



HAL
open science

Asymptotic distribution of GLR for impropriety of complex signals

Jean-Pierre Delmas, Abdelkader Oukaci, Pascal Chevalier

► **To cite this version:**

Jean-Pierre Delmas, Abdelkader Oukaci, Pascal Chevalier. Asymptotic distribution of GLR for impropriety of complex signals. ICASSP 2010 : 35th International Conference on Acoustics, Speech, and Signal Processing, Mar 2010, Dallas, United States. pp.3594 - 3597, 10.1109/ICASSP.2010.5495920 . hal-01308689

HAL Id: hal-01308689

<https://hal.science/hal-01308689>

Submitted on 16 Dec 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ASYMPTOTIC DISTRIBUTION OF GLR FOR IMPROPRIETY OF COMPLEX SIGNALS

Jean-Pierre Delmas, Abdelkader Oukaci

Telecom SudParis, UMR CNRS 5157
91011 Evry, France

Pascal Chevalier

Thales-Communications, EDS/SPM
92704 Colombes, France

ABSTRACT

In this paper, we consider the problem of testing impropriety (i.e., second-order noncircularity) of a complex-valued random variable (RV) based on the generalized likelihood ratio test (GLRT) derived under Gaussian distributions. Asymptotic (w.r.t. the data length) distributions of the GLR are given under the hypothesis that data are proper or improper, and under the true, not necessarily Gaussian distribution of the data. This enables us to derive in particular the receiver operating characteristics (ROC) of this test, an issue previously overlooked. Finally illustrative examples are presented in order to strengthen the obtained theoretical results.

Index Terms— Generalized likelihood ratio (GLR), receiver operating characteristics (ROC), asymptotic distribution of noncircularity coefficients estimate, improper, second-order noncircular complex random variables.

1. INTRODUCTION

Many papers (see, e.g., [1, 2, 3]) show that significant performance gains can be achieved by second-order algorithms exploiting the statistical information contained in $\mathbf{C}_x = E(\mathbf{x}\mathbf{x}^T)$ (where \mathbf{x} is the observation), provided it is non-zero in addition to that contained in the standard covariance matrix $\mathbf{R}_x = E(\mathbf{x}\mathbf{x}^H)$, but with an additional complexity. Moreover, some such algorithms (see e.g., [4]) adapted for improper or second-order noncircular signals, i.e., with non-zero matrices \mathbf{C}_x , fail or suffer of too slow convergence when they are used for proper or second-order circular signals.

Thus, the question that arises as to how we can classify a signal as proper or improper. This problem is a binary hypothesis test $H_0: \mathbf{C}_x = \mathbf{O}$ versus $H_1: \mathbf{C}_x \neq \mathbf{O}$. In practice, the parameters \mathbf{R}_x and \mathbf{C}_x are often unknown, thus only the GLR detector can be employed. Based on independent and identically distributed Gaussian samples $(\mathbf{x}_k)_{k=1,\dots,K}$, this detector was introduced independently by Ollila and Koivunen [5] and Schreier, Scharf and Hanssen [6], its performance was illustrated by a Monte Carlo simulation only. Formulating this testing problem in terms of real-valued Gaussian random vectors, Walden and Rubin-Delanchy [7] derived recently this GLRT as well. In addition, they presented a theoretic analysis of the null asymptotic distribution of the GLR and several

numerical analysis based on Monte Carlo simulations for the alternative distribution under the Gaussian distribution of the signals.

The aim of this paper is to complement the theoretical asymptotic analysis of [7] by considering the null and alternative asymptotic distribution of the GLR derived under the Gaussian distribution, but used in practice under arbitrary not necessarily Gaussian distributions. This paper is organized as follows. In Section 2, the GLRT is recalled for the convenience of the reader. Using the asymptotic distributions of the noncircularity coefficients given in [8], the asymptotic distribution of the GLR under the hypothesis that data are proper or improper, and under the true, not necessarily Gaussian distribution of the data are given in the scalar case, and then extended to the multidimensional case in Section 3. Due to the simplicity of the asymptotic distribution of the GLR, an interpretable closed-form expression of the ROC is given in the scalar case. Finally some illustrative examples are given in Section 4.

2. GENERALIZED LIKELIHOOD RATIO DECISION RULE

Assume that $(\mathbf{x}_k)_{k=1,\dots,K} \in \mathbb{C}^N$ is a realization of K independent identically zero-mean complex Gaussian distributed RVs of covariance matrices $\mathbf{R}_x = E(\mathbf{x}\mathbf{x}^H)$ and $\mathbf{C}_x = E(\mathbf{x}\mathbf{x}^T)$. Consider the following binary composite hypothesis testing problem:

$$\begin{aligned} H_0 &: \mathbf{C}_x = \mathbf{O}, \quad \mathbf{R}_x \\ H_1 &: \mathbf{C}_x \neq \mathbf{O}, \quad \mathbf{R}_x. \end{aligned}$$

The GLR replaces the unknown parameters \mathbf{R}_x and \mathbf{C}_x by their maximum likelihood (ML) estimates in the likelihood ratio. It is straightforward to derive its expression which is given by [5] [6]

$$L(\mathbf{x}, K) \stackrel{\text{def}}{=} \frac{p\left((\mathbf{x}_k)_{k=1,\dots,K}; \hat{\mathbf{R}}_x, \hat{\mathbf{C}}_x, H_1\right)}{p\left((\mathbf{x}_k)_{k=1,\dots,K}; \hat{\mathbf{R}}_x, \mathbf{O}, H_0\right)} = \frac{\det(\hat{\mathbf{R}}_x)^K}{\det(\hat{\mathbf{R}}_{\tilde{x}})^{K/2}} \quad (1)$$

with $\hat{\mathbf{R}}_x \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k \mathbf{x}_k^H$ and $\hat{\mathbf{R}}_{\tilde{x}} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_k^H$ where, $\tilde{\mathbf{x}}_k \stackrel{\text{def}}{=} [\mathbf{x}_k^T, \mathbf{x}_k^H]^T$. The GLRT decides H_1 if

$$L(\mathbf{x}, K) > \lambda \quad (2)$$

and otherwise H_0 . Note that no uniformly most powerful (UMP) linearly¹ invariant test for impropriety exists for $N > 1$ [7]. In the scalar case $N = 1$, the GLRT is the UMP linearly invariant test [7]. It becomes especially simple

$$L(\mathbf{x}, K) = (1 - \hat{\gamma}_x^2)^{-K/2}$$

with $\hat{\gamma}_x = \frac{1}{K} \sum_{k=1}^K x_k^2 / \frac{1}{K} \sum_{k=1}^K |x_k|^2$ is the ML estimates [8] of the noncircularity coefficient $\gamma_x \stackrel{\text{def}}{=} |E(x_k^2)|/E|x_k|^2$ and the GLRT decides H_1 if

$$\hat{\gamma}_x > \lambda', \quad (3)$$

which is quite intuitive.

3. ASYMPTOTIC DISTRIBUTION OF GLR

Throughout in the sequel, we consider this GLRT is used for independent identically zero-mean non necessarily Gaussian distributed data $(\mathbf{x}_k)_{k=1, \dots, K}$. For nonGaussian data, the decision rule (2) is no longer a GLRT. However, it is simple to implement and generally provides good performance in practice (see e.g., for the detection of a known signal corrupted by noncircular interference [9]).

3.1. Scalar complex random variable

In the scalar-valued case, the following result is proved in the Appendix.

Result 1 *Under the respective hypothesis H_0 and H_1 , the following convergences in distribution hold when $K \rightarrow \infty$*

$$\sqrt{K}\hat{\gamma}_x \xrightarrow{\mathcal{L}} \mathcal{R}(1 + \frac{\kappa_x}{2}) \quad (4)$$

$$\sqrt{K}(\hat{\gamma}_x - \gamma_x) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_\gamma^2) \text{ if } \gamma_x < 1, \quad (5)$$

where $\mathcal{R}(1 + \frac{\kappa_x}{2})$ and $\mathcal{N}(0, \sigma_\gamma^2)$ denote the Rayleigh distribution with parameter $1 + \frac{\kappa_x}{2}$ and the zero-mean Gaussian distribution with variance σ_γ^2 respectively, with

$$\sigma_\gamma^2 = 1 - 2\gamma_x^2 + \gamma_x^4 + \gamma_x^2 \kappa_x + \frac{\kappa_x}{2} + \frac{\gamma_x^2 \Re(\kappa'_x)}{2} - 2\gamma_x^2 \Re(\kappa''_x) \quad (6)$$

if $\sigma_\gamma^2 \neq 0$, where under H_0 and H_1 , κ_x , κ'_x and κ''_x are the normalized-like cumulants $\frac{\text{Cum}(x_k, x_k, x_k^*, x_k^*)}{(E(|x_k|^2))^2}$, $\frac{\text{Cum}(x_k, x_k, x_k, x_k)}{(E(x_k^2))^2}$ and $\frac{\text{Cum}(x_k, x_k, x_k, x_k^*)}{E(|x_k|^2)E(x_k^2)}$ respectively which are invariant to any rotation of the distribution of x_k .

Remark 1: This theoretical result means in practice that for $K \gg 1$, the estimate $\hat{\gamma}_x$ is approximately Rayleigh (of parameter $\frac{1 + \frac{\kappa_x}{2}}{K}$) or Gaussian $\mathcal{N}(\gamma_x, \frac{\sigma_\gamma^2}{K})$ distributed under H_0 and H_1 , respectively. We note that the domain of validity of

¹Linear w.r.t. \mathbb{C} , which thus includes rotation and scaling, but not widely linear operation.

this approximation depends on γ_x and σ_γ^2 through the rough relation $\gamma_x - \frac{2\sigma_\gamma}{\sqrt{K}} > 0$. Concerning the practical use of this result, i.e., for $P_D \neq 1$ and $P_{FA} \neq 0$, we note that the distribution of $\hat{\gamma}_x$ under H_0 and H_1 must overlap. This is roughly achieved for $\gamma_x - \frac{2\sigma_\gamma}{\sqrt{K}} < \frac{4\sqrt{1 + \kappa_x/2}}{\sqrt{K}}$.

Remark 2: In the particular case of rectilinear RVs for which $\gamma_x = 1$, we have $x_k = r_k e^{i\phi}$ with r_k is a real-valued RV and with ϕ fixed. Consequently, the noncircularity coefficient γ_x is perfectly estimated, i.e., $\hat{\gamma}_x = 1$. In this case, the detection problem is singular and for a threshold λ' close to 1, the probability of detection P_D and false alarm P_{FA} are equal to 1 and 0 respectively.

Remark 3: It is possible that $\sigma_\gamma^2 = 0$ with $\gamma_x < 1$ (such a situation is given in [8]). In this case, the sequence $K(\hat{\gamma}_x - \gamma_x)$ converges in distribution [10, Th.B, p.124] to an Hermitian form $\mathbf{z}^H \mathbf{\Omega} \mathbf{z}$, where \mathbf{z} is a two dimensional zero-mean complex Gaussian RV, whose distribution is defined by the right hand side of (10), and where our first order analysis does not allow one to specify the matrix $\mathbf{\Omega}$.

Remark 4: We note that for γ_x close to zero and $K \gg 1$, $2 \ln L(\mathbf{x}, K) = -K \ln(1 - \hat{\gamma}_x^2) \approx K \hat{\gamma}_x^2$. Furthermore for x_k Gaussian distributed, $\kappa_x = 0$. In these cases (4) gives

$$2 \ln L(\mathbf{x}, K) \xrightarrow{\mathcal{L}} \chi_2^2, \text{ under } H_0.$$

This convergence in distribution is consistent with the constant false alarm rate (CFAR) detector where the number 2 of degree of freedom of the chi-square distribution is equal to the number of real-valued components of $c_x \stackrel{\text{def}}{=} E(x_k^2)$, given by the Wilk's theorem [13, p.132].

Remark 5: For Gaussian distributed data, $\kappa_x = \kappa'_x = \kappa''_x = 0$ and the variance σ_γ^2 of the asymptotic distribution of $\hat{\gamma}_x$ under the hypothesis H_1 equal to $1 - 2\gamma_x^2 + \gamma_x^4$, is a decreasing function of γ_x . Consequently for a fixed P_{FA} , i.e., for fixed threshold λ' , P_D is an increasing function of γ_x that does not depend on the power of x_k , which is very intuitive.

For an arbitrary, not necessarily Gaussian distribution of x_k , Result 1 allows us to derive

$$P_{FA} = P(\hat{\gamma}_x > \lambda' / H_0) \approx Q_{\chi_2^2} \left(\frac{K \lambda'^2}{1 + \frac{\kappa_x}{2}} \right)$$

$$P_D = P(\hat{\gamma}_x > \lambda' / H_1) \approx Q_{\mathcal{N}} \left(\frac{\sqrt{K}(\lambda' - \gamma_x)}{\sigma_\gamma} \right),$$

where $Q_{\chi_2^2}(\cdot)$ and $Q_{\mathcal{N}}(\cdot)$ denote the complementary cumulative distribution functions of the chi-square distribution with 2 degrees of freedom and the zero-mean, unit-variance Gaussian distribution respectively, and where σ_γ is given by (6). This gives the following closed form expression of the ROC of GLR detector (3).

$$P_D \approx Q_{\mathcal{N}} \left(\frac{\sqrt{(1 + \frac{\kappa_x}{2}) Q_{\chi_2^2}^{-1}(P_{FA})} - \sqrt{K} \gamma_x}{\sigma_\gamma} \right). \quad (7)$$

We clearly see from this expression that for fixed $P_{\text{FA}}, P_{\text{D}}$ is an increasing function of the data length K and for Gaussian distributed data, an increasing function of the noncircularity coefficient γ_x .

3.2. Multidimensional complex random variable

In the multidimensional case ($N > 1$), no simple interpretable expression of P_{FA} and P_{D} are available, but the following result is proved in the Appendix.

Result 2 Under hypothesis H_0 and H_1 , the following convergences in distribution hold when $K \rightarrow \infty$ for the decision statistic $\ell(\mathbf{x}, K) \stackrel{\text{def}}{=} [L(\mathbf{x}, K)]^{-2/K}$

$$\sqrt{K}(\ell(\mathbf{x}, K) - \ell_i) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_i^2), \text{ under } H_i \quad i = 0, 1, \quad (8)$$

where $\ell_0 = 1$ and the expression of $\ell_1 < 1$ is given in the Appendix.

Remark 6: Note that Wilk's theorem [13, p.132] gives

$$2 \ln L(\mathbf{x}, K) \xrightarrow{\mathcal{L}} \chi_{N(N+1)}^2 \text{ under } H_0,$$

where the number of degrees of freedom of the chi-squared distributions is equal to $N(N+1)$ of real-valued independent parameters in \mathbf{C}_x . Furthermore under H_1 , in the case of \mathbf{C}_x is "close" to \mathbf{O} (see a more formal definition in [12, Ch.23.7]), the analysis of [11, Sec.II] is valid and gives the following approximation of distribution when $K \gg 1$:

$$2 \ln L(\mathbf{x}, K) \stackrel{a}{\sim} \chi_{N(N+1)}^{\prime 2}(\mu) \text{ under } H_1,$$

where $\chi_{N(N+1)}^{\prime 2}(\mu)$ represents a noncentral chi-squared distribution with $N(N+1)$ degrees of freedom and noncentral parameter μ . This parameter is a measure of the discrimination between the two hypothesis, whose general expression is given by [11, exp.(4)].

4. ILLUSTRATIVE EXAMPLES

We consider the following MIMO channel (extension of the example given in [6]) that transmits Q independent equiprobable BPSK symbols $a_{q,k} \in \{-1, +1\}$ over an additive noise channel, that also rotates independently the phase of the transmitted symbols $a_{q,k}$ by $\phi_{q,k}$

$$\mathbf{x}_k = \sum_{q=1}^Q \sigma_q a_{q,k} e^{i\phi_{q,k}} \mathbf{s}_q + \mathbf{n}_k, \quad (9)$$

where σ_q and \mathbf{s}_q are Q unknown amplitudes and steering vectors with unit first component, and where the components of \mathbf{n}_k are independent zero-mean complex circular Gaussian RV of variance σ_n^2 . Under H_0 and H_1 , we assume that the phase $(\phi_{q,k})_{k=1, \dots, K, q=1, \dots, Q}$ are independent and respectively uniformly distributed on $[0, 2\pi]$ or Gaussian distributed with

mean ϕ_{q_0} and variance $\sigma_{\phi_q}^2$. So we are interested in classifying this channel as either incoherent or partially coherent, which is a binary composite hypothesis testing problem. We easily deduce that

$$\mathbf{R}_x = \sum_{q=1}^Q \sigma_q^2 \mathbf{s}_q \mathbf{s}_q^H + \sigma_n^2 \mathbf{I}_Q,$$

$$\mathbf{C}_x = \mathbf{O}, \left[\text{resp.}, \sum_{q=1}^Q \sigma_q^2 e^{2i\phi_{q_0}} e^{-2\sigma_{\phi_q}^2} \mathbf{s}_q \mathbf{s}_q^T \right]$$

under H_0 [resp. H_1]. For $Q = 1$, $\kappa_x = -\frac{1}{(1+\rho_x^{-1})^2}$ under H_0 and $\gamma_x = \frac{e^{-2\sigma_{\phi_1}^2}}{1+\rho_x^{-1}}$, $\kappa_x = -\frac{1+e^{-4\sigma_{\phi_1}^2}}{(1+\rho_x^{-1})^2}$, $\kappa'_x = e^{-4\sigma_{\phi_1}^2} - 3$ and $\kappa''_x = -\frac{2}{1+\rho_x^{-1}}$ under H_1 , with a signal to noise ratio (SNR) of $\rho_x \stackrel{\text{def}}{=} \sigma_1^2 / \sigma_n^2$.

This model is compared to the Gaussian model obtained when $\phi_{q,k}$ does not depend on k and $a_{q,k}$ are independent zero-mean complex circular or real-valued Gaussian RVs under H_0 and H_1 respectively. Fig.1 shows the northwest corner of the ROC curve for the GLRT detector for $Q = 1$, $K = 100$ and $\rho_x = 0.63$ (-2dB) for BPSK model with a coherent channel (i.e., $\sigma_{\phi_1} = 0$) and Gaussian model, and thus associated with the same value of $\gamma_x = 0.387$.

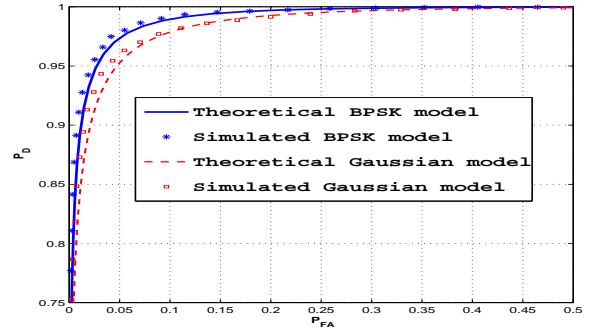


Fig.1 Asymptotic theoretical and empirical (with 10000 Monte Carlo runs) ROC curve associated with BPSK and Gaussian model.

We note that the ROC curve is sensitive to the distribution of the data x_k and that the empirical ROC fits the asymptotic theoretical ROC for the relatively small data length $K = 100$.

Fig.2 shows the detection performance P_{D} for different fixed P_{FA} for model (9) as a function of the SNR for two values of σ_{ϕ_1} .

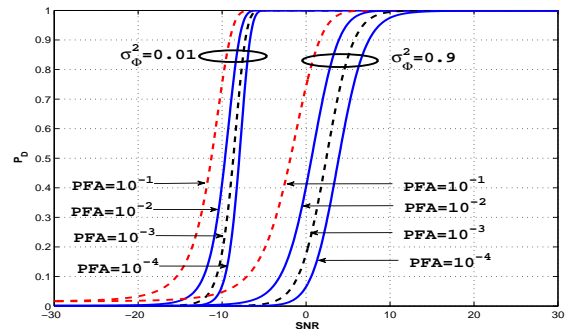


Fig.2 P_{D} for different fixed P_{FA} for model (9) as a function of SNR.

We see that the P_D for fixed P_{FA} is very sensitive to the coherence of the channel. When σ_{ϕ_1} increases for a fixed SNR, the noncircularity coefficient γ_x decreases and detection worsens.

5. CONCLUSION

In this paper, we have given the asymptotic distribution of the GLR for impropriety of complex signals, derived under the Gaussian distribution, but used under independent and identically arbitrary non necessarily Gaussian distributions of the data. Extension of this work to independent non identically distributed data will enable us to consider in a future contribution, more practical situations such that BPSK signals with nonnull carrier residues.

6. APPENDIX

Proof of Result 1 Under H_1 , (5) is directly issued from [8, Result 3]. Under H_0 , [8, Result 3] is not valid, but the analysis of [8] applies. The standard central limit² applied to the independent identically distributed bidimensional complex RVs $\begin{pmatrix} \hat{r}_x \\ \hat{c}_x \end{pmatrix}$ with $\hat{r}_x = \frac{1}{K} \sum_{k=1}^K |x_k^2|$ and $\hat{c}_x = \frac{1}{K} \sum_{k=1}^K x_k^2$ yields

$$\sqrt{K} \begin{pmatrix} \hat{r}_x - r_x \\ \hat{c}_x - c_x \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}_C \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_x^4 \begin{pmatrix} 1+\kappa_x & 0 \\ 0 & 2+\kappa_x \end{pmatrix}, \sigma_x^4 \begin{pmatrix} 1+\kappa_x & 0 \\ 0 & 0 \end{pmatrix} \right) \quad (10)$$

where $\sigma_x^2 \stackrel{\text{def}}{=} E|x_k^2|$. Then, considering the mapping

$$(\hat{r}_x, \hat{c}_x) \mapsto \hat{m}_x = \frac{\hat{c}_x}{\hat{r}_x} \mapsto \hat{\gamma}_x = |\hat{m}_x|,$$

whose differential of the first step is $dm = \frac{1}{c} dr$ under H_0 , the standard theorem of continuity (see e.g., [10, Th.A, p.122]) on regular functions of asymptotically Gaussian statistics applies. Consequently, we obtain the following convergence in distribution to a complex circular zero-mean Gaussian distribution of variance $2(1 + \frac{\kappa_x}{2})$

$$\sqrt{K} (\hat{m}_x - 0) \xrightarrow{\mathcal{L}} \mathcal{N}_C \left(0, \frac{1}{\sigma_x^4} \sigma_x^4 (2 + \kappa_x), 0 \right).$$

With $\hat{\gamma}_x^2 = |\hat{m}_x|^2$, convergence in distribution (4) is proved.

Proof of Result 2

With $\ell(\mathbf{x}, K) = \det[\mathbf{I} - (\hat{\mathbf{R}}_x^{-1} \hat{\mathbf{C}}_x)^* \hat{\mathbf{R}}_x^{-1} \hat{\mathbf{C}}_x]$, where $\hat{\mathbf{C}}_x \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k \mathbf{x}_k^T$, the proof of Result 2 follows the same steps that for Result 1. Deriving the asymptotic distribution of $\ell(\mathbf{x}, K)$ under H_0 and H_1 is based on the following mapping:

$$\begin{aligned} (\hat{\mathbf{R}}_x, \hat{\mathbf{C}}_x) &\mapsto \hat{\mathbf{\Omega}}_x = \hat{\mathbf{R}}_x^{-1} \hat{\mathbf{C}}_x \mapsto \hat{\mathbf{\Xi}}_x = \hat{\mathbf{\Omega}}_x^* \hat{\mathbf{\Omega}}_x \\ &\mapsto \ell(\mathbf{x}, K) = \det[\mathbf{I} - \hat{\mathbf{\Xi}}_x]. \quad (11) \end{aligned}$$

² $\mathcal{N}_C(\mathbf{m}, \mathbf{R}, \mathbf{C})$ denotes the Gaussian complex distribution with mean \mathbf{m} , and covariances \mathbf{R} and \mathbf{C} .

Using the asymptotic Gaussian distribution of $(\hat{\mathbf{R}}_x, \hat{\mathbf{C}}_x)$ [3] derived from the standard central limit theorem, the differential³ of the different sub mappings of (11), the chain rule and standard properties of the vec operator [14, Ch.2.4], the standard theorem of continuity (see e.g., [10, p. 122]) on regular functions of asymptotically Gaussian statistics applies and Result 2 follows, with $\ell_0 = 1$ and $\ell_1 = \det[\mathbf{I} - (\mathbf{R}_x^{-1} \mathbf{C}_x)^* \mathbf{R}_x^{-1} \mathbf{C}_x] < 1$ and where expressions of σ_0^2 and σ_1^2 are not given here due to space limitation.

7. REFERENCES

- [1] B. Picinbono, P. Chevalier "Widely Linear Estimation with Complex Data," *IEEE Trans. Signal Process.*, vol. 43, no. 8, pp. 2030-2033, August 1995.
- [2] P.J. Schreier, L. Scharf and C.T. Mullis, "Detection and estimation of improper complex random signals," *IEEE Trans. Inform. Theory*, vol. 51, no. 1, pp. 306-312, January 2005.
- [3] J.P. Delmas, "Asymptotically minimum variance second-order estimation for non-circular signals with application to DOA estimation," *IEEE Trans. Signal Process.*, vol. 52, no. 5, pp. 1235-1241, May 2004.
- [4] H. Abeida, J.P. Delmas, "MUSIC-like estimation of direction of arrival for non-circular sources," *IEEE Trans. Signal Process.*, vol. 54, no. 7, pp. 2678-2690, July 2006.
- [5] E. Ollila, V. Koivunen, "Generalized complex elliptical distributions," in *Proc. 3rd Sensor Array Multichannel Signal Processing Workshop*, Sitges, Spain, July 2004.
- [6] P.J. Schreier, L. Scharf and A. Hanssen, "A generalized likelihood ratio test for impropriety of complex signals," *IEEE Signal Process. Letters*, vol. 13, no. 7, pp. 433-436, July 2006.
- [7] A.T. Walden, P. Rubin-Delanchy, "On testing for impropriety of complex-valued Gaussian vectors," *IEEE Trans. Signal Process.*, vol. 57, no. 3, pp. 825-834, March 2009.
- [8] J.P. Delmas, H. Abeida, "Asymptotic distribution of circularity coefficients estimate of complex random variables," *Signal Processing (Elsevier)*, vol. 89, pp. 2311-2698, December 2009.
- [9] P. Chevalier, A. Blin, F. Pipon and F. Delaveau, "GLRT-Based array receivers to detect a known signal corrupted by noncircular interferences," in *Proc. EUSIPCO*, Poznan, Poland, Sept. 2007.
- [10] R.J. Serfling, *Approximation Theorems of Mathematical Statistics*, John Wiley and Sons, 1980.
- [11] S.M. Kay, "Asymptotically optimal detection in incompletely characterized non-Gaussian noise," *IEEE Trans. ASSP*, vol. 37, no. 5, pp. 627-633, May 1989.
- [12] A. Stuart, J.K. Ord, *Advanced Theory of Statistics*, fifth edition, vol.2, Edward Arnold, 1991.
- [13] G.A. Young, R.L. Smith, *Essentials of Statistical Inference*, Cambridge Series in Statistical and Probabilistic Mathematics, 2005.
- [14] J.R. Magnus, H. Neudecker, *Matrix differential calculus with applications in statistics and econometrics*, Wiley series in probability and statistics, revised edition, 1999.

³ $d\Omega = -\mathbf{R}_x^{-1} d\mathbf{R}_x \mathbf{R}_x^{-1} \mathbf{C}_x + \mathbf{R}_x^{-1} d\mathbf{C}_x$, $d\Xi = d\Omega_x^* \Omega_x + \Omega_x^* d\Omega_x$ with $\Omega_x = \mathbf{R}_x^{-1} \mathbf{C}_x$ and $d\ell = -\det[\mathbf{I} - \Xi] \text{Tr}[(\mathbf{I} - \Xi)^{-1} d\Xi]$ (from [14, Th.1, p.149]) with $\Xi = (\mathbf{R}_x^{-1} \mathbf{C}_x)^* \mathbf{R}_x^{-1} \mathbf{C}_x$.