Relative controllability of linear difference equations

Guilherme Mazanti

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Abstract

In this paper, we study the relative controllability of linear difference equations with multiple delays in the state by using a suitable formula for the solutions of such systems in terms of their initial conditions, their control inputs, and some matrix-valued coefficients obtained recursively from the matrices defining the system. Thanks to such formula, we characterize relative controllability in time T in terms of an algebraic property of the matrix-valued coefficients, which reduces to the usual Kalman controllability criterion in the case of a single delay. Relative controllability is studied for solutions in the set of all functions and in the function spaces L^p and $\mathcal{C}^k$. We also compare the relative controllability of the system for different delays in terms of their rational dependence structure, proving that relative controllability for some delays implies relative controllability for all delays that are "less rationally dependent" than the original ones, in a sense that we make precise. Finally, we provide an upper bound on the minimal controllability time for a system depending only on its dimension and on its largest delay.

1 Introduction

This paper characterizes the relative controllability of the controlled difference equation

$$\Sigma(A,B,\Lambda): \qquad x(t) = \sum_{j=1}^{N} A_j x(t-\Lambda_j) + Bu(t), \tag{1.1}$$

where $x(t) \in \mathbb{C}^d$ is the state, $u(t) \in \mathbb{C}^m$ is the control input, $N,d,m \in \mathbb{N}^*$, $\Lambda = (\Lambda_1,\dots,\Lambda_N) \in (0,+\infty)^N$ is the vector of positive delays, $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$ is a N-tuple of $d \times d$ complex-valued matrices, and $B \in \mathcal{M}_{d,m}(\mathbb{C})$ is a $d \times m$ complex-valued matrix.

The study of the autonomous difference equation

$$x(t) = \sum_{j=1}^{N} A_j x(t-\Lambda_j) \tag{1.2}$$

has a long history and its analysis through spectral methods has led to important stability criteria [1, 7, 8, 12, 15, 19] (see also [13, Chapter 9] and references therein). In particular, it has been shown

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*CMAP, École Polytechnique, and Team GECO, Inria Saclay, Palaiseau, France.

that uniform asymptotic stability and exponential stability for (1.2) are equivalent, and they are both equivalent to the condition

$$\sup\left\{\operatorname{Re}\lambda \;\middle|\; \det\left(\operatorname{Id}_d - \sum_{j=1}^{N} e^{-\lambda\Lambda_j}A_j\right) = 0\right\} < 0 \tag{1.3}$$

(see, e.g., [7, Lemma 3.1], [13, Chapter 9, Theorem 3.5], or [15, Theorem 3.2]). The left-hand side of (1.3) is in general not continuous with respect to the delays $\Lambda_1,\dots,\Lambda_N$, which means that stability of (1.2) is not preserved in general under perturbations of the delays [15, 19]. A well-known criterion by J. K. Hale and R. A. Silkowski characterizes the N-tuples of matrices $A = (A_1,\dots,A_N)$ for which (1.2) is stable and its stability is preserved under arbitrary perturbations on the delays (a property known as *strong stability*, see, e.g., [1, Theorem 5.2], [13, Chapter 9, Theorem 6.1], or [28]), stating that this is the case if and only if $\rho_{\mathrm{HS}}(A) < 1$, where

$$\rho_{\mathrm{HS}}(A) = \max_{(\theta_1,\dots,\theta_N)\in[0,2\pi]^N} \rho\left(\sum_{j=1}^{N} e^{i\theta_j}A_j\right) \tag{1.4}$$

and ρ denotes the spectral radius. Hale–Silkowski criterion actually says more, namely the striking fact that uniform asymptotic stability of (1.2) for some choice of rationally independent delays $\Lambda_1,\dots,\Lambda_N$ is equivalent to strong stability. A generalization of the Hale–Silkowski criterion to the case of delays satisfying some rational dependence structure has been provided in [4, 20]. The stability of (1.2) with time-varying matrices A_j has been considered in [4, 21], with a generalization of the Hale–Silkowski criterion to this case provided in [4, Corollaries 3.29 and 3.35]. Stability issues for time-varying delays Λ_j have been considered in [9].

A major motivation for analyzing the stability of (1.2) is that it is deeply related to properties of more general neutral functional differential equations of the form

$$\frac{d}{dt}\left(x(t) - \sum_{j=1}^{N} A_j x(t-\Lambda_j)\right) = f(x_t)$$

where $x_t : [-r,0] \to \mathbb{C}^d$ is given by $x_t(s) = x(t+s)$, $r \geq \max_{j\in\{1,\dots,N\}}\Lambda_j$, and f is some function defined on a certain space (typically $\mathcal{C}^k([-r,0],\mathbb{C}^d)$ or $W^{k,p}((-r,0),\mathbb{C}^d)$) [7, 8, 12, 21], [13, Section 9.7]. Another important motivation is that, using d'Alembert decomposition and classical transformations of hyperbolic PDEs into differential or difference equations with delays based mainly on the method of characteristics, some hyperbolic PDEs can be put under the form (1.2) [4, 6, 11, 18, 29]. In particular, this has been done in [4] in order to obtain stability results for transport and wave equations on networks with time-varying parameters from corresponding stability results for (1.2) with time-varying matrices A_j.

Several works in the literature have addressed the questions of control and stabilization of neutral functional differential equations [14, 22, 23, 25, 27], including for controlled difference equations of the form (1.1). In [14], for instance, the authors prove that (1.1) can be stabilized by a linear feedback control $u(t) = \sum_{j=1}^{N} K_j x(t-\Lambda_j)$, with the resulting system being strongly stable, if and only if there exists $\varepsilon > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\varepsilon$, one has

$$\operatorname{rk}\begin{pmatrix} B & \operatorname{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda\Lambda_j} \end{pmatrix} = d. \tag{1.5}$$

This criterion is a reminiscent of Hautus test for controllability (see, e.g., [30, Lemma 3.3.7]).

Concerning the controllability problem, due to the infinite-dimensional nature of the dynamics of neutral functional differential equations, several different notions of controllability can be used, such

as exact, approximate, spectral, or relative controllability [5, 27]. Relative controllability has been originally introduced in the study of control systems with delays in the control input [5, 17, 24], but this notion has later been extended and used to study also systems with delays in the state [10, 26] and in more general frameworks, such as for stochastic control systems [16] or fractional integro-differential systems [2]. The main idea of relative controllability is that, instead of controlling the state $x_t : [-r,0] \to \mathbb{C}^d$ of (1.1), defined by $x_t(s) = x(t+s)$, in a certain function space such as $\mathcal{C}^k([-r,0],\mathbb{C}^d)$ or $L^p((-r,0),\mathbb{C}^d)$, where $r \geq \max_{j\in\{1,\dots,N\}} \Lambda_j$, one controls only the final state $x(t) = x_t(0)$. We defer the precise definition of relative controllability used in this paper to Definition 3.4, after having proved in Theorems 3.1 and 3.2 criteria for several equivalent or closely related notions of relative controllability.

Despite the long history of the study of relative controllability, up to the author's knowledge, no general criterion allowing to characterize the relative controllability of (1.1) is available in the literature. The goal of this paper is to fill this gap by providing necessary and sufficient conditions for the relative controllability of (1.1) in some different function spaces. We also discuss the dependence of such controllability on the delays $\Lambda_1,\dots,\Lambda_N$, and, more precisely, on their rational dependence structure, and provide an upper bound on the minimal time for controllability in terms of the dimension d of the system and its largest delay. Notice that some of these questions have already been addressed for particular systems under the form (1.1) in the literature (see, e.g., [10, 26]). The main results of this paper generalize those of these works.

The main tool used in the analysis of the relative controllability of (1.1) in this paper is a suitable representation formula for its solutions, describing a solution in time t in terms of its initial condition, the control input, and some matrix-valued coefficients computed recursively (see Proposition 2.8). A similar formula has been used in [3] to analyze the stability of a system of transport equations on a network under intermittent damping and in [4] to obtain stability criteria for (1.2) with time-varying matrices A_j, which in particular provide a generalized version of the Hale–Silkowski stability criterion.

The plan of the paper is as follows. After some general discussion on the well-posedness of (1.1) and the derivation of the explicit representation formula for its solutions in Section 2, we characterize relative controllability for some fixed final time $T > 0$ in Section 3 in the set of all functions and in the function spaces L^p and $\mathcal{C}^k$. For given $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$ and $B \in \mathcal{M}_{d,m}(\mathbb{C})$, Section 4 compares the relative controllability of (1.1) for different delays $\Lambda_1,\dots,\Lambda_N$ and $L_1,\dots,L_N$ in terms of their rational dependence structure. Finally, Section 5 provides a uniform upper bound on the minimal time for the relative controllability of (1.1).

Notice that all the results in this paper also hold, with the same proofs, if one assumes $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{R})^N$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ with the state $x(t) \in \mathbb{R}^d$ and the control $u(t) \in \mathbb{R}^m$. We choose complex-valued matrices, states, and controls for (1.1) in this paper following the approach of [4], which is mainly motivated by the fact that classical spectral conditions for difference equations such as (1.3), (1.4), or (1.5) are more naturally written down in such framework.

Notations. We denote by $\mathbb{N}$ and $\mathbb{N}^*$ the sets of nonnegative and positive integers, respectively. For $a,b \in \mathbb{R}$, we write the set of all integers between a and b as $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$, with the convention that $[a,b] = \emptyset$ if $a > b$. The cardinality of a set $\mathcal{N}$ is denoted by $\#\mathcal{N}$. For $\xi \in \mathbb{R}^N$, we use $\xi_{\min}$ and $\xi_{\max}$ to denote the smallest and the largest components of ξ, respectively. For $\xi \in \mathbb{R}$, the symbol $\lfloor \xi \rfloor$ is used to the denote the integer part of ξ, i.e., the unique integer such that $\xi - 1 < \lfloor \xi \rfloor \leq \xi$.

The set of $d \times m$ matrices with coefficients in $K \subset \mathbb{C}$ is denoted by $\mathcal{M}_{d,m}(K)$, or simply by $\mathcal{M}_d(K)$ when $m = d$. The identity matrix in $\mathcal{M}_d(\mathbb{C})$ is denoted by Id_d and the zero matrix in $\mathcal{M}_{d,m}(\mathbb{C})$ is denoted by $0_{d,m}$, or simply by 0 when its dimensions are clear from the context. We use $e_1,\dots,e_d$ to denote the canonical basis of $\mathbb{C}^d$. For $p \in [1,+\infty]$, $|\cdot|_p$ indicates both the ℓ^p-norm in $\mathbb{C}^d$ and the induced matrix norm in $\mathcal{M}_{d,m}(\mathbb{C})$. The range of a matrix $M \in \mathcal{M}_{d,m}(\mathbb{C})$ is denoted by $\mathrm{Ran}\, M$, and

rk M denotes the dimension of Ran M.

2 Well-posedness and explicit representation of solutions

This sections establishes the well-posedness of (1.1) and provides an explicit representation formula for its solutions. The proofs of the main results of this section, Propositions 2.2 and 2.8, are very similar to the ones given in [4] for the corresponding uncontrolled system. We start by providing the definition of solution used in this paper.

Definition 2.1. Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, $T>0$, $x_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$, and $u:[0,T]\to\mathbb{C}^m$. We say that $x:[-\Lambda_{\max},T]\to\mathbb{C}^d$ is a *solution* of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u if it satisfies (1.1) for every $t\in[0,T]$ and $x(t)=x_0(t)$ for $t\in[-\Lambda_{\max},0)$. In this case, we set, for $t\in[0,T]$, $x_t=x(t+\cdot)|_{[-\Lambda_{\max},0)}$.

Notice that this definition of solution contains no regularity assumptions on x_0, u, or x. Nonetheless, this weak framework is enough to guarantee existence and uniqueness of solutions.

Proposition 2.2. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, $T>0$, $x_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$, and $u:[0,T]\to\mathbb{C}^m$. Then $\Sigma(A,B,\Lambda)$ admits a unique solution $x:[-\Lambda_{\max},T]\to\mathbb{C}^d$ with initial condition x_0 and control u.*

The proof of Proposition 2.2 is very similar to that of [4, Proposition 3.2], we provide it here for the sake of completeness.

Proof. Let $T^*>0$ be such that $T^*\le T$ and $T^*<\Lambda_{\min}$. It suffices to build the solution x on $[-\Lambda_{\max},T^*]$ and then complete its construction on $(T^*,T]$ by a standard inductive argument.

Suppose that $x:[-\Lambda_{\max},T^*]\to\mathbb{C}^d$ is a solution of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u. Then

$$x(t)=\begin{cases}\displaystyle\sum_{j=1}^N A_jx_0(t-\Lambda_j)+Bu(t), & \text{if } 0\le t\le T^*,\\ x_0(t), & \text{if } -\Lambda_{\max}\le t<0.\end{cases}\tag{2.1}$$

Since the right-hand side is uniquely determined by x_0, u, A, and B, we obtain the uniqueness of the solution. Conversely, if $x:[-\Lambda_{\max},T^*]\to\mathbb{C}^d$ is defined by (2.1), then (1.1) clearly holds for $t\in[0,T^*]$ and thus x is a solution of $\Sigma(A,B,\Lambda)$. ■

Remark 2.3. Let $T>0$. If $x_0,\widetilde{x_0}:[-\Lambda_{\max},0)\to\mathbb{C}^d$ and $u,\widetilde{u}:[0,T]\to\mathbb{C}^m$ are such that $x_0=\widetilde{x_0}$ and $u=\widetilde{u}$ almost everywhere on their respective domains, it follows from (2.1) that the solutions $x,\widetilde{x}:[-\Lambda_{\max},T]\to\mathbb{C}^d$ of $\Sigma(A,B,\Lambda)$ associated respectively with x_0, u, and $\widetilde{x_0}$, $\widetilde{u}$, satisfy $x=\widetilde{x}$ almost everywhere on $[-\Lambda_{\max},T]$. In particular, one still obtains existence and uniqueness of solutions of $\Sigma(A,B,\Lambda)$ (in the sense of functions defined almost everywhere) for initial conditions in $L^p((-\Lambda_{\max},0),\mathbb{C}^d)$ and controls in $L^p((0,T),\mathbb{C}^m)$ for some $p\in[1,+\infty]$. Moreover, it follows easily from (2.1) that, in this case, solutions x of $\Sigma(A,B,\Lambda)$ satisfy $x\in L^p((-\Lambda_{\max},T),\mathbb{C}^d)$, and hence $x_t\in L^p((-\Lambda_{\max},0),\mathbb{C}^d)$ for every $t\in[0,T]$.

Remark 2.4. If $x_0\in\mathcal{C}^k([-\Lambda_{\max},0),\mathbb{C}^d)$ and $u\in\mathcal{C}^k([0,T],\mathbb{C}^m)$ for some $k\in\mathbb{N}$, it follows from (2.1) that the corresponding solution x of $\Sigma(A,B,\Lambda)$ belongs to $\mathcal{C}^k([-\Lambda_{\max},T],\mathbb{C}^d)$ if and only if x_0 and u satisfy the compatibility condition

$$\lim_{t\to0}x_0^{(r)}(t)=\sum_{j=1}^N A_jx_0^{(r)}(-\Lambda_j)+Bu^{(r)}(0),\qquad\forall r\in[\![0,k]\!],\tag{2.2}$$

where $x_0^{(r)}$ and $u^{(r)}$ denote the r-th derivatives of x_0 and u, respectively.

Due to the compatibility condition (2.2) required for obtaining solutions x in the space $\mathcal{C}^k([-\Lambda_{\max}, T], \mathbb{C}^d)$, we find it useful to introduce the following definition.

Definition 2.5. Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $k \in \mathbb{N}$. We say that x_0 is $\mathcal{C}^k$-*admissible* for system $\Sigma(A, B, \Lambda)$ if $x_0 \in \mathcal{C}^k([-\Lambda_{\max}, 0), \mathbb{C}^d)$ and, for every $r \in [\![0, k]\!]$, $\lim_{t \to 0} x_0^{(r)}(t)$ exists and

$$\lim_{t \to 0} x_0^{(r)}(t) - \sum_{j=1}^{N} A_j x_0^{(r)}(-\Lambda_j) \in \operatorname{Ran} B.$$

In order to provide an explicit representation for the solutions of $\Sigma(A, B, \Lambda)$, we first provide a recursive definition of the matrix coefficients $\Xi_{\mathbf{n}}$ appearing in such representation.

Definition 2.6. For $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$ and $\mathbf{n} \in \mathbb{Z}^N$, we define the matrix $\Xi_{\mathbf{n}} \in \mathcal{M}_d(\mathbb{C})$ inductively by

$$\Xi_{\mathbf{n}} = \begin{cases} 0, & \text{if } \mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N, \\ \mathrm{Id}_d, & \text{if } \mathbf{n} = 0, \\ \displaystyle\sum_{k=1}^{N} A_k \Xi_{\mathbf{n} - e_k}, & \text{if } \mathbf{n} \in \mathbb{N}^N \setminus \{0\}. \end{cases} \tag{2.3}$$

Remark 2.7. It follows from [4, Proposition 3.8] that, for $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N \setminus \{0\}$, the matrices $\Xi_{\mathbf{n}}$ also satisfy the recurrence relation

$$\Xi_{\mathbf{n}} = \sum_{k=1}^{N} \Xi_{\mathbf{n} - e_k} A_k$$

and they can be explicitly computed from $A = (A_1, \dots, A_N)$ by

$$\Xi_{\mathbf{n}} = \sum_{v \in V_{\mathbf{n}}} A_{v_1} A_{v_2} \cdots A_{v_{|\mathbf{n}|_1}},$$

where $V_{\mathbf{n}} = \left\{ v \in [\![1, N]\!]^{|\mathbf{n}|_1} \;\middle|\; \text{for every } k \in [\![1, N]\!],\ \#\{ j \in [\![1, |\mathbf{n}|_1]\!] \mid v_j = k \} = n_k \right\}$.

We now provide an explicit representation for the solutions of $\Sigma(A, B, \Lambda)$, which is a generalization of [4, Lemma 3.13] to the case of the controlled difference equation (1.1).

Proposition 2.8. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $u : [0, T] \to \mathbb{C}^m$. The corresponding solution $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ of $\Sigma(A, B, \Lambda)$ is given for $t \in [0, T]$ by*

$$x(t) = \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \le t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \le t}} \Xi_{\mathbf{n}} B u(t - \Lambda \cdot \mathbf{n}). \tag{2.4}$$

Proof. By linearity, it suffices to show that the function x_1 defined by

$$x_1(t) = \begin{cases} \displaystyle\sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \le t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}), & \text{if } 0 \le t \le T, \\ x_0(t), & \text{if } -\Lambda_{\max} \le t < 0, \end{cases}$$

is the solution of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control 0, and that the function x_2 defined by

$$x_2(t) = \begin{cases} \displaystyle\sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{n}\le t}} \Xi_{\mathbf{n}} B u(t-\Lambda\cdot\mathbf{n}), & \text{if } 0\le t\le T, \\ 0, & \text{if } -\Lambda_{\max}\le t<0, \end{cases} \tag{2.5}$$

is the solution of $\Sigma(A,B,\Lambda)$ with initial condition 0 and control u. The first part has already been shown in [4, Lemma 3.13], we are thus left to show that x_2 satisfies (1.1) for $t\in[0,T]$.

Let $j\in[\![1,N]\!]$. For $t\in[0,T]$, we have

$$x_2(t-\Lambda_j) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{n}\le t-\Lambda_j}} \Xi_{\mathbf{n}} B u(t-\Lambda_j-\Lambda\cdot\mathbf{n}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{m}\le t,\, m_j\ge 1}} \Xi_{\mathbf{m}-e_j} B u(t-\Lambda\cdot\mathbf{m}),$$

where we extend u by zero outside the interval $[0,T]$. Hence, using (2.3), we obtain that

$$\begin{aligned} \sum_{j=1}^N A_j x_2(t-\Lambda_j) &= \sum_{j=1}^N A_j \sum_{\substack{\mathbf{m}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{m}\le t,\, m_j\ge 1}} \Xi_{\mathbf{m}-e_j} B u(t-\Lambda\cdot\mathbf{m}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{m}\le t}} \sum_{\substack{j=1 \\ m_j\ge 1}}^N A_j \Xi_{\mathbf{m}-e_j} B u(t-\Lambda\cdot\mathbf{m}) \\ &= \sum_{\substack{\mathbf{m}\in\mathbb{N}^N\setminus\{0\} \\ \Lambda\cdot\mathbf{m}\le t}} \Xi_{\mathbf{m}} B u(t-\Lambda\cdot\mathbf{m}) = x_2(t) - Bu(t), \end{aligned}$$

which shows that x_2 satisfies (1.1). ■

Remark 2.9. When $A=(A_1,\dots,A_N)$ and B are time-varying matrices, i.e. $A:[0,T]\to\mathcal{M}_d(\mathbb{C})^N$ and $B:[0,T]\to\mathcal{M}_{d,m}(\mathbb{C})$, the counterpart of Proposition 2.2 also holds with the same proof, and Remark 2.3 also applies, in the sense that solutions corresponding to A,B and $\widetilde{A},\widetilde{B}$ are equal almost everywhere if $A=\widetilde{A}$ and $B=\widetilde{B}$ almost everywhere. The conclusion that $x\in L^p((-\Lambda_{\max},T),\mathbb{C}^d)$ when $x_0\in L^p((-\Lambda_{\max},0),\mathbb{C}^d)$ and $u\in L^p((0,T),\mathbb{C}^m)$ holds under the extra assumption that $A\in L^\infty((0,T),\mathcal{M}_d(\mathbb{C})^N)$ and $B\in L^\infty((0,T),\mathcal{M}_{d,m}(\mathbb{C}))$. Moreover, the explicit formula from Proposition 2.8 becomes

$$x(t) = \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -\Lambda_j\le t-\Lambda\cdot\mathbf{n}<0}} \Xi^\Lambda_{\mathbf{n}-e_j,t} A_j(t-\Lambda\cdot\mathbf{n}+\Lambda_j) x_0(t-\Lambda\cdot\mathbf{n}) + \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{n}\le t}} \Xi^\Lambda_{\mathbf{n},t} B(t-\Lambda\cdot\mathbf{n}) u(t-\Lambda\cdot\mathbf{n}), \tag{2.6}$$

where the matrix coefficients $\Xi^\Lambda_{\mathbf{n},t}$ are defined for $A=(A_1,\dots,A_N):\mathbb{R}\to\mathcal{M}_d(\mathbb{C})^N$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, $\mathbf{n}\in\mathbb{Z}^N$, and $t\in\mathbb{R}$ as in (2.3), the third case being replaced by $\Xi^\Lambda_{\mathbf{n},t}=\sum_{k=1}^N A_k(t)\Xi^\Lambda_{\mathbf{n}-e_k,t-\Lambda_k}$.

Remark 2.10. Let $p\in[1,+\infty]$. For $t\ge 0$, we define the bounded linear operator $S(t):L^p((-\Lambda_{\max},0),\mathbb{C}^d)\to L^p((-\Lambda_{\max},0),\mathbb{C}^d)$ by

$$(S(t)x_0)(s) = \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -\Lambda_j\le t+s-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j} A_j x_0(t+s-\Lambda\cdot\mathbf{n}).$$

The operator $S(t)$ maps an initial condition x_0 to the state $x_t=x(t+\cdot)|_{(-\Lambda_{\max},0)}$, where x is the solution of $\Sigma(A,B,\Lambda)$ at time t with initial condition x_0 and control 0. For $p\in[1,+\infty)$, the family $\{S(t)\}_{t\ge 0}$ is a strongly continuous semigroup in $L^p((-\Lambda_{\max},0),\mathbb{C}^d)$ (see, e.g., [4, Theorem 3.5]).

The controllability results we establish in Section 3 are based on the explicit representation for the solutions from Proposition 2.8. Notice that the control u only affects the second term of (2.4). Since, in this term, u is evaluated only at times $t-\Lambda\cdot\mathbf{n}$, one should pack together coefficients $\Xi_{\mathbf{n}}$ corresponding to different $\mathbf{n},\mathbf{n}'\in\mathbb{N}$ for which $\Lambda\cdot\mathbf{n}=\Lambda\cdot\mathbf{n}'$, in the same manner as in [4, Definition 3.10].

Definition 2.11. Let $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$. We partition $\mathbb{N}^N$ according to the equivalence relation $\sim$ defined by writing $\mathbf{n}\sim\mathbf{n}'$ if $\Lambda\cdot\mathbf{n}=\Lambda\cdot\mathbf{n}'$. We use $[\cdot]_\Lambda$ to denote the equivalence classes of $\sim$ and we set $\mathcal{N}_\Lambda=\mathbb{N}^N/\sim$. The index Λ is omitted from the notation of $[\cdot]_\Lambda$ when the delay vector Λ is clear from the context. We define

$$\widehat{\Xi}^\Lambda_{[\mathbf{n}]}=\sum_{\mathbf{n}'\in[\mathbf{n}]}\Xi_{\mathbf{n}'}. \tag{2.7}$$

Thanks to Definition 2.11, the representation formula (2.4) for the solutions of $\Sigma(A,B,\Lambda)$ can be written as

$$x(t)=\sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\le t-\Lambda\cdot\mathbf{n}<0}}\Xi_{\mathbf{n}-e_j}A_jx_0(t-\Lambda\cdot\mathbf{n})+\sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\le t}}\widehat{\Xi}^\Lambda_{[\mathbf{n}]}Bu(t-\Lambda\cdot\mathbf{n}). \tag{2.8}$$

3 Relative controllability criteria

This section presents the main relative controllability criteria from the paper, Theorems 3.1 and 3.2 below. Theorem 3.1 provides a criterion for relative controllability in the set of all functions and in the L^p spaces, whereas the criterion in Theorem 3.2 characterizes relative controllability in the $\mathcal{C}^k$ spaces. Both algebraic criteria we obtain are expressed in terms of the coefficients $\widehat{\Xi}^\Lambda_{[\mathbf{n}]}$ and the matrix B and are generalizations of the usual Kalman condition for the controllability of a discrete-time system. Their proofs are based on the explicit representation for solutions (2.8).

Theorem 3.1. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, $T>0$, and $p\in[1,+\infty]$. Define $\widehat{\Xi}^\Lambda_{[\mathbf{n}]}$ as in (2.7). Then the following assertions are equivalent.*

(a) *One has*
$$\operatorname{Span}\left\{\widehat{\Xi}^\Lambda_{[\mathbf{n}]}Bw\;\middle|\;[\mathbf{n}]\in\mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n}\le T,\ w\in\mathbb{C}^m\right\}=\mathbb{C}^d. \tag{3.1}$$

(b) *For every $x_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$ and $x_1\in\mathbb{C}^d$, there exists $u:[0,T]\to\mathbb{C}^m$ such that the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x(T)=x_1$.*

(c) *There exists $\varepsilon_0>0$ such that, for every $\varepsilon\in(0,\varepsilon_0)$, $x_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$, and $x_1:[0,\varepsilon]\to\mathbb{C}^d$, there exists $u:[0,T+\varepsilon]\to\mathbb{C}^m$ such that the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x(T+\cdot)|_{[0,\varepsilon]}=x_1$.*

(d) *There exists $\varepsilon_0>0$ such that, for every $\varepsilon\in(0,\varepsilon_0)$, $x_0\in L^p((-\Lambda_{\max},0),\mathbb{C}^d)$, and $x_1\in L^p((0,\varepsilon),\mathbb{C}^d)$, there exists $u\in L^p((0,T+\varepsilon),\mathbb{C}^m)$ such that the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x\in L^p((-\Lambda_{\max},T+\varepsilon),\mathbb{C}^d)$ and $x(T+\cdot)|_{[0,\varepsilon]}=x_1$.*

Proof. For $T>0$, let $\mathcal{N}^T=\{[\mathbf{n}]\in\mathcal{N}_\Lambda\mid\Lambda\cdot\mathbf{n}\le T\}$ and $n_T=\#\mathcal{N}^T$. The proof is carried out as follows. Clearly, (c) $\Longrightarrow$ (b). We will show the equivalences by proving that (b) $\Longrightarrow$ (a), (a) $\Longrightarrow$ (c) and (d), and (d) $\Longrightarrow$ (a).

Assume that (b) is satisfied, which shows, using (2.8) and considering a zero initial condition, that, for every $x_1\in\mathbb{C}^d$, there exists $u:[0,T]\to\mathbb{C}^m$ such that

$$\left(\widehat{\Xi}^\Lambda_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}\left(u(T-\Lambda\cdot\mathbf{n})\right)_{[\mathbf{n}]\in\mathcal{N}^T}=\sum_{[\mathbf{n}]\in\mathcal{N}^T}\widehat{\Xi}^\Lambda_{[\mathbf{n}]}Bu(T-\Lambda\cdot\mathbf{n})=x_1, \tag{3.2}$$

where $\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}$ denotes the $d\times mn_T$ matrix composed of the n_T blocks $\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B$ of size $d\times m$ and $\big(u(T-\Lambda\cdot\mathbf{n})\big)_{[\mathbf{n}]\in\mathcal{N}^T}$ denotes the $mn_T\times 1$ matrix composed of the n_T blocks $u(T-\Lambda\cdot\mathbf{n})$ of size $m\times 1$. This means that the map $\mathbb{C}^{mn_T}\ni U\mapsto\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}U\in\mathbb{C}^d$ is surjective, and thus (a) is satisfied.

Assume now that (a) is satisfied and let

$$\varepsilon_0=\min\left\{\min_{\substack{[\mathbf{n}'],[\mathbf{n}]\in\mathcal{N}^T\\ [\mathbf{n}']\neq[\mathbf{n}]}}\left|\Lambda\cdot\mathbf{n}-\Lambda\cdot\mathbf{n}'\right|,\ \min_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}>T}}(\Lambda\cdot\mathbf{n}-T)\right\}>0.$$

Let $\varepsilon\in(0,\varepsilon_0)$, $x_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$, and $x_1:[0,\varepsilon]\to\mathbb{C}^d$. Thanks to (a), the map $\mathbb{C}^{mn_T}\ni U\mapsto\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}U\in\mathbb{C}^d$ is surjective, and hence the $d\times mn_T$ matrix $\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}$ admits a right inverse $M\in\mathcal{M}_{mn_T,d}(\mathbb{C})$. Let $U=\big(U_{[\mathbf{n}]}\big)_{[\mathbf{n}]\in\mathcal{N}^T}:[0,\varepsilon]\to\mathbb{C}^{mn_T}=(\mathbb{C}^m)^{\mathcal{N}^T}$ be given by

$$U(t)=M\left(x_1(t)-\sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\leq T+t-\Lambda\cdot\mathbf{n}<0}}\Xi_{\mathbf{n}-e_j}A_jx_0(T+t-\Lambda\cdot\mathbf{n})\right).\tag{3.3}$$

Define $u:[0,T+\varepsilon]\to\mathbb{C}^m$ by

$$u(t)=\begin{cases}U_{[\mathbf{n}]}(\Lambda\cdot\mathbf{n}+t-T), & \text{if } t\in[T-\Lambda\cdot\mathbf{n},T-\Lambda\cdot\mathbf{n}+\varepsilon] \text{ for some } [\mathbf{n}]\in\mathcal{N}^T,\\ 0, & \text{otherwise.}\end{cases}\tag{3.4}$$

Thanks to the definition of ε_0, u is well-defined, and one has $u(T+t-\Lambda\cdot\mathbf{n})=U_{[\mathbf{n}]}(t)$ for every $[\mathbf{n}]\in\mathcal{N}^T$ and $t\in[0,\varepsilon]$. Hence, it follows from (3.3) that, for every $t\in[0,\varepsilon]$,

$$\begin{aligned}x_1(t)-\sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\leq T+t-\Lambda\cdot\mathbf{n}<0}}\Xi_{\mathbf{n}-e_j}A_jx_0(T+t-\Lambda\cdot\mathbf{n})&=\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}\big(u(T+t-\Lambda\cdot\mathbf{n})\big)_{[\mathbf{n}]\in\mathcal{N}^T}\\ &=\sum_{[\mathbf{n}]\in\mathcal{N}^T}\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bu(T+t-\Lambda\cdot\mathbf{n})=\sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\leq T+t}}\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bu(T+t-\Lambda\cdot\mathbf{n}),\end{aligned}\tag{3.5}$$

where we use that, thanks to the definition of ε_0, one has

$$\mathcal{N}^T=\{[\mathbf{n}]\in\mathcal{N}_\Lambda\,|\,\Lambda\cdot\mathbf{n}\leq T+t\},\qquad\forall t\in[0,\varepsilon].\tag{3.6}$$

It now follows from (2.8) and (3.5) that the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x(T+\cdot)|_{[0,\varepsilon]}=x_1$, and hence (c) holds. Notice moreover that, if we assume $x_0\in L^p((-\Lambda_{\max},0),\mathbb{C}^d)$ and $x_1\in L^p((0,\varepsilon),\mathbb{C}^d)$, it follows from (3.3) that $U\in L^p((0,\varepsilon),\mathbb{C}^{mn_T})$, and thus, by (3.4), $u\in L^p((0,T+\varepsilon),\mathbb{C}^m)$. Hence, the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x\in L^p((-\Lambda_{\max},T+\varepsilon),\mathbb{C}^d)$, thanks to Remark 2.3, and $x(T+\cdot)|_{[0,\varepsilon]}=x_1$, which shows that (d) also holds.

Finally, assume that (d) holds, take $\varepsilon_0>0$ as in (d) and fix $\varepsilon\in(0,\varepsilon_0)$. Then, considering a zero initial condition, for every constant final state $x_1\in\mathbb{C}^d$, there exists $u\in L^p((0,T+\varepsilon),\mathbb{C}^m)$ such that, for almost every $t\in(0,\varepsilon)$, one has, as in (3.2),

$$\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}\big(u(T+t-\Lambda\cdot\mathbf{n})\big)_{[\mathbf{n}]\in\mathcal{N}^T}=x_1,$$

where we use that (3.6) holds, up to choosing a smaller $\varepsilon\in(0,\varepsilon_0)$. Hence, as in (3.2), one also obtains that the map $\mathbb{C}^{mn_T}\ni U\mapsto\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}^T}U\in\mathbb{C}^d$ is surjective, and thus (a) is satisfied. ■

The next result presents a relative controllability criterion for $\mathcal{C}^k$ solutions of $\Sigma(A,B,\Lambda)$, which is slightly different from (a) in Theorem 3.1 due to the compatibility condition (2.2) required for the existence of $\mathcal{C}^k$ solutions.

Theorem 3.2. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, $T>0$, and $k\in\mathbb{N}$. Define $\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}$ as in (2.7). Then the following assertions are equivalent.*

(a) *One has*
$$\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bw \;\middle|\; [\mathbf{n}]\in\mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n}<T,\ w\in\mathbb{C}^m\right\}=\mathbb{C}^d. \tag{3.7}$$

(b) *For every x_0 $\mathcal{C}^k$-admissible for $\Sigma(A,B,\Lambda)$ and $x_1\in\mathbb{C}^d$, there exists $u\in\mathcal{C}^k([0,T],\mathbb{C}^m)$ such that the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x\in\mathcal{C}^k([-\Lambda_{\max},T],\mathbb{C}^d)$ and $x(T)=x_1$.*

(c) *There exists $\varepsilon_0>0$ such that, for every $\varepsilon\in(0,\varepsilon_0)$, x_0 $\mathcal{C}^k$-admissible for $\Sigma(A,B,\Lambda)$, and $x_1\in\mathcal{C}^k([0,\varepsilon],\mathbb{C}^d)$, there exists $u\in\mathcal{C}^k([0,T+\varepsilon],\mathbb{C}^m)$ such that the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x\in\mathcal{C}^k([-\Lambda_{\max},T+\varepsilon],\mathbb{C}^d)$ and $x(T+\cdot)|_{[0,\varepsilon]}=x_1$.*

Proof. Let $\mathcal{N}_*^T=\{[\mathbf{n}]_\Lambda\in\mathcal{N}_\Lambda\mid\Lambda\cdot\mathbf{n}<T\}$ and $n_T^*=\#\mathcal{N}_*^T$. We begin the proof by noticing that (c) implies (b). Assume now that (b) holds and let us show that (a) is satisfied. For every $x_1\in\mathbb{C}^d$, there exists $u\in\mathcal{C}^k([0,T],\mathbb{C}^m)$ such that the solution x of $\Sigma(A,B,\Lambda)$ with zero initial condition and control u satisfies $x\in\mathcal{C}^k([-\Lambda_{\max},T],\mathbb{C}^d)$ and, from (2.8),
$$\sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\le T}}\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bu(T-\Lambda\cdot\mathbf{n})=x_1. \tag{3.8}$$
Moreover, since $x\in\mathcal{C}^k([-\Lambda_{\max},T],\mathbb{C}^d)$, it follows from Remark 2.4 that (2.2) is satisfied, and thus, for every $r\in[\![0,k]\!]$, $Bu^{(r)}(0)=0$. Thus (3.8) becomes
$$\sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}<T}}\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bu(T-\Lambda\cdot\mathbf{n})=x_1,$$
and we conclude, as in the proof of Theorem 3.1, that the map $\mathbb{C}^{mn_T^*}\ni U\mapsto\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}U\in\mathbb{C}^d$ is surjective, and thus (a) is satisfied.

Finally, assume that (a) is satisfied and let
$$\varepsilon_0=\frac{1}{2}\min\left\{\min_{\substack{[\mathbf{n}'],[\mathbf{n}]\in\mathcal{N}_*^T\\ [\mathbf{n}']\neq[\mathbf{n}]}}\left|\Lambda\cdot\mathbf{n}-\Lambda\cdot\mathbf{n}'\right|,\ \min_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}\neq T}}|\Lambda\cdot\mathbf{n}-T|\right\}>0.$$
Let $\varepsilon\in(0,\varepsilon_0)$, x_0 $\mathcal{C}^k$-admissible for $\Sigma(A,B,\Lambda)$, and $x_1\in\mathcal{C}^k([0,\varepsilon],\mathbb{C}^d)$. Since x_0 is $\mathcal{C}^k$-admissible, there exists $\mu\in\mathcal{C}^k([0,\varepsilon],\mathbb{C}^m)$, with a compact support inside $[0,\varepsilon)$, such that, for every $r\in[\![0,k]\!]$,
$$\lim_{t\to0}x_0^{(r)}(t)=\sum_{j=1}^N A_jx_0^{(r)}(-\Lambda_j)+B\mu^{(r)}(0). \tag{3.9}$$
If $T=\Lambda\cdot\mathbf{n}$ for some $\mathbf{n}\in\mathbb{N}^N$, we set $\delta_T=1$ and $\tau=[\mathbf{n}]$; otherwise, we set $\delta_T=0$ and $\tau=[0]$. As in the proof of Theorem 3.1, it follows from (a) that the $d\times mn_T^*$ matrix $\left(\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}B\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}$ admits a right

inverse $M \in \mathcal{M}_{mn_T^*,d}(\mathbb{C})$. Let $U = \left(U_{[\mathbf{n}]}\right)_{[\mathbf{n}]\in\mathcal{N}_*^T} : [0,\varepsilon] \to \mathbb{C}^{mn_T^*} = (\mathbb{C}^m)^{\mathcal{N}_*^T}$ be given by

$$U(t) = M\left(x_1(t) - \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\le T+t-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j}A_j x_0(T+t-\Lambda\cdot\mathbf{n}) - \delta_T\widehat{\Xi}_\tau^\Lambda B\mu(t)\right). \tag{3.10}$$

Notice that the sum in (3.10) can be taken over the set

$$G_1(t) = \{(\mathbf{n}=(n_1,\dots,n_N),j)\in\mathbb{N}^N\times[\![1,N]\!] \mid -\Lambda_j \le T+t-\Lambda\cdot\mathbf{n}<0,\ n_j\ge 1\},$$

since $\Xi_{\mathbf{n}} = 0$ if $\mathbf{n}\in\mathbb{Z}^N\setminus\mathbb{N}^N$. Moreover, thanks to the definition of ε_0, one has $G_1(t) = G_1(0)$ for every $t\in[0,\varepsilon]$, and thus U can be written for $t\in[0,\varepsilon]$ as

$$U(t) = M\left(x_1(t) - \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\le T-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j}A_j x_0(T+t-\Lambda\cdot\mathbf{n}) - \delta_T\widehat{\Xi}_\tau^\Lambda B\mu(t)\right).$$

In particular, one obtains that $U\in\mathcal{C}^k([0,\varepsilon],\mathbb{C}^{mn_T^*})$. We extend U into a $\mathcal{C}^k$ function on the interval $\left[-\frac{\varepsilon}{2},\frac{3\varepsilon}{2}\right]$ with a compact support in $\left(-\frac{\varepsilon}{2},\frac{3\varepsilon}{2}\right)$. Define $u:[0,T+\varepsilon]\to\mathbb{C}^m$ by

$$u(t) = \begin{cases} U_{[\mathbf{n}]}(\Lambda\cdot\mathbf{n}+t-T), & \text{if } t\in\left[T-\Lambda\cdot\mathbf{n}-\frac{\varepsilon}{2},T-\Lambda\cdot\mathbf{n}+\frac{3\varepsilon}{2}\right] \text{ for some } [\mathbf{n}]\in\mathcal{N}_*^T,\\ \mu(t), & \text{if } t\in[0,\varepsilon],\\ 0, & \text{otherwise,}\end{cases}$$

which is well-defined thanks to the choice of ε_0, and satisfies $u\in\mathcal{C}^k([0,T+\varepsilon],\mathbb{C}^m)$ thanks to the construction of U and μ. Moreover, one has $u(T+t-\Lambda\cdot\mathbf{n}) = U_{[\mathbf{n}]}(t)$ for every $[\mathbf{n}]\in\mathcal{N}_*^T$ and, thanks to (3.9), it follows from Remark 2.4 that the unique solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x\in\mathcal{C}^k([-\Lambda_{\max},T+\varepsilon],\mathbb{C}^d)$. It follows from (3.10) that, for every $t\in[0,\varepsilon]$,

$$\begin{aligned} &x_1(t) - \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\le T+t-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j}A_j x_0(T+t-\Lambda\cdot\mathbf{n})\\ &= \delta_T\widehat{\Xi}_\tau^\Lambda B\mu(t) + \left(\widehat{\Xi}_{[\mathbf{n}]}^\Lambda B\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}\left(u(T+t-\Lambda\cdot\mathbf{n})\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}\\ &= \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\le T}} \widehat{\Xi}_{[\mathbf{n}]}^\Lambda Bu(T+t-\Lambda\cdot\mathbf{n}) = \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\le T+t}} \widehat{\Xi}_{[\mathbf{n}]}^\Lambda Bu(T+t-\Lambda\cdot\mathbf{n}),\end{aligned}$$

and thus the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x(T+\cdot)|_{[0,\varepsilon]} = x_1$, which shows that (c) holds. ■

Remark 3.3. When $N=1$, the controlled difference equation (1.1) becomes $x(t) = Ax(t-\Lambda)+Bu(t)$, with $A=A_1$ and $\Lambda=\Lambda_1$. It follows from Definitions 2.6 and 2.11 that, for $\mathbf{n}=n\in\mathbb{N}$, one has $\widehat{\Xi}_{[\mathbf{n}]}^\Lambda = A^n$, and thus condition (a) from Theorem 3.1 reduces to $\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lfloor T/\Lambda\rfloor}B\end{pmatrix} = d$, which is the usual Kalman condition for controllability of discrete-time linear systems (see, e.g., [30, Theorem 2]). Moreover, condition (a) from Theorem 3.2 reduces to $\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lceil T/\Lambda\rceil-1}B\end{pmatrix} = d$, which is the same as the previous one when $T/\Lambda\notin\mathbb{N}^*$.

Notice that (b), (c), and (d) from Theorem 3.1 and (b) and (c) from Theorem 3.2 could all be used to define relative controllability in different function spaces. Motivated by the equivalences established in Theorems 3.1 and 3.2, we provide the following definition.

Definition 3.4. Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda \in (0,+\infty)^N$, and $T > 0$.

(a) We say that $\Sigma(A,B,\Lambda)$ is *relatively controllable* in time T if

$$\operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m \right\} = \mathbb{C}^d.$$

(b) If $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$, we define the *minimal controllability time* $T_{\min}$ for $\Sigma(A,B,\Lambda)$ by $T_{\min} = \inf\{T > 0 \,|\, \Sigma(A,B,\Lambda) \text{ is relatively controllable in time } T\}$.

Remark 3.5. Contrarily to the situation for linear control systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$ or $x(t) = Ax(t-1) + Bu(t)$, relative controllability for some time $T > 0$ does not imply stabilizability by a linear feedback law. Indeed, as recalled in Section 1, [14, Theorem 3.1] proves that $\Sigma(A,B,\Lambda)$ can be strongly stabilized by a linear feedback law $u(t) = \sum_{j=1}^N K_j x(t-\Lambda_j)$ if and only if there exists $\varepsilon > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\varepsilon$, (1.5) is satisfied. For $N = d = 2$ and $m = 1$, consider the system $\Sigma(A,B,\Lambda)$ with $A = (A_1, A_2)$, B, and $\Lambda = (\Lambda_1, \Lambda_2)$ given by

$$A_1 = \begin{pmatrix} \alpha & -\alpha^{1-\ell} \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\Lambda_1 = 1, \qquad \Lambda_2 = \ell,$$

with $\ell \in (0,1)$ and $\alpha > 1$. Clearly, $\Sigma(A,B,\Lambda)$ is relatively controllable in time $T \geq \ell$ since $\operatorname{Span}\{B, A_2 B\} = \mathbb{C}^2$. However, for $\lambda \in \mathbb{C}$, one has

$$\operatorname{Id}_2 - A_1 e^{-\lambda} - A_2 e^{-\lambda\ell} = \begin{pmatrix} 1 - \alpha e^{-\lambda} & \alpha^{1-\ell} e^{-\lambda} - e^{-\lambda\ell} \\ 0 & 1 \end{pmatrix},$$

and the first row of this matrix is zero for $\lambda = \ln\alpha$. Hence (1.5) does not hold for $\lambda = \ln\alpha > 0$, which shows in particular that $\Sigma(A,B,\Lambda)$ cannot be strongly stabilized by a linear feedback law.

4 Rational dependence of the delays

This section compares relative controllability of $\Sigma(A,B,\Lambda)$ for different delay vectors Λ in terms of their rational dependence structure. We start by recalling the definition of rational dependence and commensurability.

Definition 4.1. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in \mathbb{R}^N$.

(a) We say that the components of Λ are *rationally dependent* if there exists $\mathbf{n} \in \mathbb{Z}^N \setminus \{0\}$ such that $\Lambda \cdot \mathbf{n} = 0$. Otherwise, the components of Λ are said to be *rationally independent*.

(b) We say that the components of Λ are *commensurable* if there exist $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}^N$ such that $\Lambda = \lambda k$.

Notice that the set $\mathbb{Z}^N$ can be replaced by $\mathbb{Q}^N$ in Definition 4.1 without changing the definitions of rational dependence and commensurability. We next introduce a preorder in the set of all possible delay vectors $(0,+\infty)^N$, which describes when one delay vector is "less rationally dependent" than another.

Definition 4.2. For $\Lambda \in (0,+\infty)^N$, we define $Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda \cdot \mathbf{n} = 0\}$. For $\Lambda, L \in (0,+\infty)^N$, we write $\Lambda \preccurlyeq L$ or, equivalently, $L \succcurlyeq \Lambda$, if $Z(\Lambda) \subset Z(L)$. We write $\Lambda \approx L$ if $\Lambda \preccurlyeq L$ and $L \preccurlyeq \Lambda$.

If $\Lambda \in (0,+\infty)^N$ has rationally independent components, then one immediately computes $Z(\Lambda) = \{0\}$, and hence $\Lambda \preccurlyeq L$ for every $L \in (0,+\infty)^N$, that is, delay vectors with rationally independent components are minimal for the preorder $\preccurlyeq$. Notice also that, for $\Lambda \in (0,+\infty)^N$, the set $Z(\Lambda)$ encodes the structure of the equivalence classes $[\mathbf{n}]_\Lambda$ for $\mathbf{n} \in \mathbb{N}^N$, in the sense that, for $\mathbf{n}' \in \mathbb{N}^N$, one has $\mathbf{n}' \in [\mathbf{n}]_\Lambda$ if and only if $\mathbf{n}' - \mathbf{n} \in Z(\Lambda)$, which shows that $[\mathbf{n}]_\Lambda = (\mathbf{n} + Z(\Lambda)) \cap \mathbb{N}^N$. We recall the following result from [4].

Proposition 4.3 ([4], Proposition 3.9)**.** *Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0,+\infty)^N$. There exist $h \in [\![1, N]\!]$, $\ell = (\ell_1, \dots, \ell_h) \in (0,+\infty)^h$ with rationally independent components, and $M \in \mathcal{M}_{N,h}(\mathbb{N})$ with $\operatorname{rk} M = h$ such that $\Lambda = M\ell$. Moreover, for every M as before, one has*

$$\operatorname{Ran} M = \left\{ L \in \mathbb{R}^N \;\middle|\; \text{for every } \mathbf{n} \in Z(\Lambda),\ \text{one has } L \cdot \mathbf{n} = 0 \right\}.$$

In particular, it follows from Proposition 4.3 that the set of all $L \in (0,+\infty)^N$ such that $L \succcurlyeq \Lambda$ is $\operatorname{Ran} M \cap (0,+\infty)^N$. The next proposition gathers some immediate properties that follow from Definition 4.2.

Proposition 4.4. *Let $\Lambda, L \in (0,+\infty)^N$. If $\Lambda \preccurlyeq L$, then, for every $\mathbf{n} \in \mathbb{N}^N$, one has $[\mathbf{n}]_\Lambda \subset [\mathbf{n}]_L$ and*

$$\widehat{\Xi}^L_{[\mathbf{n}]} = \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}^\Lambda_\tau. \tag{4.1}$$

In particular, if $\Lambda \approx L$, then, for every $\mathbf{n} \in \mathbb{N}^N$, one has $[\mathbf{n}]_\Lambda = [\mathbf{n}]_L$ and $\widehat{\Xi}^\Lambda_{[\mathbf{n}]} = \widehat{\Xi}^L_{[\mathbf{n}]}$.

Proof. If $\Lambda \preccurlyeq L$ and $\mathbf{n} \in \mathbb{N}^N$, the inclusion $[\mathbf{n}]_\Lambda \subset [\mathbf{n}]_L$ follows immediately from the fact that $Z(\Lambda) \subset Z(L)$ and that $[\mathbf{n}]_\lambda = (\mathbf{n} + Z(\lambda)) \cap \mathbb{N}^N$ for every $\mathbf{n} \in \mathbb{N}^N$ and $\lambda \in (0,+\infty)^N$. Moreover, the set $\{\tau \in \mathcal{N}_\Lambda \mid \tau \subset [\mathbf{n}]_L\}$ is a partition of $[\mathbf{n}]_L$, since, for every $\mathbf{n}' \in [\mathbf{n}]_L$, one has $[\mathbf{n}']_\Lambda \subset [\mathbf{n}']_L = [\mathbf{n}]_L$ and all equivalence classes in $\mathcal{N}_\Lambda$ are disjoint. Hence

$$\sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}^\Lambda_\tau = \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \sum_{\mathbf{n}' \in \tau} \Xi_{\mathbf{n}'} = \sum_{\mathbf{n}' \in [\mathbf{n}]_L} \Xi_{\mathbf{n}'} = \widehat{\Xi}^L_{[\mathbf{n}]}.$$

The statements in the case $\Lambda \approx L$ follow immediately. ∎

The first main result of this section is the following theorem.

Theorem 4.5. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda, L \in (0,+\infty)^N$, and $T > 0$ be such that $\Lambda \preccurlyeq L$. Set $\kappa = \max_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$. If $\Sigma(A,B,L)$ is relatively controllable in time T, then $\Sigma(A,B,\Lambda)$ is relatively controllable in time κT.*

Proof. Notice that, for every $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N \setminus \{0\}$, one has $\frac{\Lambda \cdot \mathbf{n}}{L \cdot \mathbf{n}} = \sum_{j=1}^N \frac{\Lambda_j}{L_j} \frac{L_j n_j}{L \cdot \mathbf{n}} \leq \kappa$, and thus $\Lambda \cdot \mathbf{n} \leq \kappa L \cdot \mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^N$. Using Proposition 4.4, one obtains that

$$\begin{aligned} &\operatorname{Span}\left\{ \widehat{\Xi}^L_{[\mathbf{n}]} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_L,\ L \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m \right\} \\ &= \operatorname{Span}\left\{ \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}^\Lambda_\tau B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_L,\ L \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m \right\} \end{aligned}$$

$$
\begin{aligned}
&\subset \operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{\tau} Bw \;\middle|\; \tau \in \mathcal{N}_{\Lambda},\ \tau \subset [\mathbf{n}]_L,\ [\mathbf{n}]_L \in \mathcal{N}_L,\ L\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\}\\
&= \operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ L\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\}\\
&\subset \operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda\cdot\mathbf{n} \leq \kappa T,\ w \in \mathbb{C}^m\right\},
\end{aligned}
$$

which proves the statement. ■

Theorem 4.5 proves that relative controllability of $\Sigma(A,B,L)$ implies that of $\Sigma(A,B,\Lambda)$ for all delay vectors Λ such that $\Lambda \preccurlyeq L$ (with different controllability times). The converse of this result does not hold, as illustrated in the following example.

Example 4.6. Consider the system $\Sigma(A,B,\Lambda)$ with $N=2$, $d=3$, $m=1$, $\Lambda=(1,\lambda)$ for some $\lambda \in (0,1)$, and

$$
A_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
$$

One has $A_1 = -A_2^2$ and hence one immediately computes

$$
\Xi_{\mathbf{n}} = \begin{cases} \mathrm{Id}_3, & \text{if } \mathbf{n}=(0,0),\\ A_1, & \text{if } \mathbf{n}=(1,0),\\ A_2, & \text{if } \mathbf{n}=(0,1),\\ A_2^2, & \text{if } \mathbf{n}=(0,2),\\ 0, & \text{otherwise.} \end{cases}
$$

If $\lambda \notin \mathbb{Q}$, one has $\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} = \Xi_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{N}^2$, and thus, for every $T \geq 1$,

$$
\begin{aligned}
\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}\right\} &= \operatorname{Span}\left\{\Xi_{\mathbf{n}} B \;\middle|\; \mathbf{n}=(n_1,n_2) \in \mathbb{N}^2,\ n_1+\lambda n_2 \leq T\right\}\\
&\supset \operatorname{Span}\{\Xi_{(0,0)}B, \Xi_{(1,0)}B, \Xi_{(0,1)}B\} = \mathbb{C}^3,
\end{aligned}
$$

which shows that $\Sigma(A,B,\Lambda)$ is relatively controllable for every $T \geq 1$ when $\lambda \notin \mathbb{Q}$. However, for $\lambda = \frac{1}{2}$, one computes

$$
\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} = \begin{cases} \mathrm{Id}_3, & \text{if } [\mathbf{n}]=[(0,0)],\\ A_2, & \text{if } [\mathbf{n}]=[(0,1)],\\ 0, & \text{otherwise.} \end{cases}
$$

Thus, for every $T > 0$,

$$
\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}\right\} \subset \operatorname{Span}\{B, A_2B\} \subsetneq \mathbb{C}^3,
$$

and hence $\Sigma(A,B,\Lambda)$ is not relatively controllable for any $T>0$ when $\lambda = \frac{1}{2}$.

Even if the converse of Theorem 4.5 does not hold in general, one can still obtain that relative controllability with a delay vector $\Lambda \in (0,+\infty)^N$ implies relative controllability for another delay vector $L \succcurlyeq \Lambda$ with commensurable components and sufficiently close to Λ.

Theorem 4.7. *Let $A=(A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N) \in (0,+\infty)^N$, and $T>0$. For every $\varepsilon>0$, there exists $L=(L_1,\dots,L_N) \in (0,+\infty)^N$ with commensurable components satisfying $L \succcurlyeq \Lambda$ and $1 \leq \frac{\Lambda_j}{L_j} < 1+\varepsilon$ for every $j \in [\![1,N]\!]$ such that, if $\Sigma(A,B,\Lambda)$ is relatively controllable in time T, then $\Sigma(A,B,L)$ is also relatively controllable in time T.*

Before proving Theorem 4.7, let us show the following result.

Lemma 4.8. *Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0,+\infty)^N$ and $T > 0$. For every $\varepsilon > 0$, there exists $L = (L_1, \dots, L_N) \in (0,+\infty)^N$ with commensurable components such that $L \succcurlyeq \Lambda$, $1 \leq \frac{\Lambda_j}{L_j} < 1+\varepsilon$ for every $j \in [\![1,N]\!]$, and, for every $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n} \leq T$, one has $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$ if and only if $L \cdot \mathbf{n} = L \cdot \mathbf{n}'$.*

Proof. Write $\Lambda = M\ell$, with $M = \left(m_{jk}\right)_{j \in [\![1,N]\!], k \in [\![1,h]\!]} \in \mathcal{M}_{N,h}(\mathbb{N})$ for some $h \in [\![1,N]\!]$ and $\ell = (\ell_1, \dots, \ell_h) \in (0,+\infty)^h$ with rationally independent components, chosen according to Proposition 4.3. For $n \in \mathbb{N}^*$, we define $L^{(n)} = \left(L_1^{(n)}, \dots, L_N^{(n)}\right) \in [0,+\infty)^N$ by $L^{(n)} = \frac{1}{n} M \lfloor n\ell \rfloor$, where $\lfloor n\ell \rfloor = (\lfloor n\ell_1 \rfloor, \dots, \lfloor n\ell_h \rfloor)$. We claim that $L^{(n)}$ satisfies the required properties for $n \in \mathbb{N}^*$ large enough.

Notice first that, if $n \geq 1/\ell_{\min}$, then all the components of $\lfloor n\ell \rfloor$ are positive, and hence $L^{(n)} \in (0,+\infty)^N$. Moreover, $L^{(n)} \in \mathbb{Q}^N$, and thus $L^{(n)}$ has commensurable components. If $\mathbf{n} \in Z(\Lambda)$, one has $\Lambda \cdot \mathbf{n} = 0$, which yields $\mathbf{n}^{\mathrm{T}} M \ell = 0$ and, since ℓ has rationally independent components and the row vector $\mathbf{n}^{\mathrm{T}} M$ has integer components, one obtains that $\mathbf{n}^{\mathrm{T}} M = 0$, which implies that $L^{(n)} \cdot \mathbf{n} = \frac{1}{n} \mathbf{n}^{\mathrm{T}} M \lfloor n\ell \rfloor = 0$, and hence $\mathbf{n} \in Z(L^{(n)})$, proving that $L^{(n)} \succcurlyeq \Lambda$.

For $j \in [\![1,N]\!]$, since $n\ell_j - 1 < \lfloor n\ell_j \rfloor \leq n\ell_j$, one obtains from the definition of $L^{(n)}$ that $L_j^{(n)} = \frac{1}{n} \sum_{k=1}^h m_{jk} \lfloor n\ell_k \rfloor \leq \Lambda_j$ and that $L_j^{(n)} \geq \Lambda_j - \frac{1}{n} \sum_{k=1}^h m_{jk} \geq \Lambda_j - |M|_\infty / n$. Hence, for $n \geq 1/\ell_{\min}$, one has $1 \leq \frac{\Lambda_j}{L_j^{(n)}} \leq 1 + \frac{|M|_\infty}{n L_j^{(n)}}$. Notice that, by construction, for every $j \in [\![1,N]\!]$, one has $L_j^{(n)} \to \Lambda_j$ as $n \to +\infty$. Hence there exists $N_1 \geq 1/\ell_{\min}$ such that, for $n \geq N_1$, $L_j^{(n)} \geq \Lambda_j/2$ for every $j \in [\![1,N]\!]$. Thus, for $n \geq N_1$, one has $1 \leq \frac{\Lambda_j}{L_j^{(n)}} \leq 1 + \frac{2|M|_\infty}{n\Lambda_j} \leq 1 + \frac{2|M|_\infty}{n\Lambda_{\min}}$. Letting $N_2 \geq N_1$ be such that $N_2 > \frac{2|M|_\infty}{\varepsilon \Lambda_{\min}}$, one obtains that $1 \leq \frac{\Lambda_j}{L_j^{(n)}} < 1+\varepsilon$ for every $j \in [\![1,N]\!]$ and $n \geq N_2$.

To prove the last part of the lemma, notice that, for every $n \geq 1/\ell_{\min}$, since $\Lambda \preccurlyeq L^{(n)}$, if $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$, then $\mathbf{n} - \mathbf{n}' \in Z(\Lambda)$ and thus $L^{(n)} \cdot \mathbf{n} = L^{(n)} \cdot \mathbf{n}'$. Let $\mathcal{F}$ denote the finite set $\mathcal{F} = \{\mathbf{n} \in \mathbb{N}^N \mid \Lambda \cdot \mathbf{n} \leq (1+\varepsilon)T\}$ and define

$$\delta = \min\left\{ \left| \Lambda \cdot \mathbf{n} - \Lambda \cdot \mathbf{n}' \right| \,\middle|\, \mathbf{n}, \mathbf{n}' \in \mathcal{F},\ \Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}' \right\} > 0.$$

Since $L^{(n)} \to \Lambda$ as $n \to +\infty$ and $\mathcal{F}$ is finite, there exists $N_3 \geq N_2$ such that, for $n \geq N_3$, one has $\left| L^{(n)} \cdot \mathbf{n} - \Lambda \cdot \mathbf{n} \right| < \frac{\delta}{3}$ for every $\mathbf{n} \in \mathcal{F}$. Let $n \geq N_3$. Assume, to obtain a contradiction, that $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda \cdot \mathbf{n} \leq T$, $\Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}'$, and $L^{(n)} \cdot \mathbf{n} = L^{(n)} \cdot \mathbf{n}'$. Then, using that $1 \leq \frac{\Lambda_j}{L_j^{(n)}} < 1+\varepsilon$ for every $j \in [\![1,N]\!]$, one computes $\Lambda \cdot \mathbf{n}' < (1+\varepsilon) L^{(n)} \cdot \mathbf{n}' = (1+\varepsilon) L^{(n)} \cdot \mathbf{n} \leq (1+\varepsilon) \Lambda \cdot \mathbf{n} \leq (1+\varepsilon) T$, which shows that $\mathbf{n}' \in \mathcal{F}$. But

$$\delta \leq \left| \Lambda \cdot \mathbf{n} - \Lambda \cdot \mathbf{n}' \right| \leq \left| \Lambda \cdot \mathbf{n} - L^{(n)} \cdot \mathbf{n} \right| + \left| L^{(n)} \cdot \mathbf{n} - L^{(n)} \cdot \mathbf{n}' \right| + \left| L^{(n)} \cdot \mathbf{n}' - \Lambda \cdot \mathbf{n}' \right| < \frac{2\delta}{3},$$

which is a contradiction since $\delta > 0$. Hence, if $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda \cdot \mathbf{n} \leq T$ and $\Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}'$ one has $L^{(n)} \cdot \mathbf{n} \neq L^{(n)} \cdot \mathbf{n}'$. ∎

Proof of Theorem 4.7. Let $\varepsilon > 0$ and take L as in Lemma 4.8. If $\mathbf{n} \in \mathbb{N}^N$ is such that $\Lambda \cdot \mathbf{n} \leq T$, then $[\mathbf{n}]_\Lambda = [\mathbf{n}]_L$, since it follows from Proposition 4.4 that $[\mathbf{n}]_\Lambda \subset [\mathbf{n}]_L$ and, if $\mathbf{n}' \in [\mathbf{n}]_L$, Lemma 4.8 shows that $\mathbf{n}' \in [\mathbf{n}]_\Lambda$ since $\Lambda \cdot \mathbf{n} \leq T$. In particular, the only equivalence class from $\mathcal{N}_\Lambda$ contained in $[\mathbf{n}]_L$ is $[\mathbf{n}]_\Lambda$. Hence, Proposition 4.4 shows that, for $\mathbf{n} \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n} \leq T$, one has

$$\widehat{\Xi}^L_{[\mathbf{n}]} = \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}^\Lambda_\tau = \widehat{\Xi}^\Lambda_{[\mathbf{n}]},$$

and thus

$$\begin{aligned}\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bw \;\middle|\; [\mathbf{n}]\in\mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n}\le T,\ w\in\mathbb{C}^m\right\} &= \operatorname{Span}\left\{\widehat{\Xi}^{L}_{[\mathbf{n}]}Bw \;\middle|\; \mathbf{n}\in\mathbb{N}^N,\ \Lambda\cdot\mathbf{n}\le T,\ w\in\mathbb{C}^m\right\}\\ &\subset \operatorname{Span}\left\{\widehat{\Xi}^{L}_{[\mathbf{n}]}Bw \;\middle|\; \mathbf{n}\in\mathbb{N}^N,\ L\cdot\mathbf{n}\le T,\ w\in\mathbb{C}^m\right\},\end{aligned}$$

since $L\cdot\mathbf{n}\le\Lambda\cdot\mathbf{n}$ for every $\mathbf{n}\in\mathbb{N}^N$. Hence relative controllability of $\Sigma(A,B,\Lambda)$ in time T implies relative controllability of $\Sigma(A,B,L)$ in time T. ■

5 Minimal time for relative controllability

As stated in Remark 3.3, when $N=1$ and (1.1) is written as $x(t)=Ax(t-\Lambda)+Bu(t)$, relative controllability in time T is equivalent to Kalman condition $\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lfloor T/\Lambda\rfloor}B\end{pmatrix}=d$. Thanks to Cayley–Hamilton Theorem, $\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lfloor T/\Lambda\rfloor}B\end{pmatrix}=\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{d-1}B\end{pmatrix}$ for every $T\ge(d-1)\Lambda$. Hence, if the system is relatively controllable for some time $T>0$, it is also relatively controllable in time $T=(d-1)\Lambda$, which proves that its minimal controllability time $T_{\min}$ satisfies $T_{\min}\le(d-1)\Lambda$. The uniformity of this upper bound on the matrices A and B is important for practical applications, since, if one is interested in finding out whether a given system is relatively controllable for some time $T>0$, it suffices to verify whether it is relatively controllable in time $T=(d-1)\Lambda$, which can be done algorithmically in a finite number of steps upper bounded by a constant independent of A and B. The goal of this section is to generalize this upper bound on the minimal controllability time $T_{\min}$ for systems with larger N.

We start by considering the case of systems with commensurable delays. In this case, by considering an augmented system in higher dimension, one can characterize the relative controllability of $\Sigma(A,B,\Lambda)$ in terms of a certain output controllability of the augmented system, as shown in the next lemma.

Lemma 5.1. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, and $T>0$. Assume that Λ has commensurable components and let $\lambda>0$ and $k_1,\dots,k_N\in\mathbb{N}^*$ be such that $(\Lambda_1,\dots,\Lambda_N)=\lambda(k_1,\dots,k_N)$. Denote $K=\max_{j\in[\![1,N]\!]}k_j$. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in time $T>0$ if and only if, for every $X_0:[-\lambda,0)\to\mathbb{C}^{Kd}$ and $x_1\in\mathbb{C}^d$, there exists $u:[0,T]\to\mathbb{C}^m$ such that the unique solution $X:[-\lambda,T]\to\mathbb{C}^{Kd}$ of*

$$\begin{cases}X(t)=\widehat{A}X(t-\lambda)+\widehat{B}u(t), & t\in[0,T],\\ X(t)=X_0(t), & t\in[-\lambda,0),\end{cases}\tag{5.1}$$

satisfies $\widehat{C}X(T)=x_1$, where the matrices $\widehat{A}\in\mathcal{M}_{Kd}(\mathbb{C})$, $\widehat{B}\in\mathcal{M}_{Kd,m}(\mathbb{C})$, and $\widehat{C}\in\mathcal{M}_{d,Kd}(\mathbb{C})$ are given by

$$\widehat{A}=\begin{pmatrix}\widehat{A}_1 & \widehat{A}_2 & \widehat{A}_3 & \cdots & \widehat{A}_K\\ \mathrm{Id}_d & 0 & 0 & \cdots & 0\\ 0 & \mathrm{Id}_d & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathrm{Id}_d & 0\end{pmatrix}\in\mathcal{M}_{Kd}(\mathbb{C}),\qquad \widehat{B}=\begin{pmatrix}B\\0\\0\\\vdots\\0\end{pmatrix}\in\mathcal{M}_{Kd,m}(\mathbb{C}),$$

$$\widehat{C}=\begin{pmatrix}\mathrm{Id}_d & 0 & 0 & \cdots & 0\end{pmatrix}\in\mathcal{M}_{d,Kd}(\mathbb{C}),\qquad \widehat{A}_k=\sum_{\substack{j=1\\k_j=k}}^{N}A_j\quad\text{for } k\in[\![1,K]\!],\tag{5.2}$$

Proof. It is immediate to verify that $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ is the solution of $\Sigma(A,B,\Lambda)$ with initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and control $u : [0,T] \to \mathbb{C}^m$ if and only if the function $X : [-\lambda, T] \to \mathbb{C}^{Kd}$ defined by

$$X(t) = \begin{pmatrix} x(t) \\ x(t-\lambda) \\ x(t-2\lambda) \\ \vdots \\ x(t-(K-1)\lambda) \end{pmatrix}$$

is the solution of (5.1) with control u and with initial condition $X_0 : [\lambda, 0) \to \mathbb{C}^{Kd}$ given by

$$X_0(t) = \begin{pmatrix} x_0(t) \\ x_0(t-\lambda) \\ x_0(t-2\lambda) \\ \vdots \\ x_0(t-(K-1)\lambda) \end{pmatrix}.$$

Since $\widehat{C}X(t) = x(t)$ for every $t \in [-\lambda, T]$, the statement of the lemma follows immediately from Theorem 3.1. ■

Since (5.1) is a controlled difference equation with a single delay, we use Lemma 5.1 to characterize the relative controllability of $\Sigma(A,B,\Lambda)$ in terms of a Kalman rank condition.

Corollary 5.2. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, and $T > 0$. Assume that Λ has commensurable components. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in time T if and only if*

$$\operatorname{rk}\begin{pmatrix} \widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{A}^2\widehat{B} & \cdots & \widehat{C}\widehat{A}^{\lfloor T/\lambda \rfloor}\widehat{B} \end{pmatrix} = d, \tag{5.3}$$

where $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and λ are as in the statement of Lemma 5.1.

Proof. Notice that, by Proposition 2.8, the solution $X : [-\lambda, T] \to \mathbb{C}^{Kd}$ of (5.1) with initial condition $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$ and control $u : [0,T] \to \mathbb{C}^m$ is given by

$$X(t) = \widehat{A}^{1+\lfloor t/\lambda \rfloor} X_0\left(t - \left(1 + \left\lfloor \frac{t}{\lambda} \right\rfloor\right)\lambda\right) + \sum_{n=0}^{\lfloor t/\lambda \rfloor} \widehat{A}^n \widehat{B} u(t - n\lambda).$$

Hence

$$\widehat{C}X(T) = \widehat{C}\widehat{A}^{1+\lfloor T/\lambda \rfloor} X_0\left(T - \left(1 + \left\lfloor \frac{T}{\lambda} \right\rfloor\right)\lambda\right) + \sum_{n=0}^{\lfloor T/\lambda \rfloor} \widehat{C}\widehat{A}^n \widehat{B} u(T - n\lambda). \tag{5.4}$$

If $\Sigma(A,B,\Lambda)$ is relatively controllable in time T, then, by Lemma 5.1, taking $X_0 = 0$, one obtains that, for every $x_1 \in \mathbb{C}^d$, there exists $u : [0,T] \to \mathbb{C}^m$ such that $\sum_{n=0}^{\lfloor T/\lambda \rfloor} \widehat{C}\widehat{A}^n \widehat{B} u(T - n\lambda) = x_1$, which shows that (5.3) holds. Conversely, if (5.3) holds, then the matrix $\begin{pmatrix} \widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \cdots & \widehat{C}\widehat{A}^{\lfloor T/\lambda \rfloor}\widehat{B} \end{pmatrix}$ admits a right inverse $M \in \mathcal{M}_{(\lfloor T/\lambda \rfloor + 1)m, d}(\mathbb{C})$. For $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$ and $x_1 \in \mathbb{C}^d$, let $U = \left(U_j\right)_{j=0}^{\lfloor T/\lambda \rfloor} \in \mathbb{C}^{(\lfloor T/\lambda \rfloor + 1)m}$ be given by

$$U = \begin{pmatrix} U_0 \\ \vdots \\ U_{\lfloor T/\lambda \rfloor} \end{pmatrix} = M\left[x_1 - \widehat{C}\widehat{A}^{1+\lfloor T/\lambda \rfloor} X_0\left(T - \left(1 + \left\lfloor \frac{T}{\lambda} \right\rfloor\right)\lambda\right)\right]$$

and take $u : [0,T] \to \mathbb{C}^m$ satisfying $u(T - n\lambda) = U_n$ for every $n \in [\![0, \lfloor T/\lambda \rfloor]\!]$. It follows immediately from (5.4) that the solution of (5.1) with initial condition X_0 and control u satisfies $\widehat{C}X(T) = x_1$, and hence, by Lemma 5.1, $\Sigma(A,B,\Lambda)$ is relatively controllable in time T. ■

Thanks to Cayley–Hamiltion Theorem, Corollary 5.2 allows one to obtain an upper bound on the minimal controllability time for $\Sigma(A,B,\Lambda)$ with commensurable delays.

Lemma 5.3. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$. Assume that Λ has commensurable components. If there exists $T>0$ such that $\Sigma(A,B,\Lambda)$ is relatively controllable in time T, then its minimal controllability time $T_{\min}$ satisfies $T_{\min}\leq(d-1)\Lambda_{\max}$.*

Proof. For $j\in[\![1,K]\!]$, set

$$\widehat{C}_j=\begin{pmatrix}0_{d,(j-1)d} & \mathrm{Id}_d & 0_{d,(K-j)d}\end{pmatrix}\in\mathcal{M}_{d,Kd}(\mathbb{C}).$$

In particular, $\widehat{C}_1=\widehat{C}$. For every $j\in[\![2,K]\!]$, one has $\widehat{C}_j\widehat{A}=\widehat{C}_{j-1}$, and thus $\widehat{C}=\widehat{C}_K\widehat{A}^{K-1}$. Hence, for every $k\in\mathbb{N}$, one has

$$\begin{pmatrix}\widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{A}^2\widehat{B} & \cdots & \widehat{C}\widehat{A}^k\widehat{B}\end{pmatrix}=\begin{pmatrix}\widehat{C}_K\widehat{A}^{K-1}\widehat{B} & \widehat{C}_K\widehat{A}^K\widehat{B} & \widehat{C}_K\widehat{A}^{K+1}\widehat{B} & \cdots & \widehat{C}_K\widehat{A}^{K+k-1}\widehat{B}\end{pmatrix}.$$

Moreover, since $\widehat{C}_K\widehat{A}^j=\widehat{C}_{K-j}$ for every $j\in[\![0,K-1]\!]$, one computes, for $j\in[\![0,K-2]\!]$, $\widehat{C}_K\widehat{A}^j\widehat{B}=\widehat{C}_{K-j}\widehat{B}=0$, which shows that

$$\operatorname{rk}\begin{pmatrix}\widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{A}^2\widehat{B} & \cdots & \widehat{C}\widehat{A}^k\widehat{B}\end{pmatrix}=\operatorname{rk}\begin{pmatrix}\widehat{C}_K\widehat{B} & \widehat{C}_K\widehat{A}\widehat{B} & \widehat{C}_K\widehat{A}^2\widehat{B} & \cdots & \widehat{C}_K\widehat{A}^{K+k-1}\widehat{B}\end{pmatrix}. \tag{5.5}$$

Let $T>0$ be such that $\Sigma(A,B,\Lambda)$ is relatively controllable in time T. If $T\leq(d-1)\Lambda_{\max}$, one has immediately that $T_{\min}\leq(d-1)\Lambda_{\max}$. If $T>(d-1)\Lambda_{\max}$, one has, by Corollary 5.2 and (5.5), that

$$\operatorname{rk}\begin{pmatrix}\widehat{C}_K\widehat{B} & \widehat{C}_K\widehat{A}\widehat{B} & \widehat{C}_K\widehat{A}^2\widehat{B} & \cdots & \widehat{C}_K\widehat{A}^{K+\lfloor T/\lambda\rfloor-1}\widehat{B}\end{pmatrix}=d.$$

By Cayley–Hamilton Theorem, since $\widehat{A}\in\mathcal{M}_{Kd}(\mathbb{C})$, this implies that

$$d=\operatorname{rk}\begin{pmatrix}\widehat{C}_K\widehat{B} & \widehat{C}_K\widehat{A}\widehat{B} & \widehat{C}_K\widehat{A}^2\widehat{B} & \cdots & \widehat{C}_K\widehat{A}^{K+\lfloor T/\lambda\rfloor-1}\widehat{B}\end{pmatrix}=\operatorname{rk}\begin{pmatrix}\widehat{C}_K\widehat{B} & \widehat{C}_K\widehat{A}\widehat{B} & \widehat{C}_K\widehat{A}^2\widehat{B} & \cdots & \widehat{C}_K\widehat{A}^{Kd-1}\widehat{B}\end{pmatrix}$$

since $K+\lfloor T/\lambda\rfloor-1\geq Kd-1$. Hence, by Corollary 5.2 and (5.5), it follows that $\Sigma(A,B,\Lambda)$ is relatively controllable in time $T=K(d-1)\lambda=(d-1)\Lambda_{\max}$, which yields $T_{\min}\leq(d-1)\Lambda_{\max}$. ■

Now that Lemma 5.3 has established a uniform upper bound on the minimal controllability time for $\Sigma(A,B,\Lambda)$ with commensurate delays, one can use Theorems 4.5 and 4.7 in order to deduce a uniform upper bound for all delay vectors $\Lambda\in(0,+\infty)^N$.

Theorem 5.4. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$. If there exists $T>0$ such that $\Sigma(A,B,\Lambda)$ is relatively controllable in time T, then its minimal controllability time $T_{\min}$ satisfies $T_{\min}\leq(d-1)\Lambda_{\max}$.*

Proof. Let $\varepsilon>0$ and choose $L\in(0,+\infty)^N$ according to Theorem 4.7. Then $\Sigma(A,B,L)$ is relatively controllable in time T. Thanks to Lemma 5.3, the minimal controllability time $T_{\min}^{(L)}$ for $\Sigma(A,B,L)$ satisfies $T_{\min}^{(L)}\leq(d-1)L_{\max}$, and, in particular, $\Sigma(A,B,L)$ is relatively controllable in time $(d-1)L_{\max}$. Hence, by Theorem 4.5, $\Sigma(A,B,\Lambda)$ is relatively controllable in time $(1+\varepsilon)(d-1)L_{\max}$, which proves that the minimal controllability time $T_{\min}$ for $\Sigma(A,B,\Lambda)$ satisfies $T_{\min}\leq(1+\varepsilon)(d-1)L_{\max}\leq(1+\varepsilon)(d-1)\Lambda_{\max}$. Since $\varepsilon>0$ is arbitrary, one concludes that $T_{\min}\leq(d-1)\Lambda_{\max}$. ■

Remark 5.5. The statements and proofs of the results from this section and the previous one can be slightly modified to show that, for every $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, and $T\geq(d-1)\Lambda_{\max}$, one has

$$\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}Bw\;\middle|\;[\mathbf{n}]\in\mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n}\leq T,\ w\in\mathbb{C}^m\right\}$$

$$= \operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m \right\}.$$

The set $\mathsf{V} = \operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m \right\}$ is the set of all states $x_1 \in \mathbb{C}^d$ that can be reached by the system $\Sigma(A,B,\Lambda)$ after time $T \geq (d-1)\Lambda_{\max}$ starting from a zero initial condition.

When $N=1$ and the controlled difference equation (1.1) becomes $x(t) = Ax(t-\Lambda) + Bu(t)$ with $A = A_1$ and $\Lambda = \Lambda_1$, Kalman decomposition (see, e.g., [30, Lemma 3.3.3]) states that there exists an invertible matrix $P \in \mathcal{M}_d(\mathbb{C})$ such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad PB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

with $A_{11} \in \mathcal{M}_r(\mathbb{C}), A_{22} \in \mathcal{M}_{d-r}(\mathbb{C}), B_1 \in \mathcal{M}_{r,m}(\mathbb{C})$, where $r = \dim \mathsf{V}$, the pair (A_{11}, B_1) is controllable, and $P\mathsf{V} = \mathbb{C}^r \times \{0\}^{d-r} = \operatorname{Span}\{e_1, \dots, e_r\}$.

Such decomposition does not hold for larger N in general, i.e., one cannot find in general, for $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda \in (0,+\infty)^N$ for which $\Sigma(A,B,\Lambda)$ is not relatively controllable in any time $T > 0$, a matrix $P \in \mathcal{M}_d(\mathbb{C})$ for which one would have, for every $j \in [\![1, N]\!]$,

$$PA_jP^{-1} = \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ 0 & A_{22}^{(j)} \end{pmatrix}, \qquad PB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \tag{5.6}$$

with $A_{11}^{(j)} \in \mathcal{M}_r(\mathbb{C}), A_{22}^{(j)} \in \mathcal{M}_{d-r}(\mathbb{C}), B_1 \in \mathcal{M}_{r,m}(\mathbb{C})$, with $r \in [\![0, d-1]\!]$ and such that $\Sigma(A_{11}^{(1)}, \dots, A_{11}^{(N)}, B_1, \Lambda)$ is relatively controllable in time $T \geq (r-1)\Lambda_{\max}$. Indeed, consider the case $N=2$, $d=4$, $m=1$, $\Lambda = (1, \ell)$ for some $\ell \in \left(\frac{3}{4}, 1\right)$, and

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & \pi & 0 & 1 \\ -3 & \sqrt{2} & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{2} & 0 & -1 & 0 \\ 0 & \pi+1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \sqrt{3} & 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice that

$$\begin{aligned}
&\operatorname{Span}\{\Xi_{\mathbf{n}} B \,|\, \mathbf{n} = (n_1, n_2) \in \mathbb{N}^2,\ n_1 + \ell n_2 \leq 3\} \\
&= \operatorname{Span}\{\Xi_{(0,0)}B, \Xi_{(0,1)}B, \Xi_{(0,2)}B, \Xi_{(0,3)}B, \Xi_{(1,0)}B, \Xi_{(1,1)}B, \Xi_{(1,2)}B, \Xi_{(2,0)}B, \Xi_{(2,1)}B, \Xi_{(3,0)}B\} \\
&= \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi+3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi^2+4\pi+7 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi+3 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ \pi^2+4\pi+7 \\ (5+\pi)\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \right\} \\
&= \{0\} \times \mathbb{C}^3,
\end{aligned}$$

and thus, by the definition of relative controllability and Theorem 4.5, one obtains that $\Sigma(A,B,\Lambda)$ is not relatively controllable in any time $T > 0$. We claim that this system cannot be decomposed under the form (5.6). If it were the case, one immediately verifies from (5.6) that the vector space $\mathsf{V} = P^{-1}(\mathbb{C}^r \times \{0\}^{4-r})$ would contain B and be invariant under left multiplication by A_1 and A_2. Such invariance implies in particular that $\Xi_{\mathbf{n}} B \in \mathsf{V}$ for every $\mathbf{n} \in \mathbb{N}^2$, and thus $\{0\} \times \mathbb{C}^3 \subset \mathsf{V}$. Such invariance then also implies that

$$\mathsf{V} \ni A_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \pi \\ \sqrt{2} \end{pmatrix},$$

which shows that $\mathsf{V} = \mathbb{C}^4$, contradicting the fact that $\mathsf{V} = P^{-1}(\mathbb{C}^r \times \{0\}^{4-r})$ for an invertible $P \in \mathcal{M}_4(\mathbb{C})$ and $r \in [\![0,3]\!]$. Hence $\Sigma(A,B,\Lambda)$ cannot be put under the form (5.6).

Theorem 5.4 shows that, given $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda \in (0,+\infty)^N$, if one wants to check whether $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$, it suffices to verify whether it is relatively controllable in time $(d-1)\Lambda_{\max}$, i.e., if

$$\operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \le (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m \right\} = \mathbb{C}^d$$

or, equivalently, if

$$\operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \le (d-1)\Lambda_{\max},\ j \in [\![1,m]\!] \right\} = \mathbb{C}^d, \tag{5.7}$$

where $e_1,\dots,e_m$ is the canonical basis of $\mathbb{C}^m$. The set whose span is evaluated in the left-hand side of (5.7) is finite, its cardinality being upper bounded by $m\#\{\mathbf{n} \in \mathbb{N}^N \mid |\mathbf{n}|_1 \le (d-1)\Lambda_{\max}/\Lambda_{\min}\}$, which is large when $\Lambda_{\max}/\Lambda_{\min}$ is large. The next results provides a way of improving such upper bound, and hence reducing the number of elements to be evaluated in order to study the relative controllability of $\Sigma(A,B,\Lambda)$.

Theorem 5.6. *Let $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda, L \in (0,+\infty)^N$ with $\Lambda \preccurlyeq L$. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$ if and only if*

$$\operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L\cdot\mathbf{n} \le (d-1)L_{\max},\ j \in [\![1,m]\!] \right\} = \mathbb{C}^d. \tag{5.8}$$

Proof. If (5.8) is satisfied, then, since $\Lambda\cdot\mathbf{n} \le \frac{\Lambda_{\max}}{L_{\min}} L\cdot\mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^N$, one obtains that

$$\begin{aligned}
\mathbb{C}^d &= \operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L\cdot\mathbf{n} \le (d-1)L_{\max},\ j \in [\![1,m]\!] \right\} \\
&\subset \operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \le (d-1)\Lambda_{\max}\frac{L_{\max}}{L_{\min}},\ j \in [\![1,m]\!] \right\}
\end{aligned}$$

which proves that $\Sigma(A,B,\Lambda)$ is relatively controllable in time $T = (d-1)\Lambda_{\max}\frac{L_{\max}}{L_{\min}}$, and thus also in time $T = (d-1)\Lambda_{\max}$ thanks to Theorem 5.4.

Let $\varepsilon > 0$. Write $\Lambda = M\ell$, with $M \in \mathcal{M}_{N,h}(\mathbb{N})$ for some $h \in [\![1,N]\!]$ and $\ell = (\ell_1,\dots,\ell_h) \in (0,+\infty)^h$ with rationally independent components, chosen according to Proposition 4.3. Since $\Lambda \preccurlyeq L$, it follows from Proposition 4.3 that $L \in \operatorname{Ran} M$, and thus there exists $r \in \mathbb{R}^h$ such that $L = Mr$. Take $r_\varepsilon \in \mathbb{R}^h$ with rationally independent components satisfying $|r - r_\varepsilon|_\infty < \varepsilon/|M|_\infty$, and set $L_\varepsilon = Mr_\varepsilon$. Then $|L - L_\varepsilon|_\infty < \varepsilon$ and, in particular, $L_\varepsilon \in (0,+\infty)^N$ for ε small enough. Notice that $L_\varepsilon \approx \Lambda$, since $\Lambda \preccurlyeq L_\varepsilon$ by construction and, if $\mathbf{n} \in \mathbb{N}^N$ is such that $L_\varepsilon\cdot\mathbf{n} = 0$, then $\mathbf{n}^{\mathrm{T}} M r_\varepsilon = 0$, which implies, from the fact that r_ε has rationally independent components and that $\mathbf{n}^{\mathrm{T}} M$ is a row vector of integers, that $\mathbf{n}^{\mathrm{T}} M = 0$, yielding $\Lambda\cdot\mathbf{n} = \mathbf{n}^{\mathrm{T}} M\ell = 0$, and thus $L_\varepsilon \preccurlyeq \Lambda$. Since $\Lambda \approx L_\varepsilon$, it follows from Theorem 4.5 that $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$ if and only if $\Sigma(A,B,L_\varepsilon)$ is relatively controllable in some time, i.e.,

$$\operatorname{Span}\left\{ \widehat{\Xi}^{L_\varepsilon}_{[\mathbf{n}]} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{L_\varepsilon},\ L_\varepsilon\cdot\mathbf{n} \le (d-1)L_{\varepsilon\,\max},\ j \in [\![1,m]\!] \right\} = \mathbb{C}^d.$$

By Proposition 4.4, this is equivalent to

$$\operatorname{Span}\left\{ \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L_\varepsilon\cdot\mathbf{n} \le (d-1)L_{\varepsilon\,\max},\ j \in [\![1,m]\!] \right\} = \mathbb{C}^d. \tag{5.9}$$

Notice that, if ε is small enough, then, for every $\mathbf{n} \in \mathbb{N}^N$, $L_\varepsilon\cdot\mathbf{n} \le (d-1)L_{\varepsilon\,\max}$ implies $L\cdot\mathbf{n} \le (d-1)L_{\max}$. Indeed, assume that, for every $\varepsilon > 0$, there exists $\mathbf{n}_\varepsilon \in \mathbb{N}^N$ such that $L_\varepsilon\cdot\mathbf{n}_\varepsilon \le (d-1)L_{\varepsilon\,\max}$

and $L\cdot\mathbf{n}_\varepsilon > (d-1)L_{\max}$. Then $(d-1)L_{\max} < L\cdot\mathbf{n}_\varepsilon \leq (d-1)L_{\varepsilon\,\max} + (L-L_\varepsilon)\cdot\mathbf{n}_\varepsilon$, which implies that $(d-1)L_{\max} < L\cdot\mathbf{n}_\varepsilon \leq (d-1)L_{\max} + \varepsilon(d-1+|\mathbf{n}_\varepsilon|_1)$ and so

$$(d-1)L_{\max} < L\cdot\mathbf{n}_\varepsilon \leq (d-1)L_{\max} + \varepsilon(d-1)\left(1+\frac{L_{\varepsilon\,\max}}{L_{\varepsilon\,\min}}\right) \tag{5.10}$$

Since the set $\{L\cdot\mathbf{n} \mid \mathbf{n}\in\mathbb{N}^N\}\cap[0,\tau]$ is finite for every $\tau\geq 0$, one obtains that, for every $K\geq 0$, the set $\{\mathbf{n}\in\mathbb{N}^N \mid K < L\cdot\mathbf{n}\leq K+\delta\}$ is empty if $\delta>0$ is small enough. Hence, since $L_{\varepsilon\,\max}/L_{\varepsilon\,\min}\to L_{\max}/L_{\min}$ as $\varepsilon\to 0$, one obtains that, for $\varepsilon>0$ small enough, (5.10) cannot be satisfied, which proves that $L_\varepsilon\cdot\mathbf{n}\leq(d-1)L_{\varepsilon\,\max}$ implies $L\cdot\mathbf{n}\leq(d-1)L_{\max}$ for $\varepsilon>0$ small enough.

If $\Sigma(A,B,\Lambda)$ is relatively controllable in some time, then (5.9) is satisfied. Hence, for $\varepsilon>0$ small enough,

$$\begin{aligned}\mathbb{C}^d &= \mathrm{Span}\left\{\widehat{\Xi}^\Lambda_{[\mathbf{n}]}Be_j \;\middle|\; [\mathbf{n}]\in\mathcal{N}_\Lambda,\ L_\varepsilon\cdot\mathbf{n}\leq(d-1)L_{\varepsilon\,\max},\ j\in[\![1,m]\!]\right\}\\ &\subset \mathrm{Span}\left\{\widehat{\Xi}^\Lambda_{[\mathbf{n}]}Be_j \;\middle|\; [\mathbf{n}]\in\mathcal{N}_\Lambda,\ L\cdot\mathbf{n}\leq(d-1)L_{\max},\ j\in[\![1,m]\!]\right\},\end{aligned}$$

which proves (5.8). ■

Notice that the set whose span is evaluated on the left-hand side of (5.8) has at most $m\#\{\mathbf{n}\in\mathbb{N}^N \mid |\mathbf{n}|_1\leq(d-1)L_{\max}/L_{\min}\}$ elements, which is an improvement with respect to the upper bound obtained previously for the set whose span is evaluated on the left-hand side of (5.7) as soon as $L_{\max}/L_{\min} < \Lambda_{\max}/\Lambda_{\min}$. Hence Theorem 5.6 allows one to algorithmically check whether $\Sigma(A,B,\Lambda)$ is relatively controllable in less steps than by using (5.7). In particular, since we have $\Lambda\preccurlyeq(1,1,\ldots,1)$ for every $\Lambda\in(0,+\infty)^N$ with rationally independent components, one obtains the following improvement of (5.7) in this case.

Corollary 5.7. *Let $A=(A_1,\ldots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda\in(0,+\infty)^N$. Assume that Λ has rationally independent components. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T>0$ if and only if*

$$\mathrm{Span}\left\{\Xi_{\mathbf{n}}Be_j \;\middle|\; \mathbf{n}\in\mathbb{N}^N,\ |\mathbf{n}|_1\leq d-1,\ j\in[\![1,m]\!]\right\}=\mathbb{C}^d.$$

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