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# Potentials for Hyper-Kähler Metrics with Torsion

Bertrand Banos      Andrew Swann

## Abstract

We prove that locally any hyper-Kähler metric with torsion admits an HKT potential.

## Introduction

A hypercomplex manifold is a manifold endowed with three (integrable) complex structures  $I$ ,  $J$  and  $K$  satisfying the quaternion identities  $IJ = -JI = K$ . A metric  $g$  compatible with these three complex structures is said to be hyper-Kähler with torsion if the three corresponding Kähler forms satisfy the identities

$$IdF_I = JdF_J = KdF_K. \quad (1)$$

This is equivalent to saying that there exists a connection  $\nabla$  preserving the metric ( $\nabla g = 0$ ) and the complex structures ( $\nabla I = \nabla J = \nabla K = 0$ ) and whose torsion tensor

$$c(X, Y, Z) = g(X, \nabla_Y Z - \nabla_Z Y - [Y, Z])$$

is totally skew. This connection is necessarily unique and its torsion tensor is exactly the 3-form defined by (1).

The terminology hyper-Kähler with torsion is quite misleading since the underlying metric is in general not Kähler at all. We will prefer then the terminology HKT.

HKT metrics were introduced by Howe and Papadopoulos. They explain in [HP] how HKT geometry, and other geometries with torsion, arise as the target spaces of some two-dimensional sigma models in string theory.

Grantcharov and Poon give in [GP] the corresponding mathematical background. They define in particular the concept of *HKT potential* which is a natural generalization of the concept of Kähler or hyper-Kähler potential. Unlike the Kähler case, hyper-Kähler metrics do not admit in general a hyper-Kähler potential, even locally (see [Sw]) but they always admit, locally, an HKT potential. It is actually easy to check that on a hyper-Kähler manifold, any Kähler potential for one of the three complex structures is an HKT potential. We think this simple remark is sufficient to justify and motivate the question: *do HKT potentials always exist locally?* We know from Michelson and Strominger that the answer is yes in the flat case, that is, any HKT metric on  $\mathbb{H}^n$  (with the standard complex structures) admits locally an

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HKT potential (see [MS]). It is proved in [PSw] that an HKT manifold with a special homothety also admits an HKT potential. We prove here the general result.

Our strategy is based upon the following observation: any compatible metric on a quaternionic curve (that is, a 4-dimensional hypercomplex manifold) is HKT but not necessarily hyper-Kähler. This indicates that HKT geometry is a better quaternionic generalization of Kähler geometry, the torsion being a direct consequence of the non-commutativity of the quaternion division ring. We actually remark that an HKT structure is essentially a *closed (1,1)-form in the sense of Salamon*, that is, a 2-form compatible with the three complex structures and closed with respect to the differential  $D$  introduced by Salamon in [Sa]. This remark, which already appears in [V] but in other spirit and other formalism, combined with the properties of the operator  $D$  and the twistor space described by Salamon in [Sa] and [MCS] give us then directly the wished result. Indeed the Salamon differential operator used here provides further analogies with complex geometry and we give hypercomplex analogues of the local and global  $\partial\bar{\partial}$ -lemmas.

## 1 HKT metrics and Salamon (1, 1)-forms

An almost hypercomplex manifold is a (smooth) manifold endowed with 3 almost complex structures  $I$ ,  $J$  and  $K$  satisfying the quaternion identities

$$IJ = -JI = K.$$

Note that on a almost hypercomplex manifold there is actually a 2-sphere worth of almost complex structures:

$$S^2 = \{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}.$$

The integrability of  $I$ ,  $J$  and  $K$  is equivalent to the existence of a (unique) torsion-free connection  $\nabla^{\text{Ob}}$  preserving the quaternion action, the so-called Obata connection.

### 1.1 Exterior forms

Let  $(M, I, J, K)$  be an almost hypercomplex manifold and let  $\Lambda^k$  be the bundle of  $k$ -forms on  $M$ . We denote by  $\Lambda_{\mathcal{I}}^{p,q}$  the subbundle of forms of type  $(p, q)$  with respect to the almost complex structure  $\mathcal{I} \in S^2$ .

Studying the action of  $GL(n, \mathbb{H})$  on  $\Lambda^k(T^*)$ , Salamon introduces in [Sa] the subbundle

$$A^k = \sum_{\mathcal{I} \in S^2} (\Lambda_{\mathcal{I}}^{k,0} \oplus \Lambda_{\mathcal{I}}^{0,k}).$$

This bundle can be understood as the analogue for hypercomplex manifolds of the bundle  $\Lambda^{k,0} \oplus \Lambda^{0,k}$  for complex manifolds (see [W]).

It will be convenient for us to choose a preferred complex structure, say  $I$ . Although this choice is not really natural (all the complex structures have the same status and should be studied together), we will see that it is useful for understanding HKT geometry as a quaternionic analogue of Kähler geometry. When  $M$  is considered as a complex manifold, *this is always with respect to the complex structure  $I$* . We will write for example  $\Lambda^{p,q}$  for  $\Lambda_I^{p,q}$ . The Hodge decomposition of  $\Lambda^2$  with respect to  $I$  induces the decomposition

$$A^2 = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus A^{1,1},$$

with

$$A^{1,1} = \{\omega \in \Lambda^2 : I\omega = \omega \text{ and } J\omega = -\omega\}.$$

For example, if  $g$  is a hyperhermitian metric on  $M$  then the Kähler form  $F_I = g(I\cdot, \cdot)$  is a smooth section of  $A^{1,1}$  and conversely any smooth section  $F_I$  of  $A^{1,1}$  defines an (possibly indefinite and/or degenerate) hyperhermitian metric  $g = -F_I(I\cdot, \cdot)$ . We will call such a form a  $(1, 1)$ -form in the sense of Salamon.

## 1.2 The Salamon differential

There is an orthogonal projection  $\eta: \Lambda^k \rightarrow A^k$  whose kernel is the subbundle

$$B^k = \bigcap_{\mathcal{I} \in S^2} (\Lambda_{\mathcal{I}}^{k-1,1} \oplus \Lambda_{\mathcal{I}}^{k-2,2} \oplus \dots \oplus \Lambda_{\mathcal{I}}^{1,k-1}).$$

Let  $\mathcal{A}$  denote the space of smooth sections of  $A$ . The Salamon differential

$$D: \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$$

is simply the composition of the projection  $\eta$  with the de Rham differential  $d$ :

$$D = \eta \circ d.$$

For example, if  $\theta$  is a 1-form on  $M$  then

$$D\theta = (d\theta)^{2,0} + (d\theta)^{0,2} + \frac{1}{2}((d\theta)^{1,1} - J(d\theta)^{1,1}). \quad (2)$$

Salamon shows in [Sa] the following:

**THEOREM (Salamon).** *An almost hypercomplex structure is integrable if and only if  $D^2 = 0$ .*

This result is completely analogous to the corresponding statement involving an almost complex structure and the Dolbeault operator  $\bar{\partial}$ .

## 1.3 The twistor space

If  $(M, I, J, K)$  is a hypercomplex manifold then the manifold  $Z = M \times S^2$  admits an integrable complex structure  $\mathbb{I}$  defined by

$$\mathbb{I}_{(x,\bar{a})} = \begin{pmatrix} a_1 I_x + a_2 J_x + a_3 K_x & 0 \\ 0 & I_{\bar{a}} \end{pmatrix},$$

where  $I_{\bar{a}}: X \mapsto \bar{a} \times X$  is the usual complex structure on  $T_{\bar{a}}S^2$ . The space  $Z$  endowed with the complex structure  $\mathbb{I}$  is called the twistor space of the hypercomplex manifold  $M$ .

The cohomology of the twistor space can be related to the cohomology of the Salamon's complex as follows ([MS]):

**THEOREM (Mamone Capria, Salamon).** *Let  $M^{4n}$  be a hypercomplex manifold with twistor space  $Z$ . Then*

$$H^k(Z, \mathcal{O}) \cong \begin{cases} \frac{\text{Ker}(D: \mathcal{A}^k \rightarrow \mathcal{A}^{k+1})}{\text{Im}(D: \mathcal{A}^{k-1} \rightarrow \mathcal{A}^k)}, & 0 \leq k \leq 2n, \\ 0, & k = 2n + 1. \end{cases}$$

## 1.4 HKT metrics

Let  $(M, I, J, K)$  be a hypercomplex manifold and let  $g$  be a hyperhermitian metric on  $M$ , that is

$$g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y),$$

for all tangent vectors  $X$  and  $Y$ . We will denote by  $F_{\mathcal{I}}$  the Kähler form associated with the complex structure  $\mathcal{I}$ :

$$F_{\mathcal{I}} = g(\mathcal{I}\cdot, \cdot).$$

Note that  $g$  can be indefinite in what follows.

In the physics literature, this hyperhermitian metric is said to be HKT if there exists a hyperhermitian connection whose torsion tensor is totally antisymmetric (see [HP]). We will rather use the reformulation introduced by Grantcharov and Poon:

**DEFINITION.** The hyperhermitian metric  $g$  is *HKT* if

$$IdF_I = JdF_J = KdF_K.$$

For example, any compatible metric on a quaternionic curve is HKT:

**LEMMA 1.** *Any hyperhermitian metric on a 4-dimensional hypercomplex manifold is HKT.*

*Proof.* Let  $(M, I, J, K)$  be a 4-dimensional manifold and let  $g$  be a hyperhermitian metric on it. It is noted in [PS] that  $g$  is necessarily Einstein-Weyl with respect to the Obata connection  $\nabla^{\text{Ob}}$ . In particular, there exists a 1-form  $\omega$  such that

$$\nabla^{\text{Ob}}g = \omega \otimes g.$$

(This may be seen directly, by noting that on a four-manifold the conformal class of  $g$  is uniquely determined by  $I, J$  and  $K$ .) Since  $\nabla^{\text{Ob}}$  is torsion-free and compatible with  $I, J$  and  $K$  this implies that

$$dF_I = \omega \wedge F_I, \quad dF_J = \omega \wedge F_J, \quad dF_K = \omega \wedge F_K.$$

Define now  $\alpha$  by  $\omega = \lambda\alpha$  and  $\|\alpha\| = 1$ . Since  $M$  is 4-dimensional we get

$$\begin{cases} F_I = \alpha \wedge I\alpha + J\alpha \wedge K\alpha, \\ F_J = \alpha \wedge J\alpha + K\alpha \wedge I\alpha, \\ F_K = \alpha \wedge K\alpha + I\alpha \wedge J\alpha. \end{cases}$$

And then

$$IdF_I = JdF_J = KdF_K = \lambda I\alpha \wedge J\alpha \wedge K\alpha.$$

□

Many of the explicitly known HKT examples in higher dimensions are homogeneous and come from the Joyce hypercomplex structures associated to any compact semi-simple Lie group (see [J] and [GP]). For example, the Killing-Cartan metric on  $SU(3)$  is HKT for the (non-trivial) invariant hypercomplex structure on  $SU(3)$  constructed by Joyce. It is worth mentioning that the Lie bracket on  $su(3)$  is exactly the torsion of the HKT structure. In particular, due to the Jacobi identity, the torsion form is closed:  $SU(3)$  is a strong HKT manifold.

## 1.5 HKT forms

When one is more interested in complex and symplectic properties than in Riemannian ones, one can define a Kähler structure as a non-degenerate closed  $(1, 1)$ -form. It is possible to have a similar approach for HKT structures. The following result is due to Verbitsky [V], but we prefer to give a direct proof using the Obata connection.

**LEMMA 2.** *Let  $F \in \mathcal{A}^{1,1}$  be a non-degenerate Salamon  $(1,1)$ -form on a hypercomplex manifold  $(M, I, J, K)$ . The (pseudo) metric*

$$g = -F(I, \cdot)$$

*is HKT if and only if  $F$  is  $D$ -closed:*

$$DF = 0.$$

*Such a form is called an HKT form.*

*Proof.* Suppose that  $g$  is HKT. For any complex structure  $\mathcal{I} \in S^2$  the form  $dF_{\mathcal{I}}$  has type  $(2, 1) + (1, 2)$  with respect to the complex structure  $\mathcal{I}$ . But since  $IdF_{\mathcal{I}} = JdF_J = KdF_K$  we deduce that  $dF_{\mathcal{I}}$  has type  $(2, 1) + (1, 2)$  with respect to the three complex structures:  $dF_{\mathcal{I}} \in \mathcal{B}^3$  that is  $DF_{\mathcal{I}} = 0$ . Since  $F_{\mathcal{I}} = F$ , we obtain the result.

Suppose now that  $DF = 0$ . This is equivalent to the relation

$$dF(U, V, W) = dF(\mathcal{I}U, \mathcal{I}V, W) + dF(\mathcal{I}U, V, \mathcal{I}W) + dF(U, \mathcal{I}V, \mathcal{I}W),$$

for all  $\mathcal{I}$  in  $S^2$ . In particular

$$dF(IU, IV, IW) = dF(KU, KV, IW) + dF(KU, IV, KW) + dF(IU, KV, KW). \quad (3)$$

Since the Obata connection is torsion-free, the following holds:

$$dF(X, Y, Z) = \nabla^{\text{Ob}}F(X, Y, Z) + \nabla^{\text{Ob}}F(Y, Z, X) + \nabla^{\text{Ob}}F(Z, X, Y). \quad (4)$$

Moreover, since  $F \in \mathcal{A}^{1,1}$ , we have

$$\begin{cases} F(X, Y) = F(IX, IY) = -F(KX, KY), \\ F(IX, KY) = F(KX, IY). \end{cases} \quad (5)$$

Using (4) and (5) in (3) we obtain

$$\begin{aligned} & dF(IU, IV, IW) \\ &= 2\nabla^{\text{Ob}}F(KU, KV, IW) + 2\nabla^{\text{Ob}}F(KV, KW, IU) + 2\nabla^{\text{Ob}}F(KW, KU, IV) \\ &\quad - \nabla^{\text{Ob}}F(IU, V, W) - \nabla^{\text{Ob}}F(IV, W, U) - \nabla^{\text{Ob}}F(IW, U, V) \end{aligned}$$

and thus

$$\begin{aligned} dF(IU, IV, IW) &= \nabla^{\text{Ob}}F(KU, KV, IW) + \nabla^{\text{Ob}}F(KV, KW, IU) \\ &\quad + \nabla^{\text{Ob}}F(KW, KU, IV). \end{aligned}$$

Define now  $G = F(J, \cdot)$ .

$$\begin{aligned} & dG(KU, KV, KW) \\ &= \nabla^{\text{Ob}}G(KU, KV, KW) + \nabla^{\text{Ob}}G(KV, KW, KU) + \nabla^{\text{Ob}}G(KW, KU, KV) \\ &= \nabla^{\text{Ob}}F(KU, IV, KW) + \nabla^{\text{Ob}}F(KV, IW, KU) + \nabla^{\text{Ob}}F(KW, IU, KV) \\ &= \nabla^{\text{Ob}}F(KU, KV, IW) + \nabla^{\text{Ob}}F(KV, KW, IU) + \nabla^{\text{Ob}}F(KW, KU, IV) \end{aligned}$$

and therefore  $dF(IU, IV, IW) = dG(KU, KV, KW)$ . In other words,  $IdF_{\mathcal{I}} = KdF_K$  with  $F_{\mathcal{I}} = F$  and  $F_K = G$ .  $\square$

**Remark.** We know from Fino and Grantcharov ([FG]) that there exists some hypercomplex manifolds which do not admit an HKT metric. Lemma 2 seems to indicate that the question of existence of an HKT metric on a given hypercomplex manifold is highly non-trivial.

## 2 HKT potentials

Let  $(M, I, J, K)$  be a hypercomplex manifold. Following [GP] we define the action of  $\mathcal{I} \in S^2$  on  $k$ -forms by

$$\mathcal{I}\omega(X_1, \dots, X_k) = (-1)^k \omega(\mathcal{I}X_1, \dots, \mathcal{I}X_k)$$

and the differential  $d_k$  is

$$d_{\mathcal{I}}\omega = (-1)^k \mathcal{I}d\mathcal{I}\omega.$$

Note that  $d, d_I, d_J$  and  $d_K$  all anti-commute.

Recall that a hyperhermitian metric  $g$  on  $M$  is said to be hyper-Kähler if it is Kähler for each complex structure. A possibly locally defined function  $\mu$  is a hyper-Kähler potential for this metric  $g$  if it is a Kähler potential for each complex structure, that is,

$$F_I = dd_I\mu, \quad F_J = dd_J\mu, \quad F_K = dd_K\mu.$$

It is proved in [Sw] that such a potential does not exist in general but it is straightforward to check that if  $\nu$  is a Kähler potential for the complex structure  $I$  then

$$F_I = dd_I\nu, \quad F_J = \frac{1}{2}(dd_J + d_Kd_I)\nu, \quad F_K = \frac{1}{2}(dd_K + d_Id_J)\nu.$$

We say then that any hyper-Kähler metric admits an HKT potential:

**DEFINITION (Grantcharov, Poon).** A possibly locally defined function  $\mu$  is an *HKT potential* for an HKT metric  $g$  if

$$F_I = \frac{1}{2}(dd_I + d_Jd_K)\mu, \quad F_J = \frac{1}{2}(dd_J + d_Kd_I)\mu, \quad F_K = \frac{1}{2}(dd_K + d_Id_J)\mu.$$

**Remark.** Note that on an HKT manifold the following identities are actually equivalent:

1.  $F_I = \frac{1}{2}(dd_I + d_Jd_K)\mu,$
2.  $F_J = \frac{1}{2}(dd_J + d_Kd_I)\mu,$
3.  $F_K = \frac{1}{2}(dd_K + d_Id_J)\mu,$
4.  $g = \frac{1}{4}(1 + I + J + K)(\nabla^{\text{Ob}})^2\mu.$

### 2.1 The four-dimensional case

In this dimension one can check directly that HKT metrics always admit an HKT potential:

**LEMMA 3.** *Let  $g$  be an HKT metric on a 4-dimensional hypercomplex manifold and let  $\omega$  be the 1-form defined by the Obata connection via  $\nabla^{\text{Ob}}g = \omega \otimes g$ .*

*A function  $\mu$  is an HKT potential for  $g$  if and only if it is solution of the elliptic equation*

$$\Delta\mu - \omega^\sharp(\mu) + 4 = 0,$$

*where  $\Delta$  is the Laplacian of the Riemannian metric  $g$ .*

Local existence of HKT potentials now follows from the general theory for the Laplace operator, see for example [GT].

*Proof.* Let  $\nabla^{\text{LC}}$  be the Levi-Civita connection and define  $a = \nabla^{\text{Ob}} - \nabla^{\text{LC}}$ . Since  $\nabla^{\text{Ob}}g = \omega \otimes g$  and  $\nabla^{\text{LC}}g = 0$  we get

$$\omega(U)g(V, W) = -g(a_U V, W) - g(a_U W, V),$$

for all vector fields  $U, V$  and  $W$ . Moreover,  $a_U V = a_V U$  holds for all  $U$  and  $V$  since  $\nabla^{\text{Ob}}$  and  $\nabla^{\text{LC}}$  are torsion free. We now obtain

$$g(a_U V, W) = \frac{1}{2}(\omega(W)g(U, V) - \omega(U)g(V, W) - \omega(V)g(U, W)),$$

for all  $U, V$  and  $W$ . In particular, if  $X$  is a (local) unit vector field, then

$$g(a_X X, Y) = \frac{1}{2}\omega(Y) - \omega(X)g(X, Y)$$

and

$$g(a_X X + a_{IX} IX + a_{JX} JX + a_{KX} KX, Z) = \omega(Z)$$

for all  $Z$ .

The metric  $g$  is the unique hyperhermitian metric satisfying  $g(X, X) = 1$ . Therefore  $\mu$  is an HKT potential if and only if

$$\frac{1}{4}(1 + I + J + K)(\nabla^{\text{Ob}})^2 \mu(X, X) = 1,$$

that is,

$$\text{Trace}(\nabla^{\text{Ob}} d\mu) = 4.$$

Note that the Laplacian  $\Delta\mu$  is by definition  $-\text{Trace}(\nabla^{\text{LC}} d\mu)$ . Thus  $\mu$  is a HKT potential for  $g$  if and only if

$$-\Delta\mu + d\mu(a_X X + a_{IX} IX + a_{JX} JX + a_{KX} KX) = 4.$$

□

## 2.2 The local $DD_I$ -lemma

The easiest way to show that a Kähler metric admits a local Kähler potential is to apply the local  $dd_I$ -lemma to the closed (and therefore locally exact) Kähler form. This is exactly the same for HKT potentials if one now uses the Salamon differential:

**LEMMA 4.** *A HKT metric locally admits a potential if and only if the corresponding HKT form is locally  $D$ -exact.*

*Proof.* Suppose that  $F = \frac{1}{2}(dd_I + d_J d_K)\mu$ . Then  $F = \frac{1}{2}(d\theta - Jd\theta)$  with  $\theta = Id\mu$ . Note that  $d\theta$  is a  $(1, 1)$ -form (for  $I$ ) since  $d\theta = dd_I\mu$ . Therefore, according to (2)  $F = D\theta$ .

Conversely, suppose that  $F = D\theta$  for some 1-form  $\theta$ . Since  $F$  is a  $(1, 1)$ -form for  $I$ , we obtain from (2)

$$\begin{cases} d\theta \in \Lambda^{1,1}, \\ F = \frac{1}{2}(d\theta - Jd\theta). \end{cases}$$

Since  $I$  is an integrable complex structure, the local  $dd_I$ -lemma holds: locally there exists  $\mu$  such that  $d\theta = dd_I\mu$ . We get then

$$F = \frac{1}{2}(dd_I - Jdd_I)\mu = \frac{1}{2}(dd_I + d_J d_K)\mu.$$

□



**THEOREM.** *Any HKT metric admits locally an HKT potential.*

*Proof.* Let  $g$  be an HKT metric on a hypercomplex manifold  $(M, I, J, K)$  and let  $F = F_I$  be the corresponding HKT form. This form is  $D$ -closed and according to the theorem of Mamone Capria and Salamon it implies that it is locally  $D$ -exact. The idea of the proof is the following:

Let  $Z = M \times S^2$  be the twistor space of  $M$  and  $p: Z \rightarrow M$  the natural projection. Define the  $(0, 2)$ -form  $G$  on  $Z$  by

$$G_{(x, \vec{a})} = (F_x)_{\vec{a}}^{0,2}.$$

The form  $G$  in the point  $(x, \vec{a})$  is the  $(0, 2)$ -part of the form  $F$  in the point  $x$  with respect to the complex structure  $\vec{a} = a_1 I_x + a_2 J_x + a_3 K_x$ . Grantcharov and Poon have proved in [GP] that  $g$  is HKT if and only if the form  $(F)_{\vec{a}}^{0,2}$  is a  $\bar{\partial}_{\vec{a}}$ -closed form on  $M$ . Moreover  $(F)_{\vec{a}}^{0,2}$  is holomorphic in  $\vec{a}$ . This implies that  $G$  is a  $\bar{\partial}$ -closed  $(0, 2)$ -form on  $Z$ . Now a 1-pseudo-convexity argument says that one can always choose a neighbourhood  $U$  of a point  $x \in M$  such that

$$H_{\bar{\partial}}^{0,2}(p^{-1}(U)) = H^2(p^{-1}(U), \mathcal{O}) = 0.$$

It implies that it exists a 1-form  $\phi$  on  $p^{-1}U$  such that  $G = \bar{\partial}\phi$ . Moreover one can choose this form without part on  $S^2$ : for any point  $x \in U$ ,  $\phi_x$  is a holomorphic section of the bundle over  $S^2$  with fibre

$$\mathcal{B}_{\vec{a}} = \{\omega + i\vec{a}\omega : \omega \in T_x^*M\}.$$

Using now the compactness of  $S^2$  we deduce that, in any point  $\vec{a}$  of  $S^2$ ,  $\phi = \theta + i\vec{a}\theta$  with  $\theta$  a 1-form on  $U$ . We get then  $\text{Re}(G_{\vec{a}}) = \frac{1}{2}(d\theta - \vec{a}\theta)$  for any  $\vec{a} \in S^2$ . Taking  $\vec{a} = I$  and  $\vec{a} = J$  we obtain

$$\begin{cases} d\theta - Id\theta = 0, \\ F = \frac{1}{2}(d\theta - Jd\theta), \end{cases}$$

that is,  $F = D\theta$  on the neighbourhood  $U$  of a fixed point  $x$ .  $\square$

**Remark.** This actually shows that the local  $DD_I$ -lemma holds on hypercomplex manifolds with  $D_I = (-1)^k IDI$ .

### 2.3 The global $DD_I$ -lemma

Let  $(M, g, I, J, K)$  be an HKT manifold with HKT form  $F$ . As  $F$  is  $D$ -closed it defines a Salamon cohomology class  $[F] \in H_D^2(M)$  which we can call the *HKT class*. Assume that  $F'$  is another HKT form in the same HKT class, that is  $F - F' = D\theta$ . Since  $F - F'$  is a  $(1, 1)$ -form (for  $I$ ), we get

$$F - F' = \frac{1}{2}(d\theta - Jd\theta),$$

with  $d\theta \in \Lambda^{1,1}$ . Therefore, if the global  $dd_I$ -lemma holds on  $M$  then there exists a global function  $\phi$  on  $M$  such that  $F' = F + DD_I\phi$ .

Note that if  $DD_I\phi = 0$  then  $\phi$  is harmonic with respect the complex Laplacian  $\Delta^c$  defined by

$$\bar{\partial}^* \bar{\partial} f = \Delta^c f = g(dd_I f, F_I).$$

Indeed if  $DD_I\phi = 0$  then  $dd_I\phi = Jdd_I\phi$  and then

$$\Delta^c f = g(dd_I\phi, F_I) = g(Jdd_I\phi, F_I) = g(dd_I, JF_I) = -g(dd_I\phi, F_I) = -\Delta^c f.$$

Finally we get the following:

**THEOREM.** *Let  $(M, I, J, K)$  be a compact hypercomplex manifold on which the global  $dd_I$ -lemma holds and let  $g$  and  $g'$  be two HKT metrics with same HKT class  $[F] = [F']$ . Then there exists a smooth real function  $\phi$  on  $M$  such that  $F' = F + DD_I\phi$ . This function is unique up to a constant.  $\square$*

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