \lambda) h(u) \varphi dx dW(t) + \frac{1}{2} \int_Q h^2(u) \eta''(u - \lambda) \varphi dx dt.$$

For technical reasons, as in [2], we also need to consider a generalized notion of  
 100 entropy solution. In fact, in a first step, we will only prove the convergence of the approximate solution to a measure-valued entropy solution. Then, thanks to the result of uniqueness stated in Theorem 2.3, we will be able to deduce the convergence of the approximate solution to the unique stochastic entropy solution of Problem (1).

105 **Definition 2.2. (Measure-valued entropy solution)**

A function  $\mathbf{u}$  of  $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d \times (0, 1)))$  is a measure-valued entropy solution of the stochastic scalar conservation law (1)



with the initial condition  $u_0 \in L^2(\mathbb{R}^d)$ , if P-a.s in  $\Omega$ , for any  $\eta \in \mathcal{E}$  and for any  $(\lambda, \varphi) \in \mathbb{R} \times \mathcal{D}^+([0, T[ \times \mathbb{R}^d)$

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^d} \eta(u_0 - \lambda) \varphi(0, x) dx + \int_Q \int_0^1 \eta(\mathbf{u}(\cdot, \alpha) - \lambda) \partial_t \varphi d\alpha dx dt \\
&\quad + \int_Q \int_0^1 F^\eta(\mathbf{u}(\cdot, \alpha), \lambda) \cdot \nabla_x \varphi d\alpha dx dt \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha) - \lambda) h(\mathbf{u}(\cdot, \alpha)) \varphi d\alpha dx dW(t) \\
&\quad + \frac{1}{2} \int_Q \int_0^1 h^2(\mathbf{u}(\cdot, \alpha)) \eta''(\mathbf{u}(\cdot, \alpha) - \lambda) \varphi d\alpha dx dt.
\end{aligned}$$

110 And the main result of [2] is

**Theorem 2.3.** *Under assumptions  $H_1$  to  $H_3$  there exists a unique measure-valued entropy solution for the Problem (1) and this solution is obtained by viscous approximation. Moreover, it is the unique stochastic entropy solution in the sense of Definition 2.1.*

### 115 3. Time-splitting method

#### 3.1. Introduction

Our aim is to approximate Problem (1) under Assumptions  $H_1$  to  $H_5$ . We introduce the method proposed by Holden-Risebro in [10], which consists in splitting the effect of the source term. This technique allows us to construct a  
120 sequence to approximate the solution of (1). In few words, this approach is based on considering the equation in two parts, solving first a stochastic differential

equation, and then using the obtained solution as an initial condition for a homogeneous hyperbolic scalar conservation law. As an extension of [10], we propose in this paper to generalize their estimates on the approximate sequence  
125 to the  $\mathbb{R}^d$ -case, in the idea of Chen-Ding-Karlsen [4] concerning BV estimates. Following the notations introduced in [10] we define here two operators for  $s, t \in [0, T]$  and associated results. Let  $R(t, s)$  be the operator which takes a number  $\bar{u}$  to the solution  $u$  at time  $t$  of the stochastic differential equation,  $\forall t \in [s, T]$

$$\begin{cases} du(t) &= h(u)dW(t) \\ u(t=s) &= \bar{u}, \end{cases} \quad (2)$$

130 *i.e*  $u(t) = R(t, s)\bar{u} = \bar{u} + \int_s^t h(u)dW.$

And  $S(t-s)$  denotes the operator which takes an initial function  $u(x, s)$  at time  $s$  to the weak entropy solution  $u$  at time  $t$  of the first-order hyperbolic equation

$$\begin{cases} u_t + \operatorname{div} \vec{\mathbf{f}}(u) &= 0 & \text{in } ]0, T[ \times \mathbb{R}^d, \\ u(t=s) &= u(x, s), \end{cases} \quad (3)$$

*i.e*  $u(x, t) = S(t-s)u(x, s).$

**Lemma 3.1.** *Consider  $s \in [0, T]$ . Then  $P$ -a.s in  $\Omega$  and for all  $t \in [s, T]$ ,  
135  $R(t, s)$  will take  $[-M, M]$  into itself and be the identity outside this interval, where  $M > 0$  is defined in  $H_4$ .*

*Proof.* Consider the process  $u$  defined for all  $t \in [s, T]$  by  $u(t) = R(t, s)u(s)$ . Applying the Itô formula to a regular function  $\Psi$  independent of the time variable  $t$ , vanishing in  $[-M, M]$  and increasing outside this interval, one gets, P-a.s:

$$\begin{aligned} \Psi(u(t)) &= \Psi(u(s)) + \int_s^t \underbrace{\Psi_t(u(\sigma))}_{=0} d\sigma + \int_s^t \underbrace{\Psi_x(u(\sigma))h(u(\sigma))}_{=0} dW(\sigma) \\ &\quad + \frac{1}{2} \int_s^t \underbrace{\Psi_{xx}(u(\sigma))h^2(u(\sigma))}_{=0} d\sigma, \quad \forall t \in [s, T]. \end{aligned}$$

140 Consider  $\omega \in \tilde{\Omega}$ , where  $\tilde{\Omega}$  is a full measure subset of  $\Omega$  and  $t \in [s, T]$ . Thus, if  $u(s, \omega) \in [-M, M]$ ,  $\Psi(u(s, \omega)) = 0 = \Psi(u(t, \omega))$  and  $u(t, \omega) \in [-M, M]$ . Else,  $\Psi(u(t, \omega)) = \Psi(u(s, \omega))$ , by injectivity of  $\Psi$  in  $\mathbb{R} - [-M, M]$ ,  $u(t, \omega) = u(s, \omega)$  and  $R(t, s) = I_d$ .  $\square$

**Lemma 3.2.** Consider  $s \in [0, T]$ ,  $v_0 \in L^1(\Omega \times \mathbb{R}^d) \cap L^2(\Omega \times \mathbb{R}^d)$  a  $\mathcal{F}_s$ -measurable  
145 process such that

$$E[TV_x(v_0)] < \infty.$$

Define the process  $v$  for all  $t \in [s, T]$  by  $v(t) = R(s, t)v_0$ . Then for all  $t \in [s, T]$

$$E[TV_x(v(t))] \leq E[TV_x(v_0)].$$

**Remark 2.** Let us mention that using the lower semi-continuity property and the positivity of the total variation  $TV_x$  on  $L^1(\mathbb{R}^d)$ , for all  $v$  in  $L^1(\Omega \times \mathbb{R}^d)$ ,  $E[TV_x(v)]$  has a sense.

150 *Proof.* Consider  $s \in [0, T]$  and let  $v_0 \in L^1(\Omega \times \mathbb{R}^d) \cap L^2(\Omega \times \mathbb{R}^d)$  be a  $\mathcal{F}_s$ -measurable process with  $E[TV_x(v_0)] < \infty$ . Define for all  $t \in [s, T]$   $v(t) =$

$R(s, t)v_0$  and consider  $\eta_\delta \in \mathcal{E}$ . Applying Itô's formula with the process  $v$  and the function  $\eta_\delta$ , one gets by taking the integral over  $\mathbb{R}^d$  and the expectation, for every  $t \in [s, T]$

$$E \int_{\mathbb{R}^d} \eta_\delta(v(t)) dx = E \int_{\mathbb{R}^d} \eta_\delta(v_0) dx + \frac{1}{2} E \int_{\mathbb{R}^d} \int_s^t \eta_\delta''(v(t)) h^2(v) d\sigma dx.$$

155 Passing to the limit on  $\delta$  to 0 to get for every  $t \in [s, T]$

$$E \|v(t)\|_{L^1(\mathbb{R}^d)} = E \|v_0\|_{L^1(\mathbb{R}^d)}.$$

Following the idea of Chen-Ding-Karlsen [4] we consider  $v_0^\epsilon$  a smooth approximation of  $v_0$  such that

$$v_0^\epsilon \rightarrow v_0 \quad \text{in } L^p(\Omega \times \mathbb{R}^d), \quad p = 1, 2 \quad (4)$$

$$E \int_{\mathbb{R}^d} |\nabla v_0^\epsilon|^2 dx < \infty, \quad \text{for each fixed } \epsilon \quad (5)$$

$$E[TV_x(v_0^\epsilon)] \leq E[TV_x(v_0)]. \quad (6)$$

The proof of these results relies on approximation of BV-function in the deterministic setting, we refer the reader to Evans-Gariepy [7] p.172.

160 Let us define for all  $t$  in  $[s, T]$   $v_\epsilon(t) = R(s, t)v_0^\epsilon$ . Now we need estimate on  $\partial_{x_i} v_\epsilon$  in order to obtain BV estimate for  $v$ . Applying Itô's formula to the process  $d(v_\epsilon - v) = [h(v_\epsilon) - h(v)]dW$  and the function  $\eta_\delta$ , taking the integral over  $\mathbb{R}^d$  and the expectation, we obtain for every  $t \in [s, T]$

$$\begin{aligned} E \int_{\mathbb{R}^d} \eta_\delta(v_\epsilon - v)(t) dx &= E \int_{\mathbb{R}^d} \eta_\delta(v_0^\epsilon - v_0) dx \\ &\quad + \frac{1}{2} E \int_{\mathbb{R}^d} \int_s^t \eta_\delta''(v_\epsilon - v) [h(v_\epsilon) - h(v)]^2 d\sigma dx. \end{aligned}$$

Passing to the limit on  $\delta$  to 0 to get for every  $t \in [s, T]$

$$E\|(v_\epsilon - v)(t)\|_{L^1(\mathbb{R}^d)} = E\|v_0^\epsilon - v_0\|_{L^1(\mathbb{R}^d)}.$$

165 Thus, for every  $t \in [s, T]$ ,  $v_\epsilon(t) \rightarrow v(t)$  in  $L^1(\Omega \times \mathbb{R}^d)$ .

As P-a.s and for all  $t \in [0, T]$ ,  $v_\epsilon(t) = v_\epsilon(0) + \int_0^t h(v_\epsilon)dW$  in  $W^{1,2}(\mathbb{R}^d)$ , using the linear-continuity of the derivation operator  $\partial_{x_i} : W^{1,2}(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  for all  $i \in \{1, \dots, d\}$  and the chain-rule derivation formula, we get for all  $i \in \{1, \dots, d\}$   $\partial_{x_i}v_\epsilon(0) = \partial_{x_i}v_0^\epsilon$  and:

$$\begin{aligned} \partial_{x_i}v_\epsilon(t) &= \partial_{x_i}v_\epsilon(0) + \partial_{x_i} \int_0^t h(v_\epsilon)dW \\ &= \partial_{x_i}v_\epsilon(0) + \int_0^t h'(v_\epsilon)\partial_{x_i}v_\epsilon dW, \text{ in } L^2(\mathbb{R}^d). \end{aligned}$$

170 Applying Itô's formula with such a process and the function  $\eta_\delta$  to get that, after taking the integral over  $\mathbb{R}^d$ , the expectation and passing to the limit on  $\delta$ , for all  $t \in [s, T]$

$$E \int_{\mathbb{R}^d} |\partial_{x_i}v_\epsilon| dx = E \int_{\mathbb{R}^d} |\partial_{x_i}v_0^\epsilon| dx < \infty. \quad (7)$$

Thus, for all  $t \in [s, T]$  and P-a.s,  $v_\epsilon(t) \in BV(\mathbb{R}^d)$ . As for all  $t \in [s, T]$   $v_\epsilon(t) \rightarrow v(t)$  in  $L^1(\Omega \times \mathbb{R}^d)$ , for a subsequence denoted in the same way, for all  $t \in [s, T]$

175 and P-a.s,  $v_\epsilon(t) \rightarrow v(t)$  in  $L^1(\mathbb{R}^d)$ . According to Malek-Necas-Otto-Rokyta-Ruzicka [12] p.36, we thus have for all  $t \in [s, T]$  and P-a.s

$$TV_x(v(t)) \leq \liminf_\epsilon TV_x(v_\epsilon(t)).$$

Consequently, taking the expectation thanks to Remark 2, using Fatou's Lemma, (7) then (6), one gets that for every  $t \in [s, T]$

$$E[TV_x(v(t))] \leq \liminf_{\epsilon} E[TV_x(v_{\epsilon}(t))] = \liminf_{\epsilon} E[TV_x(v_{\epsilon}^s)] \leq E[TV_x(v_0)],$$

and the result holds.  $\square$

**180 Lemma 3.3.** *Let  $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ ,  $t > 0$ , and  $u(t) = S(t)u_0$ . Then: there exists a constant  $C > 0$  such that for almost every  $t > 0$ ,*

- i)  $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$ ,*
- ii)  $\int_{\mathbb{R}^d} |u(t_1, x) - u(t_2, x)| dx \leq CTV_x(u_0)|t_1 - t_2|$ , for all  $t_1, t_2 \in [0, T]$ ,*
- iii)  $TV_x(u(t)) \leq TV_x(u_0)$ .*

*Proof.* These results are classical ones and the proof would be outside the scope of the present work, we refer the reader to Malek-Necas-Otto-Rokyta-Ruzicka [12] p.68.  $\square$

**185 3.2. Construction of the approximate solution**

Let us now explain the construction of the approximate solution as introduced in Holden-Risebro [10]. We consider a positive integer  $N$ , denote by  $\Delta = \frac{T}{N}$  and split the time interval by denoting  $t_n = n\Delta$ ,  $n \in \{0, \dots, N\}$  each point of the time discretization. For each step of discretization  $\Delta$ , we consider

**190** the function defined for all  $t \in [0, T]$  and  $x \in \mathbb{R}^d$

$$u^\Delta(t, x) = \begin{cases} u^n(x) & \text{if } t = t_n \\ R(t, t_n)u^n(x) & \text{if } t \in ]t_n, t_{n+1}[ , \end{cases}$$

where the sequence  $(u^n)_{n \in \mathbb{N}}$  is defined by

$$\begin{cases} u^0(x) & = u_0(x) \\ u^{n+1}(x) & = S(\Delta)R(t_{n+1}, t_n)u^n(x). \end{cases}$$

**Notations:**  $\forall n \in \{0, \dots, N-1\}$ ,  $t \in [0, T]$  and  $x \in \mathbb{R}^d$ :

.  $u_-^{n+1}(x) := R(t_{n+1}, t_n)u^n(x)$ .

.  $\tilde{u}(t, x) := S(t - t_n)R(t_{n+1}, t_n)u^n(x) = S(t - t_n)u_-^{n+1}(x)$ .

195 **Proposition 1.** (*A priori estimate*) *There exists a constant  $M_1$  independent of  $n$  and  $\Delta$  such that P-a.s in  $\Omega$  and for all  $t \in [0, T]$*

$$\|u^\Delta(t)\|_{L^\infty(\mathbb{R}^d)} \leq M_1 := \max(M, \|u_0\|_{L^\infty(\mathbb{R}^d)}).$$

*Proof.* Let us mention that the construction of  $u^\Delta$  is done by induction, so the proofs of the associated results also rely on inductive reasoning. Consider  $n \in \{0, \dots, N-1\}$ , and  $u^{n+1} = S(\Delta)u_-^{n+1}$ . Thanks to Lemma 3.3 i),

$$\|u^{n+1}\|_{L^\infty(\mathbb{R}^d)} \leq \|u_-^{n+1}\|_{L^\infty(\mathbb{R}^d)}, \text{ P-a.s.}$$

200 Moreover, thanks to Lemma 3.1, P-a.s and  $\forall t \in [t_n, t_{n+1}]$

$$\|R(t, t_n)u^n\|_{L^\infty(\mathbb{R}^d)} \leq \max(M, \|u^n\|_{L^\infty(\mathbb{R}^d)})$$

and particularly for  $t = t_{n+1}$ , one has P-a.s

$$\begin{aligned} \|u_-^{n+1}\|_{L^\infty(\mathbb{R}^d)} &= \|R(t_{n+1}, t_n)u^n\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \max(M, \|u^n\|_{L^\infty(\mathbb{R}^d)}) \\ &\leq \max(M, \|u^0\|_{L^\infty(\mathbb{R}^d)}) := M_1. \end{aligned}$$

Notice that the construction of  $u^\Delta$  is countable, so  $P$ -a.s, for all  $t \in [0, T]$  and all possible discretization parameter  $N \in \mathbb{N}^*$ :

$$\|u^\Delta(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq M_1,$$

where  $M_1$  does not depend on  $\Delta$  and the result holds.  $\square$

205 **Proposition 2.**  $E[TV_x(u^i)] \leq TV_x(u_0)$ , for every  $i \in \{0, \dots, N\}$ .

*Proof.* Consider  $i \in \{0, \dots, N-1\}$ . As  $u^i = S(\Delta)u_-^i$ , and  $u_-^i = R(t_i, t_{i-1})u^{i-1}$ , using Lemma 3.3 then Lemma 3.2 one gets

$$E[TV_x(u^i)] \leq E[TV_x(u_-^i)] \leq E[TV_x(u^{i-1})],$$

a reasoning by induction gives us the result.  $\square$

**Lemma 3.4.** Let  $n \in \{1, \dots, N\}$  and consider  $t \in [t_n, t_{n+1}[$ . Then for any  
210  $\phi \in \mathcal{D}(\mathbb{R}^d)$ :

$$E \int_{\mathbb{R}^d} |u^\Delta(t_{n+1}, x) - u^\Delta(t, x)| \cdot |\phi(x)| dx \leq C\Delta TV_x(u_0) + \tilde{C}\sqrt{\Delta},$$

where  $C$  and  $\tilde{C}$  are independent of  $\Delta$ .

*Proof.* Let  $n \in \{1, \dots, N\}$  and consider  $t \in [t_n, t_{n+1}[$ . For all  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} u^\Delta(t, x) &= R(t, t_n)u^n(x) = u^n(x) + \int_{t_n}^t h(u^\Delta(\sigma))dW(\sigma) \\ u^\Delta(t_{n+1}, x) &= u^{n+1}(x). \end{aligned}$$

Thus,

$$\begin{aligned} E \int_{\mathbb{R}^d} |u^\Delta(t_{n+1}, x) - u^\Delta(t, x)| \cdot |\phi(x)| dx &\leq E \int_{\mathbb{R}^d} |u^{n+1}(x) - u^n(x)| \cdot |\phi(x)| dx \\ &\quad + E \int_{\mathbb{R}^d} \left| \int_{t_n}^t h(u^\Delta(\sigma))dW(\sigma) \right| \cdot |\phi(x)| dx \end{aligned}$$



Thanks to Lemma 3.3 ii) and Proposition 2 one shows that

$$\begin{aligned}
& E \int_{\mathbb{R}^d} |u^{n+1}(x) - u^n(x)| dx \\
\leq & E \int_{\mathbb{R}^d} |u^{n+1}(x) - u_-^{n+1}(x)| + |u_-^{n+1}(x) - u^n(x)| \cdot |\phi(x)| dx \\
= & E \int_{\mathbb{R}^d} |S(\Delta)u_-^{n+1}(x) - u_-^{n+1}(x)| \cdot |\phi(x)| + |R(t_{n+1}, t_n)u^n(x) - u^n(x)| \cdot |\phi(x)| dx \\
\leq & C\Delta \cdot E[TV_x(u_-^{n+1})] + E \int_{\mathbb{R}^d} \left| \int_{t_n}^{t_{n+1}} h(u^\Delta(s, x)) dW(s) \right| \cdot |\phi(x)| dx \\
\leq & C\Delta \cdot TV_x(u_0) + E \int_{\mathbb{R}^d} \left| \int_{t_n}^{t_{n+1}} h(u^\Delta(s, x)) dW(s) \right| \cdot |\phi(x)| dx.
\end{aligned}$$

215 Consider  $K$  a compact subset of  $\mathbb{R}^d$  such that  $\text{supp}\phi \subset K$  and notice that  $|t_n - t| \leq \Delta$ . Thus by using Cauchy-Schwarz inequality on  $\Omega \times K$  and then Itô isometry, one gets

$$\begin{aligned}
E \int_{\mathbb{R}^d} \left| \int_{t_n}^t h(u^\Delta(s, x)) dW(s) \right| \cdot |\phi(x)| dx & \leq \|\phi\|_{L^\infty(\mathbb{R}^d)} E \int_K \left| \int_{t_n}^t h(u^\Delta(s, x)) dW(s) \right| dx \\
& \leq c \left( E \int_K \left| \int_{t_n}^t h(u^\Delta(s, x)) dW(s) \right|^2 dx \right)^{\frac{1}{2}} \\
& = c \left( E \int_K \int_{t_n}^t h^2(u^\Delta(s, x)) ds dx \right)^{\frac{1}{2}} \\
& \leq \tilde{c} \sqrt{\Delta}.
\end{aligned}$$

Similarly  $E \int_{\mathbb{R}^d} \left| \int_{t_n}^{t_{n+1}} h(u^\Delta(s, x)) dW(s) \right| \cdot |\phi(x)| dx \leq \tilde{c} \sqrt{\Delta}$ , and so  $\tilde{C} = 2\tilde{c}$ .  $\square$

### 3.3. Entropy formulation

220 We follow the idea of Peyrouet [14] for introducing the entropy formulation satisfied by the approximate solution. In order to do this, consider

$$\tilde{u}(t, x) = S(t - t_n)u_-^{n+1}(x), t \in [t_n, t_n + 1].$$

As a weak entropy solution of a conservation law,  $\tilde{u}$  satisfies the following condition, for any  $\eta \in \mathcal{E}$  and any  $(k, \varphi) \in \mathbb{R} \times \mathcal{D}^+([0, T] \times \mathbb{R}^d)$  :

$$\begin{aligned} & \int_{\mathbb{R}^d} \eta(\tilde{u}(t_n) - k) \varphi(t_n) dx - \int_{\mathbb{R}^d} \eta(\tilde{u}(t_{n+1}) - k) \varphi(t_{n+1}) dx \\ & \quad + \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(\tilde{u} - k) \partial_t \varphi + F^\eta(\tilde{u}, k) \nabla \varphi dt dx \geq 0. \end{aligned}$$

Consider  $(k, \varphi, \eta) \in \mathbb{R} \times \mathcal{D}^+([0, T] \times \mathbb{R}^d) \times \mathcal{E}$ ,  $K$  a compact set of  $\mathbb{R}^d$  such that  
225  $\text{supp} \varphi(t, \cdot) \subset K$  and denote for  $s \in [t_n, t_{n+1}]$ ,  $v(s) := R(s, t_n) u^n$  the solution in  $[t_n, t_{n+1}]$  of the stochastic differential equation

$$\begin{cases} dv &= h(v) dW \\ v(t = t_n) &= u^n. \end{cases}$$

Applying the Itô formula to the process  $v$  and the regular function  $\Psi(t, \lambda) = \eta(\lambda - k)$ , one gets  $P$ -a.s.:

$$\begin{aligned} \eta(v(t_{n+1}) - k) &= \eta(v(t_n) - k) + \int_{t_n}^{t_{n+1}} \eta'(v(t) - k) h(v(t)) dW(t) \\ & \quad + \frac{1}{2} \int_{t_n}^{t_{n+1}} \eta''(v(t) - k) h^2(v(t)) dt. \end{aligned}$$

Remark that  $v(t) = u^\Delta(t)$  for all  $t \in [t_n, t_{n+1}[$  and  $v(t_{n+1}) = \tilde{u}(t_n)$ , in this way,

230  $P$ -a.s.:

$$\begin{aligned} & \int_{\mathbb{R}^d} \eta(\tilde{u}(t_n, x) - k) \varphi(t_n, x) dx - \int_{\mathbb{R}^d} \eta(u^\Delta(t_n, x) - k) \varphi(t_n, x) dx \\ &= \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t, x) - k) h(u^\Delta(t, x)) dW(t) \varphi(t_n, x) dx \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t, x) - k) h^2(u^\Delta(t, x)) dt \varphi(t_n, x) dx. \end{aligned}$$

Moreover,

$$\int_{\mathbb{R}^d} \eta(\tilde{u}(t_{n+1}, x) - k) \varphi(t_{n+1}, x) dx = \int_{\mathbb{R}^d} \eta(u^\Delta(t_{n+1}, x) - k) \varphi(t_{n+1}, x) dx.$$

Thus one first gets, for any P-measurable set  $A$

$$\begin{aligned}
& E \left( \int_{\mathbb{R}^d} \eta(u^\Delta(t_n, x) - k) \varphi(t_n) dx 1_A - \int_{\mathbb{R}^d} \eta(u^\Delta(t_{n+1}, x) - k) \varphi(t_{n+1}, x) dx 1_A \right) \\
& + E \left( \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t, x) - k) h(u^\Delta(t, x)) dW(t) \varphi(t_n, x) dx 1_A \right) \\
& + \frac{1}{2} E \left( \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t, x) - k) h^2(u^\Delta(t, x)) dt \varphi(t_n, x) dx 1_A \right) \\
& + E \left( \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(\tilde{u}(t, x) - k) \varphi_t(t, x) + F^\eta(\tilde{u}(t, x), k) \nabla \varphi(t, x) dt dx 1_A \right) \\
& \geq 0.
\end{aligned}$$

We propose to approximate  $E(\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(\tilde{u}(t, x) - k) \varphi_t(t, x) dt dx 1_A)$  by

235  $E(\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(u^{n+1} - k) \varphi_t(t, x) dt dx 1_A)$  making an error only of order  $\Delta^2$ . Indeed, thanks to Lemma 3.3 ii) and Proposition 2

$$\begin{aligned}
& \left| E \left( \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(\tilde{u}(t) - k) \varphi_t dt dx - \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(u^{n+1} - k) \varphi_t dt dx 1_A \right) \right| \\
& \leq C E \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} |\eta(\tilde{u}(t) - k) - \eta(u^{n+1} - k)| \cdot |\varphi_t| dt dx \\
& \leq C \|\varphi_t\|_\infty E \int_K \int_{t_n}^{t_{n+1}} |\tilde{u}(t) - u^{n+1}| dt dx. \\
& \leq C \int_{t_n}^{t_{n+1}} E[TV_x(u_-^{n+1})] |t - t_{n+1}| dt \\
& \leq C \Delta^2 TV_x(u_0).
\end{aligned}$$

In the same way, one shows by using the Lipschitz-continuity of  $F^\eta(\cdot, k)$  that

$$\begin{aligned}
& E \left( \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} F^\eta(u^{n+1}(x), k) \nabla \varphi dt dx 1_A \right) \text{ is an approximation of} \\
& E \left( \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} F^\eta(\tilde{u}(t, x), k) \nabla \varphi(t, x) dt dx 1_A \right) \text{ with an error of order } \Delta^2.
\end{aligned}$$

Finally we obtain by summing over  $n$

$$\begin{aligned}
& E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(u^\Delta(t_{n+1}, x) - k) \varphi_t(t, x) dt dx 1_A \right) \\
& + E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} F^\eta(u^\Delta(t_{n+1}, x), k) \nabla \varphi(t, x) dt dx 1_A \right) \\
& + E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t, x) - k) h(u^\Delta(t, x)) dW(t) \varphi(t_n, x) dx 1_A \right) \\
& + \frac{1}{2} E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t, x) - k) h^2(u^\Delta(t, x)) dt \varphi(t_n, x) dx 1_A \right) \\
& + \text{mes}(A) \int_{\mathbb{R}^d} \eta(u_0(x) - k) \varphi(0, x) dx - E \left( \int_{\mathbb{R}^d} \eta(u^\Delta(T, x) - k) \varphi(T, x) dx 1_A \right) \\
& \geq -\epsilon \Delta, \text{ where } \epsilon \Delta \rightarrow 0 \text{ when } \Delta \rightarrow 0.
\end{aligned}$$

240 **3.4. Convergence of the approximate solution**

Consider  $A$  a  $P$ -measurable set,  $(k, \varphi, \eta) \in \mathbb{R} \times \mathcal{D}^+([0, T] \times \mathbb{R}^d) \times \mathcal{E}$  and denote by  $K$  a compact set of  $\mathbb{R}^d$  such that  $\text{supp} \varphi(t, \cdot) \subset K$ . Our aim is to pass to the limit in:

$$\begin{aligned}
E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t, x) - k) h(u^\Delta(t, x)) dW(t) \varphi(t_n, x) dx 1_A \right) & := I_1^\Delta \\
+ \frac{1}{2} E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t, x) - k) h^2(u^\Delta(t, x)) dt \varphi(t_n, x) dx 1_A \right) & := I_2^\Delta \\
+ E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta(u^\Delta(t_{n+1}, x) - k) \varphi_t(t, x) dt dx 1_A \right) & := I_3^\Delta \\
+ E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} F^\eta(u^\Delta(t_{n+1}, x), k) \nabla \varphi(t, x) dt dx 1_A \right) & := I_4^\Delta \\
+ \text{mes}(A) \int_{\mathbb{R}^d} \eta(u_0(x) - k) \varphi(0, x) dx & \geq -\epsilon \Delta, \quad (8)
\end{aligned}$$

Due to the random variable, even if strong estimates with respect to variables  $t$  and  $x$  hold, we are not able to use classical results of compactness. The one

given by the concept of Young measures is appropriate here and the technique is based on the notion of narrow convergence of Young measures (or entropy processes). We refer to Eymard-Gallouët-Herbin [8] and Panov [13]. Since  $(u^\Delta)$  is a bounded sequence in  $L^\infty(Q \times \Omega)$ , the associated Young measure sequence  
250  $(u^\Delta)$  converges (up to a subsequence still indexed in the same way) to a Young measure denoted  $\mathbf{u} \in L^\infty(Q \times \Omega \times ]0, 1[)$ . Furthermore, according to Eymard-Gallouët-Herbin [8], for any Carathéodory function  $\Psi$  such that  $\Psi(\cdot, u^\Delta)$  is uniformly integrable:

$$E \int_Q \Psi(u^\Delta(t, x)) dt dx \rightarrow E \int_Q \int_0^1 \Psi(\mathbf{u}(t, x, \alpha)) d\alpha dt dx \text{ when } \Delta \rightarrow 0.$$

Let us analyze separately terms of (8).

255 
$$\cdot I_1^\Delta \rightarrow E \left( \int_{\mathbb{R}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\alpha) - k) h(\mathbf{u}(\cdot, \alpha)) d\alpha \varphi dW(t) dx 1_A \right) := I_1.$$

$$\begin{aligned} |I_1^\Delta - I_1| &= \left| E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta'(u^\Delta - k) h(u^\Delta) [\varphi(t_n) - \varphi(t)] dW(t) dx 1_A \right) \right. \\ &+ E \left( \int_{\mathbb{R}^d} \int_0^T \left[ \eta'(u^\Delta - k) h(u^\Delta) - \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha) - k) h(\mathbf{u}(\cdot, \alpha)) d\alpha \right] \varphi(t) dW(t) dx 1_A \right) \left. \right| \\ &:= |I_{1,1}^\Delta + I_{1,2}^\Delta|. \end{aligned}$$

Using Cauchy-Schwarz inequality on  $\Omega \times K$  and Itô isometry one gets

$$\begin{aligned}
|I_{1,1}^\Delta| &= \left| E \left( \sum_{n=0}^{N-1} \int_K \int_{t_n}^{t_{n+1}} \eta'(u^\Delta - k) h(u^\Delta) [\varphi(t_n) - \varphi(t)] dW(t) dx 1_A \right) \right| \\
&\leq C \sum_{n=0}^{N-1} \left[ E \int_K \left( \int_{t_n}^{t_{n+1}} \eta'(u^\Delta - k) h(u^\Delta) [\varphi(t_n) - \varphi(t)] dW(t) \right)^2 dx \right]^{\frac{1}{2}} \\
&= C \sum_{n=0}^{N-1} \left[ E \int_K \int_{t_n}^{t_{n+1}} [\eta'(u^\Delta - k) h(u^\Delta) [\varphi(t_n) - \varphi(t)]]^2 dt dx \right]^{\frac{1}{2}} \\
&\leq C \sum_{n=0}^{N-1} \left[ E \int_{t_n}^{t_{n+1}} \text{mes}(K) \times \Delta^2 dt \right]^{\frac{1}{2}} \\
&\leq C \sum_{n=0}^{N-1} \Delta^{\frac{3}{2}} = C\sqrt{\Delta} \rightarrow 0
\end{aligned}$$

Let us show that  $I_{1,2}^\Delta \rightarrow 0$ . Denote  $v^\Delta = \eta'(u^\Delta - k) h(u^\Delta) \varphi$ . Thanks to Proposition 1,  $v^\Delta$  is bounded in  $L^2(Q \times \Omega)$  and there exists  $v \in L^2(Q \times \Omega)$  such that  $v^\Delta \rightharpoonup v$  in the same space. Moreover,  $\Psi : (t, x, \omega, \lambda) \mapsto \eta'(\lambda - k) h(\lambda) \varphi(t, x)$ ,  $(t, x, \omega, \lambda) \in Q \times \Omega \times \mathbb{R}$  is a Carathéodory function and  $\Psi(\cdot, u^\Delta)$  is uniformly integrable as it is bounded in  $L^2(Q \times \Omega)$ . By identification,  $v = \int_0^1 \Psi(\mathbf{u}(\cdot, \alpha)) d\alpha$ . Furthermore, for all  $t \in [0, T]$ ,

$$\begin{aligned}
I_t : L^2(Q \times \Omega) &\rightarrow L^2(\mathbb{R}^d \times \Omega) \\
\bar{u} &\mapsto \int_0^t \bar{u}(t, x, \omega) dW(t)
\end{aligned}$$

is a linear continuous function, and so it is a weakly continuous function from  $L^2(Q \times \Omega)$  to  $L^2(\mathbb{R}^d \times \Omega)$ .

265 Consequently,  $I_t(v^\Delta) \rightarrow I_t(v)$  in  $L^2(\mathbb{R}^d \times \Omega)$ . In this manner,

$$\begin{aligned} & E \left( \int_{\mathbb{R}^d} \int_0^T \eta'(u^\Delta - k) h(u^\Delta) \varphi dW(t) dx 1_A \right) \\ \rightarrow & E \left( \int_{\mathbb{R}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha) - k) h(\mathbf{u}(\cdot, \alpha)) d\alpha \varphi dW(t) dx 1_A \right) \end{aligned}$$

and  $|I_{1,2}^\Delta| \rightarrow 0$ .

$$I_2^\Delta \rightarrow \frac{1}{2} E \left( \int_Q \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha) - k) h^2(\mathbf{u}(\cdot, \alpha)) d\alpha \varphi dt dx 1_A \right) := I_2.$$

$$\begin{aligned} |I_2^\Delta - I_2| &= \frac{1}{2} \left| E \left( \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \eta''(u^\Delta - k) h^2(u^\Delta) [\varphi(t_n) - \varphi(t)] dt dx 1_A \right) \right. \\ &\quad + E \int_Q \eta''(u^\Delta - k) h^2(u^\Delta) \varphi(t) dt dx 1_A \\ &\quad \left. - E \int_Q \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha) - k) h^2(\mathbf{u}(\cdot, \alpha)) d\alpha \varphi(t) dt dx 1_A \right| \\ &:= \frac{1}{2} |I_{2,1}^\Delta + I_{2,2}^\Delta|. \end{aligned}$$

$$\begin{aligned} |I_{2,1}^\Delta| &\leq E \left( \sum_{n=0}^{N-1} \int_K \int_{t_n}^{t_{n+1}} |\eta''(u^\Delta - k) h^2(u^\Delta) [\varphi(t_n) - \varphi(t)]| dt dx \right) \\ &\leq C \sum_{n=0}^{N-1} \Delta^2 \rightarrow 0. \end{aligned}$$

Note that  $\Psi(t, x, \omega, \lambda) = \eta''(\lambda - k) h^2(\lambda) \varphi(t, x) 1_A$  is a Carathéodory function  
270 such that  $\Psi(\cdot, u^\Delta)$  is uniformly integrable, thus  $I_{2,2}^\Delta \rightarrow 0$  and the result holds.

Using the same techniques we show the following convergences.

$$I_3^\Delta \rightarrow E \left( \int_Q \int_0^1 \eta(\mathbf{u}(\cdot, \alpha) - k) d\alpha \varphi_t dt dx 1_A \right) := I_3.$$

$$I_4^\Delta \rightarrow E \left( \int_Q \int_0^1 F^\eta(\mathbf{u}(\cdot, \alpha), k) d\alpha \nabla \varphi dt dx 1_A \right) := I_4.$$

Finally, for all  $(k, \varphi, \eta) \in \mathbb{R} \times \mathcal{D}^+([0, T] \times \mathbb{R}^d) \times \mathcal{E}$  and any  $P$ -measurable set  $A$ :

$$\begin{aligned} & E \left( \int_{\mathbb{R}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\alpha) - k) h(\mathbf{u}(\cdot, \alpha)) d\alpha \varphi dW(t) dx 1_A \right) \\ & + \frac{1}{2} E \left( \int_Q \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha) - k) h^2(\mathbf{u}(\cdot, \alpha)) d\alpha \varphi dt dx 1_A \right) \\ & + E \left( \int_Q \int_0^1 [\eta(\mathbf{u}(\cdot, \alpha) - k) \varphi_t + F^\eta(\mathbf{u}(\cdot, \alpha), k) \nabla \varphi] d\alpha dt dx 1_A \right) \\ & + \text{mes}(A) \int_{\mathbb{R}^d} \eta(u_0(x) - k) \varphi(0, x) dx \geq 0. \end{aligned}$$

275 Hence  $\mathbf{u}$  is a measure-valued entropy solution in the sense of Definition 2.2.

Thanks to Theorem 2.3,  $\mathbf{u}$  is the unique stochastic entropy solution in the sense of Definition 2.1 and we denote it by  $u$ . Hence, all the sequence of approximate solution  $u^\Delta$  converges strongly to  $u$  in  $L^1_{loc}(\Omega \times Q)$ . In addition, since  $u^\Delta$  is bounded in  $L^\infty(\Omega \times Q)$ , all the sequence converges strongly in  $L^p_{loc}(\Omega \times Q)$  for

280 any finite  $p$  and the proof of Theorem 1.1 is complete.

#### 4. Numerical experiments

We propose here an application of this splitting method to the stochastic burgers' equation in the one-dimensional case:

$$du + f(u)_x dt = \lambda h(u) dw \text{ in } \Omega \times ]0, 1[ \times \mathbb{R},$$

where  $\lambda \in \mathbb{R}$  is a parameter,  $f(u) = u^2$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$h(x) = \begin{cases} (x+1)(1-x) & \text{if } -1 < x < 1 \\ 0 & \text{else.} \end{cases}$$



**Remark 3.** *As the flux function  $f$  is locally Lipschitz-continuous in  $\mathbb{R}$  and as we work with solution explicitly bounded by constant independent of  $f$ , following Remark 1 we are in the framework presented in the previous section.*

The algorithm of discretization is the following one: with an Euler-Maruyama  
 290 method, we solve the stochastic differential equation (2) and, for solving the  
 conservation law (3), we use a Godunov scheme. We implement simulations  
 with the following initial condition:

$$u_0(x) = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ 0 & \text{else.} \end{cases} \quad (9)$$

To illustrate our proposal, we give a simulation of the solution in the determin-  
 istic case (*i.e.* when  $\lambda = 0$ ) and for various values of the noise (respectively for  
 295  $\lambda$  equal to 0.1, 0.3 and 0.8). We get the following graphics in the  $(x, t)$  plane  
 when  $x$  varies between  $-1$  and  $1$ , with  $\Delta_x = 0.005$  and  $\Delta_t = \frac{\Delta_x}{6}$ .

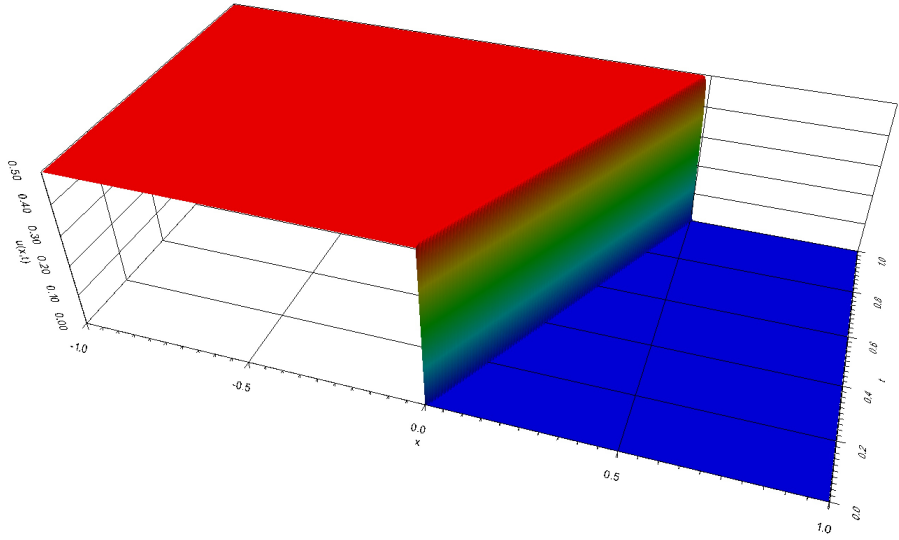


Figure 1: Determinist case :  $\lambda = 0$

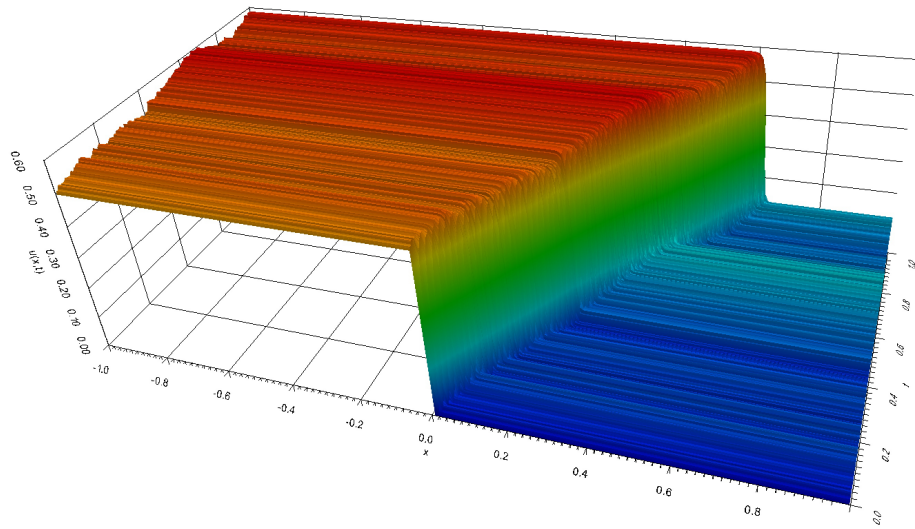


Figure 2: Stochastic case :  $\lambda = 0.1$

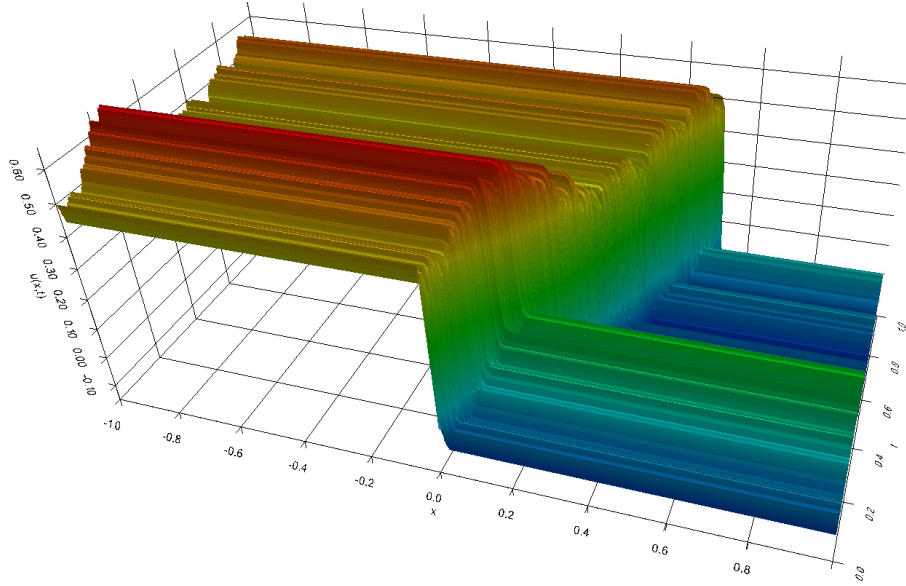


Figure 3: Stochastic case :  $\lambda = 0.3$

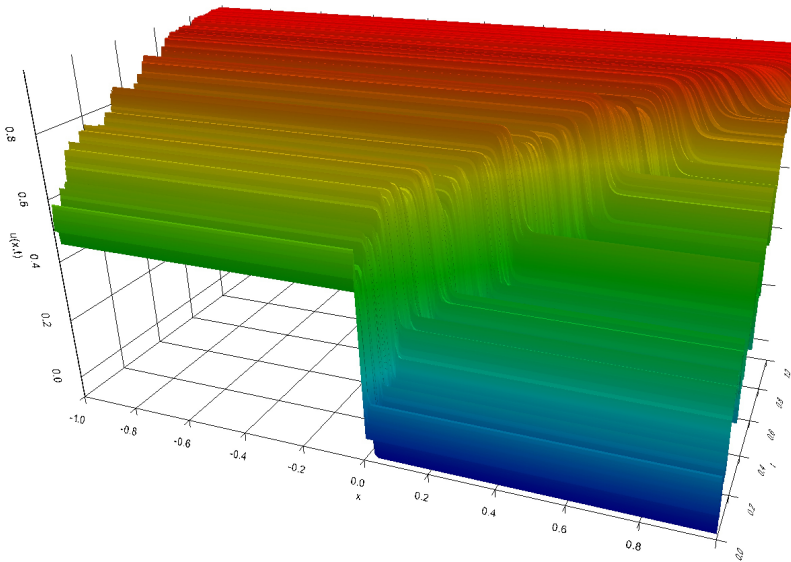


Figure 4: Stochastic case :  $\lambda = 0.8$

**Remark 4.** *Note that we conserve in the stochastic case the propagation of a single choc wave and that the stochastic perturbation seems to act only on the slope coefficient of such a choc. These simulations have been implemented with the free software Scilab.*

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