Effectivity in the Kobayashi and Debarre Conjectures

Ya Deng

Effectivity in the Kobayashi and Debarre Conjectures

Ya Deng*

Institut Fourier, Université Grenoble Alpes

May 22, 2016

Abstract

The aim of this work is to deal with effective problems in the Kobayashi and Debarre conjectures, based on the work by Damian Brotbek and Lionel Darondeau. We first show that if the line bundle L generates k-jets, the k-th Wronskian ideal sheaf for L is stable. Then we obtain an effective estimate in the "almost" Nakamaye Theorem for the universal Grassmannian. Finally we get explicit effective bounds for the Kobayashi and Debarre conjectures.

1 Introduction

The famous Kobayashi conjecture stated that a general hypersurface in $\mathbb{P}^n$ of sufficient large degree is hyperbolic. In the last 15 years, at least three important techniques were introduced to study this problem: Siu's slanted vector fields on higher order jet spaces [7], Demailly's strongly of general type for studying Green-Griffiths-Lang conjecture [5] and Brotbek's construction of families of hypersurfaces which are deformations of Fermat type [3]. In [3] Brotbek has given a formula for the degree bound d_0, which depends on two constants M and m_∞ appearing in using noetherianity. In this paper we will determine these two constants and hence give the effective bound by applying Brotbek's formula directly. Since the problems are more or less independent, we will extact them from the work by Brotbek and suggest the readers who are interested in these connections to refer [3]. Nevertheless we will adopt the same notations in [3] for the reader's convenience.

For the constant M depending on n, it arises in the Nakamaye theorem: Let X be a Kähler manifold and L a big and nef line bundle on X. Then the null locus $\mathrm{Null}(L)$ of L coincides with the augmented base locus $\mathbb{B}_+(L)$. Fix an ample divisor A on X, by noetherian property there exists a positive integer m_0 such that for $m \geq m_0$ we have $\mathrm{Bs}(mL - A) = \mathbb{B}_+(L) = \mathrm{Null}(L)$. Now we have the following effectivity problem:

Problem 1.1. *Can we get an effective bound for m_0 which depends on some numerical constants related to L and A?*

*Email address: Ya.Deng@univ-grenoble-alpes.fr

In general, this problem is hard to answer. Fortunately, in order to compute M, we only need to consider the concrete projective manifold X which is a universal Grassmannian, and L is the pull back of the tautological line bundle. Our first main result is Theorem 3.1, which is an "almost" Nakamaye Theorem. Namely although we could not get the effective bound in the Nakamaye Theorem, we can obtain a bound such that the corresponding base locus is good enough to ensure the arguments in [3] are still valid. The method can also be used to study the effective problem for the ampleness of the cotangent bundles of general complete intersections (ref. [2]).

The constant m_∞ appears in the Wronskian ideal sheaf of Demailly-Semple tower. Let L be an very ample line bundle on the Kähler manifold X, and for any holomorphic sections $s_0, \ldots, s_k \in H^0(X, L)$, we have the Wronskian for these sections

$$W(s_0, s_1, \ldots, s_k) \in H^0(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi_{0,k}^*(L^{k+1})),$$

where X_k is the k-stage Demailly-Semple tower for the directed variety (X, T_X). We set

$$\mathbb{W}(X_k, L) := \mathrm{Span}\{W(s_0, \ldots, s_n) | s_0, \ldots, s_n \in H^0(X, L)\} \subset H^0(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi_{0,k}^*(L^{k+1})),$$

to be the sublinear system of $H^0(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi_{0,k}^*(L^{k+1}))$ and $\mathfrak{w}(X_k, L)$ is denoted to be the base ideal of the linear system $\mathbb{W}(X_k, L)$. By Lemma 2.5 in [3] if L is very ample we have

$$\mathfrak{w}(X_k, L) \subset \mathfrak{w}(X_k, L^2) \subset \ldots \subset \mathfrak{w}(X_k, L^m) \subset \ldots.$$

Then the noetherian property shows that this increasing sequence stabilizes for some $m_0 := m_\infty(X_k, L)$, and the asymptotic ideal sheaf is denoted by $\mathfrak{m}_\infty(X_k, L)$. As was also proved in [3], $\mathfrak{m}_\infty(X_k, L)$ does not depend on the very ample line bundle L and is called *asymptotic Wronskian ideal sheaf*, denoted by $\mathfrak{m}_\infty(X_k)$. Now we have the following problem:

Problem 1.2. *Determine the constant $m_\infty(X_k, L)$.*

We have the following theorem which solves Problem 1.2:

Theorem 1.1. *If L separates k-jets at each point of X, then $m_\infty(X_k, L) = 1$. In particular, if L is known to be only very ample, we have $m_\infty(X_k, L) = k$.*

By Theorem 1.1 we have $m_\infty = n - 1$ if $k = n - 1$. Hence by the formula in [3] we have the following effective bound for degree of generic hypersurfaces in $\mathbb{P}^n$ which are hyperbolic:

Main Theorem 1. *Let X be a projective manifold of dimension n and L is a very ample line bundle on X. Then a generic hypersurface in $|H^0(X, L^d)|$ with $d \geq 3^n n(4n-2) + 4n + 2$ is Kobayashi hyperbolic. In particular, a generic hypersurface in $\mathbb{P}^n$ of degree $d \geq 3^n n(4n-2) + 4n + 2$ is Kobayashi hyperbolic.*

The technique can also be used to study the effectivity in the Debarre conjecture, and we have the following result:

Main Theorem 2. *Set $c_0 := \llcorner \frac{N+1}{2} \lrcorner$. For all $d \geq 2c_0 4^{c_0+1} + 5$, the complete intersection of general hypersurfaces $H_1, \ldots, H_c$ with $H_1, \ldots, H_{c_0}$ in $\mathcal{O}_{\mathbb{P}^N}(d)$ and other H_i any degree, have ample cotangent bundle.*

2 Asymptotic Wronskian Ideal Sheaf in the Demailly-Semple Tower

We denote by (X_k, V_k) the Demailly-Semple tower for (X, T_X), where X is a complex manifold, and $D_j := P(T_{X_{j-1}/X_{j-2}})$ (by an abuse of notation, in our context D_j is the divisor in X_k given by $\pi_{j,k}^{-1}(D_j)$). Let U be an open set with coordinates $(z_1, \ldots, z_n)$ such that $\pi_{0,k}^{-1}(U)$ is a trivial product of $U \times \mathcal{R}_{n,k}$, where $\mathcal{R}_{n,k}$ is the universal rational variety $\mathbb{C}^{nk}/\mathbb{G}_n$ [4]. We denote by $g_k : \pi_{0,k}^{-1}(U) \to U \times \mathcal{R}_{n,k}$ the natural biholomorphism. Then each $D_i \cap \pi_{0,k}^{-1}(U)$ is also a trivial product $U \times E_j$ under the biholomorphism g_k, where E_j is a smooth prime divisor in $\mathcal{R}_{n,k}$. Let $p \in U$ be the point corresponding to the origin in the coordinate $(z_1, \ldots, z_n)$. Therefore, we have the following biholomorphism $q_k \circ g_k : \pi_{0,k}^{-1}(p) \cap D_j \xrightarrow{\approx} E_j$, where $q_k : U \times \mathcal{R}_{n,k} \to \mathcal{R}_{n,k}$ is the projection to the second factor. In the paper, we will make the convention that any germ of holomorphic curve is defined over p, and we will not distinguish the points in E_j and $\pi_{0,k}^{-1}(p) \cap D_j$.

For any germ of holomorphic curve $\nu(t)$ with $\nu(0) = p$, its lift to the $k-1$-th Demailly-Semple tower $\nu_{[k-1]}(t)$ induces a section $\nu'_{[k-1]}(t)$ of the line bundle $\nu_{[k]}^* \mathcal{O}_{X_k}(-1)$. Then by the definition of the Wronskian, for any $f_0, \ldots, f_n \in \mathcal{O}_{X,p}$, $W(f_0, f_1, \ldots, f_k)$ can be seen as a section in $H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m))$ with $m = \frac{k(k+1)}{2}$, given by

$$W(f_0, \ldots, f_k)(\nu'^m_{[k-1]}(0)) = \begin{vmatrix} f_0 \circ \nu(t) & \ldots & f_k \circ \nu(t) \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}^k f_0 \circ \nu(t)}{\mathrm{d}t^k} & \ldots & \frac{\mathrm{d}^k f_k \circ \nu(t)}{\mathrm{d}t^k} \end{vmatrix}_{t=0}. \tag{2.1}$$

We set

$$S := \mathrm{Span}\{W(f_0, \ldots, f_n) | f_0, \ldots, f_n \in \mathcal{O}_{X,p}\} \subset H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m)),$$

and $\mathcal{I}_{n,k} \subset \mathcal{O}_{\mathcal{R}_{n,k}}$ is denoted to be the base ideal of the linear system S. By the homogenous property we have

$$g_k^* \circ q_k^* \mathcal{I}_{n,k} = \mathfrak{w}_\infty(X_k) \tag{2.2}$$

at $\pi_{0,k}^{-1}(U)$, where $\mathfrak{m}_\infty(X_k)$ is the asymptotic Wronskian ideal sheaf introduced in Section 1.

Theorem 2.1. *Let $f_0 \in m_p^{k+1}, f_1, \ldots, f_k \in \mathcal{O}_{X,p}$, then $W(f_0, \ldots, f_k) = 0$ as a section in $H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m))$.*

Proof. For any $w \in \mathcal{R}_{n,k}$, by a theorem in [4], there exists a germ of curve $\nu : (\mathbb{C}, 0) \to (X, p)$ such that $\nu_{[k]}(0) = w$ and $\nu'_{[k-1]}(0) \neq 0$. Since $\nu'_{[k-1]}(0) \in \mathcal{O}_{\mathcal{R}_{n,k}}(-1)|_w$, then if we can show that $W(f_0, \ldots, f_k)(\nu'^m_{[k-1]}(0)) = 0$, by the arbitrariness of w, we will prove the theorem.

Since $f_0 \in m_p^{k+1}$, then the multiplicity of $f_0 \circ \nu(t)$ at 0 must be no less than $k+1$, and thus $\frac{(f_0 \circ \nu(t))^{(i)}}{t}$ is still holomorphic for $i = 0, \ldots, k$. Thus

$$W(f_0, \ldots, f_k)(\nu'^m_{[k-1]}(0)) = t \cdot \begin{vmatrix} \frac{f_0 \circ \nu(t)}{t} & \ldots & f_k \circ \nu(t) \\ \vdots & \ddots & \vdots \\ \frac{1}{t} \cdot \frac{\mathrm{d}^k f_0 \circ \nu(t)}{\mathrm{d}t^k} & \ldots & \frac{\mathrm{d}^k f_k \circ \nu(t)}{\mathrm{d}t^k} \end{vmatrix}_{t=0} = 0.$$

□

Since the Wronskian operator $W : \mathcal{O}_{X,p} \times \ldots \times \mathcal{O}_{X,p} \to H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m))$ is bilinear, hence we have

Theorem 1.1*bis*. *If L separates k-jets at each point of X, then $m_\infty(X_k, L) = 1$, where $m_\infty(X_k, L) = 1$ is the positive integer defined in Section 1. In particular, if L is known to be only very ample, we have $m_\infty(X_k, L) = k$.*

3 Effectivity of "Good" Base Locus for Universal Grassmannian

We denote by $V := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta))$, which is the set of homogenous polynomials of degree δ in $\mathbb{C}[z_0, \ldots, z_n]$, and for any $J \subset \{0, \ldots, N\}$, set

$$\mathbb{P}_J := \{[z_0, \ldots, z_N] \in \mathbb{P}^N | z_j = 0 \text{ if } j \in J\}.$$

Given any $\Delta \in \mathrm{Gr_{k+1}(V)}$ and $[z] \in \mathbb{P}^N$ we write $\Delta([z]) = 0$ if $P(z) = 0$ for all $P \in \Delta \subset V$. We then define the family

$$\mathscr{Y} := \{(\Delta, [z]) \in \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N | \Delta([z]) = 0\},$$

and for any $J \subset \{0, \ldots, N\}$, set

$$\mathscr{Y}_J := \mathscr{Y} \cap (\mathrm{Gr}_{k+1}(V) \times \mathbb{P}_J) \subset \mathscr{Y} \cap \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N.$$

Let $k + 1 \geq N$, then $q_1 : \mathscr{Y} \to \mathrm{Gr_{k+1}(V)}$ is a generic finite to one morphism. We also define $q_2 : \mathscr{Y} \to \mathbb{P}^N$ to be the projection on the second factor. Let $\mathscr{L}$ be the very ample line bundle on $\mathrm{Gr}_{k+1}(V)$ which is the pull back of $\mathcal{O}(1)$ under the Plücker embedding. Then $q_1^*\mathscr{L}|_{\mathscr{Y}_J}$ is a big and nef line bundle on $\mathscr{Y}_J$ for any J. For any $J \subset \{0, \ldots, N\}$ we denote by $p_J : \mathscr{Y}_J \to \mathrm{Gr}_{k+1}(V)$, and $\hat{p}_J : \mathscr{Y}_J \to \mathbb{P}_J$ the second projection. Similarly we set

$$E_J := \{y \in \mathscr{Y} | \dim_y(p_J^{-1}(p_J(y))) > 0\}$$

$$G_J^\infty := p_J(E_J) \subset \mathrm{Gr}_{k+1}(V),$$

then $E_J = \mathrm{Null}(\mathrm{q_1^*}\mathscr{L}|_{\mathscr{Y}_\mathrm{J}})$. For $J = \emptyset$ we have $\mathscr{Y}_J = \mathscr{Y}$ and denote by $E := E_\emptyset$ and $G^\infty := G_\emptyset^\infty$. By the theorem of Nakamaye we have

$$\mathrm{Null}(q_1^*\mathscr{L}|_{\mathscr{Y}_J}) = \mathbb{B}_+(q_1^*\mathscr{L}|_{\mathscr{Y}_J}).$$

Since $q_1^*\mathscr{L} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathscr{Y}_J}$ is an ample line bundle on $\mathscr{Y}_J$, by noetherianity there exists $m_J \in \mathbb{N}$ such that

$$E_J = \mathbb{B}_+(q_1^*\mathscr{L}|_{\mathscr{Y}_J}) = \mathrm{Bs}(q_1^*\mathscr{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}_J}(-1)|_{\mathscr{Y}_J})$$

for every $m \geq m_J$.

From now on we assume that $k + 1 = N$ (since this is enough in Brotbek's work), and thus $q_1 : \mathscr{Y} \to \mathrm{Gr_{k+1}(V)}$ is a generically finite-to-one surjective morphism. First we will work on $J = \emptyset$.

We first find a smooth rational curve C in $\mathrm{Gr}_{k+1}(V)$ of degree 1, which is given by

$$\Delta([t_0,t_1]) := \mathrm{Span}(z_1^\delta, z_2^\delta, \dots z_{N-1}^\delta, t_0 z_N^\delta + t_1 z_0^\delta),$$

where $[t_0,t_1] \in \mathbb{P}^1$. We are going to show that $\Delta([t_0,t_1]) : \mathbb{P}^1 \to \mathrm{Gr}_{k+1}(V)$ is a smooth embedding from $\mathbb{P}^1$ to $\mathrm{Gr}_{k+1}(V)$, and satisfies the degree condition $\mathscr{L}\cdot C = 1$.

Set $\mathbb{I} := \{I = (i_0,\dots,i_N) || I| = \delta\}$. Then $\{z^I\}_{I\in\mathbb{I}}$ is a basis for V, and $w_{(I_0,\dots,I_k)}$ with $I_0 < I_1 < \dots < I_k$ is a basis for $\Lambda^{k+1}V$, where $I < J$ is defined by lexicographical order in $\mathbb{I}$. Now we set $J_i \in \mathbb{I}$ to be $z^{J_i} := z_i^\delta$ for $0 \le i \le k$. Then the curve $\tilde{C}$ defined in $\mathbb{P}(\Lambda^{k+1}V)$ given by the equations $w_{(I_0,\dots,I_k)} = 0$ for all $(I_0,\dots,I_k) \neq (J_1,\dots,J_N)$ and $(J_0,\dots,J_{N-1})$, is of degree 1 with respect to $\mathscr{L}$. $\tilde{C}$ is totally contained in the image of the Plücker embedding of $\mathrm{Gr}_{k+1}(V)$, whose inverse image is C.

Now choose a hyperplane D which is given by $\{[z_0,\dots,z_N] | z_0 + z_N = 0\}$ in $\mathbb{P}^N$. Then we have

Lemma 3.1. *The intersection number of the curve q_1^*C and the divisor q_2^*D in $\mathscr{Y}$ is δ^{N-1}. Moreover, $q_{1*}q_2^*D \equiv \delta^{N-1}\mathscr{L}$, where " $\equiv$ " stands for linear equivalent.*

Proof. An easy computation shows that q_1^*C and q_2^*D intersect only at the point $\mathrm{Span}(z_1^\delta, z_2^\delta, \dots z_{N-1}^\delta, z_N^\delta + (-1)^{\delta+1} z_0^\delta) \times [1,0\dots,0,-1] \in \mathscr{Y}$ with multiplicity δ^{N-1}. The first statement follows.

By projection formula we have

$$q_{1*}q_2^*D \cdot C = q_{1*}(q_2^*D \cdot q_1^*C) = \delta^{N-1}.$$

Since $\mathrm{Pic}(\mathrm{Gr_{k+1}(V)}) \approx \mathbb{Z}$ with the generator $\mathscr{L}$, then $q_{1*}q_2^*D \equiv \delta^{N-1}\mathscr{L}$ by the fact that $\mathscr{L}\cdot C = 1$. □

We first observe that $q_1^*q_{1*}q_2^*D - q_2^*D$ is an effective divisor of $\mathscr{Y}$, and from Lemma 3.1 we know that $q_1^*\mathscr{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)$ is effective. We also have a good control of the base locus as follows:

Claim:

$$\mathrm{Bs}(q_1^*\mathscr{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) \subset q_1^{-1}(G^\infty) \qquad \text{for any } m \ge \delta^{N-1}.$$

Proof: Indeed, for any $y_0 \notin q_1^{-1}(G^\infty)$, $q_1^{-1}(q_1(y_0))$ is a finite set, and one can choose a hyperplane $D' \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ such that $D' \cap q_2(q_1^{-1}(q_1(y_0))) = \emptyset$. From the result above we know that the divisor $q_1^*q_{1*}q_2^*D' - q_2^*D'$ is effective and lies in the linear system $|q_1^*\mathscr{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|$ of $\mathscr{Y}$. Now we show that $y_0 \notin \mathrm{Supp}(q_1^*q_{1*}q_2^*D' - q_2^*D')$. Indeed, for any $\Delta \in \mathrm{Gr}_{k+1}(V)$, if we denote by $\mathrm{Int}(\Delta) := \{[z] \in \mathbb{P}_N | \Delta([z]) = 0\}$, then $q_2(q_1^{-1}(q_1(y_0))) = \mathrm{Int}(\Delta_0)$, where $\Delta_0 := q_1(y_0)$, hence the condition $D' \cap q_2(q_1^{-1}(q_1(y_0))) = \emptyset$ is equivalent to that $\mathrm{Int}(\Delta_0) \cap D' = \emptyset$. However, for any $\Delta \in q_1(q_2^{-1}(D'))$, we must have $\mathrm{Int}(\Delta) \cap D' \neq \emptyset$, therefore $\Delta_0 \notin q_1(q_2^{-1}(D'))$ and thus $y_0 \notin q_1^{-1}(q_1(q_2^{-1}(D'))) \supset \mathrm{Supp}(q_1^*q_{1*}q_2^*D' - q_2^*D')$. Thus from the arbitrariness of y_0 we have

$$\mathrm{Bs}(q_1^*\mathscr{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) \subset q_1^{-1}(G^\infty).$$

Since $\mathscr{L}$ is very ample on $\mathrm{Gr}_{k+1}(V)$, and q_1 is finite to one morphism outside $q_1^{-1}(G^\infty)$, we conclude that for any $m \geq \delta^{N-1}$,

$$\mathrm{Bs}(q_1^*\mathscr{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) \subset q_1^{-1}(G^\infty).$$

■

Now we work on general $J \subset \{0, \ldots, N\}$. First recall that $p_J : \mathscr{Y}_J \to \mathrm{Gr}_{k+1}(V)$, and $\hat{p}_J : \mathscr{Y}_J \to \mathbb{P}_J$ the second projection. The method is repeating the arguments above. For any $y_0 \notin p_J^{-1}(G_J^\infty)$, the set $p_J^{-1}(p_J(y_0))$ is a finite one, and thus one can choose a generic hyperplane $D \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ such that $D \cap \mathbb{P}_J \cap \hat{p}_J(p_J^{-1}(p_J(y_0))) = \emptyset$, which means that $\mathrm{Int}(\Delta_0) \cap D \cap \mathbb{P}_J = \emptyset$, where $\Delta_0 := p_J(y_0)$. However, for any $\Delta \in p_J(\hat{p}_J^{-1}(D \cap \mathbb{P}_J))$ we must have $\mathrm{Int}(\Delta) \cap D \cap \mathbb{P}_J \neq \emptyset$, and thus $\Delta_0 \notin p_J(\hat{p}_J^{-1}(D \cap \mathbb{P}_J))$, *a fortiori* $y_0 \notin p_J^{-1}(p_J(\hat{p}_J^{-1}(D \cap \mathbb{P}_J))) = q_1^{-1}(q_1(q_2^{-1}(D'))) \cap \mathscr{Y}_J$. Thus the restriction of the effective divisor $q_1^*q_{1*}q_2^*D - q_2^*D$ to $\mathscr{Y}_J$ is well-defined and $y_0 \notin q_1^*q_{1*}q_2^*D - q_2^*D|_{\mathscr{Y}_J}$. From the arbitrariness of y_0 we know that the base locus of the linear system $|q_1^*\mathscr{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathscr{Y}_J}|$ is contained in $p_J^{-1}(G_J^\infty)$. Thus by the same argument we have

$$\mathrm{Bs}(q_1^*\mathscr{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathscr{Y}_J}) \subset p_J^{-1}(G_J^\infty)$$

for $m \geq \delta^{N-1}$.

In conclusion, we have the following theorem:

Theorem 3.1. *For any $J \subset \{0, \ldots, N\}$, and $k+1 = N$, we have*

$$\mathrm{Bs}(q_1^*\mathscr{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathscr{Y}_J}) \subset p_J^{-1}(G_J^\infty)$$

for any $m \geq \delta^{N-1}$.

Since $p_J^{-1}(G_J^\infty)$ may strictly contain the null locus $\mathrm{Null}(q_1^*\mathscr{L}|_{\mathscr{Y}_J}) = E_J$, Theorem 3.1 does not imply the Nakamaye Theorem needed in [3]. However, as was pointed out to us by Brotbek, this theorem is enough to obtain the effective bound appearing in Theorem 3.1 in [3]. Indeed, his proof is still valid under the weaker condition

$$\mathrm{Bs}(q_1^*\mathscr{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathscr{Y}_J}) \subset p_J^{-1}(G_J^\infty)$$

for any $m \geq M$ and any $J \subset \{0, \ldots, N\}$. By Theorem 3.1 we can take the constant M to be δ^{N-1}.

The above theorem can be generalized to the cases for products of Grassmannians. Now set $N = c(k+1)$, and denote by $\mathbf{G}_{k+1}(\delta_i) := \mathbf{G}_{k+1}\Big(|H^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_i)\big)|\Big)$ and $\mathbf{G} := \prod_{i=0}^{k+1} \mathbf{G}_{k+1}(\delta_i)$ for simplicity. Set $\mathscr{Y}$ to be the universal Grassmannian defined by

$$\mathscr{Y} := \{(\Delta_1, \ldots, \Delta_c, z) \in \mathbf{G} \times \mathbb{P}^N | \forall\ i, \Delta_i([z]) = 0\}.$$

Let $p_1 : \mathscr{Y} \to \mathbf{G}$, $p_2 : \mathscr{Y} \to \mathbb{P}^N$ and $q_i : \mathscr{Y} \to \mathbf{G}_{k+1}(\delta_i)$ be the canonical projections to each factor, then p_1 is a generically finite to one morphism. Set $\mathcal{L}$ to be the tautological ample line bundle

$$\mathcal{L} := \mathcal{O}_{\mathbf{G}_{k+1}(\delta_1)}(1) \boxtimes \ldots \boxtimes \mathcal{O}_{\mathbf{G}_{k+1}(\delta_c)}(1),$$

and

$$\mathcal{L}(\mathbf{a}) := \mathcal{O}_{\mathbf{G}_{k+1}(\delta_1)}(a_1) \boxtimes \ldots \boxtimes \mathcal{O}_{\mathbf{G}_{k+1}(\delta_c)}(a_c)$$

for any c-tuple of positive integers $\mathbf{a} = (a_1, \ldots, a_c)$.

We then define a smooth rational curve C_i in $\mathbf{G}$ of degree 1, which is given by

$$\begin{aligned}\Delta([t_0, t_1]) := \mathrm{Span}(z_1^{\delta_1}, z_{c+1}^{\delta_1} \ldots, z_{kc+1}^{\delta_1}) \times \mathrm{Span}(z_2^{\delta_2}, z_{c+2}^{\delta_2} \ldots, z_{kc+2}^{\delta_2}) \times \ldots \\ \times \mathrm{Span}(t_1 z_i^{\delta_i} + t_2 z_0^{\delta_i}, z_{c+i}^{\delta_i}, \ldots, z_{kc+i}^{\delta_i}) \times \ldots \\ \times \mathrm{Span}(z_c^{\delta_c}, z_{2c}^{\delta_c}, \ldots, z_{(k+1)c}^{\delta_c}).\end{aligned}$$

where $[t_0, t_1] \in \mathbb{P}^1$. It is easy to check that $\Delta([t_1, t_2])$ is an smooth embedding from $\mathbb{P}^1$ to $\mathbf{G}$, and satisfies the degree condition $\mathcal{L}(\mathbf{a}) \cdot C_i = a_i$ for each i. Choose a hyperplane D_i which is given by $\{[z_0, \ldots, z_N] | z_i + z_0 = 0\}$ in $\mathbb{P}^N$. Then we have

Lemma 3.2. *The intersection number of the curve $p_1^* C_i$ and the divisor $p_2^* D_i$ in $\mathcal{Y}$ is $b_i := \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$. Moreover, $p_{1*} p_2^* \mathcal{O}_{\mathbb{P}^N}(1) \equiv \mathscr{L}(\mathbf{b})$, where $\mathbf{b} = (b_1, \ldots, b_c)$.*

Proof. It is easy to show that $p_1^* C_i$ and $p_2^* D_i$ intersect only at one point with multiplicity b_i.

By projection formula we have

$$p_{1*} p_2^* D_i \cdot C_i = p_{1*}(p_2^* D_i \cdot p_1^* C_i) = b_i.$$

Since $\mathcal{L}(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}^c \xrightarrow{\sim} \mathrm{Pic}(\mathbf{G})$ is an isomorphism, then $p_{1*} p_2^* D_i \equiv p_{1*} p_2^* \mathcal{O}_{\mathbb{P}^N}(1) \equiv \mathcal{L}(\mathbf{b})$, due to the fact that $\mathcal{L} \cdot C_i = 1$. □

Thus the line bundle $p_1^* \mathcal{L}(\mathbf{b}) \otimes p_2^* \mathcal{O}_{\mathbb{P}^N}(-1)$ is effective, and by the similar arguments in the Section 3 we also have

$$\mathrm{Bs}(p_1^* \mathcal{L}(\mathbf{b}) \otimes p_2^* \mathcal{O}_{\mathbb{P}^N}(-1)) \subset G^\infty,$$

where G^∞ to be the set of points in $\mathbf{G}$ such that is fiber of p_1 is not a finite set. We run the same methods above to show that the same estimate also holds for all the stratams $\mathscr{Y}_I$ of $\mathscr{Y}$.

4 The Codimension of the Exceptional Locus

In this section we will give a better estimate for the codimension of the exceptional locus, appearing in Section 3 of [2]. First we state the following lemma due to Olivier Benoist [1].

Lemma 4.1. *Let $X \subset \mathbb{P}^N$ be a (irreducible) subvariety of dimension n. Let G be the set of all hypersurfaces of degree δ containing X. Then*

$$\mathrm{codim}(G, |\mathcal{O}_{\mathbb{P}^N}(\delta)|) \geq \binom{n+\delta}{\delta}.$$

For $k+1 \geq N$ and a $k+1$-tuple of positive integers $\boldsymbol{\delta} := (\delta_0, \ldots, \delta_k)$ with each $\delta_i \geq \delta$ for some δ, if we denote by $T_{\boldsymbol{\delta},k+1,N} := \prod_{i=0}^k |\mathcal{O}_{\mathbb{P}^N}(\delta_i)|$ and $T^\infty_{\boldsymbol{\delta},k+1,N}$ the subvariety of $T_{\boldsymbol{\delta},k+1,N}$ parametrizing equations that do not define a 0-dimensional scheme. Then we have

Theorem 4.1. *With the above notation,*

$$\operatorname{codim}(T^{\infty}_{\boldsymbol{\delta},k+1,N}, T_{\boldsymbol{\delta},k+1,N}) \geq \binom{N+\delta}{\delta}^{k} \cdot (\delta+1).$$

In particular, if N varies in $1,\dots,k+1$, then we have a rough uniform estimate

$$\operatorname{codim}\Big(T^{\infty}_{\boldsymbol{\delta},k+1,N}, T_{\boldsymbol{\delta},k+1,N}\Big) \geq (\delta+1)^{k+1}. \tag{4.1}$$

Proof. First we will assume that $k+1=N$. For any elements $(P_1,\dots,P_N) \in \prod_{i=1}^{N} |\mathcal{O}_{\mathbb{P}^N}(\delta_i)|$, if the subscheme defined by $P_1,\dots,P_N$ fails to be a 0-dimensional subscheme, there exist a unique $1 \leq i < N-1$ such that the subschemes defined by $P_1,\dots,P_i$ and $P_1,\dots,P_{i+1}$ are both of codimension i, and these elements form a subvariety of $T^{\infty}_{\boldsymbol{\delta},N,N}$ denoted by Z_i. Thus we have a decomposition $T^{\infty}_{\boldsymbol{\delta},N,N} = \cup_{i=1}^{N-1} Z_i$. If we denote by $p_i : \prod_{j=1}^{N} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)| \to \prod_{j=1}^{i} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)|$ and $q_i : \prod_{j=1}^{N} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)| \to \mathcal{O}_{\mathbb{P}^N}(\delta_i)$ the projection map respectively, then the image $p_i(Z_i)$ is Zariski dense in $\prod_{j=1}^{i} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)|$. We set U_i to te the Zariski open set in $\prod_{j=1}^{i} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)|$ parametrizing smooth irreducible varieties, and U_i is contained in $p_i(Z_i)$. Take any $\boldsymbol{P}_i := (P_1,\dots,P_i) \in U_i$, and we have

$$\operatorname{codim}\Big(q_{i+1}\big(Z_i \cap p_i^{-1}(\boldsymbol{P}_i)\big), |\mathcal{O}_{\mathbb{P}^N}(\delta_{i+1})|\Big) \geq \binom{\delta_{i+1}+N-i}{N-i}. \tag{4.2}$$

Indeed, if we denote by X_i the smooth variety of codimension i defined by $P_1,\dots,P_i$, then by the definition of Z_i, P_{i+1} must contain X_i. By Lemma 4.1, P_{i+1} must be contained in a subvariety of $|\mathcal{O}_{\mathbb{P}^N}(\delta_{i+1})|$ of codimension $\binom{\delta_{i+1}+N-i}{N-i}$. From (4.2) we conclude that, when p_i restricts to Z_i, the generic fiber has codimension more than $\binom{\delta_{i+1}+N-i}{N-i} \cdot \prod_{j=i+2}^{N} \binom{N+\delta_j}{N}$ in $\prod_{j=i+1}^{N} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)|$, and thus by (4.2) we have

$$\operatorname{codim}\Big(Z_i, \prod_{j=1}^{N} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)|\Big) \geq \binom{\delta_{i+1}+N-i}{N-i} \cdot \prod_{j\neq i+1} \binom{N+\delta_j}{N} \geq \binom{N+\delta}{\delta}^{N-1} \cdot \binom{\delta_{i+1}+N-i}{N-i},$$

and by $T^{\infty}_{\boldsymbol{\delta},N,N} = \cup_{i=1}^{N-1} Z_i$ we have

$$\operatorname{codim}\Big(T^{\infty}_{\boldsymbol{\delta},N,N}, T_{\boldsymbol{\delta},N,N}\Big) \geq \min_{1\leq i<N-1} \binom{N+\delta}{\delta}^{N-1} \cdot \binom{\delta_{i+1}+N-i}{N-i} \geq \binom{N+\delta}{\delta}^{N-1} \cdot (\delta+1),$$

here we use the assumptions that $\delta_i \geq \delta$ for any i.

Now we turn to the case $k+1 > N$. Set $p_N : T_{\boldsymbol{\delta},k+1,N} \to \prod_{j=0}^{N-1} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)| =: T_{\boldsymbol{\delta},N,N}$ be the projection map, with the fibers $\prod_{j=N}^{k} |\mathcal{O}_{\mathbb{P}^N}(\delta_j)|$. Then we can easily find that $p_N(T^{\infty}_{\boldsymbol{\delta},k+1,N}) \subset T^{\infty}_{\boldsymbol{\delta},N,N}$, and thus

$$\operatorname{codim}\Big(T^{\infty}_{\boldsymbol{\delta},k+1,N}, T_{\boldsymbol{\delta},k+1,N}\Big) \geq \binom{N+\delta}{\delta}^{k} \cdot (\delta+1).$$

□

5 Effectivity in the Kobayashi Conjecture

We still use the notations in [3]. Let X be a projective manifold of dimension n and L a very ample line bundle on X. First we will give a better estimate for δ. Set $\boldsymbol{\delta}$ to be a $k+1$-tuple of the form $(\delta,\dots,\delta)$. In Lemma 3.8 of [3], in order to avoid the exceptional locus, δ needs to satisfy the following condition:

$$\mathrm{codim}_{(\mathbb{C}^{\mathbb{I}_J})^{k+1}}\mathbb{V}^{\infty}_{2,J}=\mathrm{codim}\Big(T^{\infty}_{\boldsymbol{\delta},k+1,\#J-1},T_{\boldsymbol{\delta},k+1,\#J-1}\Big)>\dim(\hat{X}_k)=n+(n-1)k$$

for any $\mathbb{P}^J:=\{[z]\in\mathbb{P}^N|z_j=0 \text{ for any } j\notin J\}$ with $J\subset\{0,\dots,N\}$. By [3] we can fix $N=n,k=n-1$. We replace the estimates for $\mathrm{codim}_{(\mathbb{C}^{\mathbb{I}_J})^{k+1}}\mathbb{V}^{\infty}_{2,J}$ in (34) of [3] by (4.1) in Theorem 4.1, δ only needs to satisfy

$$(\delta+1)^n\geq(n-1)n,$$

and thus if $n\geq 3$, $\delta\geq 1$ is enough to verify this condition.

In order to control the indeterminacy, from Proposition 3.8 in [3] the condition for δ is

$$\dim\hat{X}_k+k<\#\mathbb{I}_x$$

for *generic* x in X. Since for generic x we have $\#\mathbb{I}_x=\binom{N+\delta}{\delta}$, then

$$(k+1)n<\binom{N+\delta}{\delta}.$$

Since we have fixed $N=n$ and $k=n-1$, then $\delta\geq 3$ when $n\geq 2$. In conclusion, if $n\geq 3$, any $\delta\geq 3$ is valid in controling the indeterminacy and avoiding the exceptional locus.

Let X_{n-1} be the $n-1$-th Demailly-Semple tower for X. By Theorem 1.1 we have

$$\mathfrak{w}_{\infty}(X_{n-1},L)=\mathfrak{w}(X_{n-1},L^{n-1}),$$

which implies that $m_{\infty}=n-1$. By Theorem 3.1 we can take $M:=\delta^{N-1}=3^{n-1}$, if we fix $\delta=3$. Thus by the definition in [3] we have

$$r(u,v,M,N,k,\epsilon,\delta)=3^{n-1}n\big(\epsilon+3(n-1)\big)+1.$$

Since $R:=\max\{r(u,v,M,N,k,\epsilon,\delta)|m_{\infty}\leq\epsilon<m_{\infty}+\delta\}$, thus we have

$$R=3^{n-1}n(4n-2)+1.$$

and the degree bound should be

$$d_0:=3^n n(4n-2)+4n+2.$$

Thus we prove the Main Theorem 1.

6 Effectivity for the Ampleness of the Complete Intersection

Let M be a projective manifold of dimension N and L a very ample line bundle on it. First we will assume that $N = 2c$. In order to obtain the effective estimate needed in [2], by arguments for the universal Grassmannian $\mathscr{Y}$ defined by $\prod_{i=1}^{c} \mathbf{G}_2(\delta_i)$ in Section 3, we replace $Q^{\otimes m(\delta)}$ in Formula $(*_2)$ in [2] by

$$\mathcal{L}(\mathbf{b}) := \mathcal{O}_{\mathbf{G}_2(\delta_1)}(b_1) \boxtimes \ldots \boxtimes \mathcal{O}_{\mathbf{G}_2(\delta_c)}(b_c),$$

where $b_i := \frac{\prod_{i=j}^{c} \delta_j^{2c}}{\delta_i}$, then we get the required condition in the Corollary 2.8 in [2]:

$$2\sum_{p=1}^{c}(\epsilon_p + \delta_p)b_p - r < 0. \tag{6.1}$$

If $2c > N$, we can not apply the results in [2] to get the effectivity directly. However, compared to their construction of the morphism from universal families of c complete intersections to the universal Grassmannian, we can fix $2c+1$ general sections $\zeta_0, \ldots, \zeta_{2c}$ in the linear system $|L|$, where L is a very ample line bundle, in place of their $N+1$ sections $\zeta_0, \ldots, \zeta_N$. Then the universal Grassmannian is replaced the following one

$$\mathcal{Y} := \{(\Delta_1, \ldots, \Delta_c, z) \in \mathbf{G}_2(\delta_1) \times \ldots \mathbf{G}_2(\delta_c) \times \mathbb{P}^{2c} | \forall\ i, \Delta_i([z]) = 0\},$$

which enables us to apply our methods to get the effectivity. Luckily, although we add more general sections, as the details of the proofs in [2] show, we still have the same estimates for $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_c)$ if we would like to also avoid the indeterminacy locus of the rational map and the exceptional locus which are the union of the positive dimensional fibers in the universal Grassmannian (i.e. determine the lower bound for $\boldsymbol{\delta}$ as that in Proposition 2.4 and Lemma 2.6 in [2]). Furthermore, in order to avoid the exceptional locus, $\boldsymbol{\delta}$ needs to satisfy

$$\dim\big(\mathbb{P}(\Omega_M)\big) < \operatorname{codim}(\mathbf{A}_I^{\infty}, \mathbf{A}_I) = \operatorname{codim}\Big(T^{\infty}_{\boldsymbol{\delta},2c,\#I-1}, T_{\boldsymbol{\delta},2c,\#I-1}\Big),$$

for any $\mathbb{P}^I := \{[z] \in \mathbb{P}^N | z_i = 0 \text{ for any } i \notin I\}$ with $I \subset \{0, \ldots, 2c\}$. If we use our estimates for codimension of the exceptional locus Theorem 4.1 instead, the condition for $\boldsymbol{\delta}$ is

$$N(N-1) < (\min\ \delta_j + 1)^{2c},$$

and thus $\delta_j \geq 1$. In conclusion, we have a refined results for the Corollary 2.8 in [2]:

Corollary 6.1. *On a N-dimensional smooth projective variety M, equipped with a very ample line bundle L, for any degrees $(d_1, \ldots, d_c) \in (\mathbb{N})^c$ satisfying*

$$\exists \boldsymbol{\delta}_{(\delta_p \geq 2)},\ \exists \boldsymbol{\epsilon}_{(\epsilon_p \geq 1)},\ \exists r > 2c\prod_{i=1}^{c}\delta_i^2 + 2\sum_{i=1}^{c}\frac{\epsilon_i \prod_{j=1}^{c}\delta_j^2}{\delta_i},\ \ s.t.$$
$$d_p = \delta_p(r+1) + \epsilon_p \qquad (p = 1, \ldots, c),$$

the complete intersection $X := H_1 \cap \ldots H_c$ of general hypersurfaces $H_1 \in |L^{d_1}|, \ldots, H_c \in |L^{d_c}|$ has ample cotangent bundle.

Since any subvariety of a variety with ample cotangent bundle also has ample cotangent bundle, we only need to deal with the case $c_0 := \llcorner \frac{N+1}{2} \lrcorner$. If we take $\delta_1 = \ldots = \delta_{c_0} = 2$, then any $d \geq 2c_0 4^{c_0+1} + 5$ can be decomposed as

$$d = 2(r+1) + \epsilon$$

with $1 \leq \epsilon \leq 2$ and $r > c_0 4^{c_0+1}$, satisfying the conditions in Corollary 6.1. In particular, if M is taken to be $\mathbb{P}^N$, then we can take L to be $\mathcal{O}_{\mathbb{P}^N}(1)$, and thus for all $d \geq 2c_0 4^{c_0+1} + 5$, the complete intersection of general hypersurfaces $H_1, \ldots, H_c$ with $H_1, \ldots, H_{c_0}$ in $\mathcal{O}_{\mathbb{P}^N}(d)$ and other H_i any degree have ample cotangent bundle, and we prove the Main Theorem 2.

Remark 6.1. *It is worth to mention that in [8] Songyan Xie first gave an effective bound N^{N^2} by using different methods.*

Acknowledgements I would like to warmly thank my thesis supervisor Professor Jean-Pierre Demailly for his very fruitful discussions and suggestions, and also for his patience and disponibility. The effective bounds in the work were improved substantially during the "Kähler geometry Workshop" held at Institut Fourier in May 2016, and I would like to also acknowledge the perfect working conditions provided by him, who was the only organizer. A thousand thanks to Damian Brotbek for his very careful reading of the manuscript, useful suggestions and comments, to Songyan Xie for explaining his work and ideas patiently, and also to Lionel Darondeau for his interests and remarks on the work.

References

[1] Benoist O, Le Théoreme de Bertini en Famille. Bulletin de la Société Mathématique de France, 2011, 139(4): 555-569.

[2] Brotbek D, Darondeau L, Complete intersection varieties with ample cotangent bundles. arXiv preprint arXiv:1511.04709.

[3] Brotbek D, On the hyperbolicity of general hypersurfaces, arXiv:1604.00311 [math.AG].

[4] Demailly J P. Algebraic criteria for Kobayashi hyperbolicity and jet differentials. Algebraic geometry (Santa Cruz 1995). Part, 2.

[5] Demailly J P. Proof of the Kobayashi conjecture on the hyperbolicity of very general hypersurfaces. arXiv preprint arXiv:1501.07625, 2015.

[6] Lazarsfeld R. Positivity in algebraic geometry II. Springer-Verlag. 2004.

[7] Siu Y T. Hyperbolicity of generic high-degree hypersurfaces in complex projective space. Inventiones mathematicae, 2015, 202(3): 1069-1166.

[8] Xie S Y. On the ampleness of the cotangent bundles of complete intersections. arXiv preprint arXiv:1510.06323.