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Effectiveness in Kobayashi Conjecture and Nakamaye Theorem for Universal Grassmannian

Ya Deng*

Institut Fourier, Université Grenoble Alpes

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Abstract

Our main goal in this paper is to give an effective bound for the degree of generic hypersurfaces in \mathbb{P}^n which are Kobayashi hyperbolic, based on the recent work of Brotbek [2].

1 Introduction

The famous Kobayashi conjecture stated that a general hypersurface in \mathbb{P}^n of sufficient large degree is hyperbolic. Recently there are three methods in studying this problem: Siu's slanted vector fields on higher order jet spaces [5], Demailly's strongly of general type for studying Green-Griffiths-Lang conjecture [4] and Brotbek's construction of families of hypersurfaces which are deformations of Fermat type [2]. In [2] Brotbek has given a formula for the degree bound d_0 , which depends on two constants $M(n)$ and m_∞ appearing in using noetherianity. In this paper we will determine these two constants and hence give the effective bound by applying Brotbek's formula directly. Since the problems are more or less independent, we will extract them from the work by Brotbek and suggest the readers who are interested in these connections to refer [2]. Nevertheless we will adopt the same notations in [2] for the reader's convenience.

For the constant $M(n)$ depending on n , it arises in the Nakamaye theorem: Let X be a Kähler manifold and L a big and nef line bundle on X . Then the null loci $\text{Null}(L)$ of L coincides with the augmented base locus $\mathbb{B}_+(L)$. Fix an ample divisor A on X , by noetherian property there exists a positive integer m_0 such that for $m \geq m_0$ we have $\text{Bs}(mL - A) = \mathbb{B}_+(L) = \text{Null}(L)$. Now we have the following effectiveness problem:

Problem 1.1. *Can we get an effective bound for m_0 which depends on some numerical constants related to L and A ?*

In general, this problem is hard to answer. Fortunately, in order to compute $M(n)$, we only need to consider the concrete projective manifold X which is a universal Grassmannian, and

*Email address: Ya.Deng@univ-grenoble-alpes.fr

L is the pull back of the tautological line bundle. Our first main result is Theorem 2.1, and we have the constant $M(n) = n^n(n+1)^n$. The method can also be used to study the effective problem for the ampleness of the cotangent bundles of general complete intersections (ref. [1]).

The constant m_∞ appears in the Wronskian ideal sheaf of Demailly-Semple tower. Let L be an very ample line bundle on the Kähler manifold X , and for any holomorphic sections $s_0, \dots, s_k \in H^0(X, L)$, we have the Wronskian for these sections

$$\omega(s_0, s_1, \dots, s_k) \in H^0(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi_{0,k}^*(L^{k+1})),$$

where X_k is the k -stage Demailly-Semple tower for the directed variety (X, T_X) . We set

$$\mathbb{W}(X_k, L) := \text{Span}\{\omega(s_0, \dots, s_n) | s_0, \dots, s_n \in H^0(X, L)\} \subset H^0(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi_{0,k}^*(L^{k+1})),$$

and $\mathfrak{w}(X_k, L)$ is denoted to be the base ideal of the linear system $\mathbb{W}(X_k, L)$. By Lemma 2.5 in [2] if L is very ample we have

$$\mathfrak{w}(X_k, L) \subset \mathfrak{w}(X_k, L^2) \subset \dots \subset \mathfrak{w}(X_k, L^m) \subset \dots$$

Then the noetherian property shows that this increasing sequence stabilizes for some $m_0 := m_\infty(X_k, L)$, and we have the following problem:

Problem 1.2. *Determine the constant $m_\infty(X_k, L)$.*

We have the following theorem which solves Problem 1.2:

Theorem 1.1. *If L separates k -jets at each point of X , then $m_\infty(X_k, L) = 1$. In particular, if L is known to be only very ample, we have $m_\infty(X_k, L) = k$.*

By Theorem 1.1 we have $m_\infty = n$ if $k = n$. Hence by the formula in [2] we have the following effective bound for degree of generic hypersurfaces in \mathbb{P}^n which are hyperbolic:

Main Theorem. *Let X be a projective manifold of dimension n and L is a very ample line bundle on X . Then a generic hypersurface in $|H^0(X, L^d)|$ with $d \geq n^{n+1}(n+1)^{n+2}(n^3 + 2n^2 + 2n - 1) + n^3 + 3n^2 + 3n$ is Kobayashi hyperbolic. In particular, a generic hypersurface in \mathbb{P}^n of degree $d \geq n^{n+1}(n+1)^{n+2}(n^3 + 2n^2 + 2n - 1) + n^3 + 3n^2 + 3n$ is Kobayashi hyperbolic.*

2 Effectiveness in the Nakamaye Theorem for Universal Grassmannian

We denote by $V := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta))$, which is the set of homogenous polynomials of degree δ in $\mathbb{C}[z_0, \dots, z_N]$, and for any $J \subset \{0, \dots, N\}$, set

$$\mathbb{P}_J := \{[z_0, \dots, z_N] \in \mathbb{P}^N | z_j = 0 \text{ if } j \in J\}.$$

Given any $\Delta \in \text{Gr}_{k+1}(V)$ and $[z] \in \mathbb{P}^N$ we write $\Delta([z]) = 0$ if $P(z) = 0$ for all $P \in \Delta \subset V$. We then define the family

$$\mathscr{B} := \{(\Delta, [z]) \in \text{Gr}_{k+1}(V) \times \mathbb{P}^N | \Delta([z]) = 0\},$$

and for any $J \subset \{0, \dots, N\}$, set

$$\mathcal{Y}_J := \mathcal{Y} \cap (\mathrm{Gr}_{k+1}(V) \times \mathbb{P}_J) \subset \mathcal{Y} \cap \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N.$$

Let $k+1 \geq N$, then $q_1 : \mathcal{Y} \rightarrow \mathrm{Gr}_{k+1}(V)$ is a generic finite to one morphism. We also define $q_2 : \mathcal{Y} \rightarrow \mathbb{P}^N$ to be the projection on the second factor. Let \mathcal{L} be the very ample line bundle on $\mathrm{Gr}_{k+1}(V)$ which is the pull back of $\mathcal{O}(1)$ under the Plücker embedding. Then $q_1^* \mathcal{L}|_{\mathcal{Y}_J}$ is a big and nef line bundle on \mathcal{Y}_J for any J . For any $J \subset \{0, \dots, N\}$ we denote by $p_J : \mathcal{Y}_J \rightarrow \mathrm{Gr}_{k+1}(V)$, and $\hat{p}_J : \mathcal{Y}_J \rightarrow \mathbb{P}_J$ the second projection. Similarly we set

$$E_J := \{y \in \mathcal{Y} \mid \dim_y(p_J^{-1}(p_J(y))) > 0\}$$

$$G_J^\infty := p_J(E_J) \subset \mathrm{Gr}_{k+1}(V),$$

then $E_J = \mathrm{Null}(q_1^* \mathcal{L}|_{\mathcal{Y}_J})$. For $J = \emptyset$ we have $\mathcal{Y}_J = \mathcal{Y}$ and denote by $E := E_\emptyset$ and $G^\infty := G_\emptyset^\infty$. By the Theorem of Nakamaye we have

$$\mathrm{Null}(q_1^* \mathcal{L}|_{\mathcal{Y}_J}) = \mathbb{B}_+(q_1^* \mathcal{L}|_{\mathcal{Y}_J}).$$

By noetherianity there exists $m_J \in \mathbb{N}$ such that

$$E_J = \mathbb{B}_+(q_1^* \mathcal{L}|_{\mathcal{Y}_J}) = \mathrm{Bs}(q_1^* \mathcal{L}^{m_J} \otimes q_2^* \mathcal{O}_{\mathbb{P}_J}(-1)|_{\mathcal{Y}_J})$$

for every $m \geq m_J$. Our goal is to get an effective estimate of m_J . From now on we assume that $k+1 = N$ (since this is enough in Brotbek's work), and thus $q_1 : \mathcal{Y} \rightarrow \mathrm{Gr}_{k+1}(V)$ is an generic finite to one surjective morphism. First we will work on $J = \{0, \dots, N\}$.

We first find a smooth rational curve C in $\mathrm{Gr}_{k+1}(V)$ of degree 1, which is given by

$$\Delta([t_0, t_1]) := \mathrm{Span}(z_1^\delta, z_2^\delta, \dots, z_{N-1}^\delta, t_0 z_N^\delta + t_1 z_0^\delta),$$

where $[t_0, t_1] \in \mathbb{P}^1$. We will show that $\Delta([t_0, t_1]) : \mathbb{P}^1 \rightarrow \mathrm{Gr}_{k+1}(V)$ is a smooth embedding from \mathbb{P}^1 to $\mathrm{Gr}_{k+1}(V)$, and satisfies the degree condition $\mathcal{L} \cdot C = 1$.

Set $\mathbb{I} := \{I = (i_0, \dots, i_N) \mid |I| = \delta\}$. Then $\{z^I\}_{I \in \mathbb{I}}$ is a basis for V , and $w_{(I_0, \dots, I_k)}$ with $I_0 < I_1 < \dots < I_k$ is a basis for $\Lambda^{k+1}V$, where $I < J$ is defined by lexicographical order in \mathbb{I} . Now we set $J_i \in \mathbb{I}$ to be $z^{J_i} := z_i^\delta$ for $0 \leq i \leq k$. Then the curve \tilde{C} defined in $\mathbb{P}(\Lambda^{k+1}V)$ given by the equations $w_{(I_0, \dots, I_k)} = 0$ for all $(I_0, \dots, I_k) \neq (J_1, \dots, J_N)$ and (J_0, \dots, J_{N-1}) , is of degree 1 with respect to \mathcal{L} . \tilde{C} is totally contained in the image of the Plücker embedding of $\mathrm{Gr}_{k+1}(V)$, whose inverse image is C .

Now choose a hyperplane D which is given by $\{[z_0, \dots, z_N] \mid z_0 + z_N = 0\}$ in \mathbb{P}^N . Then we have

Lemma 2.1. *The intersection number of the curve $q_1^* C$ and the divisor $q_2^* D$ in \mathcal{Y} is δ^{N-1} . Moreover, $q_{1*} q_2^* D \equiv \delta^{N-1} \mathcal{L}$, where “ \equiv ” stands for linear equivalent.*

Proof. An easy computation shows that $q_1^* C$ and $q_2^* D$ intersect only at the point $\mathrm{Span}(z_1^\delta, z_2^\delta, \dots, z_{N-1}^\delta, z_N^\delta + (-1)^{\delta+1} z_0^\delta) \times [1, 0, \dots, 0, -1] \in \mathcal{Y}$ with multiplicity δ^{N-1} . The first statement follows.

By projection formula we have

$$q_{1*} q_2^* D \cdot C = q_{1*} (q_2^* D \cdot q_1^* C) = \delta^{N-1}.$$

Since $\mathrm{Pic}(\mathrm{Gr}_{k+1}(V)) \approx \mathbb{Z}$ with the generator \mathcal{L} , then $q_{1*} q_2^* D \equiv \delta^{N-1} \mathcal{L}$ by the factor that $\mathcal{L} \cdot C = 1$. \square

Since $q_1^*q_{1*}q_2^*D - q_2^*D$ is an effective divisor of \mathcal{Y} , from Lemma 2.1 we know that $q_1^*\mathcal{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)$ is effective. Moreover, we have

$$\text{Bs}(q_1^*\mathcal{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) = E = \mathbb{B}_+(q_1^*\mathcal{L}).$$

Indeed, for any $y_0 \notin E$, $q_1^{-1}(q_1(y_0))$ is a finite set, and one can choose a hyperplane $D' \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ such that $D' \cap q_2(q_1^{-1}(q_1(y_0))) = \emptyset$. From the result above we know that the divisor $q_1^*q_{1*}q_2^*D' - q_2^*D'$ is effective and lies in the linear system $|q_1^*\mathcal{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|$ of \mathcal{Y} . Now we show that $y_0 \notin \text{Supp}(q_1^*q_{1*}q_2^*D' - q_2^*D')$. Indeed, for any $\Delta \in \text{Gr}_{k+1}(V)$, if we denote by $\text{Int}(\Delta) := \{[z] \in \mathbb{P}^N \mid \Delta([z]) = 0\}$, then $q_2(q_1^{-1}(q_1(y_0))) = \text{Int}(\Delta_0)$, where $\Delta_0 := q_1(y_0)$, hence the condition $D' \cap q_2(q_1^{-1}(q_1(y_0))) = \emptyset$ is equivalent to that $\text{Int}(\Delta_0) \cap D' = \emptyset$. However, for any $\Delta \in q_1(q_2^{-1}(D'))$, we must have $\text{Int}(\Delta) \cap D' \neq \emptyset$, therefore $\Delta_0 \notin q_1(q_2^{-1}(D'))$ and thus $y_0 \notin q_1^{-1}(q_1(q_2^{-1}(D')))$ and thus $y_0 \notin \text{Supp}(q_1^*q_{1*}q_2^*D' - q_2^*D')$.

Thus from the arbitrariness of y_0 we have

$$\text{Bs}(q_1^*\mathcal{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) \subset E.$$

Since \mathcal{L} is very ample on $\text{Gr}_{k+1}(V)$, we see that for any $m \geq \delta^{N-1}$,

$$\text{Bs}(q_1^*\mathcal{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) = \mathbb{B}_+(q_1^*\mathcal{L}).$$

Now we work on general $J \subset \{0, \dots, N\}$. First recall that $p_J : \mathcal{Y}_J \rightarrow \text{Gr}_{k+1}(V)$, and $\hat{p}_J : \mathcal{Y}_J \rightarrow \mathbb{P}_J$ the second projection. The method is repeating the arguments above. For any $y_0 \notin E_J$, the set $p_J^{-1}(p_J(y_0))$ is a finite one, and thus one can choose a generic hyperplane $D \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ such that $D \cap \mathbb{P}_J \cap \hat{p}_J^{-1}(p_J(y_0)) = \emptyset$, which means that $\text{Int}(\Delta_0) \cap D \cap \mathbb{P}_J = \emptyset$, where $\Delta_0 := p_J(y_0)$. However, for any $\Delta \in p_J(\hat{p}_J^{-1}(D \cap \mathbb{P}_J))$ we must have $\text{Int}(\Delta) \cap D \cap \mathbb{P}_J \neq \emptyset$, and thus $\Delta_0 \notin p_J(\hat{p}_J^{-1}(D \cap \mathbb{P}_J))$, afortiori $y_0 \notin p_J^{-1}(p_J(\hat{p}_J^{-1}(D \cap \mathbb{P}_J))) = q_1^{-1}(q_1(q_2^{-1}(D'))) \cap \mathcal{Y}_J$. Thus the restriction of the effective divisor $q_1^*q_{1*}q_2^*D - q_2^*D$ to \mathcal{Y}_J is well-defined and $y_0 \notin q_1^*q_{1*}q_2^*D - q_2^*D|_{\mathcal{Y}_J}$. From the arbitrariness of y_0 we know that the base locus of the linear system $|q_1^*\mathcal{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}_J}$ is contained in E_J . Thus we have

$$\text{Bs}(q_1^*\mathcal{L}^{\delta^{N-1}} \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}_J}) = E_J.$$

In conclusion, we have the following theorem:

Theorem 2.1. *For any $J \subset \{0, \dots, N\}$, and $k+1 = N$, we have*

$$\text{Bs}(q_1^*\mathcal{L}^m \otimes q_2^*\mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}_J}) = \mathbb{B}_+(q_1^*\mathcal{L}|_{\mathcal{Y}_J})$$

for any $m \geq \delta^{N-1}$.

From [2] we can set $\delta = n(n+1)$ and $N = n+1$, then $M(n) := \delta^{N-1} = n^n(n+1)^n$.

3 Effectiveness For the Complete Intersection

Pursing the methods in 2 we can give an effective estimate for the ampleness of the cotangent bundles of general complete intersections, based on the results in [1]. This section is an independent part and the readers can skip if they are only interested in the proof of the Main Theorem.

We denote by $\mathbf{G}_2(\delta) := \text{Gr}_2(V_\delta)$, where $V_\delta := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta))$, and $\mathbf{G} := \mathbf{G}_2(\delta_1) \times \dots \times \mathbf{G}_2(\delta_c)$. We set

$$\mathcal{Y} := \{(\Delta_1, \dots, \Delta_c, z) \in \mathbf{G} \times \mathbb{P}^N \mid \forall i, \Delta_i([z]) = 0\}.$$

For simplicity we will assume that N is even and $c = \frac{N}{2}$. Let $p_1 : \mathcal{Y} \rightarrow \mathbf{G}$, $p_2 : \mathcal{Y} \rightarrow \mathbb{P}^N$ and $q_i : \mathcal{Y} \rightarrow \mathbf{G}_2(\delta_i)$ be the canonical projections to each factor. We denote by \mathcal{L} the tautological ample line bundle

$$\mathcal{L} := \mathcal{O}_{\mathbf{G}_2(\delta_1)}(1) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbf{G}_2(\delta_c)}(1),$$

and

$$\mathcal{L}(\mathbf{a}) := \mathcal{O}_{\mathbf{G}_2(\delta_1)}(a_1) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbf{G}_2(\delta_c)}(a_c)$$

for any c -tuple of positive integers $\mathbf{a} = (a_1, \dots, a_c)$.

We then define a smooth rational curve C_i in \mathbf{G} of degree 1, which is given by

$$\begin{aligned} \Delta([t_0, t_1]) &:= \text{Span}(z_1^{\delta_1}, z_2^{\delta_1}) \times \text{Span}(z_3^{\delta_1}, z_4^{\delta_1}) \times \dots \\ &\times \text{Span}(z_{2i-1}^{\delta_i}, t_1 z_{2i}^{\delta_i} + t_2 z_0^{\delta_i}) \times \text{Span}(z_{2i+1}^{\delta_{i+1}}, z_{2i+2}^{\delta_{i+1}}) \times \dots \\ &\times \text{Span}(z_{2c-1}^{\delta_c}, z_{2c}^{\delta_c}). \end{aligned}$$

where $[t_0, t_1] \in \mathbb{P}^1$. It is easy to check that $\Delta[t_1, t_2]$ is a smooth embedding from \mathbb{P}^1 to \mathbf{G} , and satisfies the degree condition $\mathcal{L}(\mathbf{a}) \cdot C_i = a_i$ for each i . Choose a hyperplane D_i which is given by $\{[z_0, \dots, z_N] \mid z_{2i} + z_0 = 0\}$ in \mathbb{P}^N . Then we have

Lemma 3.1. *The intersection number of the curve $p_1^*C_i$ and the divisor $p_2^*D_i$ in \mathcal{Y} is $b_i := \frac{\prod_{j=1}^c \delta_j^2}{\delta_i}$. Moreover, $p_{1*}p_2^*\mathcal{O}_{\mathbb{P}^N}(1) \equiv \mathcal{L}(\mathbf{b})$, where $\mathbf{b} = (b_1, \dots, b_c)$.*

Proof. It is easy to show that $p_1^*C_i$ and $p_2^*D_i$ intersect only at one point with multiplicity b_i .

By projection formula we have

$$p_{1*}p_2^*D_i \cdot C_i = p_{1*}(p_2^*D_i \cdot p_1^*C_i) = b_i.$$

Since $\mathcal{L}(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}^c \xrightarrow{\sim} \text{Pic}(\mathbf{G})$ is an isomorphism, then $p_{1*}p_2^*D_i \equiv p_{1*}p_2^*\mathcal{O}_{\mathbb{P}^N}(1) \equiv \mathcal{L}(\mathbf{b})$, due to the fact that $\mathcal{L} \cdot C_i = 1$. \square

Thus the line bundle $\mathcal{L}(\mathbf{b}) \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(-1)$ is effective, and the similar arguments in the section above we also have

$$\text{Bs}(\mathcal{L}(\mathbf{b}) \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(-1)) = \mathbb{B}_+(\mathcal{L}).$$

Hence we can get the effective estimate needed in [1].

4 Wronskian Ideal Sheaf in the Demailly-Semple Tower

We denote by (X_k, V_k) the Demailly-Semple tower for (X, T_X) , where X is a complex manifold, and $D_j := P(T_{X_{j-1}/X_{j-2}})$ (by an abuse of notation, in our context D_j is the divisor in X_k given by $\pi_{j,k}^{-1}(D_j)$). We would like to build the following parametrizing theorem.

Theorem 4.1. *Let U be an open set with coordinates (z_1, \dots, z_n) such that $\pi_{0,n}^{-1}(U)$ is a trivial product of $U \times \mathcal{R}_{n,k}$, where $\mathcal{R}_{n,k}$ is the universal rational homogeneous variety $\mathbb{C}^{nk}/\mathbb{G}_n$ [3]. If $w \in D_j \cap \pi_{0,n}^{-1}(U)$ such that $w \notin D_i$ for any other $i \in \{2, \dots, k\}$. Then there exists an open set $W \subset U \times \mathbb{C}^{(n-1)k-1}$ and a holomorphic family of germs of holomorphic curves $f(t, \mathbf{a})$ with $\mathbf{a} \in W$, such that the lift of these curves to the k -th Demailly-Semple tower $f_{[k]}(0, \mathbf{a})$ gives a biholomorphism between W and an open set in $D_j \setminus \{\cup_{i \neq j} D_i\}$ containing w , and $f_{[k]}(t, \mathbf{a})$ is also a local biholomorphism to some open set in $\pi_{0,k}^{-1}(U)$, with $\{t = 0\}$ defining the divisor D_j locally. The multiplicity of $f(t, \mathbf{a})$ is always 2 at $t = 0$.*

Proof. Since $X_1 = P(T_X)$, let $U_1 \subset \pi_{0,1}(U)$ be the affine coordinate $U \times \mathbb{C}^{n-1}$ such that for $(z_1, \dots, z_n, t_2, \dots, t_n) \in U \times \mathbb{C}^{n-1}$, it will correspond to the point $(z_1, \dots, z_n, [\frac{\partial}{\partial z_1} + t_2 \frac{\partial}{\partial z_2} + \dots + t_n \frac{\partial}{\partial z_n}])$ in $P(T_X)$. We denote by $V_1 := \pi_{1,k}^{-1}(U_1)$.

To make the theorem look more intuitive, we first prove the theorem for $j = k$. For any $w \in D_k$ such that $w \notin D_j$ for any $j \in \{2, \dots, k-1\}$, we can assume that $w \in V_1$. Then the following germs of holomorphic curves

$$f(t, \mathbf{a}) := \mathbf{a}_0 + \mathbf{a}_1 t^2 + \mathbf{a}_2 t^4 + \dots + \mathbf{a}_{k-1} t^{2k-2} + \mathbf{a}_k t^{2k-1}, \quad (4.1)$$

where \mathbf{a}_i are n -tuple in \mathbb{C}^n satisfying that $\mathbf{a}_2^{(1)} = \dots = \mathbf{a}_k^{(1)} = 0$, $\mathbf{a}_1^{(1)} = 1$ and $\mathbf{a}_k \neq (0, \dots, 0)$, whose lift to the k -th stage of Demailly-Semple tower $f_{[k]}(0, \mathbf{a})$ represents all points in $V_1 \cap D_k \setminus \{\cup_{2 \leq j < k} D_j\}$ when \mathbf{a}_0 varies in U and $\mathbf{a}_1, \dots, \mathbf{a}_k$ varies in $\mathbb{C}^{(n-1)k}$. Without loss of generality we assume that for the curve $f(t, \mathbf{b})$ such that $f_{[k]}(0, \mathbf{b}) = w$, $\mathbf{b}_k^{(2)} \neq 0$. Then we can take an affine coordinate $U \times \mathbb{C}^{(n-1)k}$ for $\pi_{0,k}^{-1}(U)$ containing the point w such that $f_{[k]}(0, \mathbf{a})$ corresponds to

$$(\mathbf{a}_0, \mathbf{a}_1^{(2)}, \dots, \mathbf{a}_1^{(n)}, \dots, \mathbf{a}_{k-1}^{(2)}, \dots, \mathbf{a}_{k-1}^{(n)}, 0, \frac{\mathbf{a}_k^{(3)}}{\mathbf{a}_k^{(2)}}, \dots, \frac{\mathbf{a}_k^{(n)}}{\mathbf{a}_k^{(2)}})$$

in this affine coordinate, where the coordinate 0 corresponds to the divisor D_k . Thus if we normalize $\mathbf{a}_k^{(2)} = 1$ in addition, the map

$$f_{[k]}(0, \mathbf{a}) : (\mathbf{a}_0, \mathbf{a}_1^{(2)}, \dots, \mathbf{a}_1^{(n)}, \mathbf{a}_2^{(2)}, \dots, \mathbf{a}_{k-1}^{(n)}, \mathbf{a}_k^{(3)}, \dots, \mathbf{a}_k^{(n)}) \in U \times \mathbb{C}^{(n-1)k-1} \rightarrow V_{12} \cap D_k \setminus \{\cup_{2 \leq j < k} D_j\}$$

is a biholomorphism. Here V_{12} is the image of the affine coordinate $U \times \mathbb{C}^{(n-1)k}$ above in $\pi_{0,k}^{-1}(U)$.

Since $f'_{[k]}(0, \mathbf{b}) \neq 0$, then $f_{[k]}(t, \mathbf{a})$ is a local biholomorphism to V_1 and $\{t = 0\}$ defines the divisor D_k locally.

In general, for any $w \in D_j$ such that $w \notin D_i$ for any other D_i , if we assume that $w \in V_1$ the following germs of holomorphic curves

$$f(t, \mathbf{a}) := \mathbf{a}_0 + \mathbf{a}_1 t^2 + \mathbf{a}_2 t^4 + \dots + \mathbf{a}_{j-1} t^{2j-2} + \mathbf{a}_j t^{2j-1} + \mathbf{a}_{j+1} t^{2j} + \dots + \mathbf{a}_k t^{k+j-1}, \quad (4.2)$$

with $\mathbf{a}_2^{(1)} = \dots = \mathbf{a}_k^{(1)} = 0$, $\mathbf{a}_1^{(1)} = 1$ and $\mathbf{a}_j \neq (0, \dots, 0)$, whose lift to the k -th stage of Demailly-Semple tower $f_{[k]}(0, \mathbf{a})$ represents all points in $V_1 \cap D_j \setminus \{\cup_{i \neq j} D_i\}$ when \mathbf{a}_0 varies in U and $\mathbf{a}_1, \dots, \mathbf{a}_k$ varies in $\mathbb{C}^{(n-1)k}$. Without loss of generality we also assume that for the curve $f(t, \mathbf{b})$ such that $f_{[k]}(0, \mathbf{b}) = w$, $\mathbf{b}_j^{(2)} \neq 0$. Then we can take an affine coordinate $U \times \mathbb{C}^{(n-1)k}$ for $\pi_{0,k}^{-1}(U)$ containing the point w such that $f_{[k]}(0, \mathbf{a})$ corresponds to

$$(\mathbf{a}_0, \mathbf{a}_1^{(2)}, \dots, \mathbf{a}_1^{(n)}, \dots, \mathbf{a}_{j-1}^{(2)}, \dots, \mathbf{a}_{j-1}^{(n)}, 0, \frac{\mathbf{a}_j^{(3)}}{\mathbf{a}_j^{(2)}}, \dots, \frac{\mathbf{a}_j^{(n)}}{\mathbf{a}_j^{(2)}}, P_{j+1}^{(2)}(\mathbf{a}), \dots, P_{j+1}^{(n)}(\mathbf{a}), \dots, P_k^{(2)}(\mathbf{a}), \dots, P_k^{(n)}(\mathbf{a}))$$

here we have

$$\begin{cases} P_i^{(2)} = (\mathbf{a}_j^{(2)})^{i-j} \mathbf{a}_{i-1}^{(2)} + Q_i(\mathbf{a}_j^{(2)}, \dots, \mathbf{a}_{i-2}^{(2)}) \\ P_i^{(l)} = (\mathbf{a}_j^{(2)})^{i-j-1} \mathbf{a}_i^{(l)} + S_i^{(l)}(\mathbf{a}_j^{(2)}, \dots, \mathbf{a}_i^{(2)}, \mathbf{a}_j^{(l)}, \dots, \mathbf{a}_{l-1}^{(l)}) \text{ if } 2 < l \leq k \end{cases}$$

where Q_i and $S_i^{(l)}$ are both polynomials. Thus we can take $\mathbf{a}_k^{(2)} = 0$ in addition such that $f_{[k]}(0, \mathbf{a})$ is a local biholomorphism to $D_j \setminus \{\cup_{i \neq j} D_i\}$, and the theorem holds for general j . \square

Remark 4.1. *The existence of the local biholomorphism induced by $f_{[k]}(0, \mathbf{a})$ above can be proved directly by implicit function theorem.*

Now we study the Wronskian ideal defined by $W(f_0, \dots, f_k)$, with $f_0, \dots, f_k \in \mathcal{O}(U)$. By the definition of the Wronskian, $W(f_0, \dots, f_k) \in H^0(\pi_{0,k}^{-1}(U), \mathcal{O}_{X_k}(m))$, where $m := \frac{k(k+1)}{2}$. For any holomorphic curve $\nu(t)$ which lies on U , its lift to the $k-1$ -th Demailly-Semple tower $\nu_{[k-1]}(t)$ induces a section $\nu'_{[k-1]}(t)$ of the line bundle $\nu_{[k]}^* \mathcal{O}_{X_k}(-1)$, and thus

$$W(f_0, \dots, f_k)(\nu'_{[k-1]}(t)^m) = \begin{vmatrix} f_0 \circ \nu(t) & \dots & f_k \circ \nu(t) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \nu(t)}{dt^k} & \dots & \frac{d^k f_n \circ \nu(t)}{dt^k} \end{vmatrix}.$$

Recall that $V_1 := \pi_{1,k}^{-1}(U_1)$, where $U_1 \subset \pi_{0,1}(U)$ be the affine coordinate $U \times \mathbb{C}^{n-1}$ corresponds to the points $(z_1, \dots, z_n, [\frac{\partial}{\partial z_1} + t_2 \frac{\partial}{\partial z_2} + \dots + t_n \frac{\partial}{\partial z_n}])$ in $P(T_X)$. We have the following result:

Theorem 4.2. *When $W(1, z_1, z_1^2, \dots, z_1^k)$ restrict to V_1 we have*

$$\text{Div}(W(1, z_1, z_1^2, \dots, z_1^k)) = \sum_{j=2}^k \frac{k(k+1)}{2} D_j.$$

Proof. Choose any $w \in P_k V^{\text{reg}} \cap V_1$ with $\pi_{0,k}(w) = p$, where p is the origin in the local coordinate (z_1, \dots, z_n) , then we have a a germ of holomorphic curve

$$\nu(t, \mathbf{a}) := \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \dots + \mathbf{a}_k t^k, \quad (4.3)$$

whose lift to k -th stage $\nu_{[k]}(0, \mathbf{a})$ is equal to w . By the definition of V_1 , $\mathbf{a}_1^{(1)} \neq 0$. Then

$$W(1, z_1, z_1^2, \dots, z_1^k)(\nu'_{[k-1]}(0, \mathbf{a})^m) = C(\mathbf{a}_1^{(1)})^k \neq 0,$$

where C is some non zero constant; hence $W(1, z_1, z_1^2, \dots, z_1^k)|_w \neq 0$, and we conclude that its zero locus lies on $\cup_{2 \leq j \leq k} D_j$.

Choose any $w_j \in V_1 \cap D_j \setminus \{\cup_{i \neq j} D_i\}$ still with $\pi_{0,k}(w) = p$, then by Theorem 4.1 there exists an open set $W \subset U \times \mathbb{C}^{(n-1)k-1}$ and a holomorphic family of germs of holomorphic curves $\nu(t, \mathbf{a})$ with $\mathbf{a} \in W$

$$\nu(t, \mathbf{a}) := \mathbf{a}_0 + \mathbf{a}_1 t^2 + \mathbf{a}_2 t^4 + \dots + \mathbf{a}_{j-1} t^{2j-2} + \mathbf{a}_j t^{2j-1} + \mathbf{a}_{j+1} t^{2j} + \dots + \mathbf{a}_k t^{k+j-1}, \quad (4.4)$$

such that $\nu_{[k]}(t, \mathbf{a})$ is a local biholomorphism to V_1 , with $\{t = 0\}$ defining the divisor D_j locally, and the multiplicity of $\nu(t, \mathbf{a})$ is always 2 at $t = 0$. Let $\mathbf{b} \in W$ such that $\nu_{[k]}(0, \mathbf{b}) = w$, and by $\pi_{0,k}(w) = p$ we have $\mathbf{b}_0 = (0, \dots, 0)$. Since $\mathbf{a}_1^{(1)} \neq 0$ for any $a \in W$, we have

$$W(1, z_1, z_1^2, \dots, z_1^k)(\nu'_{[k-1]}(t, \mathbf{a})^m) = ((\mathbf{a}_1^{(1)})^k + P(\mathbf{a}))t^{\frac{k(k+1)}{2}} + o(t^{\frac{k(k+1)}{2}}),$$

where $P(\mathbf{a})$ is the polynomial of \mathbf{a} with $P(\mathbf{b}) = 0$. Thus $\text{Ord}_{D_j}(W(1, z_1, z_1^2, \dots, z_1^k)) = \frac{k(k+1)}{2}$. Observe that $X_k = P_k V^{\text{reg}} \cup D_2 \cup \dots \cup D_k$, which concludes the proof. \square

Theorem 4.2 shows that the invertible sheaf $\mathcal{O}(\sum_{j=2}^k (-\frac{k(k+1)}{2} D_j))$ is contained in \mathfrak{w}_∞ . Now we will show that \mathfrak{w}_∞ is also invertible outside the codimension 2 variety $\cup_{2 \leq j < i \leq k} D_j \cap D_i$.

For any point $w \in D_j \setminus \cup_{i \neq j} D_i$, without loss of generality we can assume that $w \in V_1$ and $p := \pi_{0,k}(w)$ corresponds to the origin in the local coordinate (z, \dots, z_n) , then by Theorem 4.1 there exists an open set $W \subset U \times \mathbb{C}^{(n-1)k-1}$ and a holomorphic family of germs of holomorphic curves $\nu(t, \mathbf{a})$ with $\mathbf{a} \in W$

$$\nu(t, \mathbf{a}) := \mathbf{a}_0 + \mathbf{a}_1 t^2 + \mathbf{a}_2 t^4 + \dots + \mathbf{a}_{j-1} t^{2j-2} + \mathbf{a}_j t^{2j-1} + \mathbf{a}_{j+1} t^{2j} + \dots + \mathbf{a}_k t^{k+j-1}, \quad (4.5)$$

with $\mathbf{a}_2^{(1)} = \dots = \mathbf{a}_k^{(1)} = \mathbf{a}_k^{(2)} = 0$, $\mathbf{a}_1^{(1)} = 1$ and $\mathbf{a}_j^{(2)} \neq 0$, such that $\nu_{[k]}(t, \mathbf{a})$ is a local biholomorphism to V_1 , with $\{t = 0\}$ defining the divisor D_j locally. Let $\mathbf{b} \in W$ such that $\nu_{[k]}(0, \mathbf{b}) = w$, then $\mathbf{b}_0 = 0$.

If we define the following Wronskian

$$W_j := \begin{cases} W(1, z_1, z_1^2, \dots, z_1^{\frac{k+j-1}{2}}, z_2, z_2 z_1, \dots, z_2 z_1^{\frac{k-j}{2}}), & \text{if } k-j \text{ is even;} \\ W(1, z_1, z_1^2, \dots, z_1^{\frac{k+j-1}{2}}, z_2, z_2 z_1, \dots, z_2 z_1^{\frac{k-j-1}{2}}), & \text{if } k-j \text{ is odd.} \end{cases} \quad (4.6)$$

We find that in both cases we have

$$W_j(\nu'_{[k-1]}(t, \mathbf{a})^m) = C(\mathbf{a}_j^{(2)})^{\lceil \frac{k-j+1}{2} \rceil} \cdot t^{\frac{k(k+1)}{2} - \frac{(k-j)^2}{2} - \frac{3}{2}(k-j)-1} + o(t^{\frac{k(k+1)}{2} - \frac{(k-j)^2}{2} - \frac{3}{2}(k-j)-1}),$$

where C is some positive constant. Thus at each $w \in D_j \setminus \cup_{i \neq j} D_i$ the inclusion holds

$$\mathcal{O}_{X_k, w}(-(\frac{k(k+1)}{2} - \frac{(k-j)^2}{2} - \frac{3}{2}(k-j) - 1)D_j) \subset \mathfrak{w}_{\infty, w}.$$

Indeed, by the Puiseux expansion method (ref. [3]) we can prove that $\mathfrak{w}_{\infty, w} = \mathcal{O}_{X_k, w}(-(\frac{k(k+1)}{2} - \frac{(k-j)^2}{2} - \frac{3}{2}(k-j) - 1)D_j)$ at these singular points.

We put $\tau = t^2$, and the curve $t \rightarrow \nu(t, \mathbf{a})$ becomes a Puiseux expansion

$$\mu(\tau, \mathbf{a}) := \mathbf{a}_0 + \mathbf{a}_1\tau + \mathbf{a}_2\tau^2 + \dots + \mathbf{a}_{j-1}\tau^{j-1} + \mathbf{a}_j\tau^{j-\frac{1}{2}} + \dots + \mathbf{a}_k\tau^{\frac{k+j-1}{2}}. \quad (4.7)$$

Hence the derivative $\mu^{(i)}(\tau, \mathbf{a})$ may involve negative powers of τ only when $i \geq j$, and the exponent is always $\geq j - \frac{1}{2} - i$. Take any $f_0, f_1, \dots, f_k \in \mathcal{O}_{X,p}$, since the Wronskian is an invariant jet differential, from the relation of invariant jet differential we have

$$\begin{vmatrix} f_0 \circ \nu(t, \mathbf{a}) & \dots & f_k \circ \nu(t, \mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \nu(t, \mathbf{a})}{dt^k} & \dots & \frac{d^k f_n \circ \nu(t, \mathbf{a})}{dt^k} \end{vmatrix} = t^{\frac{k(k+1)}{2}} \cdot \begin{vmatrix} f_0 \circ \mu(\tau, \mathbf{a}) & \dots & f_k \circ \mu(\tau, \mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \mu(\tau, \mathbf{a})}{d\tau^k} & \dots & \frac{d^k f_n \circ \mu(\tau, \mathbf{a})}{d\tau^k} \end{vmatrix}.$$

Since the multiplicity of

$$\begin{vmatrix} f_0 \circ \mu(\tau, \mathbf{a}) & \dots & f_k \circ \mu(\tau, \mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \mu(\tau, \mathbf{a})}{d\tau^k} & \dots & \frac{d^k f_n \circ \mu(\tau, \mathbf{a})}{d\tau^k} \end{vmatrix}$$

at $\tau = 0$ is no less than that of $W(1, \tau, \tau^2, \dots, \tau^{j-1}, \tau^{j-\frac{1}{2}}, \dots, \tau^{\frac{k+j-1}{2}}) = C\tau^{-\frac{(k-j)^2+3(k-j)+2}{4}}$, where C is some positive constant. Therefore we have

$$\text{Ord}_{D_j} W(f_0, \dots, f_n)(\nu'_{[k-1]}(t, \mathbf{a})) \geq \frac{k(k+1)}{2} - \frac{(k-j)^2}{2} - \frac{3}{2}(k-j) - 1,$$

and this proves as expected that

$$\mathcal{O}_{X_k, w}(-(\frac{k(k+1)}{2} - \frac{(k-j)^2}{2} - \frac{3}{2}(k-j) - 1)D_j) = \mathfrak{w}_{\infty, w}$$

for any $w \in D_j \setminus \cup_{i \neq j} D_i$. In conclusion, we have

Theorem 4.3. *Outside the codimension 2 subvariety $Z := \cup_{2 \leq j < i \leq k} D_j \cap D_i$, the ideal sheaf \mathfrak{w}_{∞} is invertible, which is equal to*

$$\mathcal{O}_{X_k \setminus Z} \left(\sum_{j=2}^k \left(-\frac{k(k+1) - (k-j+1)(k-j+2)}{2} D_j \right) \right).$$

Moreover, at each point $w \in X_k \setminus Z$, \mathfrak{w}_{∞} is generated by k -jets $\mathcal{O}_{X,p}/m_p^{k+1}$, where $p := \pi_{0,k}(w)$. In particular, if L is an ample line bundle which separates k -jets, then $\mathfrak{w}_{\infty}(X_k, L) = \mathfrak{w}(X_k, L)$ at the place $X_k \setminus Z$.

Proof. We only need to prove the second part. As is shown above, $W(1, z_i, z_i^2, \dots, z_i^k)_{1 \leq i \leq k}$ generate the ideal \mathfrak{w}_{∞} at all regular points $P_k T_X^{\text{reg}}$. The Wronskians in the form (4.6) generate the ideal at all points in $X_k \setminus Z$, and since the holomorphic functions appearing in these Wronskians are all belong to k -jets, then we prove the theorem. \square

Now we will change a bit our parametrizing theorem to make the following proof simpler. Let U be an open set with coordinates (z_1, \dots, z_n) such that $\pi_{0,k}^{-1}(U)$ is a trivial product of $U \times \mathcal{R}_{n,k}$, and we denote by $g : \pi_{0,k}^{-1}(U) \rightarrow U \times \mathcal{R}_{n,k}$ the biholomorphism. Then each $D_i \cap \pi_{0,k}^{-1}(U)$ also has the form $U \times E_j$ under the map g , where E_j is a smooth prime divisor in $\mathcal{R}_{n,k}$. Let $p \in U$ be the point corresponding to the origin in the coordinate (z_1, \dots, z_n) . Thus the following germs of holomorphic curves

$$f(t, \mathbf{a}) := \mathbf{a}_1 t^2 + \mathbf{a}_2 t^4 + \dots + \mathbf{a}_{j-1} t^{2j-2} + \mathbf{a}_j t^{2j-1} + \mathbf{a}_{j+1} t^{2j} + \dots + \mathbf{a}_n t^{k+j-1}, \quad (4.8)$$

with \mathbf{a}_i the n -tuples in \mathbb{C}^n satisfying that $\mathbf{a}_1 \neq (0, \dots, 0)$ and $\mathbf{a}_j \neq (0, \dots, 0)$, whose lift to k -th stage $f_{[k]}(0, \mathbf{a})$ represents all points in $\pi_{0,k}^{-1}(p) \cap D_j \setminus Z$, which is biholomorphic to $E_j \setminus \cup_{i \neq j} E_i$ under the map $q_2 \circ g : \pi_{0,k}^{-1}(U) \rightarrow \mathcal{R}_{n,k}$, where $q_2 : U \times \mathcal{R}_{n,k} \rightarrow \mathcal{R}_{n,k}$ is the projection to the second factor. Thus as we have done in Theorem 4.1, we can build the parametrizing theorem at any $w \in E_j \setminus \cup_{i \neq j} E_i$ as follows:

Theorem 4.1 bis. *For any $w \in E_j \setminus \cup_{i \neq j} E_i$, then there exists a $\mathbf{b} \in \mathbb{C}^{nk}$ with $\mathbf{b}_2^{(l)} = \dots = \mathbf{b}_k^{(l)} = 0$, $\mathbf{b}_1^{(l)} = 1$ for some $1 \leq l \leq n$, and $\mathbf{b}_j^{(i)} \neq 0$, $\mathbf{b}_k^{(i)} = 0$ for some $1 \leq i \leq n$, such that $q_2 \circ g \circ f_{[k]}(0, \mathbf{b}) = w$. Meanwhile, there exists an open neighborhood W of \mathbf{b} in $\mathbb{C}^{n(k-1)-1}$ (here we treat \mathbf{b} as variables in $\mathbb{C}^{n(k-1)-1}$ by the conditions for \mathbf{b} above) such that the map $q_2 \circ g \circ f_{[k]}(0, \mathbf{a}) : \mathbf{a} \in W \rightarrow E_j$ is a biholomorphism from W to some open set in $E_j \setminus \cup_{i \neq j} E_i$. Moreover, $q_2 \circ g \circ f_{[k]}(t, \mathbf{a})$ also a local biholomorphism to some open set in $\mathcal{R}_{n,k}$, with $\{t = 0\}$ defining the divisor D_j locally.*

Thus for any $f_0, \dots, f_n \in \mathcal{O}_{X,p}$, $W(f_0, f_1, \dots, f_k)$ can be seen as a section in $H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m))$ with $m = \frac{k(k+1)}{2}$. We set

$$S := \text{Span}\{W(f_0, \dots, f_n) | f_0, \dots, f_n \in \mathcal{O}_{X,p}\} \subset H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m)),$$

and $m_{n,k} \subset \mathcal{O}_{\mathcal{R}_{n,k}}$ is denoted to be the base ideal of the linear system S . By the homogenous property we have

$$g^* \circ q_2^* m_{n,k} = \mathfrak{w}_\infty \quad (4.9)$$

at $\pi_{0,k}^{-1}(U)$.

Recall again that $V_1 := \pi_{1,k}^{-1}(U_1)$, where $U_1 \subset \pi_{0,1}(U)$ be the affine coordinate $U \times \mathbb{C}^{n-1}$ corresponds to the points $(z_1, \dots, z_n, [\frac{\partial}{\partial z_1} + t_2 \frac{\partial}{\partial z_2} + \dots + t_n \frac{\partial}{\partial z_n}])$ in $P(T_X)$. We will denote by $\tilde{V}_1 \subset \mathcal{R}_{n,k}$ the open set in $\mathcal{R}_{n,k}$ given by $q_2 \circ g(V_1)$. Then we have a similar theorem as above:

Theorem 4.4. *The ideal $m_{n,k}$ is invertible outside the codimension 2 subvariety $\tilde{Z} := \cup_{2 \leq j < i \leq k} E_i \cap E_j$, which is equal to*

$$\mathcal{O}_{\mathcal{R}_{n,k} \setminus \tilde{Z}} \left(\sum_{j=2}^k \left(-\frac{k(k+1) - (k-j+1)(k-j+2)}{2} E_j \right) \right).$$

On the Zariski open set \tilde{V}_1 in $\mathcal{R}_{n,k}$, $W(1, z_1, \dots, z_1^k) \neq 0$ at all points in $\mathcal{R}_{n,k} \setminus \cup_{2 \leq j \leq k} E_j$, and we have

$$\text{Div}(W(1, z_1, z_1^2, \dots, z_1^k)) = \sum_{j=2}^k \frac{k(k+1)}{2} E_j.$$

In particular, the ideal sheaf $\mathcal{O}(\sum_{j=2}^k(-\frac{k(k+1)}{2}E_j))$ is contained in $m_{n,k}$.

Theorem 4.5. Let $f_0 \in m_p^k, f_1, \dots, f_k \in \mathcal{O}_{X,p}$. Then on \tilde{V}_1 , we have

$$\text{Div}(W(f_0, f_1, \dots, f_k)) \geq \sum_{j=2}^k \frac{k(k+1)}{2} E_j.$$

Proof. For any point $w \in \tilde{V}_1 \cap E_j \setminus \cup_{i \neq j} E_i$, by Theorem 4.1bis there exists an open set $W \subset \mathbb{C}^{(n-1)k-1}$ and a holomorphic family of germs of holomorphic curves $\nu(t, \mathbf{a})$ with $\mathbf{a} \in W$

$$\nu(t, \mathbf{a}) := \mathbf{a}_1 t^2 + \mathbf{a}_2 t^4 + \dots + \mathbf{a}_{j-1} t^{2j-2} + \mathbf{a}_j t^{2j-1} + \mathbf{a}_{j+1} t^{2j} + \dots + \mathbf{a}_k t^{k+j-1}, \quad (4.10)$$

with $\mathbf{a}_2^{(1)} = \dots = \mathbf{a}_k^{(1)} = \mathbf{a}_k^{(i)} = 0, \mathbf{a}_1^{(1)} = 1$ and $\mathbf{a}_j^{(i)} \neq 0$ for some $2 \leq i \leq k$, such that $g_2 \circ g(\nu_{[k]}(t, \mathbf{a}))$ is a local biholomorphism to an open set in \tilde{V}_1 , with $\{t=0\}$ defining the divisor E_j locally.

We put $\tau = t^2$, and the curve $t \rightarrow \nu(t, \mathbf{a})$ becomes a Puiseux expansion

$$\mu(\tau, \mathbf{a}) := \mathbf{a}_1 \tau + \mathbf{a}_2 \tau^2 + \dots + \mathbf{a}_{j-1} \tau^{j-1} + \mathbf{a}_j \tau^{j-\frac{1}{2}} + \dots + \mathbf{a}_k \tau^{\frac{k+j-1}{2}},$$

Since $f_0 \in m_p^k$, thus the multiplicity of $f_0 \circ \mu(\tau, \mathbf{a})$ for τ is k . Thus the multiplicity of

$$\begin{vmatrix} f_0 \circ \mu(\tau, \mathbf{a}) & \dots & f_k \circ \mu(\tau, \mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \mu(\tau, \mathbf{a})}{d\tau^k} & \dots & \frac{d^k f_n \circ \mu(\tau, \mathbf{a})}{d\tau^k} \end{vmatrix}$$

at $\tau = 0$ is no less than that of $W(1, \tau, \tau^2, \dots, \tau^{j-1}, \tau^{j-\frac{1}{2}}, \dots, \tau^{\frac{k+j}{2}-1}, \tau^k) = C\tau^{\frac{(k-j+1)(k-j)}{4}}$. Thus by the relation

$$W(f_0, \dots, f_n)(\nu'_{[k-1]}(t, \mathbf{a})) = \begin{vmatrix} f_0 \circ \nu(t, \mathbf{a}) & \dots & f_k \circ \nu(t, \mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \nu(t, \mathbf{a})}{dt^k} & \dots & \frac{d^k f_n \circ \nu(t, \mathbf{a})}{dt^k} \end{vmatrix} = t^{\frac{k(k+1)}{2}} \cdot \begin{vmatrix} f_0 \circ \mu(\tau, \mathbf{a}) & \dots & f_k \circ \mu(\tau, \mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{d^k f_0 \circ \mu(\tau, \mathbf{a})}{d\tau^k} & \dots & \frac{d^k f_n \circ \mu(\tau, \mathbf{a})}{d\tau^k} \end{vmatrix},$$

which implies

$$\text{Ord}_{E_j} W(f_0, \dots, f_n)(\nu'_{[k-1]}(t, \mathbf{a})) \geq \frac{k(k+1)}{2}.$$

and the theorem is proved. \square

By Theorem 4.5, for any $f_0 \in m_p^k, f_1, \dots, f_k \in \mathcal{O}_{X,p}$, on \tilde{V}_1 we have $\frac{W(f_0, \dots, f_n)}{W(1, z_1, \dots, z_1^k)} \in \mathcal{O}(\tilde{V}_1)$, which means that on \tilde{V}_1 the linear system

$$S_1 := \text{Span}\{W(f_0, \dots, f_n) | f_0, \dots, f_n \in \mathcal{O}_{X,p}/m_p^{k+1}\} \subset S \subset H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m)),$$

is enough to generate $m_{n,k}$. Namley, the base ideal of S_1 is $m_{n,k}$ on \tilde{V}_1 . Since we can take other affine coordinates U_i for $X_1 = P(T_X)$, such that $U_1 \cup \dots \cup U_n = P(T_X)$, and $W(1, z_i, \dots, z_i^k)$ plays

the same role in $q_2 \circ g(\pi_{1,k}^{-1}(U_i))$ as $W(1, z_1, \dots, z_1^k)$ in $\tilde{V}_1 := q_2 \circ g(\pi_{1,k}^{-1}(U_1))$. Then S_1 generates $m_{n,k}$ at each point in $\mathcal{R}_{n,k}$. Since

$$g^* \circ q_2^* m_{n,k} = \mathfrak{w}_\infty \quad (4.11)$$

at $\pi_{0,k}^{-1}(U)$, then we have the following theorem

Theorem 1.1 bis. *If L separates k -jets at each point of X , then $m_\infty(X_k, L) = 1$, where $m_\infty(X_k, L) = 1$ is the positive integer defined in Section 1. In particular, if L is known to be only very ample, we have $m_\infty(X_k, L) = k$.*

5 Effectiveness in the Kobayashi Conjecture

We will use the notations in [2]. Let X be a projective manifold of dimension n and L is a very ample line bundle on X . Then L^n separates n jets at each point of X . We now take $N = n + 1, k = n, \delta = n(n + 1), u = 1, v = 1$. By Theorem 1.1 we have

$$\mathfrak{w}_\infty(X_n, L) = \mathfrak{w}(X_n, L^n),$$

which implies that $m_\infty = n$. By Theorem 2.1 we can take $M = M(n) = (n(n + 1))^n$. Thus by the definition in [2] we have

$$r(u, v, M, N, k, \epsilon, \delta) = (n(n + 1))^n (n + 1) (\epsilon + n^2(n + 1)) + 1.$$

Since $R := \max\{r(u, v, M, N, k, \epsilon, \delta) | m_\infty \leq \epsilon < m_\infty + \delta\}$, thus we have

$$R = (n(n + 1))^n (n + 1) (n + n(n + 1) - 1 + n^2(n + 1)) + 1$$

and the degree bound should be

$$d_0 := u(m_\infty + v\delta) + (R + k)v\delta = n^{n+1}(n + 1)^{n+2}(n^3 + 2n^2 + 2n - 1) + n^3 + 3n^2 + 3n.$$

Thus we prove the Main Theorem 1.

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