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Abstract. A quenched large deviation principle for Brownian motion in Poissonian potentials with long-range correlations is proved. The proofs are based on a method developed by Sznitman [30] for Brownian motion among obstacles with compact support. An important tool to do so is the existence of the so-called Lyapunov exponents. In particular, it provides large deviations for Brownian motion in random potentials with polynomially decaying correlations such as the classical potentials studied by L. Pastur, R. Fukushima [24, 13] among others and the potentials recently introduced by Lacoin [18, 19].

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1. Introduction and results

In this paper, we will establish quenched large deviation estimates for Brownian motion in long-range potentials that are of the form

$$V(x,\omega)=\sum_j W(x-\omega_j),\quad x\in\mathbb{R}^d, \tag{1.1}$$

where $\omega=(\omega_j;j\in\mathbb{N})$ is a Poisson cloud in $\mathbb{R}^d$, $d\geq 1$ and $W(x)=|x|^{-\gamma}\wedge 1$ with $\gamma>d$. In this parameter range, the potential is almost surely finite.

The *quenched and annealed survival functions up to time* t are respectively defined by

$$S_{t,\omega}:=E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds)\big],\quad t>0,\omega\in\Omega \tag{1.2}$$

$$S_t:=\mathbb{E}E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds)\big],\quad t>0, \tag{1.3}$$

where $Z=(Z_s)_{s\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$, P_x is the Wiener measure starting from $x\in\mathbb{R}^d$, E_x is the expectation with respect to P_x and $\mathbb{E}$ is the expectation with respect to the Poisson cloud.

For $\gamma>d$, the survival functions are strictly positive. Precise estimates of the asymptotic behavior of the annealed survival function were obtained by Donsker and Varadhan for $\gamma>d+2$ and by Pastur [24] and Fukushima [12] for $d<\gamma<d+2$. The case where $\gamma=d+2$ is considered by Ôkura [23] and Chen and Kulik [3, 4] worked on the case $\gamma\leq d$.

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We will also consider potentials introduced by Lacoin [18, 19]. Their interest stems from the fact that the relations verified by their scaling exponents differ substantially from those established by Wüthrich [32, 33, 34] for a potential of the form (1.1) where W has compact support.

These potentials are constructed from a Poisson Boolean model. Let $\Omega := \{\omega = (\omega_i, r_i)_{i\geq 0}, \omega_i \in \mathbb{R}^d, r_i \geq 1\}$ be a Poisson point process in $\mathbb{R}^d \times [0,\infty[$, $d \geq 1$, whose intensity measure is given by $\mathcal{L} \times \nu$, where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^d$ and ν is a probability measure on $[0,\infty[$ which depends on a parameter $\delta > 0$ and is defined by

$$\nu([r,\infty[) = r^{-\delta}, \quad r \geq 1. \tag{1.4}$$

Note that each Poisson cloud $\omega \in \Omega$ is a locally finite subset of $\mathbb{R}^d \times [0,\infty[$. Index (ω_i, r_i) so that $(|\omega_i|, i \geq 1)$ is an increasing sequence where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. See [21, section 1.4] for an alternative description of this model and [16] for results on the percolative properties of the balls centered at the points ω_j with radius r_j.

Given $\gamma > 0$, Lacoin's potential $V : \mathbb{R}^d \times \Omega \longrightarrow [0,\infty[$ is defined by

$$V(x,\omega) := \sum_{i=1}^{\infty} r_i^{-\gamma} \mathbf{1}_{\{|x-\omega_i|<r_i\}}. \tag{1.5}$$

The behavior of this model depends on the positive parameters δ and γ. For $\gamma + \delta > d$, the potential is finite a.s. and the survival functions are strictly positive.

We are interested in the *quenched path measures* of the Brownian motion in these random potentials. They are defined by

$$dQ_{t,\omega} := \frac{1}{S_{t,\omega}} \exp\big(-\int_0^t V(Z_s,\omega)ds\big) dP_0, \qquad t \geq 0, \omega \in \Omega \tag{1.6}$$

where the normalizing constants $S_{t,\omega}$ are the quenched survival functions defined in (1.2).

Our goal is to pursue previous works by obtaining large deviation estimates for the quenched path measures when the potential is of the form (1.1) or (1.5). To do so, we follow the method developed by Sznitman [30, 31].

As a first step, in Theorems 2.1 and 2.2, we show the existence of the Lyapunov exponents and a shape Theorem for Brownian motion in a stationary potential under very general assumptions. The appropriate tool in this context is provided by Björklund [1]. In a second step, although we do not give the precise asymptotics, we show that the survival functions S_t and $S_{t,\omega}$ have sub-exponential decay. Indeed, this is all that is needed to obtain large deviation estimates. See Lemma 3.6 and Proposition 3.10 for Lacoin's potential. Then combining ideas of [30, 31] and Fukushima [12], we obtain upper and lower large deviation estimates.

We will see that the sufficient conditions for the existence of the Lyapunov exponents and for a shape Theorem given in Theorems 2.1 and 2.2 are verified by potentials of the form (1.1) when $\gamma > d$ and of the form (1.5) when $\gamma + \delta - d > 0$. They are also verified by a potential of the form (1.1) where W is nonnegative and compactly supported as in [31] and by the regular potentials considered by Ruess [28].

Sznitman's method, based on the Lyapunov exponents, can also be used to obtain large deviation estimates for random walks in a random potential in [35, 11]. Mourrat [22] considered the simple random walk in an i.i.d. potential taking values in $[0;+\infty]$. See also [20].

An approach by homogenization might also be possible. This approach is described in [25] and further developed in [26] for random walks in a random potential which can be unbounded and

can depend on a few steps of the walk. For a large class of random walks, they proved a (level 3) large deviation principle under a moment condition on the potential and a mixing condition on the environment. In particular, note that for a potential of the form (1.1) or (1.5), if it is finite $\mathbb{P}$ a.s. then it has exponential moments (see (4.89) and (3.46) respectively). However by Davidov's covariance inequality [8] (see also [27, inequality (1.3)]), to verify the mixing condition of [26, lemma A.4 (d)], a necessary condition is that $\mathbb{C}\mathrm{ov}(V(0), V(x)) < C|x|^{-d}$ for some $C > 0$. By (4.85), this condition is verified by a potential of the form (1.1) whenever $\gamma > d$ but for a potential of the form (1.5), it is not verified for all values of the parameters with $\gamma + \delta > d$ as can be seen from (3.47). In this respect, the approach to large deviations by Lyapunov exponents seems to be the best suited one for models with a behavior like Lacoin's model in the parameter range $\gamma + \delta - d > 0$, $\gamma < d$ but $2\gamma + \delta - d < d$.

For $y \in \mathbb{R}^d$ and $R > 0$, let $B(y, R)$ be the Euclidean ball $\{x \in \mathbb{R}^d; |x - y| < R\}$. $B(y)$ stands for $B(y, 1)$ and $H(y)$ is the hitting time of $\bar{B}(y)$, the closure of $B(y)$,

$$H(y) = \inf\{t \geq 0 : Z_t \in \bar{B}(y)\}.$$

For an open set $A \subset \mathbb{R}^d$, $T_A = \inf\{t \geq 0, Z_t \notin A\}$.

The volume of the unit ball of $\mathbb{R}^d$ is denoted by $\mathcal{L}_d$ and λ_d is the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in the unit disk.

2. Lyapunov exponents

In this section, we work under very general assumptions. We first show the existence of Lyapunov exponents of a Brownian motion in a stationary potential and then we show that they verify a shape Theorem.

Recently, Ruess [28] considered Brownian motion in a stationary potential. Inspired by Schröder [29], he showed that the Lyapunov exponents exist and verify a variational formula for a large class of potential. However, as we will see below, from the subadditive Theorem, one can prove the existence of the Lyapunov exponents under much weaker assumptions on the potential.

Independently of the Brownian motion, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a $\mathbb{R}^d$-*action* $(\tau_z), z \in \mathbb{R}^d$ which is a family of measurable maps indexed by $\mathbb{R}^d$ such that $\mathbb{P}$ is invariant under τ_z for all $z \in \mathbb{R}^d$ and $\tau_x \circ \tau_x = \tau_{x+y}$ for all $x, y \in \mathbb{R}^d$.

Let $V(\omega)$ be a nonnegative random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the *random potential* defined by

$$V(x, \omega) = V(\tau_x \omega), \quad x \in \mathbb{R}^d, \omega \in \Omega.$$

That is, $(V(x, \omega), x \in \mathbb{R}^d)$ is a stationary potential. We will call it a stationary ergodic potential whenever the underlying dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \tau_z, z \in \mathbb{R}^d)$ is ergodic (see [5]).

Note that under condition (2.9) below, $\mathbb{P}$ a.s. $V(\cdot, \omega)$ belongs to the *the local Kato class* $\mathcal{K}_d^{\mathrm{loc}}$ which can be defined by

$$\mathcal{K}_d^{\mathrm{loc}} := \{f : \mathbb{R}^d \to \mathbb{R} \text{ measurable }; \text{for all } R \geq 1, \lim_{r \downarrow 0} \sup_{x \in B(0,R)} E_x[\int_0^{T_r} |f|(Z_s)ds] = 0\}$$

where $T_r := \inf\{s \geq 0 : |Z_s - Z_0| \geq r\}$ (see [31, p. 12] and [14, Assumption 1]).

Indeed, since V is a stationary potential, for all $R > 0$,

$$\mathbb{E}[\sup_{B(0,R)} V(\cdot,\omega)] \leq R^d \mathbb{E}[\sup_{B(0)} V(\cdot,\omega)] < \infty.$$

For $x, y \in \mathbb{R}^d, |x-y| > 1$, we define

$$e(x,y,\omega) := E_x[\exp(-\int_0^{H(y)} V(z_s,\omega)ds), H(y) < \infty] \tag{2.7}$$

$$a(x,y,\omega) := -\log e(x,y,\omega). \tag{2.8}$$

Since $V(\cdot,\omega) \in \mathcal{K}_d^{\text{loc}}$ a.s., the probabilites $e(x,y,\omega)$ are strictly positive.

We introduce the *Green measure* relative to the potential V:

$$G(x,A,\omega) := E_x\big[\int_0^\infty \mathbf{1}_A(Z_t)\exp(-\int_0^t V(Z_s,\omega)ds)dt\big]$$

where $x \in \mathbb{R}^d$, $\omega \in \Omega$ and A is a Borel subset of $\mathbb{R}^d$. G can be interpreted as the expected occupation time measure of Brownian motion killed at rate $V(\cdot,\omega)$. We define $g(x,y,\omega)$ as the density function relative to the Green measure and we call it the *Green function*. The existence of g is proved in [31, (2.2.3)].

We show in Theorems 2.1 and 2.2 below that the Green function as well as $a(x,y,\omega)$ have an exponential decay rate which is called Lyapunov exponent.

Theorem 2.1 (Existence of Lyapunov exponent)**.** *Let V be a stationary ergodic potential such that*

$$\mathbb{E}[\sup_{B(0)} V(\cdot,\omega)] < \infty. \tag{2.9}$$

For $d = 1, 2$, moreover, we suppose that there exist positive constants ρ, ε such that for all $\omega \in \Omega$, there is $u = u(\omega) \in \mathbb{R}^d$ such that

$$\mathcal{L}(\{V(\cdot,\omega) > \varepsilon\} \cap B(u,\rho)) > \varepsilon \text{ and } \mathbb{E}|u(\cdot)| < \infty \tag{2.10}$$

Then there is a non-random semi-norm $\alpha(\cdot)$ on $\mathbb{R}^d$ such that $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$, for all $x \in \mathbb{R}^d$

$$\lim_{k\to\infty} \frac{1}{k} a(0,kx,\omega) = \lim_{k\to\infty} \frac{1}{k}\mathbb{E}[a(0,kx,\omega)] = \inf_{k\in\mathbb{N}} \frac{1}{k}\mathbb{E}[a(0,kx,\omega)] = \alpha(x). \tag{2.11}$$

α is called the quenched Lyapunov exponent. $a(0,x,\omega)$ can be replaced by $-\log g(0,x,\omega)$ in (2.11).

Björklund [1] extended to a very general context the shape Theorem proved in [6] for first passage percolation with independent passage times and in [2] for stationary passage times. This theorem can be applied in our framework.

First, recall the definition of the Lorentz space (see for instance [1, p.634] or [2]).

$$\mathbb{L}^{d,1} = \{f : (\Omega,\mathcal{F}) \to (\mathbb{R},\mathcal{B}(R)) \text{ is measurable and } ||f||_{d,1} < \infty\}$$

where $||f||_{d,1} = \int_0^1 f^*(s)s^{(1/d)-1}ds$ and $f^* : [0,1] \to \mathbb{R}^+$ is the non-increasing right continuous function which has the same distribution as $|f|$. Note that $\mathbb{L}^{d,1}$ is a Banach space and there are positive constants c_1 and c_2 such that for all $\varepsilon > 0$

$$c_1||f||_d \leq ||f||_{d,1} \leq c_2||f||_{d+\varepsilon} \qquad \text{where } ||f||_p^p = \int_\Omega |f|^p d\mathbb{P}. \tag{2.12}$$

Theorem 2.2 (Shape Theorem)**.** *Let V be a stationary ergodic potential such that*

$$\sup_{x\in B(0)} V(x,\cdot) \in \mathbb{L}^{d,1}. \tag{2.13}$$

For $d = 1$ or 2, suppose moreover that there exist positive constants ρ, ε such that for all $\omega \in \Omega$, there is $u = u(\omega) \in \mathbb{R}^d$ such that

$$\mathcal{L}(\{V(\cdot,\omega) > \varepsilon\} \cap B(u,\rho)) > \varepsilon \text{ and } \mathbb{E}(|u(\cdot)|^d) < \infty \tag{2.14}$$

Then $\mathbb{P}$ a.s.

$$\lim_{x\to\infty} \frac{1}{|x|}|a(0,x,\omega) - \alpha(x)| = 0 \tag{2.15}$$

$a(0,x)$ can be replaced by $-\log g(0,x)$ in (2.15).

If the potential is stationary and regular in the sense of [28], then $\sup_\Omega V(\omega) < \infty$, and condition (2.13) is obviously verified. Therefore when $d \geq 3$, we obtain the existence of Lyapunov exponents and shape Theorem.

For the proof of Theorems 2.1 and 2.2, we need to define

$$d(x,y,\omega) := \max(-\inf_{B(x)} \log e(\cdot,y,\omega), -\inf_{B(y)} \log e(\cdot,x,\omega)), \quad x,y \in \mathbb{R}^d, \omega \in \Omega.$$

By using the strong Markov property of Brownian motion, we can see immediately that $d(\cdot,\cdot,\omega)$ is a semi-norm on $\mathbb{R}^d$. By [31, Lemma 5.2.1], $\mathbb{P}$ a.s. $d(\cdot,\cdot,\omega)$ defines a distance on$\mathbb{R}^d$ which induces the usual topology.

We shall now derive estimates which are helpful to compare the various quantities $a(x,y,\omega)$, $g(x,y,\omega)$ and $d(x,y,\omega)$. Define

$$F(\omega) := \log^+(\int_{B(0)\times B(0)} g(x,y,\omega)dxdy) + \sup_{\bar{B}(0)} V(\cdot,\omega), \quad \omega \in \Omega \tag{2.16}$$

and let $F(x,\omega) := F(\tau_x\omega)$.

Theorem 2.2 is first proved for $d(0,x,\omega)$. Condition (2.18) allows to replace $d(0,x,\omega)$ by $a(0,x,\omega)$ or $g(0,x,\omega)$ in equation (2.15).

The proof of the following lemma can be found in [31, Proposition 5.2.2]. The proof is very general as it requires only basic notions of potential theory.

Lemma 2.3. *Under condition (2.9), there exists a positive constant $C(d)$ such that for $x, y \in \mathbb{R}^d, |x-y| > 4$, $\mathbb{P}$ a.s.*

$$\max(|d(x,y,\omega) + \log g(x,y,\omega)|, |d(x,y,\omega) - a(x,y,\omega)|) \leq C(d)(1 + F(x,\omega) + F(y,\omega)).$$

Lemma 2.4. *(i) If (2.9), and (2.10) when $d = 1$ or 2, hold, then for all $x \in \mathbb{Z}^d$, $\mathbb{P}$ a.s,*

$$\lim_{k\to\infty} \frac{F(kx,\omega)}{k} = \lim_{k\to\infty} \mathbb{E}\frac{F(kx,\omega)}{k} = 0. \tag{2.17}$$

(ii) If (2.13), and (2.14) when $d = 1$ or 2, hold, then $\mathbb{P}$ a.s. and in $L^1(\mathbb{P})$,

$$\lim_{x\to\infty, x\in\mathbb{Z}^d} \frac{F(x,\omega)}{|x|} = 0. \tag{2.18}$$

Proof. 1) Assume $d \geq 3$. Then there is a positive constant C such that $\mathbb{P}$ a.s. for all $x \neq y$, $g(x,y,\omega) \leq C|x-y|^{2-d}$. Hence

$$\int_{B(0)\times B(0)} g(x,y,\omega)dxdy \leq \int_{B(0)\times B(0)} C|x-y|^{2-d}dxdy < \infty.$$

The convergence in $L^1(\mathbb{P})$ is evident. Fix $x \in \mathbb{R}^d$. Put $X_k(\omega) := \sup_{\bar{B}(kx)} V(\cdot,\omega)$. By condition (2.9), $(X_k; k \geq 0)$ is a stationary sequence of nonnegative random variables with finite expectation. For $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}(X_k \geq \varepsilon k) = \sum_{k=1}^{\infty} \mathbb{P}(X_0 \geq \varepsilon k) \leq \varepsilon^{-1}\mathbb{E}(X_0) < \infty. \tag{2.19}$$

Then by Borel Cantelli lemma, $\mathbb{P}$ a.s., $\lim_{k\to\infty} \frac{X_k}{k} = 0$. It follows that condition (2.17) is verified for $d \geq 3$.

2) Assume that $d = 1$ or 2 and that condition (2.10) is verified for some positive numbers ρ, ε and for $u : \Omega \to \mathbb{R}^d$ such that $\mathbb{E}|u| < \infty$.

Consider $D := B(0, |u| + 2\rho + 1)$. Construct two increasing sequences of stopping times with respect to the natural right continuous filtration $(\mathcal{F}_t)$ on $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$. These stopping times describe the successive times of return to $\bar{B}(0)$ and exit times from D of the Brownian motion

$$R_1 := \inf\{t \geq 0 : Z_t \in \bar{B}(0)\}, \qquad U_1 := \inf\{t \geq R_1, Z_t \notin D\}$$

and by induction for $n \geq 1$,

$$R_{n+1} = R_1 \circ \theta_{U_n} + U_n, \qquad U_{n+1} = U_1 \circ \theta_{R_n} + R_n$$

where θ_t, $t \geq 0$ is the canonical shift on $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$.

Since the Brownian motion is recurrent when $d = 1$ or 2, the stopping times are a.s. finite,

$$0 \leq R_1 < U_1 < R_2 < U_2 < \cdots < R_n < U_n \cdots \text{ and } R_n, U_n \uparrow \infty \ .$$

We now have for $x \in \mathbb{R}^d$,

$$\begin{aligned}
\int_{\bar{B}(0)} g(x,y,\omega)dy &= \int_0^\infty E_x[\mathbf{1}_{\{Z_t \in \bar{B}(0)\}} \exp(-\int_0^t V(Z_s)ds)]dt \\
&= E_x[\int_0^\infty \mathbf{1}_{\bar{B}(0)}(Z_t) \exp(-\int_0^t V(Z_s)ds)dt] \\
&= E_x[\sum_{i\geq 1} \int_{R_i}^{U_i} \mathbf{1}_{\bar{B}(0)}(Z_t) \exp(-\int_0^t V(Z_s)ds)dt] \\
&\leq \sum_{i=1}^\infty E_x[\exp(-\int_0^{R_i} V(Z_s)ds) \int_{R_i}^{U_i} \mathbf{1}_{\bar{B}(0)}(Z_t)dt] \\
&= \sum_{i=1}^\infty E_x\big[\exp(-\int_0^{R_i} V(Z_s)ds) E_{Z_{R_i}}[\int_0^{U_1} \mathbf{1}_{\bar{B}(0)}(Z_t)dt]\big] \\
&\text{by the strong Markov property,} \\
&\leq \sup_{x\in\bar{B}(0)} E_x(T_D) \sum_{i=1}^\infty E_x[\exp(-\int_0^{R_i} V(Z_s)ds)] \\
&\leq C(|u|+2\rho+1)^2 \sum_{i=1}^\infty E_x[\exp(-\int_0^{R_i} V(Z_s)ds)]. \qquad (2.20)
\end{aligned}$$

Now, for $i \geq 1$, by the strong Markov property and by induction, for all $x \in \bar{B}(0)$,

$$\begin{aligned}
E_x[\exp(-\int_0^{R_{i+1}} V(Z_s)ds)] &\leq E_x[\exp(-\int_0^{R_i} V(Z_s)ds) E_{Z_{R_i}}[\exp(-\int_0^{T_D} V(Z_s)ds)]] \\
&\leq E_x[\exp(-\int_0^{R_i} V(Z_s)ds)] \cdot c(\omega) \leq c(\omega)^i \qquad (2.21)
\end{aligned}$$

where

$$c(\omega) := \sup_{x\in\bar{B}(0)} E_x[\exp(-\int_0^{T_D} V(Z_s)ds)]. \qquad (2.22)$$

Note that a lower bound on the heat kernel in a region of $\mathbb{R}^d$ as the one obtained from [31, Lemma 2.1] (or more generally [7, Theorem 3.3.5]) is enough to deduce that given $\rho > 0$ there is $\eta = \eta(\rho) > 0$ such that for all measurable $A \subset B(0,\rho)$ and for all $x \in \overline{B}(0,\rho)$

$$E_x[\int_0^{T_{B(0,2\rho)}} \mathbf{1}_A(Z_s)ds] > \eta \mathcal{L}(A). \qquad (2.23)$$

Now let $A := \{V(\cdot,\omega) > \varepsilon\} \cap B(u,\rho)$ and let $Y := \int_0^{T_{B(0,2\rho)}} \mathbf{1}_A(Z_s)ds$. Then by (2.23), (2.10) and by Cauchy-Schwarz, there is a constant $C > 0$ such that for all $x \in \overline{B}(u,\rho)$,

$$\varepsilon\eta \leq E_x(Y) \leq E(Y; Y > \varepsilon\eta/2) + \varepsilon\eta/2 \leq (E_x(Y^2)P_x(Y > \varepsilon\eta/2))^{1/2} + \varepsilon\eta/2.$$

Hence for all $x \in \overline{B}(u,\rho)$,

$$P_x[\int_0^{T_{B(0,2\rho)}} \mathbf{1}_A(Z_s)ds > \varepsilon\eta/2] > \Big(\frac{\varepsilon\eta}{2}\Big)^2 \frac{1}{E_x(Y^2)} > \frac{C}{\rho^4}\Big(\frac{\varepsilon\eta}{2}\Big)^2. \qquad (2.24)$$

Moreover, by the tubular estimate [31, p. 198], for all $t > 0$ and $x \in \bar{B}(0)$,

$$P_x\left[\sup_{0<s<t}|Z_s - (x_1 + \frac{s}{t}(u - x_1))| < \rho\right] \geq C\exp\left(-\lambda_d\frac{t}{\rho^2} - \frac{1}{2t}|u - x_1|^2\right) \tag{2.25}$$

$$\geq C\exp\left(-\lambda_d\frac{t}{\rho^2} - \frac{1}{t}(|u|^2 + 1)\right). \tag{2.26}$$

Recall here that λ_d is the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in the unit disk. Hence, by taking $t = |u| + 1$ in (2.26),

$$P_x[T_D > H_{B(u,\rho)}] \geq P_x\left[\sup_{0<s<t}|Z_s - (x + \frac{s}{t}(u - x))| < \rho\right]$$

$$\geq C\exp\left(-(1 + \frac{\lambda_d}{\rho^2})(|u| + 1)\right). \tag{2.27}$$

Then by (2.24), (2.27) and by the strong Markov property, for all $x \in \overline{B}(0)$

$$P_x[\int_0^{T_D}\mathbf{1}_A(Z_s)ds > \eta\varepsilon/2] \geq P_x\Big[T_D > H_{B(u,\rho)}, P_{Z_{H_{B(u,\rho)}}}\big(\int_0^{T_{B(u,2\rho)}}\mathbf{1}_A(Z_s)ds > \eta\varepsilon/2\big)\Big]$$

$$> C\exp\left(-(1 + \frac{\lambda_d}{\rho^2})(|u| + 1)\right)\left(\frac{\varepsilon\eta}{2\rho^2}\right)^2 := p_0(u). \tag{2.28}$$

This provides the following upper bound for $c(\omega)$ defined in (2.22).

$$\begin{aligned} c(\omega) &= \sup_{x\in\bar{B}(0)} E_x[\exp(-\int_0^{T_D} V(Z_s)ds)] \\ &\leq \sup_{x\in\bar{B}(0)} E_x[\exp(-\varepsilon\int_0^{T_D}\mathbf{1}_A(Z_s)ds)] \\ &\leq \exp(-\eta\varepsilon^2/4)p_0(u) + 1 - p_0(u) \\ &= 1 - p_0(u)(1 - e^{-\eta\varepsilon^2/4}). \end{aligned}$$

Then by (2.20) and (2.21), for all $x \in \overline{B}(0)$,

$$\int_{\bar{B}(0)} g(x, y, \omega)dy \leq C(|u| + 2\rho + 1)^2\sum_{i=1}^{\infty} E_x[\exp(-\int_0^{R_i} V(Z_s)ds)]$$

$$\leq C(|u| + 2\rho + 1)^2\frac{1}{1 - c(\omega)}. \tag{2.29}$$

Therefore by (2.28) and (2.29),

$$\log^+\int_{B(0)\times B(0)} g(x, y, \omega)dxdy \leq C[1 + \log^+|u| - \log(1 - c(\omega))]$$

$$\leq C[1 + \log^+|u| - \log p_0(u)]$$

$$\leq C[1 + \log^+|u| + |u|]. \tag{2.30}$$

Since $\mathbb{E}(\log^+|u|) \leq \mathbb{E}(|u|) < \infty$, the lemma follows for $d = 1, 2$.

(2.18) follows from (2.30) and the fact that if $X(x), x \in \mathbb{Z}^d$ are identically distributed with $\mathbb{E}(|X(0)|^d) < \infty$, then $\lim_{x\to\infty, x\in\mathbb{Z}^d}\frac{X(x)}{|x|} = 0$, $\mathbb{P}$ a.s. This can be checked as in (2.19). ∎

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 We apply here Kingman's subadditive Theorem. Given $x \in \mathbb{R}^d \backslash \{0\}$. We define

$$X_{m,n} := d(mx, nx, \omega), \qquad 0 \le m \le n.$$

Since $d(\cdot, \cdot, \omega)$ is a distance and by the stationarity of potential V, we have that

(i) $X_{0,0} = 0, X_{m,n} \le X_{m,k} + X_{k,n}$ for all $0 \le m \le k \le n$.

(ii) $\{X_{m,n}\}_{0 \le m \le n}$ has same distribution under $\mathbb{P}$ as $\{X_{m+1,n+1}\}_{0 \le m \le n}$.

We can also show that

(iii) $\mathbb{E}[X_{0,1}] < \infty$.

Indeed, for all $t > 0$, $z \in B(0), \omega \in \Omega$, we have that

$$\begin{aligned} e(z,x,\omega) &= E_z[\exp(-\int_0^{H(x)} V(Z_s,\omega)ds), H(x) < \infty] \\ &\ge E_z[\exp(-\int_0^t V(Z_s,\omega)ds), \sup_{0\le s\le t} |Z_s - (z + \frac{s}{t}(x-z)|) < 1] \\ &\ge P_z\big[\sup_{0\le s\le t} |Z_s - (z+\frac{s}{t}(x-z)|) < 1\big] \exp\big(-t \sup_{y \in \mathcal{C}_r(z,x)} V(y,\omega)\big) \end{aligned} \tag{2.31}$$

where $\mathcal{C}_r(z,x) := \{y \in \mathbb{R}^d; \inf_{0\le s\le 1} |y - (z + s(x-z))| < r\}$.

By the tubular estimate [31, p. 198], there exists a positive constant $C = C(d)$ such that for all $t > 0$ and $r > 0$,

$$P_z\big[\sup_{0\le s\le t} |Z_s - (z+\frac{s}{t}(x-z))| < r\big] \ge C \exp\big(-t\frac{\lambda_d}{r^2} - \frac{1}{2}\frac{|x-z|^2}{t}\big). \tag{2.32}$$

Set $t = |x - z|$. Then by (2.31) and (2.32), there is a positive constant C_0 such that

$$-\log e(z,x,\omega) \le C_0(|x-z| \vee 1) + \sup_{y \in \mathcal{C}_1(z,x)} V(y,\omega).$$

Therefore, by taking the sup over $z \in B(0)$, and by exchanging the role of 0 and x,

$$d(0,x,\omega) \le C_0(|x|+1) + \sup_{y \in \mathcal{C}_2(0,x)} V(y,\omega). \tag{2.33}$$

And since, by definition, $\mathbb{E}[X_{0,1}] = \mathbb{E}[d(0,x,\omega)]$, condition (iii) follows (2.9).

Since the hypothesis of the subadditive Theorem are satisfied, and since we assumed that the dynamical system is ergodic, there exists a constant $\alpha(x)$ such that $\mathbb{P}$ a.s.

$$\lim_{k\to\infty} \frac{1}{k} d(0,kx,\omega) = \lim_{k\to\infty} \frac{1}{k}\mathbb{E}[d(0,kx,\omega)] = \inf_{k\in\mathbb{N}} \frac{1}{k}\mathbb{E}[d(0,kx,\omega)] = \alpha(x) \tag{2.34}$$

It is easy to check that $\alpha(\cdot)$ is a semi-norm on $\mathbb{R}^d$ and that for all $x \in \mathbb{R}^d$

$$\alpha(x) \le (C_0 + \mathbb{E}[\sup_{B(0,3)} V(\cdot)])\ |x|. \tag{2.35}$$

By Lemmas 2.3 and 2.4, one can replace $d(0,x,\omega)$ by either one of $a(0,x,\omega)$, $-\log g(0,x,\omega)$ in equations (2.34) and (2.35) above. ∎

Proof of Theorem 2.2

By stationarity of the potential and by translation invariance of Brownian motion, $d(x,y,\tau_z\omega) = d(x+z,y+z,\omega)$ $\mathbb{P}$ a.s. for all $z,y,z \in \mathbb{R}^d$. Moreover, by [31, Lemma 5.2.1], $d(\cdot,\cdot,\omega)$ is a.s. a distance on $\mathbb{R}^d$. Under the integrability condition (2.13), it follows from (2.33) that $d(0,x,\omega)$ is in $\mathbb{L}^{d,1}$ for all $x \in \mathbb{Z}^d$.

Hence the conditions of the shape Theorem [1, Theorem 1.2] are verified. Therefore, there exists a semi-norm L on $\mathbb{R}^d$ such that

$$\lim_{|x|\to\infty, x\in\mathbb{Z}^d} \frac{d(0,x,\omega) - L(x)}{|x|} = 0 \qquad \text{a.s.} \tag{2.36}$$

But by Theorem 2.1, $\alpha(x) = L(x)$ for all $x \in \mathbb{Z}^d$ and consequently, α.

For $x \in \mathbb{R}^d$, denote by $\hat{x}$ the nearest neighbor point in $\mathbb{Z}^d$ of x (with some rule to break ties). Then, $|x-\hat{x}| < \sqrt{d}$ and for all $x \in \mathbb{R}^d \setminus B(0)$,

$$\begin{aligned}\frac{|d(0,x,\omega)-\alpha(x)|}{|x|} &\le \frac{|d(0,x,\omega)-d(0,\hat{x},\omega)|}{|x|} + \frac{|d(0,\hat{x},\omega)-\alpha(\hat{x})|}{|x|} + \frac{|\alpha(\hat{x})-\alpha(x)|}{|x|} \\ &\le \frac{|d(\hat{x},x,\omega)|}{|\hat{x}|}\cdot\frac{|\hat{x}|}{|x|} + \frac{|d(0,\hat{x},\omega)-\alpha(\hat{x})|}{|\hat{x}|}\cdot\frac{|\hat{x}|}{|x|} + \frac{\alpha(x-\hat{x})}{|x|}\end{aligned} \tag{2.37}$$

Consider successively the terms on the right hand side of (2.37) above. As in (2.33), for all $x \in \mathbb{R}^d$,

$$\begin{aligned}d(\hat{x},x) &\le C_0(|\hat{x}-x|+1) + \sup_{y\in\mathcal{C}_2(\hat{x},x)} V(y,\omega) \\ &\le C_0(\sqrt{d}+1) + \sup_{B(\hat{x},\sqrt{d}+3)} V(\cdot,\omega) := Y(\hat{x}).\end{aligned} \tag{2.38}$$

Since $(Y(\hat{x}), \hat{x} \in \mathbb{Z}^d)$ are identically distributed and in $L^d(\mathbb{P})$, by Borel-Cantelli lemma,

$$\lim_{x\to\infty} \frac{|d(\hat{x},x|)}{|\hat{x}|}\cdot\frac{|\hat{x}|}{|x|} \le \lim_{\hat{x}\to\infty,\hat{x}\in\mathbb{Z}^d} \frac{2Y(\hat{x})}{|\hat{x}|} = 0 \quad \mathbb{P}\text{ a.s.} \tag{2.39}$$

So the first term converges a.s. to 0. From (2.36) and from (2.35) respectively, the second and third terms converge to 0 a.s. Hence $\mathbb{P}$ a.s.,

$$\lim_{|x|\to\infty} \frac{1}{|x|}|d(0,x,\omega)-\alpha(x)| = 0. \tag{2.40}$$

By using (ii) of Lemma 2.4, $d(0,x)$ can be replaced by $a(0,x)$ or $-\log g(0,x)$ in the equation above. ∎

We end this section with two lemmas which will be useful later to obtain large deviation estimates.

Lemma 2.5. *Let V be a stationary potential such that for all $\theta \in \mathbb{R}$*

$$\mathbb{E}\exp(\theta \sup_{B(0)} V(\cdot,\omega)) < \infty. \tag{2.41}$$

Then condition (2.13) is verified and

$$\limsup_{r\to\infty} \frac{1}{\log r} \sup_{B(0,r)} V(\cdot,\omega) = 0 \qquad \mathbb{P}\text{ a.s.} \tag{2.42}$$

Proof. Since $(\theta X)^n/n! \le e^{\theta X}$ for $\theta > 0$, $n \in \mathbb{N}$ and X a nonnegative random variable, condition (2.13) follows from (2.41) and the remark (2.12).

By translation invariance and Markov's inequality, for all $\varepsilon > 0$ and $\theta > 0$

$$\mathbb{P}(\sup_{B(0,r)} V(\cdot) > \varepsilon \log r) \leq r^d \mathbb{P}(\sup_{B(0)} V(\cdot) > \varepsilon \log r) \leq r^{d-\theta\varepsilon} \mathbb{E} \exp(\theta \sup_{B(0)} V(\cdot, \omega)). \tag{2.43}$$

Choose θ such that $\theta\varepsilon - d > 1$ and apply Borel-Cantelli lemma to obtain (2.42). ∎

In [31, Section 5.4], Sznitman obtained large deviation estimates for a Brownian motion in a Poissonian potential constructed from a function W with compact support. The rate function is given in terms of the Lyapunov exponents

$$I(x) := \sup_{\lambda \geq 0} (\alpha_\lambda(x) - \lambda), \qquad x \in \mathbb{R}^d \tag{2.44}$$

where $\alpha_\lambda(x)$ are the Lyapunov exponents with respect to the potential $\lambda + V$. In sections 3 and 4, we will show that how this result can be extended to large classes of long-range potentials.

Lemma 2.6. *Under conditions (2.9) and (2.10), $I(x)$ is a nonnegative convex continuous function.*

3. Lacoin's potential

In this section, large deviation estimates are obtained for the potentials introduced by Lacoin in [18], [19]. They were defined in (1.4) and (1.5) of the introduction. We first use Theorem 2.1 to show the existence of the Lyapunov exponents which are then used to express the rate function which appear in the large deviation estimates. To do so the shape theorem is used as well as a description of large time behavior of the Brownian motion.

Some basic properties of the potentials are gathered in the following lemma.

Lemma 3.1. *$\mathbb{P}$ a.s. $V(x, \omega)$ is finite for every $x \in \mathbb{R}^d$ if and only if $\gamma + \delta - d > 0$. In this case,*

$$\mathbb{E}[V(0)] = \frac{\mathcal{L}_d \delta}{\gamma + \delta - d}, \quad \mathbb{V}ar[V(0)] = \frac{\mathcal{L}_d \delta}{2\gamma + \delta - d} \tag{3.45}$$

and for all $R > 0$, $s \in \mathbb{R}$,

$$\mathbb{E}\big[\exp\big(s \sum_{i=1}^{\infty} r_i^{-\gamma} \mathbf{1}_{\{|\omega_i| \leq r_i + R\}}\big)\big] = \exp\big(\int_1^{\infty} \delta \mathcal{L}_d (r+R)^d r^{-\delta-1} (e^{sr^{-\gamma}} - 1) dr\big) \tag{3.46}$$

is finite and there are positive constants $C_1(d, \delta, \gamma)$, $C_2(d, \delta, \gamma)$ such that for all $x \in \mathbb{R}^d$, $|x| > 1$,

$$C_1 |x|^{d-\delta-2\gamma} \leq \mathbb{C}\mathrm{ov}(V(0), V(x)) \leq C_2 |x|^{d-\delta-2\gamma}. \tag{3.47}$$

In particular, Lacoin's potential verifies (2.41).

Proof. Using Campell's Theorem for the sums over Poisson point processes. A detailed proof of the lemma can be found in [20, page 78-80]. ∎

Then we always assume that $\gamma + \delta - d > 0$ in this section.

3.1. **Existence of Lyapunov exponents.** For $z \in \mathbb{R}^d, t \in \mathbb{R}$, we define

$$\tau_z^t : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}, (x,r) \mapsto (x - tz, r)$$

and

$$T_z^t : \Omega \to \Omega, (\omega) = \{(\omega_i, r_i), i \geq 0\} \mapsto \{\tau_z^t(\omega_i, r_i), i \geq 0\}. \tag{3.48}$$

By a change of variable,

$$\int_{\mathbb{R}^d} \int_1^\infty f(\tau_z^t(x,r)) dx d\nu(r) = \int_{\mathbb{R}^d} \int_1^\infty f(x,r) dx d\nu(r)$$

for all integrable functions f on $(\mathbb{R}^d \times \mathbb{R}^+, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+), \mathcal{L} \times \nu)$. Then $\mathcal{L} \times \nu$ is an invariant measure for the flow $(\tau_z^t, t \in \mathbb{R})$. Hence $\mathbb{P}$ is an invariant probability for the flow $(T_z^t, t \in \mathbb{R})$ and the potential V is stationary since, from the definition of potential, we have that $V(x+z, \omega) = V(x, T_z\omega)$ for all $x, z \in \mathbb{R}^d$, $\omega \in \Omega$ where $T_z := T_z^1$.

Lemma 3.2. *The dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T_z, z \in \mathbb{R}^d)$ is ergodic.*

Proof. If $z = (z_1, z_2, \cdots, z_d) \in \mathbb{R}^d \setminus \{0\}$, there exists $i \geq 1$ such that $z_i \neq 0$. Set

$$A_0 := \{x \in \mathbb{R}^d : |x_i| \leq \min(1, |z_i|)\} \times \mathbb{R}.$$

Then,

(1) $\cup_{t_1 \leq t \leq t_2} \tau_z^t A_0$ are measurable for all $t_1 \leq t_2 \in \mathbb{R}$,

(2) $\cup_{-\infty < t < \infty} \tau_z^t A_0 = \mathbb{R}^{d+1}$,

(3) $A_0 \cap \tau_z^t A_0 = \emptyset$ for all $|t| > 2$.

Thanks to Theorem 1 and definition 2 in [5, p. 197], for all $z \in \mathbb{R}^d \setminus \{0\}$, the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T_z^t, t \in \mathbb{R})$ is mixing and in particular, it is ergodic. ∎

Lemma 3.3. *Let $\gamma + \delta - d > 0$. There exists positive constants $c_1(d, \delta, \gamma), c_2(d, \delta, \gamma)$ such that for all $R > 0$,*

$$\mathbb{E}E_0\big[\exp\big(-\int_0^{T_{B(0,R)}} V(Z_s, \omega) ds\big)\big] \leq c_1 \exp(-c_2 R). \tag{3.49}$$

Proof. Indeed, by Campbell's Theorem (see [17, p. 28]),

$$\begin{aligned}
&\mathbb{E}E_0\big[\exp\big(-\int_0^{T_{B(0,R)}} V(Z_s, \omega) ds\big)\big] \\
&= E_0\Big[\exp\Big(-\int_{\mathbb{R}^{d+1}} \big(1 - \exp(-\int_0^{T_{B(0,R)}} r^{-\gamma} \mathbf{1}_{\{|Z_s - u| \leq r\}} ds)\big) du d\nu(r)\Big)\Big] \\
&= E_0\Big[\exp\Big(-\int_{\mathbb{R}^d} \big(1 - \int_1^\infty \delta r^{-\delta-1} \exp(-\int_0^{T_{B(0,R)}} r^{-\gamma} \mathbf{1}_{\{|Z_s - u| \leq r\}} ds) dr\big) du\Big)\Big].
\end{aligned} \tag{3.50}$$

Note that for all $r \geq 1$ and $u \in \mathbb{R}^d$,

$$r^{-\gamma} \mathbf{1}_{\{|u| \leq r\}} \geq 2^{-\gamma} \mathbf{1}_{\{|u| \leq 1\}} \mathbf{1}_{\{1 \leq r \leq 2\}}.$$

Then

$$\begin{aligned}
&\int_1^\infty \delta r^{-\delta-1} \exp(-\int_0^{T_{B(0,R)}} r^{-\gamma}\mathbf{1}_{\{|Z_s-u|\le r\}}ds)dr \\
&\le \int_1^\infty \delta r^{-\delta-1} \exp(-2^{-\gamma}\int_0^{T_{B(0,R)}} \mathbf{1}_{\{|Z_s-u|\le 1\}}\mathbf{1}_{\{1\le r\le 2\}}ds)dr \\
&= 2^{-\delta} + (1-2^{-\delta})\exp(-2^{-\gamma}\int_0^{T_{B(0,R)}} \mathbf{1}_{\{|Z_s-u|\le 1\}}ds). \qquad (3.51)
\end{aligned}$$

From (3.50) and (3.51),

$$\begin{aligned}
&\mathbb{E}E_0\big[\exp\big(-\int_0^{T_{B(0,R)}} V(Z_s,\omega)ds\big)\big] \\
&\le E_0\Big[\exp\Big(-(1-2^{-\delta})\int_{\mathbb{R}^d}\big(1-\exp(-2^{-\gamma}\int_0^{T_{B(0,R)}}\mathbf{1}_{\{|Z_s-u|\le 1\}})ds\big)du\Big)\Big]. \qquad (3.52)
\end{aligned}$$

We now use the same argument as in [31, Proposition 5.2.8] to obtain (3.49). ∎

We will now show that Lacoin's potential verify the conditions of Theorems 2.1 and 2.2.

Proposition 3.4. *Let $\gamma+\delta-d>0$. For all $x\in\mathbb{Z}^d$, there exist Lyapunov exponents $\alpha_(x)$ such that $\mathbb{P}$ a.s.*

$$\lim_{|x|\to\infty}\frac{1}{|x|}|a(0,x,\omega)-\alpha(x)| = 0 \qquad (3.53)$$

Moreover, $\alpha(\cdot)$ is a norm on $\mathbb{R}^d$.

Proof. First, note that all moments of $\sup_{B(0,1)} V(\cdot)$ are finite. Indeed,

$$\sup_{B(0,1)} V(\cdot,\omega) \le \sum_{i=1}^\infty r_i^{-\gamma}\mathbf{1}_{\{|\omega_i|\le r_i+1\}}$$

and by (4.89),

$$\mathbb{E}[\exp(\sum_{i=1}^\infty r_i^{-\gamma}\mathbf{1}_{\{|\omega_i|\le r_i+1\}})] = \exp\big(\int_1^\infty \delta\mathcal{L}_d(r+1)^d r^{-\delta-1}(e^{r^{-\gamma}}-1)dr\big) < \infty \qquad (3.54)$$

(since $e^{r^{-\gamma}}-1<2r^{-\gamma}$ when r is large enough and $\gamma+\delta-d>0$). For $\omega\in\Omega$, we set

$$u = u(\omega) := \omega_i \text{ such that } 1\le r_i<2 \text{ and if there exist } j<i : |\omega_j|<|\omega_i| \text{ then } |r_i|\ge 2. \qquad (3.55)$$

In other words, $u(\omega)=\omega_i$ where (ω_i,r_i) is the point of Poisson cloud in the set $\mathbb{R}^d\times[1,2]$ with $|\omega_i|$ minimum. If we choose $\epsilon=2^{-\gamma}$ and $\rho=1$, then for all $x\in B(u,\rho)$, we have that

$$V(x,\omega) = \sum_{k=1}^\infty r_k^{-\gamma}\mathbf{1}_{\{|x-\omega_k|<r_k\}} \ge r_i^{-\gamma}\mathbf{1}_{\{|x-\omega_i|<r_i\}} > 2^{-\gamma}.$$

Note that $(\mathcal{L}\times\nu)(B(0,t)\times[1,2]) = \int_{|y|<t}\int_1^2 \delta r^{-\delta-1}dydr = (1-2^{-\delta})\mathcal{L}_d t^d$. Then

$$\mathbb{P}(|u|>t) = \mathbb{P}(\text{no points of the Poisson cloud are in } B(0,t)\times[1,2]) = e^{-(1-2^{-\delta})\mathcal{L}_d t^d}.$$

Therefore $\mathbb{E}(|u|^d)<\infty$. The conditions of Theorem 2.2 are verified. Hence (3.53) holds. We now show that $\alpha(x)$ is a norm. For $k\ge 1, x\in\mathbb{Z}^d$ such that $k|x|>1$, using Jensen's inequality and

(3.49),

$$\mathbb{E}[\log e(0,kx,\omega)] \le \log \mathbb{E}[e(0,kx,\omega)]$$
$$\le \log \mathbb{E}E_0[\exp(-\int_0^{T_{B(0,|kx|-1)}} V(Z_s,\omega)ds)] \le \log c_1 - c_2(k|x|-1). \qquad (3.56)$$

Then for $x \in \mathbb{R}^d \setminus \{0\}$, $\alpha(x) = \lim_{k\to\infty} \frac{1}{k}\mathbb{E}[a(0,kx,\omega)] \ge c_2|x| > 0$. ∎

3.2. **Asymptotics of the quenched and annealed survival functions.** The following lemma is not needed in the proofs but it sheds some light on the intensity of the potential.

Lemma 3.5. *Assume that $\gamma+\delta-d>0$. Then, as $t\to\infty$,*

$$H(t) := \log \mathbb{E}(\exp(-tV(0))) = \begin{cases} -\mathcal{L}_d \frac{\delta}{\delta-d}(1-\Gamma((\gamma+\delta-d)/\gamma)t^{(d-\delta)/\gamma}+O(e^{-t})), & \text{if } \delta>d \\ -\mathcal{L}_d \frac{\delta}{\gamma}(C+\log t+O(e^{-t})) & \text{if } \delta=d \\ \mathcal{L}_d \frac{\delta}{d-\delta}\left(1-\Gamma((\gamma+\delta-d)/\gamma)t^{(d-\delta)/\gamma}+O(e^{-t})\right) & \text{if } \delta<d \end{cases}$$

where C does not depend on d,δ or γ. And in particular,

$$\mathbb{P}(V(0)=0) = \lim_{t\to\infty} \exp H(t) = \begin{cases} \exp(-\mathcal{L}_d\frac{\delta}{\delta-d}) & \text{if } \delta>d \\ 0 & \text{if } \delta\le d. \end{cases}$$

Note that for the potential constructed from the shape function $W(x)=\mathbf{1}(|x|<a)$ where $a>0$, we have that $H(t) = -\mathcal{L}_d a^d(1-e^{-t})$ for $t>0$.

Proof. If $\gamma+\delta-d>0$ and $\delta<d$, then $\log \mathbb{E}\exp(-tV(0))$

$$= \int_1^\infty \int_{|y|<r} (\exp(-tr^{-\gamma})-1)dy\delta r^{-\delta-1}dr = \mathcal{L}_d\delta\int_1^\infty r^{d-\delta-1}(\exp(-tr^{-\gamma})-1)dr$$
$$= \mathcal{L}_d\frac{\delta}{d-\delta}\int_1^\infty (\exp(-tr^{-\gamma})-1)dr^{d-\delta} = \mathcal{L}_d\frac{\delta}{d-\delta}\int_1^\infty (\exp(-tu^{-\frac{\gamma}{d-\delta}})-1)du$$
$$= \mathcal{L}_d\frac{\delta}{d-\delta}\int_0^1 (1-\exp(-tu^{-\frac{\gamma}{d-\delta}})du - \mathcal{L}_d\frac{\delta}{d-\delta}\int_0^\infty (1-\exp(-tu^{-\frac{\gamma}{d-\delta}})du$$
$$= \mathcal{L}_d\frac{\delta}{d-\delta} + O(e^{-t}) - \mathcal{L}_d\frac{\delta}{d-\delta}t^{(d-\delta)/\gamma}\int_0^\infty (1-\exp(-u^{-\frac{\gamma}{d-\delta}}))du \qquad (u:=t^{\frac{d-\delta}{\gamma}}U)$$
$$= \mathcal{L}_d\frac{\delta}{d-\delta} + O(e^{-t}) - \mathcal{L}_d\frac{\delta}{d-\delta}\Gamma(1-(d-\delta)/\gamma)t^{(d-\delta)/\gamma} \qquad \text{(integration by parts)}$$

If $\delta>d$, then

$$\begin{aligned} \log \mathbb{E}\exp(-tV(0)) &= -\mathcal{L}_d\frac{\delta}{\delta-d} + \mathcal{L}_d\frac{\delta}{d}\int_0^1 u^{-1+(-d+\delta)/\gamma}e^{-tu}du \\ &= -\mathcal{L}_d\frac{\delta}{\delta-d} + \mathcal{L}_d\frac{\delta}{d}t^{(d-\delta)/\gamma}\int_0^t y^{-1+(-d+\delta)/\gamma}e^{-y}dy \\ &= -\mathcal{L}_d\frac{\delta}{\delta-d} + \mathcal{L}_d\frac{\delta}{d}t^{(d-\delta)/\gamma}\Gamma((\delta-d)/\gamma) + O(e^{-t}). \end{aligned}$$

∎

We first consider the annealed survival function of the Brownian motion in the random potential introduced in (1.3). For $t > 0$,

$$\begin{aligned} S_t &:= \mathbb{E}E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds)\big] \\ &= E_0 \exp\left(\int_{\mathbb{R}^d}\int_1^\infty [\exp(-\int_0^t r^{-\gamma}\mathbf{1}_{\{|Z_s-y|<r\}}ds)-1]\delta r^{-\delta-1}drdy\right). \end{aligned} \tag{3.57}$$

Lemma 3.6. *Assume $\gamma+\delta-d>0$. Then*

$$-\log S_t \le \begin{cases} Ct^{d/(d+2)} & \text{if } d\le\delta \\ C(t^{d/(d+2)}\vee t^{(d-\delta)/\gamma}) & \text{if } d>\delta \end{cases} \tag{3.58}$$

and in particular,

$$\lim_t -\frac{1}{t}\log S_t = 0. \tag{3.59}$$

Proof. We follow the argument of [10, (1.22) of Theorem 2]. By (3.57), for all $t, R > 0$,

$$S_t \ge \exp\left(\int_{\mathbb{R}^d}\int_1^\infty [\exp(-\int_0^t \sup_{x\in B(0,R)} r^{-\gamma}\mathbf{1}_{\{(|x-y|<r\}}ds)-1]\delta r^{-\delta-1}dy\right) P_0(T_R>t).$$

Then

$$\begin{aligned} -\log S_t \le & -\int_{\mathbb{R}^d}\int_1^\infty [\exp(-\int_0^t \sup_{x\in B(0,R)} r^{-\gamma}\mathbf{1}_{\{|x-y|<r\}}ds)-1]\delta r^{-\delta-1}dr\,dy \\ & -\log P_0(T_R>t). \end{aligned} \tag{3.60}$$

The first term is bounded by $\displaystyle\int_1^\infty\int_{|y|<R+r}[1-\exp(-tr^{-\gamma})]\delta r^{-\delta-1}dr\,dy$

$$\begin{aligned} &\le \int_1^\infty (R+r)^d[1\wedge tr^{-\gamma}]\delta r^{-\delta-1}dr \\ &\le \int_1^{t^{1/\gamma}} (R+r)^d\delta r^{-\delta-1}dr + \int_{t^{1/\gamma}}^\infty (R+r)^d t\delta r^{-\delta-\gamma-1}dr \\ &\le C\int_1^{t^{1/\gamma}} (R^d+r^d)\delta r^{-\delta-1}dr + C\int_{t^{1/\gamma}}^\infty (R^d+r^d)\delta t r^{-\delta-\gamma-1}dr \\ &\le CR^d + C(1\vee t^{(d-\delta)/\gamma}) + CR^d t^{-\delta/\gamma} + Ct^{(d-\delta)/\gamma} \\ &\le CR^d + C(1\vee t^{(d-\delta)/\gamma}). \end{aligned}$$

Substituting this estimate in (3.60), we obtain that

$$-\log S_t \le CR^d + C(1\vee t^{(d-\delta)/\gamma}) + \lambda_d tR^{-2}.$$

Therefore, if $d<\delta$ then, by taking $R = t^{1/(d+2)}$,

$$-\log S_t \le CR^d + \lambda_d tR^{-2} \le Ct^{d/(d+2)}.$$

If $d>\delta$ and $\gamma > \dfrac{d+2}{d}(d-\delta)$ then, $-\log S_t \le CR^d + \lambda_d tR^{-2} \le Ct^{d/(d+2)}$,
and if $d>\delta$ and $\gamma < \dfrac{d+2}{d}(d-\delta)$ then, by taking $R = t^{(d-\delta)/d\gamma}$

$$-\log S_t \le CR^d + Ct^{(d-\delta)/\gamma} + \lambda_d tR^{-2} \le Ct^{(d-\delta)/\gamma}.$$

■

From the theory of Donsker and Varadhan, it is simple to give an expression for the liminf. We state it as a complement. The limsup however cannot be treated so simply.

Proposition 3.7. *Assume $\gamma + \delta - d > 0$. Then*

$$\liminf_t t^{-d/(d+2)} \log S_t \geq -\left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}. \tag{3.61}$$

Proof. By Jensen's inequality, for $y \in \mathbb{R}^d$,

$$\begin{aligned}
&\int_1^\infty \exp(-\int_0^t r^{-\gamma}\mathbf{1}_{\{|Z_s-y|<r\}}ds)\delta r^{-\delta-1}dr \\
&\quad \geq \exp(-\int_0^t \int_1^\infty r^{-\gamma}\mathbf{1}_{\{|Z_s-y|<r\}}\delta r^{-\delta-1}drds) \\
&\quad = \exp(-\int_0^t \int_{1\vee|Z_s-y|}^\infty \delta r^{-\gamma-\delta-1}drds) = \exp(-\int_0^t W_1(Z_s-y)ds)
\end{aligned}$$

where $W_1(y) := \dfrac{\delta}{\delta+\gamma}(1\vee|y|)^{-\gamma-\delta}$. Hence, by [10, (1.2) of Theorem 2],

$$\begin{aligned}
\liminf_t t^{-d/(d+2)} \log S_t &\geq t^{d/(d+2)} \log E_0 \exp(-\int_{\mathbb{R}^d}[1-\exp(-\int_0^t W_1(Z_s-y)ds)]dy) \\
&= -\left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}.
\end{aligned}$$

■

Proposition 3.8. *If $\gamma + \delta - d > 0$ then*

$$\int_0^\infty S_t dt < \infty. \tag{3.62}$$

Proof. By Lemma 3.3, for all $R > 0$ and $t > 0$,

$$\begin{aligned}
c_1 \exp(-c_2 R) &\geq \mathbb{E}E_0\big[\exp\big(-\int_0^{T_{B(0,R)}} V(Z_s,\omega)ds\big)\big] \\
&\geq \mathbb{E}E_0\big[\exp\big(-\int_0^t V(Z_s,\omega)ds\big), T_{B(0,R)} \leq t\big] \\
&\geq \mathbb{E}E_0\big[\exp\big(-\int_0^t V(Z_s,\omega)ds\big)\big] - P_0(T_{B(0,R)} > t).
\end{aligned} \tag{3.63}$$

Note that, as a particular case of the tubular estimate (2.25), there is a positive constant C such that for all $t, R > 0$,

$$P_0(T_{B(0,R)} > t) = P_0(\sup_{0<s<t}|Z_s| < R) \leq C\exp(-\lambda_d t/R^2). \tag{3.64}$$

Combine this inequality with (3.63), to obtain that

$$S_t \leq c_1 \exp(-c_2 R) + C\exp(-\lambda_d t/R^2).$$

In particular, for $R = t^{1/3}$, we obtain that there are positive constants C_1, C_2 such that for all $t > 0$, $S_t \leq C_1 \exp(-C_2 t^{1/3})$. Hence $\displaystyle\int_0^\infty S_t dt < \infty$. ■

To obtain a similar result for the quenched survival function which was introduced in (1.2), a weak independence property is needed. This is the role of the following lemma which is similar to [12, Lemma 6.].

Lemma 3.9. *Assume that $\gamma + \delta - d > 0$.*

Then there is a constant $C = C(\gamma, \delta, d)$ such that for all $\varepsilon > 0$, for all $R_0 > 1$ and $R > C\varepsilon^{-1/(\gamma+\delta-d)} \vee 2R_0$,

$$\mathbb{P}\left(\sup_{y \in B(0,R_0)} \sum_{\omega_j \notin B(0,R)} r_j^{-\gamma} \mathbf{1}_{\{|y-\omega_j|<r_j\}} > \varepsilon \right) \leq \exp(-2\varepsilon R^\gamma). \tag{3.65}$$

Proof. Let $R > 2R_0 > R_0 > 1$. Then for all $y \in B(0, R_0)$ and $\omega_j \notin B(0, R)$,

$$|\omega_j| < |\omega_j - y| + |y| < |\omega_j - y| + R_0 < 2|\omega_j - y|.$$

Hence

$$\sup_{y \in B(0,R_0)} \sum_{\omega_j \notin B(0,R)} r_j^{-\gamma} \mathbf{1}_{\{|y-\omega_j|<r_j\}} \leq \sum_{\omega_j \notin B(0,R)} r_j^{-\gamma} \mathbf{1}_{\{|\omega_j|<2r_j\}}. \tag{3.66}$$

Moreover, by Campbell's Theorem, for all $s > 0$,

$$\begin{aligned}
\log \mathbb{E}[\exp(s \sum_{\omega_j \notin B(0,R)} r_j^{-\gamma} \mathbf{1}_{\{|\omega_j|<2r_j\}})] &= \int_1^\infty \int_{\mathbb{R}^d} [\exp(sr^{-\gamma} \mathbf{1}_{\{R<|z|<2r\}}) - 1] dz \delta r^{-\delta-1} dr \\
&= \delta \mathcal{L}_d \int_{R/2}^\infty ((2r)^d - R^d)(\exp(sr^{-\gamma}) - 1) r^{-\delta-1} dr \\
&\leq Cs \exp(s2^\gamma R^{-\gamma}) \int_{R/2}^\infty r^{d-\gamma-\delta-1} dr \\
&\leq Cs \exp(s2^\gamma R^{-\gamma}) R^{d-\delta-\gamma}. \qquad (3.67)
\end{aligned}$$

Then by Markov's inequality, (3.66) and (3.67) with $s = 4R^\gamma$, there exists $C = C(d, \delta, \gamma)$ such that for all $\varepsilon > 0$, for all $R_0 > 1$, $R > (\frac{C}{2\varepsilon})^{\frac{1}{\gamma+\delta-d}} \vee R > 2R_0$

$$\mathbb{P}[\sup_{y \in B(0,R_0)} \sum_{\omega_j \notin B(0,R)} r_j^{-\gamma} \mathbf{1}_{\{|\omega_j - y|<r_j\}} > \varepsilon] \leq \exp(CR^{d-\delta} - 4\varepsilon R^\gamma) < \exp(-2\varepsilon R^\gamma). \tag{3.68}$$

■

Proposition 3.10. *Assume that $\gamma + \delta - d > 0$. Then $\mathbb{P}$ a.s., as $t \to \infty$,*

$$-\frac{1}{t} \log S_{t,\omega} \to 0. \tag{3.69}$$

Proof. For $x \in \mathbb{Z}^d$ and $R > 1$, write $V(x) = V_1(x) + V_2(x)$ where

$$V_1(x) = \sum_{\omega_j \in B(0,2R)} r_j^{-\gamma} \mathbf{1}_{\{|x-\omega_j|<r_j\}} \text{ and } V_2(x) = \sum_{\omega_j \notin B(0,2R)} r_j^{-\gamma} \mathbf{1}_{\{|x-\omega_j|<r_j\}}.$$

For $\varepsilon > 0$, let R be large enough so that (3.65) holds. Then by the independence property of the Poisson point process,

$$\begin{aligned}
\mathbb{P}(\sup_{x \in B(0,R)} V(x) < \varepsilon) &> \mathbb{P}(\sup_{x \in B(0,R)} V_1(x) < \varepsilon/2, \sup_{x \in B(0,R)} V_2(x) < \varepsilon/2) \\
&= \mathbb{P}(\sup_{x \in B(0,R)} V_1(x) < \varepsilon/2) \mathbb{P}(\sup_{x \in B(0,R)} V_2(x) < \varepsilon/2) \\
&> \exp(-CR^d)(1 - \exp(-\varepsilon R^\gamma)).
\end{aligned}$$

The last inequality follows from Lemma 3.9 and the fact that $V_1(x) = 0$ if no points of the Poisson cloud are in $B(0, 2R) \times [1, \infty[$. Hence, for all $\varepsilon > 0$ and for all R large enough,

$$\mathbb{P}(\sup_{x \in B(0,R)} V(x) < \varepsilon) > 0. \tag{3.70}$$

Let $(\varepsilon_\ell; \ell \in \mathbb{N})$ be a sequence of positive numbers such that $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Then there is a sequence $R_\ell \to \infty$ such that for all $\ell \in \mathbb{N}$, $\mathbb{P}(\sup_{x \in B(0,R_\ell)} V(x) < \varepsilon_\ell) > 0$.

By ergodicity, $\mathbb{P}$ a.s. for each ℓ there is $z_\ell = z_\ell(\omega) \in \mathbb{R}^d$ such that $\sup_{x \in B(z_\ell, R_\ell)} V(x) < \varepsilon_\ell$.

Denote by $\lambda_V(U)$ the principal Dirichlet eignevalue of $-\frac{1}{2}\Delta + V$ in a domain $U \subset \mathbb{R}^d$. Then (see [31, Section 3.1]), $\mathbb{P}$ a.s. for all $\ell \in \mathbb{N}$,

$$\begin{aligned} -\lim_t \frac{1}{t} \log S_{t,\omega} &= \lambda_V(\mathbb{R}^d) \\ &\leq \lambda_V(B(z_\ell, R_\ell)) \\ &= \inf\{\int_{B(z_\ell,R_\ell)} [\frac{1}{2}|\nabla u|^2 + Vu^2]dx,\ \int_{B(z_\ell,R_\ell)} u^2 dx = 1\} \\ &\leq C R_\ell^{-2} + \varepsilon_\ell. \end{aligned}$$

Let $\ell \to \infty$ to obtain (3.69). ∎

3.3. **Large deviation estimates.** In this section, we prove large deviation estimates with respect to the quenched path measure $Q_{t,\omega}(\cdot)$ introduced in (1.6). The rate function, which appears in (2.44), is expressed in terms of the Lyapunov exponents $\alpha_\lambda(\cdot)$ with respect to the potentials $\lambda + V$. Note that, if conditions (2.9), (2.10) of Theorem 2.1 are verified by a potential V then they are also verified by the potential $\lambda + V$ for $\lambda \geq 0$. Similarly for conditions (2.13) and (2.14) of Theorem 2.2.

Theorem 3.11. *Suppose that $\gamma + \delta - d > 0$. Then $\mathbb{P}$ a.s.,*

i) for all closed subset A of $\mathbb{R}^d$,

$$\limsup_{t \to \infty} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \leq -\inf_{x \in A} I(x) \tag{3.71}$$

ii) and for all open subset $\mathcal{O}$ of $\mathbb{R}^d$,

$$\liminf_{t \to \infty} \frac{1}{t} \log Q_{t,\omega}(Z_t \in t\mathcal{O}) \geq -\inf_{x \in \mathcal{O}} I(x). \tag{3.72}$$

The upper bound: *Proof of (3.71).*

We follow the arguments of [31, (4.6) of Theorem 5.4.2] (see also [35, (69) of Theorem 19]). They require the shape Theorem for the Lyapunov exponents given in Proposition 3.4 and Proposition 3.10.

First assume that A is a compact subset of $\mathbb{R}^d$. For each $t > 0$, it is possible to choose n_t points $x_1, x_2, \ldots, x_{n_t}$ in A such that n_t grows at most polynomially in t and

$$tA \subset B_t := \cup_{k=1}^{n_t} B(x_k).$$

By definition of $S_{t,\omega}$ and $Q_{t,\omega}$, $\mathbb{P}$ a.s. for all $\lambda \geq 0$,

$$\begin{aligned}\exp(-\lambda t) S_{t,\omega} Q_{t,\omega}(Z_t \in tA) &= \exp(-\lambda t) E_0[Z_t \in tA, \exp(-\int_0^t V(Z_s,\omega)ds)] \\ &\leq E_0[\exp(-\int_0^{H(B_t)} (\lambda + V)(Z_s,\omega)ds), H(tA) < \infty] \\ &\leq \sum_{k=1}^{n_t} e_\lambda(0, x_k, \omega) \leq n_t \max_{1\leq k\leq n_t} e_\lambda(0, x_k, \omega).\end{aligned}$$

Therefore for all $\lambda \geq 0$, by Propositions 3.4 and 3.10, $\mathbb{P}$ a.s.,

$$-\lambda + \lim_{t\to\infty} \frac{\log S_{t,\omega}}{t} + \limsup_{t\to\infty} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \leq -\inf_A \alpha_\lambda(x).$$

$$\text{Hence} \quad \limsup_{t\to\infty} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \leq -\sup_{\lambda\geq 0} \inf_{x\in A} (\alpha_\lambda(x) - \lambda). \tag{3.73}$$

To complete the proof, it remains to interchange the sup and the inf in (3.73). This is done by a classical argument (see for example [9] or [31, p. 250]). It does not require additional properties of the potential. Neither does the proof of the general case when A is a closed subset of $\mathbb{R}^d$ as can be seen from [31, p. 250]. ∎

The lower bound.

The main difficulty in proving a large deviation lower bound is to establish the existence of a ball near each sufficiently far $x \in \mathbb{Z}^d$ where V takes on small values. For a potential constructed from a function W with compact support this is done in [31, (4.22)]. For the classical long range potential, this will be done in the following lemma.

Even though the scaling we need for the large deviation lower bound is quite different, the analogy with the proof of [12, Proposition 3] will be very useful.

Lemma 3.12. *Assume that $\gamma + \delta - d > 0$. Let $\varepsilon > 0$ and $R_0 > 0$. Then $\mathbb{P}$ a.s. for all sufficiently large $|x|$, there is $z \in \mathbb{Z}^d$ such that $|x - z| < \sqrt{|x|}$ and $V < \varepsilon$ on $B(z, R_0)$.*

Proof. Fix $\varepsilon > 0$ and $R_0 > 0$.

For $x \in \mathbb{Z}^d$, let $R = (\frac{4d}{\varepsilon} \log |x|)^{1/\gamma}$ and consider

$$\mathcal{I}(x) := B(x, \sqrt{|x|}) \cap (2R)\mathbb{Z}^d.$$

Note that there are positive constants $C_0 = C_0(d)$ and $C_1 = C_1(d)$ such that for all $x \in \mathbb{Z}^d$ with $|x| > C_1$,

$$C_0 \left(\frac{\sqrt{|x|}}{R}\right)^d < \sharp\mathcal{I}(x) < C_1 \left(\frac{\sqrt{|x|}}{R}\right)^d \tag{3.74}$$

where $\sharp\mathcal{I}(x)$ denotes the number of vertices in $\mathcal{I}(x)$.

For $z \in \mathcal{I}(x)$, consider the events

$$A_x(z) := \{ \sup_{B(z,R_0)} \sum_{\omega_j \in B(z,R)} r_j^{-\gamma} \mathbf{1}_{\{|\omega_j - y| < r_j\}} > \varepsilon/2\} \text{ and}$$

$$D_x(z) := \{ \sup_{B(z,R_0)} \sum_{\omega_j \notin B(z,R)} r_j^{-\gamma} \mathbf{1}_{\{|\omega_j - y| < r_j\}} > \varepsilon/2\}.$$

By definition of the Poisson point process,

$$\begin{aligned}\mathbb{P}(A_x(z)) &\leq \mathbb{P}(\text{there is at least one point of } \omega=(\omega_i,r_i) \text{ such that } |\omega_i|<R, |\omega_i|<r_i+R_0)\\ &\leq 1-\exp(-\lambda). \end{aligned} \tag{3.75}$$

where $\lambda=\lambda(x)$ is the parameter of the Poissonian variable $\sum_i \mathbf{1}_{\{\omega_i|<R,|\omega_i|<r_i+R_0\}}$. For $\delta\neq d$, it is given by

$$\begin{aligned}&\int_1^\infty \delta r^{-\delta-1}\int_{\mathbb{R}^d}\mathbf{1}_{\{|u|<R,|u|<r+R_0\}}dudr\\ &\quad=\mathcal{L}_d\int_1^{R-R_0}\delta r^{-\delta-1}(r+R_0)^d dr+\mathcal{L}_d\int_{R-R_0}^\infty \delta r^{-\delta-1}R^d dr\\ &\quad=\mathcal{L}_d\int_1^{R-R_0}\delta r^{-\delta-1}(r+R_0)^d dr+\mathcal{L}_d R^d(R-R_0)^{-\delta}\\ &\quad\leq C_2\int_1^{R-R_0} r^{d-\delta-1}dr+C_2R_0^d\int_1^{R-R_0}\delta r^{-\delta-1}dr+\mathcal{L}_dR^d(R-R_0)^{-\delta}\\ &\quad\leq C_2\frac{(R-R_0)^{d-\delta}}{d-\delta}-C_2\frac{1}{d-\delta}-C_2R_0^d(R-R_0)^{-\delta}+C_2R_0^d+\mathcal{L}_dR^d(R-R_0)^{-\delta}. \end{aligned}\tag{3.76}$$

By lemma 3.9, for all $R>(\frac{C}{2\varepsilon})^{\frac{1}{\gamma+\delta-d}}$,

$$\mathbb{P}(D_x(z))\leq\exp(-\varepsilon R^\gamma). \tag{3.77}$$

Then, using (3.74), (3.75), (3.77) and the independence of the events $A_x(z)$, il follows that

$$\begin{aligned}\mathbb{P}\left(\cap_{z\in\mathcal{I}(x)}(A_x(z)\cup D_x(z))\right) &\leq \mathbb{P}(\cap_{z\in\mathcal{I}(x)}A_x(z))+\mathbb{P}(\cup_{z\in\mathcal{I}(x)}D_x(z))\\ &\leq (1-\exp(-\lambda))^{C_0(\sqrt{|x|}/R)^d}+C_1\left(\frac{\sqrt{|x|}}{R}\right)^d\exp(-\varepsilon R^\gamma). \end{aligned}\tag{3.78}$$

These are the terms of summable series in $x\in\mathbb{Z}^d$.

Indeed, for the second term of (3.78),

$$|x|^{(d-1)}\frac{|x|^{d/2}}{R^d}\exp(-\varepsilon R^\gamma)=\frac{|x|^{(3d-2)/2}}{(\frac{4d}{\varepsilon}\log|x|)^{d/\gamma}}|x|^{-4d}.$$

For the first term of (3.78), by the upper bound (3.76), when $\delta<d$, there is a positive constant C_3 such that for all R large enough, $\lambda<C_3R^{d-\delta}$. Hence since $\delta+\gamma-d>0$, for sufficiently large $|x|$,

$$\lambda<C_3R^{d-\delta}<C_3\log|x|(\log|x|)^{(d-\delta-\gamma)/\gamma}<\frac{d}{4}\log|x|$$

and by using $1-t\leq e^{-t}$ for all $t>0$, we have that the first term is bounded above by

$$\exp(-\exp(-\lambda)C_0\frac{|x|^{d/2}}{R^d})\leq\exp(-|x|^{d/6}).$$

This upper bound also holds when $\delta\geq d$, as in this case, $\lambda=\lambda(x)$ is bounded above.

Conclude with an application of Borel-Cantelli lemma. ∎

By following the strategy described in [31, Theorem 5.4.2], a large deviation lower bound can also be obtained.

Proof of the lower bound (3.72).

By Proposition 3.10 and the continuity of $I(x)$, to obtain (3.72), it is sufficient to show that for all $v \in \mathbb{Q}^d$, $v \neq 0$ and $r > 0$,

$$\liminf_{t\to\infty} \frac{1}{t} \log E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds), Z_t \in tB(v,r)\big] \geq -I(v) \quad \mathbb{P} \text{ a.s.} \tag{3.79}$$

Fix $v \in \mathbb{Q}^d$, $v \neq 0$, $r > 0$, $\varepsilon > 0$ and $R_0 > 0$.

Then by lemma 3.12, $\mathbb{P}$ a.s., there exists t_ε such that for each $t \geq t_\varepsilon$, there is $y(t) \in \mathbb{Z}^d$ such that

$$|v[t] - y(t)| < \sqrt{t|v|} \text{ and } V(\cdot,\omega) < \varepsilon \text{ on } B(y(t), R_0). \tag{3.80}$$

Now for $0 < \gamma_1 < \gamma_2 < 1$, and $n \in \mathbb{N}$, define

$$S_{n,\gamma_1} := H(nv) \circ \theta_{n\gamma_1} + n\gamma_1 \quad \text{and} \quad A_{n,\gamma_1,\gamma_2} := \{S_{n,\gamma_1} < n\gamma_2\}$$

where θ_t, $t \geq 0$ is the canonical shift on $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$.

Note that S_{n,γ_1} is a stopping time and A_{n,γ_1,γ_2} is the event that $Z.$ enters $B(nv)$ in the time interval $[n\gamma_1, n\gamma_2]$. Then for all t large enough, by (3.80),

$$\begin{aligned} &E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds), Z_t \in tB(v,r)\big] \geq \\ &\quad E_0\big[A_{[t],\gamma_1,\gamma_2}, \exp(-\int_0^{S_{[t],\gamma_1}} V(Z_s,\omega)ds)\big]\cdot \\ &\quad \inf_{z\in B(v[t])} E_z\big[\exp(-\int_0^{H(y(t))} V(Z_s,\omega)ds), H(y(t)) \leq t^{3/4}\big] \cdot e^{-\varepsilon t} P_0[T_{B(0,R_0)} > t] \end{aligned} \tag{3.81}$$

For all $z \in B(v[t])$, the second factor of (3.81) is greater than

$$\begin{aligned} &E_z\big[\exp(-\int_0^{t^{3/4}} V(Z_s,\omega)ds), \sup_{0\leq s\leq t^{3/4}} |Z_s - z - \frac{s}{t^{3/4}}(y(t)-z)| < 1\big] \\ &\quad \geq \exp(-t^{3/4} \sup_{B(v[t],\sqrt{t|v|})} V(\cdot,\omega)) P_0[\sup_{0\leq s\leq t^{3/4}} |Z_s - \frac{s}{t^{3/4}}(y(t)-z)| < 1] \end{aligned}$$

Using lemma 2.5 and (2.32), $\mathbb{P}$ a.s, for all t large enough, this term is greater than

$$\exp[-t^{3/4}\log(|v|t)] C(d) \exp(-\lambda_d t^{3/4} - t^{1/4}|v|). \tag{3.82}$$

Using (3.64), the third term of (3.81) is greater than

$$\exp(-\varepsilon t) P_0[T_{B(0,R_0)} > t] \geq C(d) \exp\big(-\varepsilon t - \lambda_d \frac{t}{R_0^2}\big). \tag{3.83}$$

Then from (3.81), (3.82) and (3.83), since $S_{[t],\gamma_1} \geq \gamma_1 t$, $\mathbb{P}$ a.s.

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t} \log E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds), Z_t \in tB(v,r)\big] \\ &\quad \geq \lambda\gamma_1 + \liminf_{t\to\infty} \frac{1}{t} \log E_0\big[A_{[t],\gamma_1,\gamma_2}, \exp(-\int_0^{S_{[t],\gamma_1}} (\lambda + V(Z_s,\omega))ds)\big] \\ &\qquad - \varepsilon - \lambda_d R_0^{-2}. \end{aligned} \tag{3.84}$$

Let $\varepsilon \to 0$ and $R_0 \to \infty$. Then use Lemma 3.13 below assuming that the parameters $0 < \gamma_1 < \gamma_2 < 1$ verify $(\gamma_1, \gamma_2) \cap [\alpha'_\lambda(v)_+, \alpha'_\lambda(v)_-] \neq \emptyset$ to obtain that

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \log E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds), Z_t \in tB(v,r)\big] \\ \geq \ \lambda\gamma_1 - \kappa_\lambda(v,\gamma_1,\gamma_2) \\ \geq \ \lambda\gamma_1 - \alpha_\lambda(v). \end{aligned}$$

Now to obtain (3.79), as in [31, page 253], it is possible to choose $\lambda > 0$ based only on the properties of $I(\cdot)$ given in Lemma 2.6 (nonnegative convex continuous function). ∎

For $v \in \mathbb{R}^d \setminus \{0\}$, $0 < \gamma_1 < \gamma_2 < \infty$, $0 \leq m \leq n$, define

$$S_{m,n,v,\gamma_1} := H(nv) \circ \theta_{(n-m)\gamma_1} + (n-m)\gamma_1,$$

$$A_{m,n,v,\gamma_1,\gamma_2} := \{S_{m,n,v,\gamma_1} < (n-m)\gamma_2\} \ \text{ and}$$

$$b_\lambda(m,n,v,\gamma_1,\gamma_2,\omega) := -\inf_{z\in B(mv)} \log E_z\big[A_{m,n,v,\gamma_1,\gamma_2}, \exp\big(-\int_0^{S_{m,n,v,\gamma_1}} (\lambda+V)(Z_s,\omega)ds\big)\big].$$

Note that S_{m,n,v,γ_1} is a stopping time and $A_{m,n,v,\gamma_1,\gamma_2}$ is the event that $Z_\cdot$ enters $B(nv)$ in the time interval $[(n-m)\gamma_1, (n-m)\gamma_2]$. The strong Markov property implies that

$$\{b_\lambda(m,n,v,\gamma_1,\gamma_2,\omega)\}_{m\geq 0, n\geq 0}$$

is a subadditive sequence. A calculation similar to (2.33) shows that $\mathbb{E}b_\lambda(0,1,v,\gamma_1,\gamma_2) < \infty$. Then the following lemma follows from Kingman's subadditive ergodic Theorem.

Lemma 3.13. *Under condition (2.9), for $v \in \mathbb{R}^d, v \neq 0$, $\lambda > 0$, $0 < \gamma_1 < \gamma_2 < \infty$,*

$$\lim_{n\to\infty} \frac{b_\lambda(0,n,v,\gamma_1,\gamma_2,\omega)}{n} = \lim_{n\to\infty} \mathbb{E}\frac{b_\lambda(0,n,v,\gamma_1,\gamma_2,\omega)}{n} = \kappa_\lambda(v,\gamma_1,\gamma_2) \in [0,\infty)$$

Moreover, if $\lambda > 0$ and $]\gamma_1,\gamma_2[\cap[\alpha'_\lambda(v)_+, \alpha'_\lambda(v)_-] \neq \emptyset$ then $\kappa_\lambda(v,\gamma_1,\gamma_2) \leq \alpha_\lambda(v)$.

Here $\alpha'_\lambda(v)_+, \alpha'_\lambda(v)_-$ are respectively the right and left derivatives of $\alpha_\lambda(v)$. The proof of the second affirmation of this lemma can be found in [31, Lemma 5.4.3]. It requires only basic notions of potential theory.

4. A classical potential : $W(x) = |x|^{-\gamma} \wedge 1$, $\gamma > d$.

In this section, we consider the potential given by $V(x,\omega) = \sum_j W(x-\omega_j)$ where $\omega = (\omega_j)$ is a Poisson point process on $\mathbb{R}^d$, $d \geq 1$ with intensity given by Lebesgue measure.

By Campbell's Theorem, we have that $\mathbb{P}$ a.s. for all $x \in \mathbb{R}^d$, $V(x,\omega)$ is finite for $\gamma > d$,

$$\mathbb{E}V(0) = \frac{\gamma}{\gamma-d}\mathcal{L}_d, \quad \mathbb{V}\mathrm{ar}\, V(0) = \frac{2\gamma}{2\gamma-d}\mathcal{L}_d \quad \text{and} \quad \mathbb{C}\mathrm{ov}(V(0),V(x)) \asymp |x|^{-\gamma}. \tag{4.85}$$

Indeed, for $|x| > 2$, on the one hand,

$$\begin{aligned} \mathbb{C}\mathrm{ov}(V(0),V(x)) = \int_{\mathbb{R}^d} (|u|^{-\gamma} \wedge 1)(|x-u|^{-\gamma} \wedge 1)du \\ = 2\int_{B(0)} |x-u|^{-\gamma}du + 2\int_{B(0)^c} |u|^{-\gamma}|x-u|^{-\gamma}\mathbf{1}_{\{|u|<|x-u|\}}du \\ < C|x|^{-\gamma} + C\int_{B(0)^c} |u|^{-\gamma}\left(\frac{|x|}{2}\right)^{-\gamma}\mathbf{1}_{\{|u|<|x-u|\}}du < C|x|^{-\gamma} \end{aligned}$$

and on the other hand, $\mathbb{C}\mathrm{ov}(V(0),V(x)) > 2\int_{B(0)} |x-u|^{-\gamma}du > C|x|^{-\gamma}$.

Note that if γ,δ are the parameters of Lacoin's potential and $\widehat{\gamma}$ is the parameter of the classical potential then their correlation decay is of the same order when $\gamma = d$ and $\widehat{\gamma} = \delta + d$. And for these values, $\gamma + \delta - d = \widehat{\gamma} - d$.

By Campbell's Theorem, if $\gamma > d$ then

$$H(t) := \log \mathbb{E}(\exp(-tV(0))) = -\mathcal{L}_d\Gamma(1-d/\gamma)t^{d/\gamma} + O(e^{-t}), \quad t\to\infty. \tag{4.86}$$

See [24]. Fukushima [12] calculated the second ordrer term of the expansion. Note that $\mathbb{P}(V(0) = 0) = \lim_{t\to\infty} \exp H(t) = 0$.

Using the Laplace transform, the annealed survival function, defined in (1.3), can be written as follows

$$S_t = \mathbb{E}E_0\big[\exp(-\int_0^t V(Z_s,\omega)ds)\big] = E_0\exp\big(\int_{\mathbb{R}^d}(\exp(-\int_0^t W(Z_s-y)ds - 1\big)dy.$$

The long time asymptotics of this Brownian functional are described in [10, 15], [24] and [23]. It is known that, as $t\to\infty$,

$$-\log S_t = \begin{cases} C_1 t^{d/\gamma}(1+o(1)) & d<\gamma<d+2, \ \text{Pastur [24]} \\ C_2 t^{d/(d+2)}(1+o(1)) & \gamma = d+2, \ \hat{\text{O}}\text{kura [23]} \\ C_3 t^{d/(d+2)}(1+o(1)) & \gamma > d+2, \ \text{Donsker and Varadhan [10]}. \end{cases}$$

Proposition 4.1. *Let $\gamma > d$. For all $x\in\mathbb{Z}^d$, there exist Lyapunov exponent $\alpha(x)$ such that $\mathbb{P}$ a.s.*

$$\lim_{|x|\to\infty}\frac{1}{|x|}|a(0,x,\omega) - \alpha(x)| = 0 \tag{4.87}$$

Proof. To verify conditions (2.13) and (2.14) which appear in Theorem 2.2. First note that

$$\sup_{x\in[-1,1]^d} V(x,\cdot) \le \sum_j \widetilde{W}(\omega_j) \tag{4.88}$$

where $\widetilde{W}(x) := \mathbf{1}_{(|x|\le 1+\sqrt{d})} + (1+\sqrt{d})^\gamma |x|^{-\gamma}\mathbf{1}_{(|x|>1+\sqrt{d})}$, $x\in\mathbb{R}^d$, and for $\theta\in\mathbb{R}^+$,

$$\int_{\mathbb{R}^d}(\exp(\theta\widetilde{W}(x)) - 1)dx < \int_{|x|\le 1+\sqrt{d}}(e^\theta - 1)dx + e^{|\theta|}\int_{|x|>1+\sqrt{d}}(1+\sqrt{d})^\gamma|x|^{-\gamma}dx < \infty.$$

Then by Campbell's Theorem for all $\theta\in\mathbb{R}^+$, the condition (2.41) of Lemma 2.5 is verified. Indeed,

$$\mathbb{E}\exp(\theta \sup_{x\in[-1,1]^d} V(x)) < \infty, \quad \mathbb{P}\ a.s. \tag{4.89}$$

This condition also appears as *Assumption 2* in [14].

Now for all Poisson cloud $\omega = (\omega_i)_{i\ge1}$ where $|\omega_i| \le |\omega_{i+1}|, i\ge 1$,

$$\begin{aligned} \mathbb{E}(|\omega_1|^d) &= d\int_0^\infty t^{d-1}\mathbb{P}(|\omega_1| > t)dt \\ &= d\int_0^\infty t^{d-1}\mathbb{P}(\text{ no points of the Poisson cloud are in } B(0,t))dt \\ &= d\int_0^\infty t^{d-1}\exp(-\mathcal{L}_d t^d)dt = d/\mathcal{L}_d < \infty \end{aligned}$$

and for all $z \in B(\omega_1, 1)$, $V(z,\omega) \geq W(z-\omega_1) = \min\{|z-\omega_1|^{-\gamma}, 1\} = 1$.

Then condition (2.14) of Theorem 2.2 is verified with $u(\omega) := \omega_1$ and $\delta_1 = \delta_2 = 1$. Therefore by Theorem 2.2, (4.87) follows and the semi-norm is given by $\alpha(x) = \lim_{k\to\infty} \frac{1}{k} a(0, kx, \omega) = \lim_{k\to\infty} \frac{1}{k}\mathbb{E}[a(0, kx, \omega)]$. To see that it is a norm, observe that $W(x) = |x|^{-\gamma} \wedge 1 \geq W_1(x) := \mathbf{1}_{\{|x|\leq 1\}}$, a nonnegative bounded measurable function with compact support. Denote by $\alpha^1(x)$ the Lyapunov exponent with respect to $V^1(x) = \sum_j W_1(x-\omega_i)$. Then $\alpha(x) \geq \alpha^1(x)$ and α^1 is a norm by [31, Theorem 5.2.5]. ∎

For these potentials, whenever they are finite a.s., we have the following quenched large deviation principle.

Theorem 4.2 (Large deviation estimates)**.** *Assume that $\gamma > d$. Then $\mathbb{P}$ a.s.*

i) for all closed subset A of $\mathbb{R}^d$

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \leq -\inf_{x\in A} I(x) \tag{4.90}$$

ii) for all open subset $\mathcal{O}$ of $\mathbb{R}^d$

$$\liminf_{t\to\infty} \frac{1}{t} \log Q_{t,\omega}(Z_t \in t\mathcal{O}) \geq -\inf_{x\in\mathcal{O}} I(x). \tag{4.91}$$

where the non-random rate function I is given in (2.44) with the Lyapunov exponents $\alpha_\lambda(\cdot)$ calculated with respect to the potentials $\lambda + V$. Their existence follows from proposition 4.1.

The proof is very similar to the proof of Theorem 3.11. We use Lemma 4.3 which provides an independence property, Proposition 4.4 which is proved as in Proposition 3.10 and Lemma 4.5 given below.

Lemma 4.3. *Assume that $\gamma > d$.*

Then there is a positive constant $C = C(\gamma, d)$ such that for all $\varepsilon > 0$, for all $r > 1$ and $R > C\varepsilon^{-1/(\gamma-d)} \vee 2r$,

$$\mathbb{P}\left(\sup_{y\in B(0,r)} \sum_{\omega_j \notin B(0,R)} |y-\omega_j|^{-\gamma} > \varepsilon\right) \leq \exp(-4\varepsilon R^\gamma). \tag{4.92}$$

Proposition 4.4. *Assume that $\gamma > d$. Then $\lim_t -\frac{1}{t} \log S_{t,\omega} = 0, \quad \mathbb{P}$ a.s.*

Lemma 4.5. *Assume that $\gamma > d$. Let $\varepsilon > 0$ and $R_0 > 0$. Then $\mathbb{P}$ a.s. for all sufficiently large $|x|$, there is $z \in \mathbb{Z}^d$ such that $|x-z| < \sqrt{|x|}$ and $V < \varepsilon$ on $B(z, R_0)$.*

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