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SMALL VOLUME OF BALLS, LARGE VOLUME ENTROPY AND THE MARGULIS CONSTANT

STÉPHANE SABOURAU

ABSTRACT. In his seminal work about bounded cohomology, M. Gromov showed that, under some topological conditions, every closed Riemannian manifold of small volume has large volume entropy. In this article, we strengthen some aspects of this result using an alternative approach. More precisely, we prove that, under some similar, yet different, topological assumptions, every closed Riemannian manifold whose volume of balls is small has large volume entropy. Along the proof of this result, we establish a new systolic inequality involving the commutator systole and a new curvature-free estimate relating the filling radius to the Margulis constant.

1. INTRODUCTION

The *volume entropy* of a closed Riemannian manifold M , denoted by $\text{Ent}(M)$, is defined as the exponential growth rate of the volume of balls in the universal Riemannian cover \tilde{M} of M . More precisely,

$$\text{Ent}(M) = \lim_{R \rightarrow \infty} \frac{\log[\text{vol}\tilde{B}(\tilde{x}, R)]}{R}$$

where $\tilde{B}(\tilde{x}, R)$ is the ball of radius R centered at any point \tilde{x} of \tilde{M} . Here, and throughout this article, all manifolds are supposed to be connected.

The following classical result of M. Gromov relates the volume entropy – normalized by the volume – to the simplicial volume, a nonnegative topological invariant which does not vanish for closed manifolds admitting a negatively curved Riemannian metric.

Theorem 1.1 ([Gr82]). *Let M be a closed Riemannian n -manifold. Then*

$$\text{Ent}(M)^n \text{vol}(M) \geq c_n \|M\|$$

where $\|M\|$ is the simplicial volume of M and c_n is an explicit positive constant depending only on n .

In short, if the volume is small then the volume entropy is large. The proof of this theorem relies on the notions of simplicial norm and bounded cohomology, and more specifically on the technique of diffusion of chains. A sharp estimate on the (scale invariant) product $\text{Ent}(M)^n \text{vol}(M)$ has been established by G. Besson, G. Courtois and S. Gallot for closed negatively curved locally symmetric Riemannian manifolds, *cf.* [BCG]. The proof of

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their result rests a different argument, namely the construction of a volume-decreasing map via the barycenter method.

In this article, we strengthen some aspects of M. Gromov's result, at least for closed manifolds admitting a negatively curved Riemannian metric, by relying on a yet different approach. More precisely, we prove the following.

Theorem 1.2. *Let M be a closed n -manifold admitting a negatively curved Riemannian metric. Then, there exists $\delta > 0$ such that for every Riemannian metric on M , if the volume of every ball $B(R)$ of radius R in M satisfies*

$$\text{vol}B(R) \leq a_n R^n$$

then

$$\text{Ent}(M) \geq \frac{\delta}{R}$$

where a_n is an explicit positive constant depending only on n .

In short, if the volume of every ball of small radius is small, then the volume entropy is large. Therefore, this condition ensures a large volume entropy even if the volume of the whole manifold is large. Note that for $R \geq \text{vol}(M)^{\frac{1}{n}}$, we recover a lower bound on the product $\text{Ent}(M)^n \text{vol}(M)$ to be compared with Theorem 1.1. Actually, the conclusion of Theorem 1.2 holds for a larger class of manifolds, including manifolds which do not admit nonpositively curved Riemannian metrics, *cf.* Corollary 4.6. This class of manifolds is described in terms of the topology of the loop space of their classifying spaces, see Definition 3.1 for further details.

The proof of this theorem relies on the notion of filling radius and filling techniques introduced by M. Gromov in the context of systolic geometry, *cf.* [Gr82]. The filling radius of a closed Riemannian manifold M is a geometric invariant denoted by $\text{FillRad}(M)$, see Definition 4.4, related to the volume through the following fundamental bounds of M. Gromov and L. Guth.

Theorem 1.3 ([Gr83, Main Theorem 1.2.A], [Gu11, Theorem 1]). *Let M be a closed Riemannian n -manifold. Then*

(1)

$$\text{vol}(M) \geq c_n \text{FillRad}(M)^n$$

where c_n is an explicit positive constant depending only on n .

(2) *For every $R \leq \text{FillRad}(M)$, there exists a ball $B(R)$ of radius R in M with*

$$\text{vol}B(R) \geq a_n R^n$$

where a_n is an explicit positive constant depending only on n .

These estimates make any lower bound on the filling radius highly desirable, as it would provide a universal lower bound on the volume of the manifold. In the context of curvature-free geometry, such lower bounds exist for the injectivity radius [Gr83] (and more generally, for the local contractibility function [GP92]), the length of the shortest noncontractible closed geodesic (*i.e.*, the *systole*) on essential manifolds [Gr83] and the length of the shortest closed geodesic on the two-sphere [Sa04].

In this article, we establish another curvature-free lower bound in terms of the commutator systole. Here, the *commutator systole* of a closed Riemannian manifold M , denoted by $\text{sys}_0(M)$, is defined as the length of the shortest loop representing a nontrivial element in the commutator subgroup $[\pi_1(M), \pi_1(M)]$ of $\pi_1(M)$. That is,

$$\text{sys}_0(M) = \inf\{\text{length}(\gamma) \mid [\gamma] \neq 0 \in [\pi_1(M), \pi_1(M)]\}.$$

Clearly, the commutator systole is bounded from below by the systole, that is, $\text{sys}_0(M) \geq \text{sys}(M)$.

We obtain the following filling radius estimate.

Theorem 1.4. *Let M be a closed n -manifold admitting a negatively curved Riemannian metric. Then every Riemannian metric on M satisfies*

$$\text{FillRad}(M) \geq \frac{1}{32}\text{sys}_0(M).$$

In particular,

$$(1) \quad \text{vol}(M) \geq c_n \text{sys}_0(M)^n \quad (1.1)$$

where c_n is an explicit positive constant depending only on n .

(2) For every $R \leq \text{sys}_0(M)$, there exists a ball $B(R)$ of radius R in M with

$$\text{vol}B(R) \geq a_n R^n$$

where a_n is an explicit positive constant depending only on n .

As previously, this result holds for a larger class of manifolds formed of essential manifolds. However, it fails for some essential manifolds, even for closed nonpositively curved manifolds (*e.g.*, the product of a closed hyperbolic surface with a small circle). Actually, this theorem is a consequence of a similar estimate, *cf.* Theorem 4.5, relating the filling radius to the Margulis constant defined in Definition 4.1.

Theorem 1.4 is related to a more general problem in systolic geometry connected to the notion of subgroup systole as follows. For every normal subgroup $H \triangleleft \pi_1(M)$, we define the H -systole of M as

$$\text{sys}_H(M) = \inf\{\text{length}(\gamma) \mid [\gamma] \neq 0 \in H\}.$$

By definition, $\text{sys}_0(M) = \text{sys}_H(M)$ when $H = [\pi_1(M), \pi_1(M)]$. We say that a closed n -manifold M satisfies a *systolic inequality for H* if there exists a positive constant c_M depending only on the topology of M such that

$$\text{vol}(M) \geq c_M \text{sys}_H(M)^n$$

for every Riemannian metric on M . From [Gr83] and [Ba93], a closed manifold M satisfies a systolic inequality for $H = [\pi_1(M), \pi_1(M)]$ if and only if M is essential (*e.g.*, aspherical). A natural question is to determine which manifolds M satisfy a systolic inequality for a given proper normal subgroup H in $\pi_1(M)$. For the notion of relative systole related to the quotients of the fundamental group (and not to its subgroups), the general theory of systolic inequality applies, *cf.* [Gr83], [Ba93], and yields necessary and sufficient topological conditions for the existence of (relative) systolic inequalities.

Unfortunately, this does not work for normal subgroup systoles and a general theory for systolic inequalities for normal subgroups still needs to be developed. Indeed, except for the commutator subgroup, see the inequality (1.1) in Theorem 1.4, we do not know when such inequalities hold. (For surfaces, we actually have a stronger systolic inequality for the commutator subgroup, which is asymptotically optimal with the genus, *cf.* [Sa08].) Thus, Theorem 1.4 can be seen as a first step in the direction of a more general theory.

2. REVIEW OF ALGEBRAIC AND GEOMETRIC ENTROPIES

In this short section, we introduce some definitions regarding algebraic and geometric entropies, and recall classical results relating the two notions without proof.

Definition 2.1. Let d be a left-invariant metric on a group $\Gamma = \langle S \rangle$ generated by a finite set S . Denote by e the neutral element of Γ . The metric d induces a norm $|\cdot|$ on Γ defined as

$$|\alpha| = d(e, \alpha)$$

for every $\alpha \in \Gamma$. Define the *entropy* of $(\Gamma, |\cdot|)$ as

$$\text{Ent}_{|\cdot|}(\Gamma) = \liminf_{R \rightarrow \infty} \frac{\log \#\{\alpha \in \Gamma \mid |\alpha| \leq R\}}{R} \quad (2.1)$$

In the special case where $\Gamma = \langle \alpha, \beta \rangle$ is a two-generator group and d is the word distance with respect to the generating set $\{\alpha, \beta\}$, we simply write $\text{Ent}(\alpha, \beta)$ and refer to it as the *algebraic entropy* of the pair α, β .

Let (M, g) be a closed Riemannian manifold. Fix $x_0 \in M$ and a lift \tilde{x}_0 of x_0 in the universal cover \tilde{M} . The group $\Gamma = \pi_1(M, x_0)$ acts on the universal Riemannian cover (\tilde{M}, \tilde{g}) by isometries. The geometric norm on Γ induced by g is defined as

$$|\alpha|_g = d_{\tilde{g}}(\alpha \cdot \tilde{x}_0, \tilde{x}_0)$$

for every $\alpha \in \Gamma$. The geometric norm $|\cdot|_g$ depends on the basepoint \tilde{x}_0 . However, it is a classical result that the *geometric entropy* $\text{Ent}_{|\cdot|_g}(\Gamma)$ agrees with the volume entropy $\text{Ent}(M, g)$ defined in the introduction. That is,

$$\text{Ent}_{|\cdot|_g}(\Gamma) = \text{Ent}(M, g).$$

We denote by $\text{Ent}_{|\cdot|_g}(\alpha, \beta)$ the entropy of the subgroup $\langle \alpha, \beta \rangle$ of Γ with respect to the restriction of the geometric norm on Γ induced by g . We refer to it as the *geometric entropy* of the subgroup $\langle \alpha, \beta \rangle$.

In the sequel, we will only consider the algebraic and geometric entropies (for which the liminf in (2.1) is a limit by submultiplicativity of the counting function up to some constant shift). We recall the following classical estimates regarding the algebraic and geometric entropies of a pair of elements in the fundamental group of a closed Riemannian manifold (the proofs are straightforward and are left as an exercise).

Proposition 2.2. *Let $\alpha, \beta \in \Gamma$ and $\phi : \Gamma \rightarrow G$ be a group homomorphism.*

(1)

$$\text{Ent}(\phi(\alpha), \phi(\beta)) \leq \text{Ent}(\alpha, \beta)$$

(2)

$$\text{Ent}_{|\cdot|_g}(\alpha, \beta) \leq \text{Ent}(M, g)$$

(3) If $|\alpha|_g \leq L$ and $|\beta|_g \leq L$ for some positive real L , then

$$\text{Ent}(\alpha, \beta) \leq L \times \text{Ent}_{|\cdot|_g}(\alpha, \beta).$$

In particular,

$$\text{Ent}(\alpha, \beta) \leq L \times \text{Ent}(M, g).$$

(4) If α and β commute then

$$\text{Ent}(\alpha, \beta) = 0.$$

3. FREE LOOP SPACE OF EILENBERG-MACLANE SPACES

In this section, we describe the topology of the loop space of the Eilenberg-MacLane spaces corresponding to discrete groups satisfying some algebraic features. The topology of these loop spaces will play a crucial role in our arguments.

We first need to introduce some group-theoretical properties.

Definition 3.1. Let G be a discrete group. Consider the following algebraic property.

(P₀) *The centralizer*

$$C_\alpha = \{\beta \in G \mid \alpha\beta = \beta\alpha\}$$

of every nontrivial element α of G is an infinite cyclic subgroup.

Consider also the following algebraic property with $\delta > 0$.

(P_δ) *The algebraic entropy of every pair α, β of elements of G which do not lie in the same infinite cyclic subgroup is at least δ . That is,*

$$\text{Ent}(\alpha, \beta) \geq \delta.$$

From Proposition 2.2.(4), the property (P_δ) implies (P₀) and can be viewed as a quantitative version of (P₀).

Example 3.2. Torsion-free Gromov hyperbolic groups satisfy the property (P₀), cf. [Gr87]. In particular, this is the case for the fundamental group $\pi_1(M)$ of every closed Riemannian n -manifold M of curvature $\kappa \leq -1$. Actually, the fundamental group $\pi_1(M)$ also satisfies the stronger property (P_δ) for some δ uniformly bounded from above in terms of natural Riemannian invariants of M . More precisely, this holds true in either of the following cases:

- (1) for every $\delta \leq \delta(n, \lambda)$, where λ is a (negative) lower bound on the curvature, cf. [BCG11];
- (2) for every $\delta \leq \delta(n, i_0)$, where i_0 is a lower bound on the injectivity radius, cf. [BCG].

Definition 3.3. A nontrivial element α in a discrete group G is *primitive* if it cannot be written as the power of any element in G . That is, if $\alpha = \omega^m$ for some $\omega \in G$ and $m \in \mathbb{Z}$ then $m = \pm 1$.

The following lemma underlines some algebraic consequences of the property (P_0) .

Lemma 3.4. *Let G be a discrete group satisfying the property (P_0) . Let α and β be two nontrivial elements in G . Then*

- (1) *The group G is torsion-free.*
- (2) *The generators of the (infinite cyclic) centralizer C_α are primitive.*
- (3) *If α and β commute then $C_\alpha = C_\beta$. In particular, every nontrivial element of G lies in a unique maximal infinite cyclic subgroup, namely its centralizer.*

Proof. The centralizers of the nontrivial elements of G are infinite cyclic subgroups and so are torsion-free. It follows that G itself is torsion-free since any nontrivial element of G lies in its own centralizer.

Let α_0 be a generator of C_α . Suppose that $\alpha_0 = \omega^m$ for some $\omega \in G$ and some integer m . Clearly, ω commutes with α , that is, $\omega \in C_\alpha = \langle \alpha_0 \rangle$. Thus, $\omega = \alpha_0^k$ for some integer k . Therefore, $\alpha_0 = \alpha_0^{km}$. Now, since G is torsion-free, we deduce that $km = 1$, and so $m = \pm 1$.

Let α_0 and β_0 be some generators of C_α and C_β . Since $\beta \in C_\alpha = \langle \alpha_0 \rangle$, we have $\beta = \alpha_0^j$ for some integer j . Thus, α_0 commutes with β , that is, $\alpha_0 \in C_\beta$. Therefore, the centralizer C_α , which is generated by α_0 , is a subgroup of C_β . Switching the roles of α and β , we derive $C_\alpha = C_\beta$. \square

Definition 3.5. Let G be a discrete group and $BG = K(G, 1)$ be the Eilenberg-MacLane space of G . Denote by $\Lambda(BG) = \text{Map}(S^1, BG)$ the *free loop space* of BG and by $\Omega(BG)$ the space of loops based at the same fixed point $*$ in BG . Both spaces are endowed with the compact-open topology.

The homotopy type of the free loop space of BG is closely related to the algebraic properties of G . We refer to [Bur89] and references therein for a description of the homology group of the free loop space in terms of the Hochschild homology of the group ring $\mathbb{Z}[G]$. In our case, we will need the following simpler description of its connected components which results from [H74].

Proposition 3.6. *Let G be a discrete group. Then the connected components of $\Lambda(BG)$ are in one-to-one correspondence with the conjugacy classes in G .*

Moreover, if G satisfies the property (P_0) then

- (1) *the connected component of $\Lambda(BG)$ containing the constant loops is homotopy equivalent to BG ;*
- (2) *the connected components of $\Lambda(BG)$ not containing the constant loops are homotopy equivalent to the circle $S^1 = \mathbb{R}/\mathbb{Z}$.*

Proof. The first statement of the proposition is classical. We will focus on the topological description of the connected components of the free loop space following the argument of [H74]. Fix a loop γ_* in $\Omega(BG)$ with base-point $* \in BG$ and consider the connected component of $\Lambda(BG)$ containing γ_* . All homotopy groups will have the point $*$ or the loop γ_* as their

basepoint. The natural Serre fibration

$$\begin{array}{ccc} \Omega(BG) & \hookrightarrow & \Lambda(BG) \xrightarrow{p} BG \\ & & \gamma \longmapsto \gamma(0) \end{array}$$

induces the long exact sequence

$$\cdots \rightarrow \pi_{n+1}(BG) \rightarrow \pi_n(\Omega(BG)) \rightarrow \pi_n(\Lambda(BG)) \rightarrow \pi_n(BG) \rightarrow \cdots \rightarrow \pi_0(BG).$$

Now, from the adjoint relation, *cf.* [Ha02, p. 395], we have $\pi_n(\Omega(BG)) = \pi_{n+1}(BG)$ for every $n \geq 0$ (hence the homotopy equivalence $\Omega(BG) \simeq G$) and $\pi_n(BG) = 0$ for $n \geq 2$. Thus, for $n \geq 2$, we obtain

$$\pi_n(\Lambda(BG)) = 0$$

while for $n = 1$, we derive the inclusion

$$p_{\#} : \pi_1(\Lambda(BG)) \hookrightarrow \pi_1(BG).$$

We claim that $\pi_1(\Lambda(BG))$ agrees with the centralizer $C_{[\gamma_*]}$ of the homotopy class $[\gamma_*]$ of γ_* in $\pi_1(BG)$. Indeed, every element α in $\pi_1(BG)$ lies in the image of $p_{\#}$ if and only if there exists a map

$$S^1 \times S^1 \rightarrow BG$$

such that

- the loop $c_1 : S^1 \times \{0\} \rightarrow BG$ agrees with γ_* ;
- the loop $c_2 : \{0\} \times S^1 \rightarrow BG$ represents α .

In this case, the homotopy classes of c_1 and c_2 commute. That is, $\alpha \in C_{[\gamma_*]}$. Conversely, given a loop c whose homotopy class α commutes with $[\gamma_*]$, we can construct a map

$$S^1 \times S^1 \rightarrow BG$$

such that $S^1 \times \{0\} \rightarrow BG$ and $\{0\} \times S^1 \rightarrow BG$ agree with γ_* and c by contracting the commutator loop $\gamma_* c \gamma_*^{-1} c^{-1}$ to the basepoint $*$.

Therefore, the connected component Λ_{γ_*} of $\Lambda(BG)$ containing γ_* is an aspherical space with fundamental group the centralizer $C_{[\gamma_*]}$.

To conclude, we need to examine two cases depending whether the homotopy class $[\gamma_*]$ of γ_* is trivial or not. In the first case, its centralizer $C_{[\gamma_*]}$ is the whole group G and the connected component Λ_{γ_*} is homotopy equivalent to $K(G, 1) = BG$. While, in the second case, its centralizer $C_{[\gamma_*]}$ is isomorphic to \mathbb{Z} and the connected component Λ_{γ_*} is homotopy equivalent to the $K(\mathbb{Z}, 1)$ -space S^1 . \square

Example 3.7. Let M be a closed negatively curved Riemannian manifold, see Example 3.2. The Birkhoff curve shortening flow gives rise to a deformation retract of the free loop space. This curve shortening flow retracts each noncontractible loop of M to the unique closed geodesic in its free homotopy class. It also retracts contractible loops of M to points.

We conclude this section with the following important consequence of Proposition 3.6, key to the proof of the filling radius estimate of Theorem 1.4.

Remark 3.8. Let G be a discrete group satisfying the property (P₀). Every noncontractible loop γ in BG is homotopic to the iterate γ_0^k of a loop γ_0 whose homotopy class $[\gamma_0]$ generates the centralizer $C_{[\gamma]}$ and so is primitive from Lemma 3.4. The primitive loop γ_0 is unique up to orientation and

homotopy. It can be chosen to depend only on the maximal infinite cyclic subgroup $C_{[\gamma]}$ generated by $[\gamma]$, *cf.* Lemma 3.4.

From Proposition 3.6, there exists a deformation retract of the unparameterized free loop space $\Pi(BG) = \Lambda(BG)/O(2)$, which homotopes contractible loops to points and noncontractible loops γ to the iterate γ_0^k of the primitive loop γ_0 modulo $O(2)$. We will fix such a homotopy once and for all, and refer to it as the *deformation retract of the (unparameterized free) loop space*.

Note that this property fails for the torus \mathbb{T}^2 , which obviously does not satisfy the property (P₀). Indeed, the free homotopy class of a noncontractible unparameterized loop γ of \mathbb{T}^2 is homotopy equivalent to S^1 (the S^1 -family of closed geodesics homotopic to γ for any given flat metric, for instance). Therefore, there is no canonical representative – up to homotopy – to which the whole class of γ deformation retracts, instead, there is a whole family of natural representatives parameterized by S^1 .

4. FILLING RADIUS AND MARGULIS CONSTANT

In this section, we establish a universal geometric inequality, *cf.* Theorem 4.5, between the filling radius, *cf.* Definition 4.4, and the Margulis constant, *cf.* Definition 4.1, for closed Riemannian manifolds satisfying some topological properties, *cf.* Proposition 3.1. As a consequence of this inequality, we derive a lower bound on the volume entropy in terms of the volume of small balls, *cf.* Corollary 4.6.

We first need to introduce the definition of the Margulis constant.

Definition 4.1. Let M be a closed Riemannian manifold and $\phi : \pi_1(M) \rightarrow G$ be a group homomorphism. For every $x \in M$ and $\ell > 0$, denote by

$$\Gamma_{\phi,x}^\ell = \langle \phi([\gamma]) \in G \mid x \in \gamma, \phi([\gamma]) \neq e \text{ and } \text{length}(\gamma) \leq \ell \rangle$$

the subgroup of G generated by the ϕ -image of the homotopy classes of the loops based at x of length at most ℓ . Observe that $\Gamma_{\phi,x}^{\ell'}$ is a subgroup of $\Gamma_{\phi,x}^\ell$ for every $\ell' \leq \ell$. Hence, if $\Gamma_{\phi,x}^\ell$ is cyclic, the same holds for $\Gamma_{\phi,x}^{\ell'}$.

Define

$$\ell_\phi(M) = \sup\{\ell \mid \Gamma_{\phi,x}^\ell \text{ is a cyclic subgroup of } G \text{ for every } x \in M\}.$$

The *Margulis constant* can be defined as

$$\ell_0(M) = \ell_{\text{id}}(M).$$

It is minimal among the $\ell_\phi(M)$. More precisely, we have the following.

Proposition 4.2. *Let M and ϕ be as above. Then*

$$\ell_0(M) \leq \ell_\phi(M).$$

Proof. Let $\ell < \ell_0(M)$. By definition of the Margulis constant, for every $x \in M$, the subgroup $\Gamma_{\text{id},x}^\ell$ is cyclic. Since every homomorphic image of a cyclic group is cyclic, the subgroup $\Gamma_{\phi,x}^\ell = \phi(\Gamma_{\text{id},x}^\ell)$ is cyclic too. Therefore, $\ell \leq \ell_\phi(M)$ and the desired inequality follows. \square

The Margulis constant is related to the commutator systole through the straightforward bound.

Proposition 4.3. *Let M be a closed Riemannian manifold whose fundamental group satisfies the property (P_0) . Then*

$$\text{sys}_0(M) \leq 4 \ell_0(M).$$

Proof. By definition of the Margulis constant, there exist two loops γ_1 and γ_2 based at the same point of length at most $\ell_0(M)$ whose homotopy classes α_1 and α_2 do not lie in the same (infinite) cyclic subgroup. From (P_0) , the homotopy classes α_1 and α_2 do not commute, that is, the commutator $[\alpha_1, \alpha_2]$ is nontrivial. By construction, this commutator can be represented by a loop of length at most $4 \ell_0(M)$, namely $\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$. Hence, the commutator systole of M satisfies the bound $\text{sys}_0(M) \leq 4 \ell_0(M)$. \square

Let us recall the notion of filling radius introduced by M. Gromov in [Gr83] to established systolic inequalities on essential manifolds.

Definition 4.4. Let (M, g) be a closed Riemannian n -manifold. Denote by d_g the distance on M induced by the Riemannian metric g . The map

$$i : (M, d_g) \hookrightarrow (L^\infty(M), \|\cdot\|)$$

defined by $i(x)(\cdot) = d_g(x, \cdot)$ is an embedding from the metric space (M, d_g) into the Banach space $L^\infty(M)$ of bounded functions on M endowed with the sup-norm $\|\cdot\|$. This natural embedding, also called the Kuratowski embedding, is an isometry between metric spaces. We will consider M isometrically embedded into $L^\infty(M)$.

The *filling radius* of (M, g) , denoted by $\text{FillRad}(M, g)$, is the infimum of the positive reals ρ such that

$$(i_\rho)_*([M]) = 0 \in H_n(U_\rho(M); \mathbb{k})$$

where $i_\rho : M \hookrightarrow U_\rho(M)$ is the inclusion into the ρ -tubular neighborhood of M in $L^\infty(M)$, and $[M] \in H_n(M; \mathbb{k})$ is the fundamental class of M . Here, the homology coefficients are in $\mathbb{k} = \mathbb{Z}$, if M is orientable, and in $\mathbb{k} = \mathbb{Z}_2$, otherwise.

The following filling radius estimate, which holds, for instance, for every closed manifold admitting a negatively curved Riemannian metric, *cf.* Example 3.2, is key to all the volume and volume entropy bounds in this article.

Theorem 4.5. *Let M be a closed n -manifold with $n \geq 2$. Consider a group homomorphism $\phi : \pi_1(M) \rightarrow G$ to a discrete group G satisfying the property (P_0) . Suppose that the classifying map $\Phi : M \rightarrow K(G, 1)$ induced by ϕ satisfies $\Phi_*[M] \neq 0 \in H_n(G; \mathbb{k})$. Then every Riemannian metric on M satisfies*

$$\text{FillRad}(M) \geq \frac{1}{8} \ell_\phi(M).$$

In particular,

(1)

$$\text{vol}(M) \geq c_n \ell_0(M)^n$$

where c_n is an explicit positive constant depending only on n .

(2) For every $R \leq \ell_0(M)$, there exists a ball $B(R)$ of radius R in M with

$$\text{vol}B(R) \geq a_n R^n$$

where a_n is an explicit positive constant depending only on n .

Proof. By definition of the filling radius, the fundamental class $[M]$ of M vanishes in the ρ -neighborhood $U_\rho(M)$ of M in $L^\infty(M)$, where $\rho > \text{FillRad}(M)$. Therefore, there exists a continuous map $\sigma : P \rightarrow U_\rho(M)$ defined on an $(n+1)$ -dimensional simplicial complex P such that the restriction $\sigma : \partial P \rightarrow M$ represents $[M]$ in $H_n(M; \mathbb{k})$, that is,

$$\sigma_*[\partial P] = [M] \in H_n(M; \mathbb{k}). \quad (4.1)$$

Suppose that $\rho < \frac{1}{8}\ell_\phi(M)$. We would like to construct a continuous map $F : P \rightarrow K$ which agrees with $\Phi \circ \sigma$ on ∂P . Combined with (4.1), this would yield the relation $\Phi_*[M] = F_*[\partial P]$. Hence a contradiction with $\Phi_*[M] \neq 0$. Actually, it might not be possible to construct such a map F . Still, we will show how to adapt the arguments to obtain a contradiction.

Denote by P^k the k -skeleton of P . Subdividing P if necessary, we can assume that the diameter of the images by σ of the simplices of P is less than $\varepsilon > 0$, with $\varepsilon < \frac{1}{4}\ell_\phi(M) - 2\rho$. We first define a map

$$\bar{f} : P^0 \cup \partial P \rightarrow M$$

with $\bar{f}|_{\partial P} = \sigma$ by sending each vertex $p_i \in P$ to a nearest point $\sigma(p_i)$ in M , as we wish. Since the inclusion $i : M \hookrightarrow U_\rho(M)$ is isometric, every pair p_i, p_j of adjacent vertices of P satisfies

$$\begin{aligned} d_M(\bar{f}(p_i), \bar{f}(p_j)) &\leq d_{L^\infty}(\bar{f}(p_i), \sigma(p_i)) + d_{L^\infty}(\sigma(p_i), \sigma(p_j)) + d_{L^\infty}(\sigma(p_j), \bar{f}(p_j)) \\ &\leq r \end{aligned}$$

with $r = 2\rho + \varepsilon < \frac{1}{4}\ell_\phi(M)$. We extend the map \bar{f} to P^1 by taking the edges of $P \setminus \partial P$ to minimizing segments in M joining the images of their endpoints, as we wish.

By construction, the boundary of every 2-simplex of P is sent by \bar{f} to a loop of length at most $3r$. If the length was less than the systole of M , the loop would be contractible in M . We could define a map $P^2 \cup \partial P \rightarrow M$ extending \bar{f} to the 2-simplices of P . The composition of this map with Φ could then be extended to a map $F : P \rightarrow K$ by asphericity of K leading to the desired contradiction. This is the original argument developed by Gromov in the proof of the systolic inequality on essential manifolds, *cf.* [Gr83]. However, in our case, the boundary of some 2-simplex of P might be mapped to a noncontractible loop of M . Indeed, we only know that it is sent to a loop of M of length at most $3r$, not necessarily shorter than the systole of M .

Consider the composite map

$$f = \Phi \circ \bar{f} : P^1 \cup \partial P \rightarrow K$$

Let us introduce the following definition. Given a k -simplex Δ of P , we will say that Δ *extends well* to K if the image $f(\partial\Delta^2)$ of the boundary $\partial\Delta^2$ of every 2-simplex Δ^2 of Δ is contractible in K . As noticed above, for such a simplex, the map $f : P^1 \cup \partial P \rightarrow K$ extends to Δ by asphericity of K .

For other simplices, we need to make use of the far-reaching consequences of the property (P_0) summarized in Proposition 3.6 and Remark 3.8. Specifically, we proceed as follows. Let Δ be a 2-simplex of P which does not extend well to K , that is, $f(\partial\Delta)$ is homotopically nontrivial in K . The deformation retract of the loop space of K given by Proposition 3.6 and Remark 3.8 gives rise to a homotopy from $f(\partial\Delta)$ to a closed curve depending only on the nontrivial homotopy class of $f(\partial\Delta)$. This closed curve is the iterate of a primitive loop γ_Δ , namely the generator (up to reparametrization) of the unique maximal infinite cyclic subgroup containing the homotopy class of $f(\partial\Delta)$. In particular, the image $f(\partial\Delta)$ is freely homotopic to γ_Δ^k for some integer k .

At this point, we want to emphasize that our construction strongly relies on the deformation retract of the loop space of K given by Proposition 3.6 and Remark 3.8. In particular, all the loops in a homotopically nontrivial class – here, the homotopy class of $f(\partial\Delta)$ – naturally deform to the same loop up to homotopy. This will allow us to extend the map f to the simplex Δ in a way compatible with an extension to the 3-skeleton P^3 , and ultimately to the whole complex P .

Define the cellular complex

$$X = K \cup \left(\bigcup_{\Delta \in \mathcal{C}} D_\Delta \right)$$

by gluing a 2-cell D_Δ along γ_Δ for every 2-simplex Δ of P such that $f(\partial\Delta)$ is noncontractible in K . By construction, every loop $f(\partial\Delta)$ is contractible in X . More precisely, every loop $f(\partial\Delta)$ is contractible in

$$K_\Delta \cup D_\Delta$$

where K_Δ is the cover of K with fundamental group the subgroup of G generated by the homotopy class of γ_Δ . By definition of γ_Δ , this subgroup is isometric to \mathbb{Z} . Thus, the space K_Δ has the same homotopy type as $S^1 = K(\mathbb{Z}; 1)$. It follows that the space $K_\Delta \cup D_\Delta$ is homotopy equivalent to D_Δ and so to a point. Therefore, the map

$$f : P^1 \cup \partial P \rightarrow K \hookrightarrow X$$

extends to

$$f : P^2 \cup \partial P \rightarrow X.$$

Now, contrary to K or $K_\Delta \cup D_\Delta$, the whole space X might not be aspherical (it is not difficult to construct such examples). Therefore, we need to rely on other arguments to show that $f : P^2 \cup \partial P \rightarrow X$ extends to P .

Let Δ_1 and Δ_2 be two 2-simplices of P lying in the same $(n+1)$ -simplex. These two simplices can be connected by an edge $e = [p_1, p_2]$ of P (which may be empty if they have a vertex in common) whose endpoints p_i are vertices of Δ_i . The edge e decomposes into two segments $e_1 = [p_1, p]$ and $e_2 = [p_2, p]$ with a common endpoint p such that the arcs $\bar{f}(e_1)$ and $\bar{f}(e_2)$ have the same length in M . By construction, the two loops

$$\bar{f}(e_i) \cup \bar{f}(\partial\Delta_i) \cup \bar{f}(e_i)$$

of M based at $\bar{f}(p)$ are of length at most $4r$. Since $4r < \ell_\phi(M)$, the image by Φ of these two loops generate a cyclic subgroup in homotopy. In particular, if none of the loops $f(\partial\Delta_i)$ is contractible in K , then $\gamma_{\Delta_1} = \gamma_{\Delta_2}$ up to reparametrization. This observation is key for our argument.

Let Δ^3 be a 3-simplex of P which does not extend well to K . As observed above, the loops γ_{Δ^2} , and so the spaces $K_{\Delta^2} \cup D_{\Delta^2}$, are the same for every 2-simplex Δ^2 of Δ^3 such that $f(\partial\Delta^2)$ is noncontractible in K . We will denote them by γ_{Δ^3} and $K_{\Delta^3} \cup D_{\Delta^3}$, since they only depend on Δ^3 and not on the 2-simplices Δ^2 . By construction, the map f takes the boundary $\partial\Delta^3$ of Δ^3 to $K_{\Delta^3} \cup D_{\Delta^3}$ before projecting it to X . As the space $K_{\Delta^3} \cup D_{\Delta^3}$ is contractible, the map

$$\partial\Delta^3 \rightarrow K_{\Delta^3} \cup D_{\Delta^3}$$

extends to Δ^3 and gives rise to a map

$$f : \Delta^3 \rightarrow K_{\Delta^3} \cup D_{\Delta^3} \rightarrow K \cup D_{\Delta^3} \hookrightarrow X$$

where the second map is given by the covering map $K_{\Delta^3} \rightarrow K$. Note that the loop γ_{Δ^3} along which the 2-cell D_{Δ^3} is attached remains fixed under this covering map.

This argument carries over by induction on the k -skeleton P^k of P as follows. By induction, the map $f : P^{k-1} \cup \partial P \rightarrow X$ extends to a map $f : P^k \cup \partial P \rightarrow X$, where the restriction of f to every k -simplex Δ^k which does not extend well to K decomposes as

$$f : \Delta^k \rightarrow K_{\Delta^k} \cup D_{\Delta^k} \rightarrow K \cup D_{\Delta^k} \hookrightarrow X \quad (4.2)$$

and the second map is given by the covering map $K_{\Delta^k} \rightarrow K$. The key point to extend the map $f : \partial\Delta^{k+1} \rightarrow X$ to Δ^{k+1} is to observe that the contractible spaces $K_{\Delta^k} \cup D_{\Delta^k}$ are the same for every k -simplex Δ^k of Δ^{k+1} which does not extend well to K , as previously noticed. We will denote this space by $K_{\Delta^{k+1}} \cup D_{\Delta^{k+1}}$. Thus, the map

$$\partial\Delta^{k+1} \rightarrow K_{\Delta^{k+1}} \cup D_{\Delta^{k+1}}$$

given by the first map of (4.2) extends to Δ^k , which concludes the proof of the induction.

In conclusion, the map

$$\partial P \xrightarrow{\sigma} M \xrightarrow{\Phi} K \hookrightarrow X$$

extends to $f : P \rightarrow X$. For $n \geq 3$, the composite map $M \xrightarrow{\Phi} K \hookrightarrow X$ induces a nontrivial homomorphism between $H_n(M; \mathbb{k})$ and $H_n(X; \mathbb{k})$. This leads to a contradiction since $f_*[\partial P]$ agrees with the nonzero homology class $\Phi_*[M]$ in $H_n(M; \mathbb{k}) = H_n(X; \mathbb{k})$. For $n = 2$, we assume that all the maps are simplicial. Let B be an open 2-cell in the 2-skeleton K^2 of K lying in the support of the homologically nontrivial 2-cycle given by the image of M under Φ . Consider the map

$$K^2 \hookrightarrow X^2 \rightarrow S^2 = B/\partial B$$

defined by collapsing all the points of X^2 lying outside B to a single point. The pre-composition of this map with Φ induces a monomorphism between $H_2(M; \mathbb{k})$ and $H_2(S^2; \mathbb{k})$. Hence a contradiction as above.

It follows that the closed Riemannian manifold M does not bound in $U_\rho(M)$ with $\rho < \frac{1}{8}\ell_\phi(M)$. Hence the inequality $\text{FillRad}(M) \geq \frac{1}{8}\ell_\phi(M)$. \square

Corollary 4.6. *Let M be a closed n -manifold as in Theorem 4.5, where the group G satisfying the property (P_δ) for some $\delta > 0$ (see for instance Example 3.2). Then every Riemannian metric on M satisfies*

$$\text{vol}(M) \text{Ent}(M)^n \geq c_n \delta^n$$

where c_n is an explicit positive constant depending only on n .

Moreover, there exists an explicit positive constant a_n depending only on n such that if the volume of every ball $B(R)$ of radius $R \leq \text{vol}(M)^{\frac{1}{n}}$ in M satisfies

$$\text{vol}B(R) \leq a_n R^n$$

then

$$\text{Ent}(M) \geq \frac{\delta}{R}.$$

Proof. By definition of $\ell_\phi(M)$, there exist two loops γ_1 and γ_2 based at the same point of M , of length at most $\ell_\phi(M)$, such that the images by ϕ of the homotopy classes $\alpha_1 = [\gamma_1]$ and $\alpha_2 = [\gamma_2]$ do not lie in the same infinite cyclic subgroup of G . From the property (P_δ) and Proposition 2.2.(1)-(3), we have

$$\delta \leq \text{Ent}(\phi(\alpha_1), \phi(\alpha_2)) \leq \text{Ent}(\alpha_1, \alpha_2) \leq \ell_\phi(M) \text{Ent}(M, g).$$

Now, the desired result follows from a combination of this inequality with Theorem 4.5 and Theorem 1.3. \square

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