

Multiscale domain decomposition method for solving high-dimensional non-linear stochastic problems with localized uncertainties and non-linearities

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Context

Uncertainty quantification

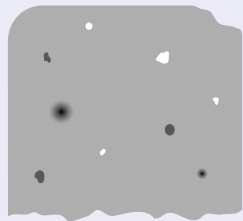
- Model problem

$$-\nabla \cdot B(u, \nabla u) + C(u, \nabla u) = f \quad \text{on } \Omega$$

with appropriate boundary conditions

- Different possible uncertainties
 - Source term f
 - Operators B, C
 - Geometrical variability Ω
- Parametric modeling
 - Finite set of m random variables $\xi = (\xi_1, \dots, \xi_m)$ (parameters of the continuous model)
 - Finite dimensional probability space $(\Xi, \mathcal{B}(\Xi), \mu)$, with $\Xi \subset \mathbb{R}^m$

Localized uncertainties
and non-linearities



Goal

- Propagate the uncertainties through the model

$$-\nabla \cdot B(u(\xi), \nabla u(\xi); \xi) + C(u(\xi), \nabla u(\xi); \xi) = f(\xi) \quad \text{on } \Omega(\xi)$$

Context

Functional representation of random variables / Stochastic spectral methods

- Functional expansion of the solution

$$u(\xi) \approx \sum_{\alpha \in \mathcal{A}} u_{\alpha} \psi_{\alpha}(\xi) \in L^p_{\mu}(\Xi; \mathcal{V})$$

- Stochastic approximation space
 - Continuous (generalized) polynomial chaos basis [Ghanem and Spanos 1991; Xiu and Karniadakis 2002; Soize and Ghanem 2004]
 - Lagrange interpolation (or collocation) polynomial basis [Xiu and Hesthaven 2005; Babuška, Nobile, and Tempone 2007]
 - Piecewise polynomial basis (stochastic finite elements [Deb, Babuška, and Oden 2001; Wan and Karniadakis 2005], multi-wavelets [Le Maître et al. 2004b; Le Maître et al. 2004a])

Advantages

- **Explicit** and **accurate** prediction of the solution in terms of the random variables
- **Quick post-processing** / **Fast evaluation** of quantities of interest
 - Statistical moments
 - Quantiles
 - Sensitivity indices
 - Confidence intervals
 - ...

Motivation

Multiscale problems with localized sources of uncertainties and possible localized non-linearities on a subdomain called a patch

- Non-linear and uncertain operators

$$B(\cdot, \cdot; \xi) = \begin{cases} B_L(\cdot, \cdot) & \text{on } \Omega \setminus \Lambda \text{ (linear and deterministic)} \\ B_L(\cdot, \cdot; \xi) + B_N(\cdot, \cdot; \xi) & \text{on } \Lambda(\xi) \text{ (non-linear and uncertain)} \end{cases}$$

$$C(\cdot, \cdot; \xi) = \begin{cases} C_L(\cdot, \cdot) & \text{on } \Omega \setminus \Lambda \\ C_L(\cdot, \cdot; \xi) + C_N(\cdot, \cdot; \xi) & \text{on } \Lambda(\xi) \end{cases}$$

- Dedicated strategy to handle high dimensionality and complexity

Challenges

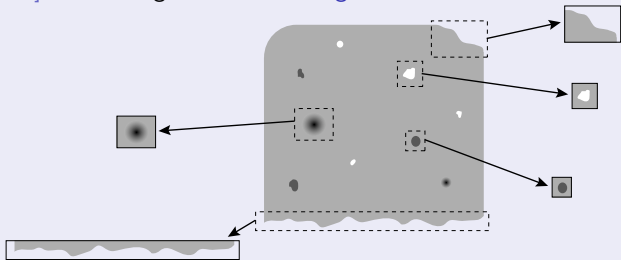
- Develop a numerical method that exploits the localized side of uncertainties by operating a separation of scales
 - Simple global problem
 - ↔ Traditional solvers / Commercial software packages available in the industry
 - More technical local problems
 - ↔ Specific solvers / In-house research codes

Motivation

Multiscale domain decomposition method with patches

- Multiscale method inspired from the **Semi-Schwarz-Lagrange method** [Gendre et al. 2009; Lozinski 2010; Gendre, Allix, and Gosselet 2011; Hager et al. 2012; Chevreuril, Nouy, and Safatly 2013] that belongs to the class of **global-local iterative methods**

“Numerical zooms”



Main ingredient - Separation of global and local problems

- Local solutions with **better approximation properties**
- **Simple global** problems
 - Linear
 - No geometrical variabilities
 - Deterministic operator
- Possible **refined local** approximations

Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method with patches
- 3 Computational aspects
- 4 Numerical illustrations
- 5 Conclusions and outlooks
- 6 Bibliography

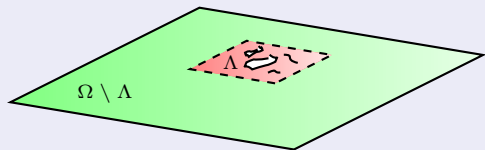
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- 2 Multiscale domain decomposition method with patches
 - Global-local formulation with patches
 - Global-local iterative algorithm
- 3 Computational aspects
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Initial formulation

Non-linear diffusion-reaction stochastic problem

- Random domain $\Omega(\xi)$ and patch $\Lambda(\xi)$ containing **localized uncertainties** and **non-linearities**



- Model example with **random** material coefficients and source terms

$$\begin{cases} -\nabla \cdot (K \nabla u) = f & \text{on } \Omega \setminus \Lambda \\ -\nabla \cdot (K(\xi) \nabla u) + R(\xi) u^3 = f(\xi) & \text{on } \Lambda(\xi) \end{cases} \text{ with } \begin{cases} u = 0 & \text{on } \Gamma_D \\ K \nabla u \cdot n = g & \text{on } \Gamma_N \end{cases}$$

Weak formulation

Find $u \in \mathcal{V}^\Xi$ such that it holds almost surely

$$a_\Omega(u(\xi), \delta u; \xi) + n_\Lambda(u(\xi), \delta u; \xi) = \ell_\Omega(\delta u; \xi) \quad \forall \delta u \in \mathcal{V}$$

with **bilinear** form $a_\Omega(u, v; \xi) = \int_\Omega K(\xi) \nabla u \cdot \nabla v$, **semi-linear** form

$$n_\Lambda(u, v; \xi) = \int_\Lambda R(\xi) u^3 v \text{ and linear form } \ell_\Omega(v; \xi) = \int_\Omega f(\xi) v + \int_{\Gamma_N} g(\xi) v.$$

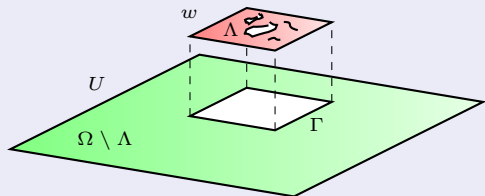
Global-local formulation with patches

Domain decomposition: introduction of a patch

- Splitting of the solution

$$u = \begin{cases} U & \text{on } \Omega \setminus \Lambda \\ w & \text{on } \Lambda \end{cases}$$

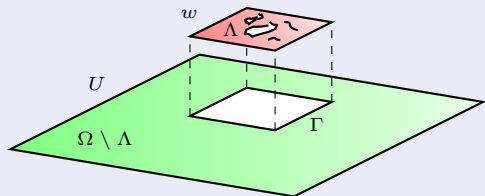
with $U = w$ on Γ (weak sense)



Global-local formulation with patches

Domain decomposition: introduction of a patch

- **Uncertain** domain Ω
- **Deterministic** subdomain $\Omega \setminus \Lambda$
- **Uncertain** patch Λ
- **Deterministic** interface Γ



Reformulation using Lagrange multiplier

Find $(U, w, \lambda) \in \mathcal{U}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$ such that it holds almost surely

- **Global** equation on subdomain $\Omega \setminus \Lambda$

$$a_{\Omega \setminus \Lambda}(U(\xi), \delta U) + b_{\Gamma}(\lambda(\xi), \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \mathcal{U}$$

- **Local** equation on patch Λ

$$a_{\Lambda}(w(\xi), \delta w; \xi) + n_{\Lambda}(w(\xi), \delta w; \xi) - b_{\Gamma}(\lambda(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) \quad \forall \delta w \in \mathcal{W}$$

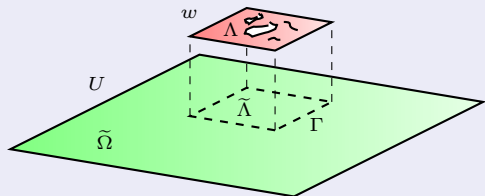
- **Coupling** equation on interface Γ

$$b_{\Gamma}(\delta \lambda, U(\xi)) - b_{\Gamma}(\delta \lambda, w(\xi)) = 0 \quad \forall \delta \lambda \in \mathcal{M}$$

Global-local formulation with patches

Overlapping domain decomposition: introduction of a fictitious patch

- **Deterministic fictitious** domain $\tilde{\Omega}$
- **Deterministic fictitious** patch $\tilde{\Lambda}$
- **Uncertain** patch Λ
- **Deterministic** interface Γ



Reformulation with overlapping domains

Find $(U, w, \lambda) \in \tilde{\mathcal{U}}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$ such that it holds almost surely

- **Reformulated global** equation on **fictitious** domain $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$

$$c_{\tilde{\Omega}}(U(\xi), \delta U; \xi) - c_{\tilde{\Lambda}}(U(\xi), \delta U; \xi) + b_{\Gamma}(\lambda(\xi), \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \tilde{\mathcal{U}}$$

- **Local** equation on patch Λ

$$a_{\Lambda}(w(\xi), \delta w; \xi) + n_{\Lambda}(w(\xi), \delta w; \xi) - b_{\Gamma}(\lambda(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) \quad \forall \delta w \in \mathcal{W}$$

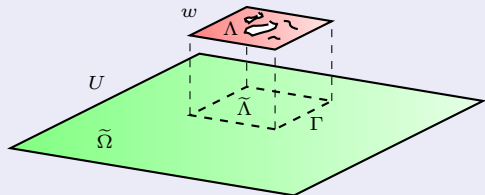
- **Coupling** equation on interface Γ

$$b_{\Gamma}(\delta \lambda, U(\xi)) - b_{\Gamma}(\delta \lambda, w(\xi)) = 0 \quad \forall \delta \lambda \in \mathcal{M}$$

Global-local formulation with patches

Overlapping domain decomposition: introduction of a fictitious patch

- Deterministic fictitious domain $\tilde{\Omega}$
- Deterministic fictitious patch $\tilde{\Lambda}$
- Uncertain patch Λ
- Deterministic interface Γ



Reformulation with overlapping domains

Find $(U, w, \lambda) \in \tilde{\mathcal{U}}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$ such that it holds almost surely

- Reformulated global equation on fictitious domain $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$

$$c_{\tilde{\Omega}}(U(\xi), \delta U; \xi) - c_{\tilde{\Lambda}}(U(\xi), \delta U; \xi) + b_{\Gamma}(\lambda(\xi), \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \tilde{\mathcal{U}}$$

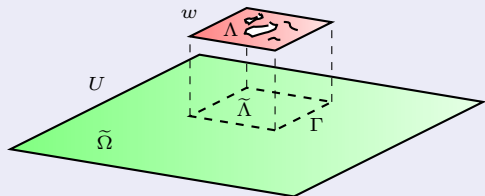
where bilinear form $c_{\mathcal{O}}(U, V; \xi) = \int_{\mathcal{O}} \tilde{K}(\xi) \nabla U \cdot \nabla V$, with $\tilde{K} = K$ on $\Omega \setminus \Lambda$.

\hookrightarrow Global-local problem admits an infinite number of solutions (U, w, λ)

Global-local formulation with patches

Overlapping domain decomposition: introduction of a fictitious patch

- Deterministic fictitious domain $\tilde{\Omega}$
- Deterministic fictitious patch $\tilde{\Lambda}$
- Uncertain patch Λ
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Reformulation with overlapping domains

Find $(U, w, \lambda) \in \tilde{\mathcal{U}}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$ such that it holds almost surely

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$$c_{\tilde{\Omega}}(U(\xi), \delta U; \xi) - c_{\tilde{\Lambda}}(U(\xi), \delta U; \xi) + b_{\Gamma}(\lambda(\xi), \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \tilde{\mathcal{U}}$$

where bilinear form $c_{\mathcal{O}}(U, V; \xi) = \int_{\mathcal{O}} \tilde{K}(\xi) \nabla U \cdot \nabla V$, with $\tilde{K} = K$ on $\Omega \setminus \Lambda$.

→ Global-local problem admits an infinite number of solutions (U, w, λ)

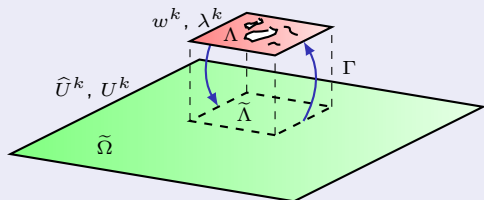
→ Choosing \tilde{K} deterministic on fictitious patch $\tilde{\Lambda}$ leads to a global problem with fully deterministic operator

Global-local iterative algorithm

Iterative coupling procedure

1

Global step

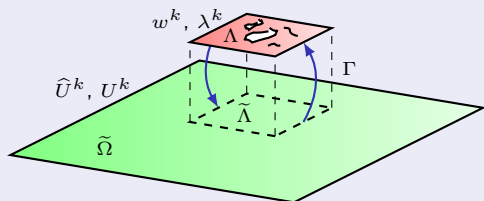
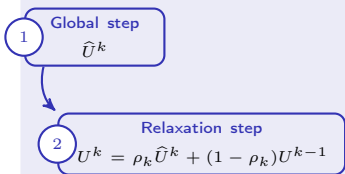
 \hat{U}^k 

- 1 Global step: Compute $\hat{U}^k \in \tilde{U}^\Xi$ such that it satisfies almost surely

$$c_{\tilde{\Omega}}(\hat{U}^k(\xi), \delta U; \xi) = c_{\tilde{\Lambda}}(U^{k-1}(\xi), \delta U; \xi) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \tilde{U}$$

Global-local iterative algorithm

Iterative coupling procedure



- 1 **Global step:** Compute $\hat{U}^k \in \tilde{\mathcal{U}}^\Xi$ such that it satisfies almost surely

$$c_{\tilde{\Omega}}(\hat{U}^k(\xi), \delta U; \xi) = c_{\tilde{\Lambda}}(U^{k-1}(\xi), \delta U; \xi) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \tilde{\mathcal{U}}$$

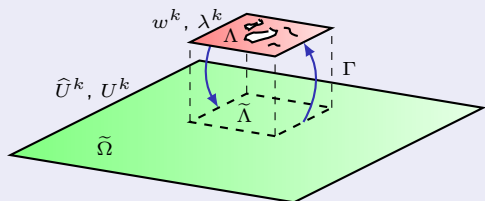
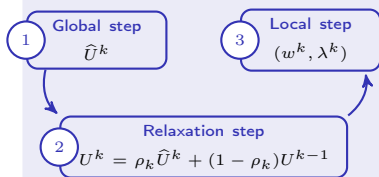
- 2 **Relaxation step:** Define $U^k \in \tilde{\mathcal{U}}^\Xi$ by

$$U^k(\xi) = \rho_k \hat{U}^k(\xi) + (1 - \rho_k) U^{k-1}(\xi) \quad \text{with relaxation parameter } \rho_k > 0$$

↔ Significant influence on the convergence properties of the algorithm

Global-local iterative algorithm

Iterative coupling procedure



- 1 **Global step:** Compute $\hat{U}^k \in \tilde{\mathcal{U}}^\Xi$ such that it satisfies almost surely
- $$c_{\tilde{\Omega}}(\hat{U}^k(\xi), \delta U; \xi) = c_{\tilde{\Lambda}}(U^{k-1}(\xi), \delta U; \xi) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \tilde{\mathcal{U}}$$

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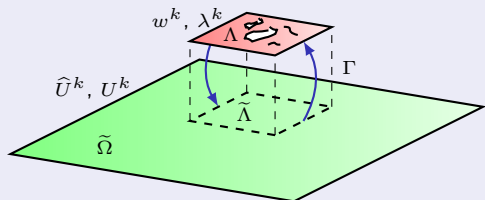
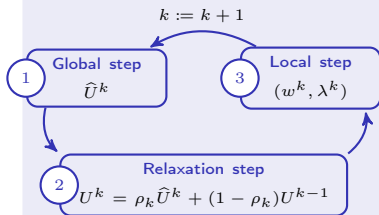
\leftrightarrow Significant influence on the convergence properties of the algorithm

- 3 **Local step:** Compute $(w^k, \lambda^k) \in \mathcal{W}^\Xi \times \mathcal{M}^\Xi$ such that it satisfies almost surely

$$\begin{cases} a_{\Lambda}(w^k(\xi), \delta w; \xi) + n_{\Lambda}(w^k(\xi), \delta w; \xi) - b_{\Gamma}(\lambda^k(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) & \forall \delta w \in \mathcal{W} \\ b_{\Gamma}(\delta \lambda, w^k(\xi)) = b_{\Gamma}(\delta \lambda, U^k(\xi)) & \forall \delta \lambda \in \mathcal{M} \end{cases}$$

Global-local iterative algorithm

Iterative coupling procedure



- 1 Global step:** Compute $\widehat{U}^k \in \widetilde{\mathcal{U}}^\Xi$ such that it satisfies almost surely

$$c_{\widetilde{\Omega}}(\widehat{U}^k(\xi), \delta U; \xi) = c_{\widetilde{\Lambda}}(U^{k-1}(\xi), \delta U; \xi) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \widetilde{\mathcal{U}}$$
- 2 Relaxation step:** Define $U^k \in \widetilde{\mathcal{U}}^\Xi$ by

$$U^k(\xi) = \rho_k \widehat{U}^k(\xi) + (1 - \rho_k) U^{k-1}(\xi) \quad \text{with relaxation parameter } \rho_k > 0$$

\hookrightarrow Significant influence on the convergence properties of the algorithm
- 3 Local step:** Compute $(w^k, \lambda^k) \in \mathcal{W}^\Xi \times \mathcal{M}^\Xi$ such that it satisfies almost surely

$$\begin{cases} a_{\Lambda}(w^k(\xi), \delta w; \xi) + n_{\Lambda}(w^k(\xi), \delta w; \xi) - b_{\Gamma}(\lambda^k(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) & \forall \delta w \in \mathcal{W} \\ b_{\Gamma}(\delta \lambda, w^k(\xi)) = b_{\Gamma}(\delta \lambda, U^k(\xi)) & \forall \delta \lambda \in \mathcal{M} \end{cases}$$

Analysis of the algorithm

Consistency

- Introduction of the closed linear subspace $\tilde{\mathcal{U}}_\star(\xi)$ of $\tilde{\mathcal{U}}$ defined by

$$\tilde{\mathcal{U}}_\star(\xi) = \{V \in \tilde{\mathcal{U}} ; c_{\tilde{\Lambda}}(V, \delta U; \xi) = 0 \text{ for all } \delta U \in H_0^1(\tilde{\Lambda})\}$$

- All global iterates $U^k(\xi)$ belong to the subspace $\tilde{\mathcal{U}}_\star(\xi)$.
- If the sequence $\{(U^k(\xi), w^k(\xi), \lambda^k(\xi))\}_{k \in \mathbb{N}}$ strongly converges to an element $(\tilde{U}(\xi), w(\xi), \lambda(\xi))$ in $\tilde{\mathcal{U}} \times \mathcal{W} \times \mathcal{M}$, then $(\tilde{U}(\xi)|_{\Omega \setminus \Lambda}, w(\xi), \lambda(\xi)) \in \mathcal{U} \times \mathcal{W} \times \mathcal{M}$ is the **unique solution** $(U(\xi), w(\xi), \lambda(\xi))$ of the initial global-local problem.
- Also, the limit $\tilde{U}(\xi)$ is the **unique extension** of $U(\xi)$ to $\tilde{\mathcal{U}}_\star(\xi)$ such that

$$\begin{cases} \tilde{U}(\xi)|_{\Omega \setminus \Lambda} = U(\xi) \\ \tilde{U}(\xi)|_{\tilde{\Lambda}} \text{ is uniquely defined by } U(\xi)|_{\Gamma} \end{cases}$$

Then, we denote by $U(\xi) \in \tilde{\mathcal{U}}_\star(\xi)$ the unique extension to $\tilde{\Omega}$ of the global solution $U(\xi) \in \mathcal{U}$.

- The extended global solution $U \in L_\mu^p(\Xi; \tilde{\mathcal{U}})$, with exponent $2 \leq p \leq +\infty$.

Analysis of the algorithm

Convergence

- One iteration of the algorithm can be written as

$$U^k(\xi) = B_{\rho_k}(U^{k-1}(\xi); \xi),$$

where $B_{\rho_k}(\cdot; \xi): \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{U}}$ is the **non-linear iteration map**.

- Assume that the sequence of relaxation parameters $\{\rho_k\}_{k \in \mathbb{N}}$ is such that

$$0 < \rho_{\inf} \leq \rho_k \leq \rho_{\sup} < +\infty,$$

for some strictly positive constants ρ_{\inf} and ρ_{\sup} independent of ξ and k .

Then, for ρ_{\sup} sufficiently small, the sequence $\{(U^k(\xi), w^k(\xi), \lambda^k(\xi))\}_{k \in \mathbb{N}}$ converges almost surely to the unique solution $(U(\xi), w(\xi), \lambda(\xi))$ of the initial global-local problem in $\tilde{\mathcal{U}}_*(\xi) \times \mathcal{W} \times \mathcal{M}$.

↪ Almost sure convergence

- Also, the sequence $\{U^k\}_{k \in \mathbb{N}}$ (resp. $\{w^k\}_{k \in \mathbb{N}}$ and $\{\lambda^k\}_{k \in \mathbb{N}}$) converges to U (resp. w and λ) in $L^p_\mu(\Xi; \tilde{\mathcal{U}})$ (resp. $L^p_\mu(\Xi; \mathcal{W})$ and $L^p_\mu(\Xi; \mathcal{M})$).

↪ Convergence in L^p_μ

Analysis of the algorithm

Robustness - Impact of approximations

- Analysis of the **sensitivity** of the iterative algorithm with respect to **approximations** introduced in the various steps of the algorithm (e.g. due to the use of iterative solvers with finite precision for solving global and/or local problems)
- Assume that approximations are **controlled** in a L_μ^p -norm (typically $p = 2$ or $p = \infty$), that means the **approximation error** $U_\varepsilon^k - U^k$ at iteration k should satisfy

$$\|U_\varepsilon^k - U^k\|_{L_\mu^p(\Xi; \tilde{u})} \leq \varepsilon \|U\|_{L_\mu^p(\Xi; \tilde{u})} + \varepsilon^* \|U_\varepsilon^{k-1} - U\|_{L_\mu^p(\Xi; \tilde{u})}$$

where ε conveys an **absolute error** with respect to the solution norm $\|U\|_{L_\mu^p(\Xi; \tilde{u})}$, while ε^* conveys an **approximation error** controlled relatively to the solution error $\|U_\varepsilon^{k-1} - U\|_{L_\mu^p(\Xi; \tilde{u})}$ at previous iteration $k - 1$.

- Suppose that the set of iteration maps $\{B_{\rho_k}(\cdot; \xi)\}_{k \geq 1}$ is uniformly contractive on $\tilde{U}_*(\xi)$, that means

$$\|B_{\rho_k}(V; \xi) - B_{\rho_k}(W; \xi)\|_{\tilde{u}} \leq \rho_B \|V - W\|_{\tilde{u}},$$

for all $V, W \in \tilde{U}_*(\xi)$, with a contractivity constant $\rho_B < 1$ independent of ξ .

Analysis of the algorithm

Robustness - Impact of approximations

- Further assume that the set of perturbed iteration maps $\{B_{\rho_k}^\varepsilon(\cdot; \xi)\}_{k \geq 1}$ is such that for all V in a δ -neighborhood \mathcal{V}_δ of the exact global solution U , defined by $\mathcal{V}_\delta = \{V \in L_\mu^p(\Xi; \tilde{\mathcal{U}}_*); \|V - U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})} < \delta \|U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})}\}$, we have

$$\|B_{\rho_k}^\varepsilon(V) - B_{\rho_k}(V)\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})} \leq \varepsilon \|U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})} + \varepsilon^* \|V - U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})},$$

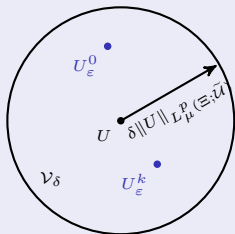
for some given tolerances $0 \leq \varepsilon^* < 1 - \rho_B$ and $0 \leq \varepsilon \leq \delta(1 - \rho_B - \varepsilon^*)$.

- Then, if the initial global iterate $U_\varepsilon^0 \in \mathcal{V}_\delta$, the **approximate sequence** $\{U_\varepsilon^k\}_{k \in \mathbb{N}}$ is such that

$$\limsup_{k \rightarrow +\infty} \|U_\varepsilon^k - U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})} \leq \gamma(\varepsilon, \varepsilon^*) \|U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})},$$

with $\gamma(\varepsilon, \varepsilon^*) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and **tends to a $\gamma(\varepsilon, \varepsilon^*)$ -neighborhood of U in $L_\mu^p(\Xi; \tilde{\mathcal{U}})$.**

- Open ball \mathcal{V}_δ of radius $\delta \|U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})}$ centered at the exact global solution U and containing all global iterates U_ε^k



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Computational aspects

Global problem

- Direct computation of global problem by solving a system of **uncoupled linear** equations with the **same deterministic** global finite element matrix
- Use of **traditional (parallel) solvers** available in **commercial deterministic** finite element **codes**

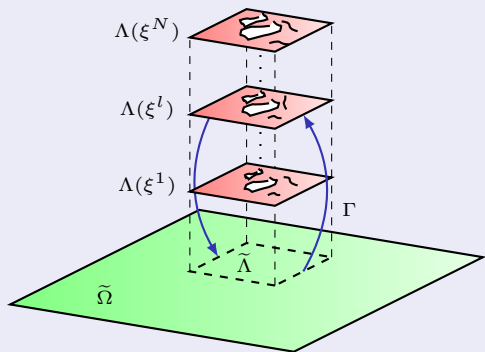
Local problems

- Resolution of stochastic local problems using a **sampling-based (non-intrusive)** approach
- Least-squares minimization technique requires the availability of **standard deterministic codes** and is well adapted to massively **parallel computation**
- Control of the accuracy of local approximate solutions by an **adaptive selection** of stochastic approximation spaces and sample sets for stable approximations
- Greedy algorithms proposed in [Chkifa et al. 2013; Chkifa, Cohen, and Schwab 2014; Chkifa, Cohen, and Schwab 2014; Chkifa, Cohen, and Schwab 2015] for the construction of **sparse** polynomial approximations of local solutions

Non-intrusive sampling-based approach

Least-squares minimization

- Evaluations of the solution of local deterministic models performed in parallel and using deterministic solvers
- N random evaluations of stochastic local model over patches $\Lambda(\xi^l)$, $l = 1 \dots N$
- Global model over fictitious domain $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$



Relaxation step

Fixed relaxation parameter

- Simplest and probably most ineffective method
- A **large** relaxation parameter may speed up the convergence but can lead to a divergence of the algorithm, while a **small** relaxation parameter enables to control the convergence but trigger more iterations in return.
- The computation of an **optimal** relaxation parameter is not obvious in the non-linear framework.
- Even in the linear case, the **optimal** fixed value of relaxation parameter ρ is problem-dependent and not known *a priori*.
- In the linear case and under some conditions on the iteration map, an optimal convergent rate of the global-local iterative algorithm can be achieved by computing the **optimal** relaxation parameter ρ_{opt} defined by [Xu 1992]

$$\rho_{\text{opt}} = \frac{2}{\lambda_{\min} + \lambda_{\max}},$$

where λ_{\min} and λ_{\max} are the lowest and largest eigenvalues of the iteration map (see [Chevreuil, Nouy, and Safatly 2013]).

Relaxation step

Convergence acceleration technique - Aitken relaxation

- Aitken's Delta-Squared method [Irons and Tuck 1969; Macleod 1986]
 - Simple and efficient convergence acceleration technique
 - Improvement of the current solution by using information gained at two previous iterations
- Update Aitken's formula for the relaxation parameter [Küttler and Wall 2008; Liu, Sun, and Fan 2014; Duval et al. 2014]

$$\begin{cases} \rho_1 = \rho_2 = 1 \\ \rho_k = -\rho_{k-1} \frac{\langle \delta^k(\xi) - \delta^{k-1}(\xi), \delta^{k-1}(\xi) \rangle_{\tilde{u}}}{\|\delta^k(\xi) - \delta^{k-1}(\xi)\|_{\tilde{u}}^2}, \quad \text{with } \delta^k(\xi) = \hat{U}^k(\xi) - U^{k-1}(\xi) \end{cases}$$

- New formula to ensure convergence of the algorithm

$$\rho_k = T_{[\rho_{\text{inf}}, \rho_{\text{sup}}]} \left(-\rho_{k-1} \frac{\langle \delta^k(\xi) - \delta^{k-1}(\xi), \delta^{k-1}(\xi) \rangle_{\tilde{u}}}{\|\delta^k(\xi) - \delta^{k-1}(\xi)\|_{\tilde{u}}^2} \right),$$

where $T_{[\rho_{\text{inf}}, \rho_{\text{sup}}]}$ is the truncate function defined by

$$T_{[\rho_{\text{inf}}, \rho_{\text{sup}}]}(\rho) = \begin{cases} \rho & \text{if } \rho_{\text{inf}} \leq \rho \leq \rho_{\text{sup}} \\ \rho_{\text{inf}} & \text{if } \rho < \rho_{\text{inf}} \\ \rho_{\text{sup}} & \text{if } \rho > \rho_{\text{sup}} \end{cases}$$

Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method with patches
- 3 Computational aspects
- 4 Numerical illustrations**
 - Isotropic case
 - Anisotropic case
- 5 Conclusions and outlooks
- 6 Bibliography

Non-linear stochastic problem - Isotropic case

Non-linear diffusion-reaction problem with localized random material heterogeneities

$$-\nabla \cdot (K \nabla u) + Ru^3 = 100 \quad \text{on } \Omega = (0, 1)^2$$

$$u = 0 \quad \text{on } \Gamma_D = \partial\Omega$$

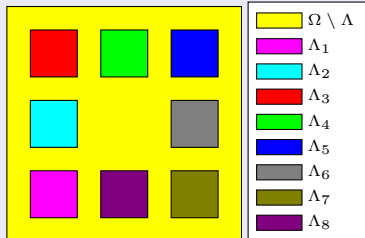
- Random diffusion and reaction fields

$$K = \begin{cases} 1 & \text{on } \Omega \setminus \left(\bigcup_{q=1}^8 \Lambda_q \right) \\ 1 + \xi_{2q-1} \chi_q(x) & \text{on } \Lambda_q \end{cases}$$

$$R = \begin{cases} 0 & \text{on } \Omega \setminus \left(\bigcup_{q=1}^8 \Lambda_q \right) \\ \xi_{2q} \chi_q(x) & \text{on } \Lambda_q \end{cases}$$

- Uniform random variables

$$\xi = (\xi_i)_{i=1}^{16} \quad \text{on } \Xi_i = (0, 1)$$



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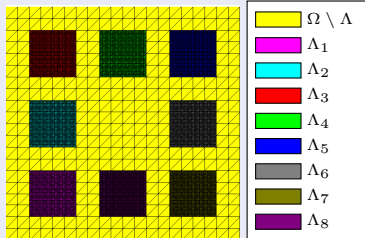
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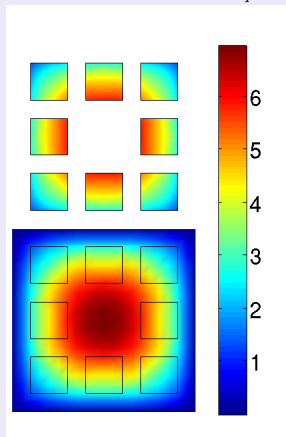
Approximation spaces

- **Spatial level:** nested global and local finite element approximation spaces
 - $\dim(\tilde{\mathcal{U}}_H) = 361$ on $\tilde{\Omega}$
 - $\dim(\mathcal{W}_h^q) = 1681$ on Λ_q
- **Stochastic level:** **sparse** Legendre polynomial approximation spaces
 - **Adaptive construction** of the multi-index set using a **greedy algorithm** [Chkifa et al. 2013]

Non-linear stochastic problem - Isotropic case

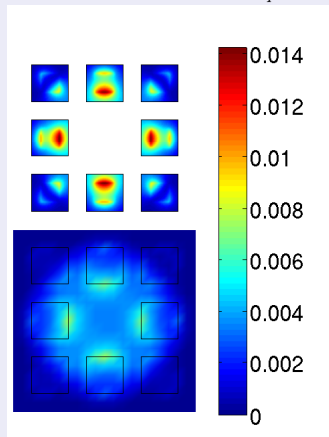
Statistical moments of global and local solutions at convergence

Mean $\mathbb{E}(w_q)$
of local solutions w_q



Mean $\mathbb{E}(U)$
of global solution U

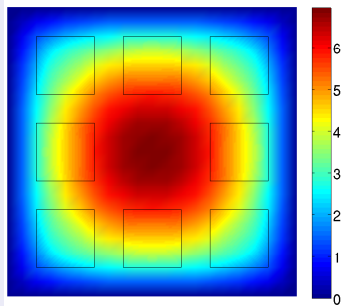
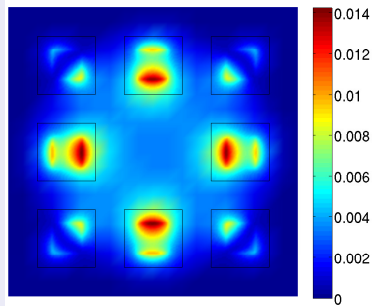
Variance $\mathbb{V}(w_q)$
of local solutions w_q



Variance $\mathbb{V}(U)$
of global solution U

Non-linear stochastic problem - Isotropic case

Statistical moments of multiscale solution at convergence

Mean $\mathbb{E}(u)$
of multiscale solution u Variance $\mathbb{V}(u)$
of multiscale solution u 

Non-linear stochastic problem - Isotropic case

Convergence

- Relative error indicator $\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}})$ in L^2_{μ} -norm used to control the accuracy of global approximations U^k with respect to a global reference solution U^{ref}

$$\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}}) = \frac{\|U^k - U^{\text{ref}}\|_{L^2_{\mu}(\Xi; L^2(\Omega))}}{\|U^{\text{ref}}\|_{L^2_{\mu}(\Xi; L^2(\Omega))}}$$

with $\|U\|_{L^2_{\mu}(\Xi; L^2(\Omega))} = \sqrt{\mathbb{E}_{\mu} \left(\|U(\xi)\|_{L^2(\Omega)}^2 \right)} = \sqrt{\int_{\Xi} \int_{\Omega} U(x, \xi)^2 \mu(d\xi)}$

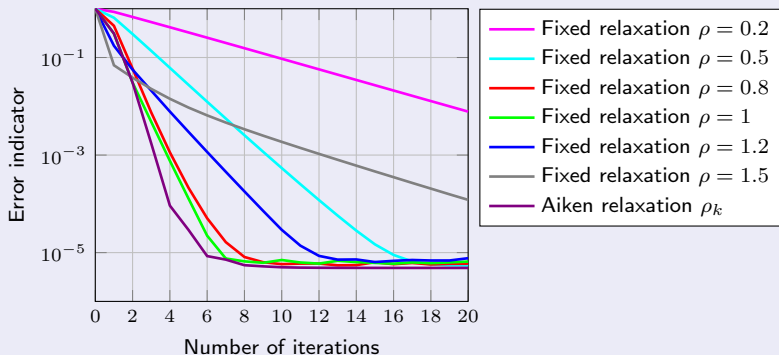
- Reference solution $(U^{\text{ref}}, w_q^{\text{ref}}, \lambda_q^{\text{ref}})$ obtained by solving the full-scale coupled problem using an adaptive least-squares minimization method
 - Deterministic non-linear coupled problems partially solved using a tangent-Newton iterative algorithm with a prescribed tolerance $\epsilon = 10^{-12}$
 - Prescribed tolerance $\varepsilon_{\text{cv}}^{\text{ref}} = 10^{-5}$ for cross-validation
 - Number of samples $N^{\text{ref}} = 9\,526$
 - Dimension of stochastic approximation spaces P^{ref}

	w_1^{ref}	w_2^{ref}	w_3^{ref}	w_4^{ref}	w_5^{ref}	w_6^{ref}	w_7^{ref}	w_8^{ref}	U^{ref}
P^{ref}	2 522	2 432	2 246	3 214	2 364	2 867	2 643	2 759	1 761
	λ_1^{ref}	λ_2^{ref}	λ_3^{ref}	λ_4^{ref}	λ_5^{ref}	λ_6^{ref}	λ_7^{ref}	λ_8^{ref}	
P^{ref}	2 421	3 577	2 056	3 978	1 865	4 060	2 451	3 627	

Non-linear stochastic problem - Isotropic case

Convergence

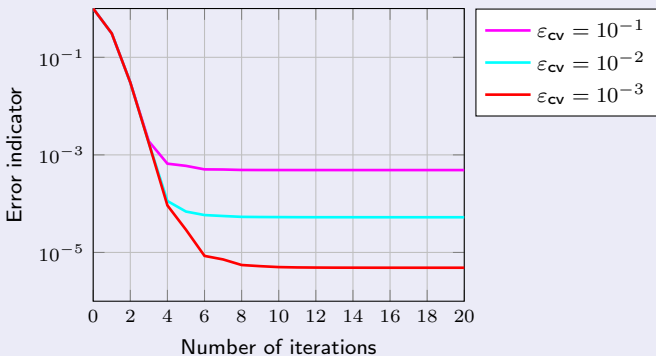
- Deterministic non-linear local problems partially solved using a tangent-Newton iterative algorithm with a prescribed tolerance $\epsilon = 10^{-12}$
- Prescribed tolerance $\epsilon_{cv} = 10^{-3}$ for cross-validation
- Evolution of relative error indicator $\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}})$ in L^2_μ -norm with respect to iteration number k for different values of relaxation parameter ρ_k



Non-linear stochastic problem - Isotropic case

Convergence

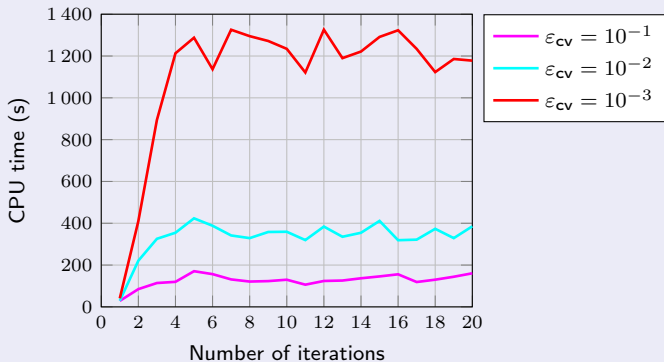
- Deterministic non-linear local problems partially solved using a tangent-Newton iterative algorithm with a prescribed tolerance $\epsilon = 10^{-12}$
- Relaxation parameter ρ_k dynamically updated using Aitken's Delta-Squared method
- Evolution of relative error indicator $\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}})$ in L^2_μ -norm with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Isotropic case

Computational efficiency

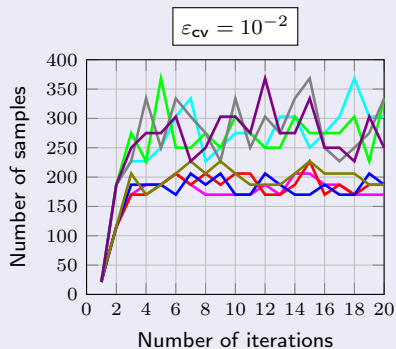
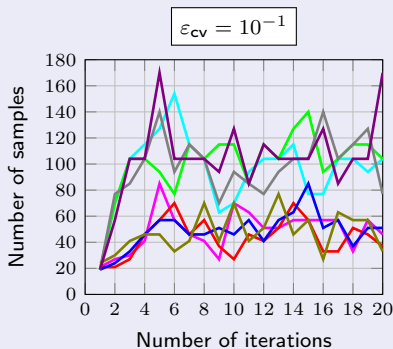
- Evolution of CPU time t with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Isotropic case

Adaptive sampling

- Initial number of samples $N_{\text{init}} = 2$
- Sample size increased by $N_{\text{fact}} = 10\%$
- Evolution of **sample size** N_q for local solution (w_q^k, λ_q^k) in each patch Λ_q with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates

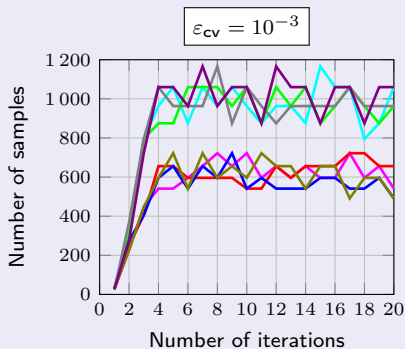


— N_1
— N_2
— N_3
— N_4
— N_5
— N_6
— N_7
— N_8

Non-linear stochastic problem - Isotropic case

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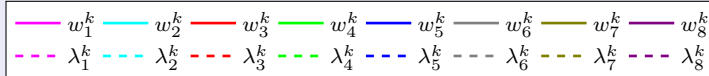
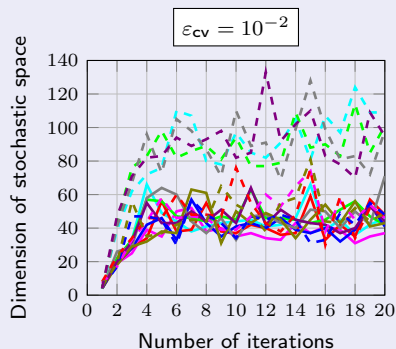
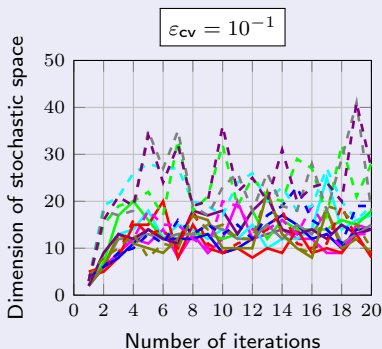


— N_1
— N_2
— N_3
— N_4
— N_5
— N_6
— N_7
— N_8

Non-linear stochastic problem - Isotropic case

Adaptive construction of sparse polynomial approximation spaces

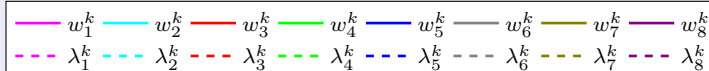
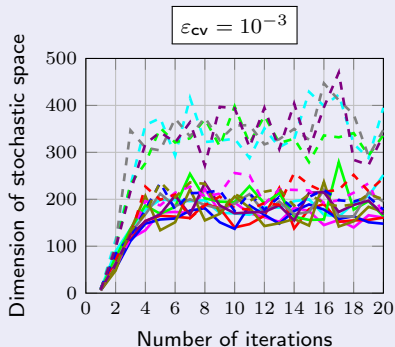
- Initial dimension $P_{\text{init}} = 0$
- Evolution of the **dimension of the stochastic approximation space** $P(w_q^k)$ (resp. $P(\lambda_q^k)$) associated to local solution w_q^k (resp. Lagrange multiplier λ_q^k) in each patch Λ_q with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Isotropic case

Adaptive construction of sparse polynomial approximation spaces

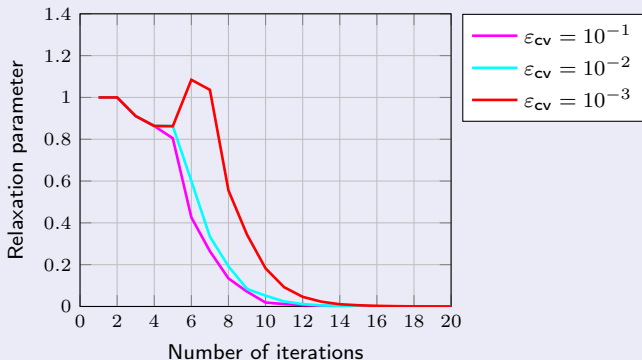
- Initial dimension $P_{\text{init}} = 0$
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Non-linear stochastic problem - Isotropic case

Aitken relaxation

- Evolution of relaxation parameter ρ_k with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates

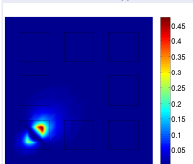


Non-linear stochastic problem - Isotropic case

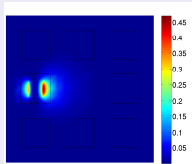
Probabilistic outputs - Quantities of interest

- Sensitivity indices $\tilde{S}_i(u) = \frac{\mathbb{V}(\mathbb{E}(u|\xi_i))}{\max_{x \in \Omega} \mathbb{V}(u)}$ quantifying the relative importance and the **impact** of each random variable ξ_i on the variance $\mathbb{V}(u)$ of multiscale solution u

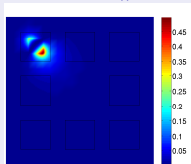
Patch #1

 $\tilde{S}_1(u)$

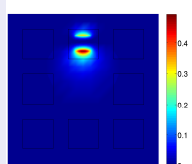
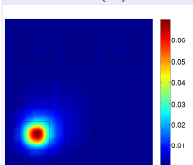
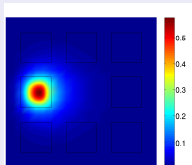
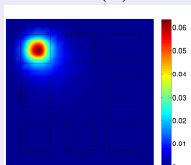
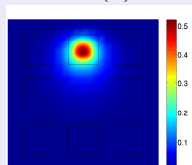
Patch #2

 $\tilde{S}_3(u)$

Patch #3

 $\tilde{S}_5(u)$

Patch #4

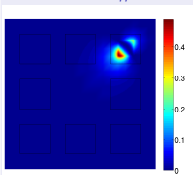
 $\tilde{S}_7(u)$  $\tilde{S}_2(u)$  $\tilde{S}_4(u)$  $\tilde{S}_6(u)$  $\tilde{S}_8(u)$

Non-linear stochastic problem - Isotropic case

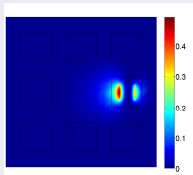
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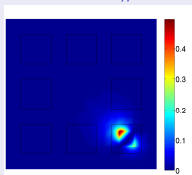
Patch #5

 $\tilde{S}_9(u)$

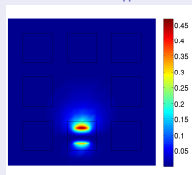
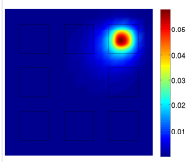
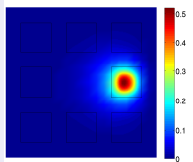
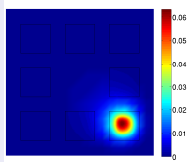
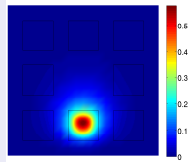
Patch #6

 $\tilde{S}_{11}(u)$

Patch #7

 $\tilde{S}_{13}(u)$

Patch #8

 $\tilde{S}_{15}(u)$  $\tilde{S}_{10}(u)$  $\tilde{S}_{12}(u)$  $\tilde{S}_{14}(u)$  $\tilde{S}_{16}(u)$

Non-linear stochastic problem - Anisotropic case

Non-linear diffusion-reaction problem with localized random material heterogeneities

$$-\nabla \cdot (K \nabla u) + Ru^3 = 100 \quad \text{on } \Omega = (0, 1)^2$$

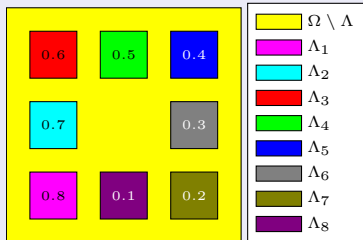
$$u = 0 \quad \text{on } \Gamma_D = \partial\Omega$$

- **Random** diffusion and reaction fields

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 $\xi = (\xi_i)_{i=1}^{16}$ on $\Xi_i = (0, 1)$



- **Real coefficients (Anisotropy)**
 $\gamma_q = (1 - 0.1(q + 1))$

Non-linear stochastic problem - Anisotropic case

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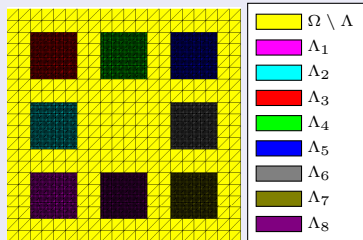
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$$R = \begin{cases} 0 & \text{on } \Omega \setminus \left(\bigcup_{q=1}^8 \Lambda_q\right) \\ \gamma_q \xi_{2q} \chi_q(x) & \text{on } \Lambda_q \end{cases}$$

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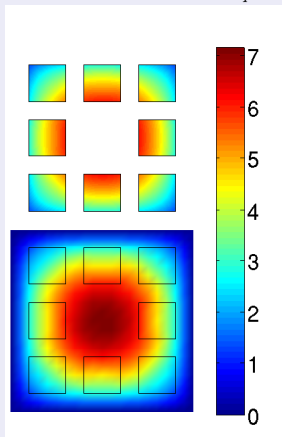
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Non-linear stochastic problem - Anisotropic case

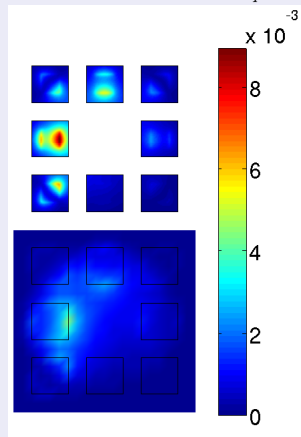
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Mean $\mathbb{E}(U)$
of global solution U

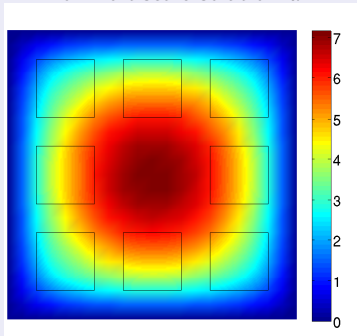
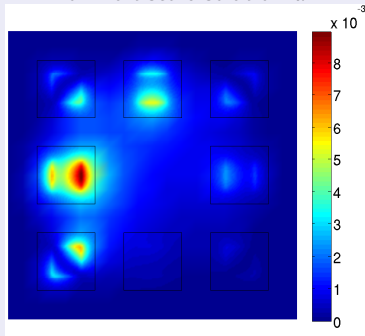
Variance $\mathbb{V}(w_q)$
of local solutions w_q



Variance $\mathbb{V}(U)$
of global solution U

Non-linear stochastic problem - Anisotropic case

Statistical moments of multiscale solution at convergence

Mean $\mathbb{E}(u)$
of multiscale solution u Variance $\mathbb{V}(u)$
of multiscale solution u 

Non-linear stochastic problem - Anisotropic case

Convergence

- Relative error indicator $\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}})$ in L^2_{μ} -norm used to control the accuracy of global approximations U^k with respect to a global reference solution U^{ref}

$$\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}}) = \frac{\|U^k - U^{\text{ref}}\|_{L^2_{\mu}(\Xi; L^2(\Omega))}}{\|U^{\text{ref}}\|_{L^2_{\mu}(\Xi; L^2(\Omega))}}$$

with $\|U\|_{L^2_{\mu}(\Xi; L^2(\Omega))} = \sqrt{\mathbb{E}_{\mu} \left(\|U(\xi)\|_{L^2(\Omega)}^2 \right)} = \sqrt{\int_{\Xi} \int_{\Omega} U(x, \xi)^2 \mu(d\xi)}$

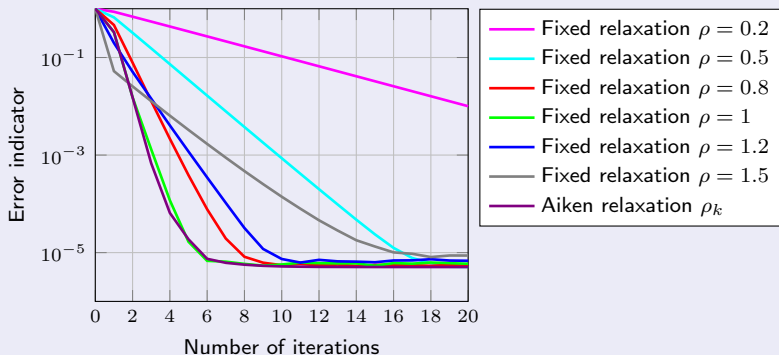
- Reference solution $(U^{\text{ref}}, w_q^{\text{ref}}, \lambda_q^{\text{ref}})$ obtained by solving the full-scale coupled problem using an adaptive least-squares minimization method
 - Deterministic non-linear coupled problems partially solved using a tangent-Newton iterative algorithm with a prescribed tolerance $\epsilon = 10^{-12}$
 - Prescribed tolerance $\varepsilon_{\text{cv}}^{\text{ref}} = 10^{-6}$ for cross-validation
 - Number of samples $N^{\text{ref}} = 7872$
 - Dimension of stochastic approximation spaces P^{ref}

	w_1^{ref}	w_2^{ref}	w_3^{ref}	w_4^{ref}	w_5^{ref}	w_6^{ref}	w_7^{ref}	w_8^{ref}	U^{ref}
P^{ref}	1 844	2 077	2 092	2 020	1 681	1 967	1 769	1 800	1 381
	λ_1^{ref}	λ_2^{ref}	λ_3^{ref}	λ_4^{ref}	λ_5^{ref}	λ_6^{ref}	λ_7^{ref}	λ_8^{ref}	
P^{ref}	1 560	3 024	2 198	2 783	1 496	2 720	1 323	2 749	

Non-linear stochastic problem - Anisotropic case

Convergence

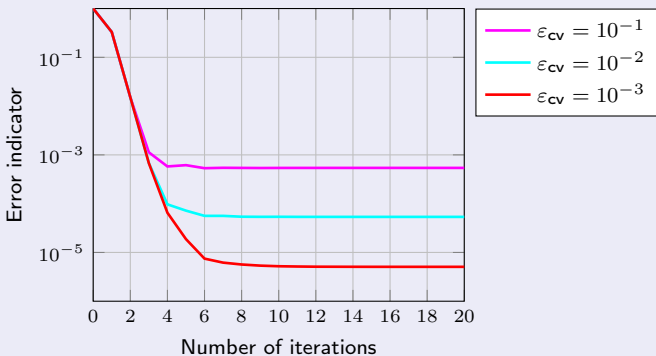
- Deterministic non-linear local problems partially solved using a tangent-Newton iterative algorithm with a prescribed tolerance $\epsilon = 10^{-12}$
- Prescribed tolerance $\epsilon_{cv} = 10^{-3}$ for cross-validation
- Evolution of relative error indicator $\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}})$ in L^2_μ -norm with respect to iteration number k for different values of relaxation parameter ρ_k



Non-linear stochastic problem - Anisotropic case

Convergence

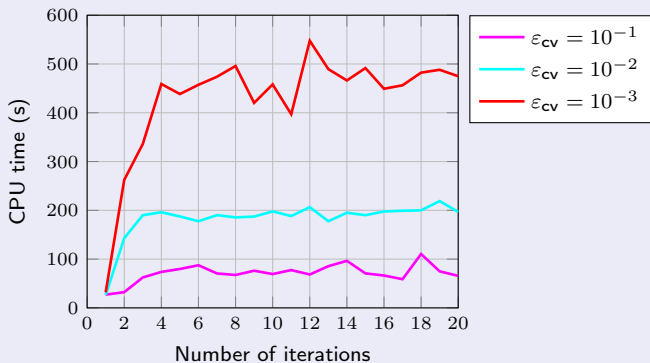
- Deterministic non-linear local problems partially solved using a tangent-Newton iterative algorithm with a prescribed tolerance $\epsilon = 10^{-12}$
- Relaxation parameter ρ_k dynamically updated using Aitken's Delta-Squared method
- Evolution of relative error indicator $\varepsilon_{\Xi, \Omega}(U^k; U^{\text{ref}})$ in L^2_μ -norm with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Anisotropic case

Computational efficiency

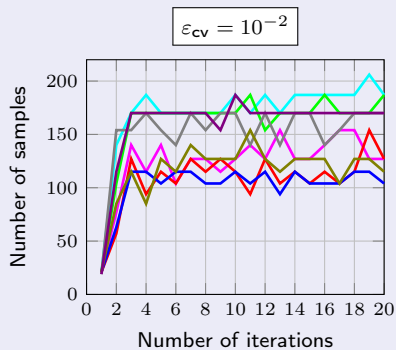
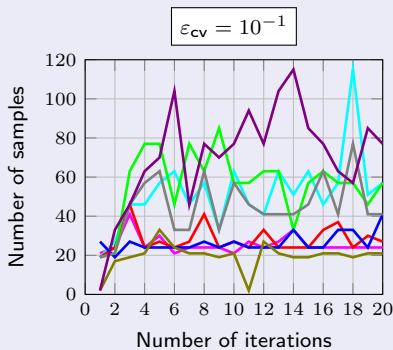
- Evolution of CPU time t with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Anisotropic case

Adaptive sampling

- Initial number of samples $N_{\text{init}} = 2$
- Sample size increased by $N_{\text{fact}} = 10\%$
- Evolution of **sample size** N_q for local solution (w_q^k, λ_q^k) in each patch Λ_q with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates

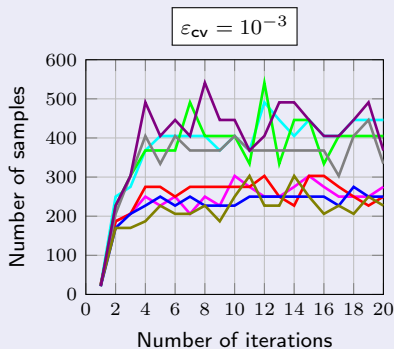


— N_1
— N_2
— N_3
— N_4
— N_5
— N_6
— N_7
— N_8

Non-linear stochastic problem - Anisotropic case

Adaptive sampling

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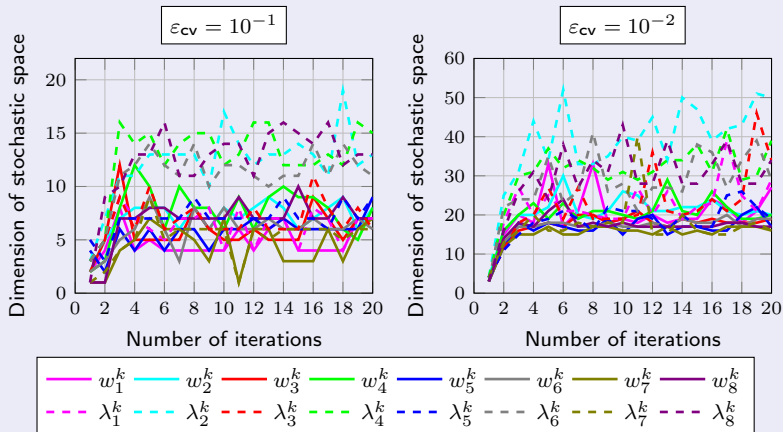


— N_1
— N_2
— N_3
— N_4
— N_5
— N_6
— N_7
— N_8

Non-linear stochastic problem - Anisotropic case

Adaptive construction of sparse polynomial approximation spaces

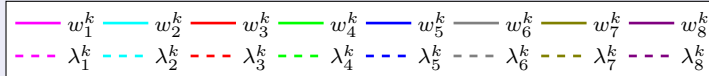
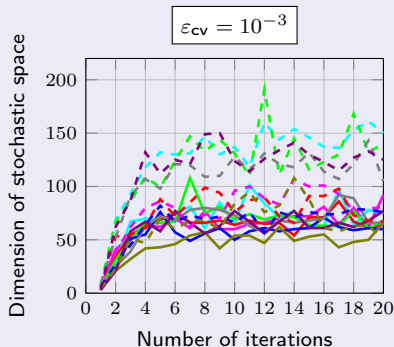
- Initial dimension $P_{\text{init}} = 0$
- Evolution of the dimension of the stochastic approximation space $P(w_q^k)$ (resp. $P(\lambda_q^k)$) associated to local solution w_q^k (resp. Lagrange multiplier λ_q^k) in each patch Λ_q with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Anisotropic case

Adaptive construction of sparse polynomial approximation spaces

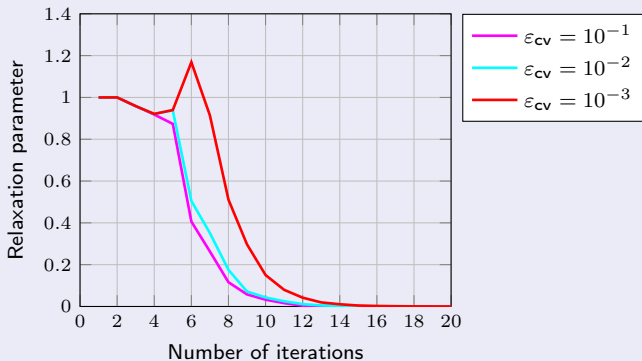
- Initial dimension $P_{\text{init}} = 0$
- Evolution of the dimension of the stochastic approximation space $P(w_q^k)$ (resp. $P(\lambda_q^k)$) associated to local solution w_q^k (resp. Lagrange multiplier λ_q^k) in each patch Λ_q with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates



Non-linear stochastic problem - Anisotropic case

Aitken relaxation

- Evolution of relaxation parameter ρ_k with respect to iteration number k for different tolerances ε_{cv} of cross-validation error estimates

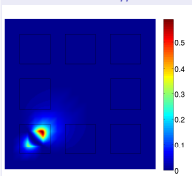


Non-linear stochastic problem - Anisotropic case

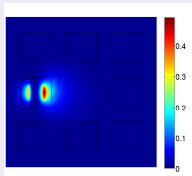
Probabilistic outputs - Quantities of interest

- Sensitivity indices $\tilde{S}_i(u) = \frac{\mathbb{V}(\mathbb{E}(u|\xi_i))}{\max_{x \in \Omega} \mathbb{V}(u)}$ quantifying the relative importance and the **impact** of each random variable ξ_i on the variance $\mathbb{V}(u)$ of multiscale solution u

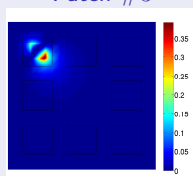
Patch #1

 $\tilde{S}_1(u)$

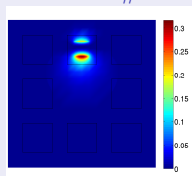
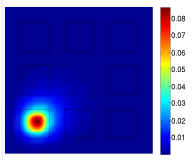
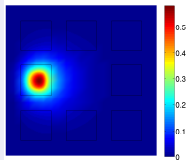
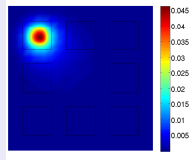
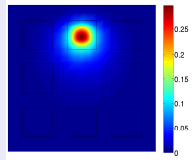
Patch #2

 $\tilde{S}_3(u)$

Patch #3

 $\tilde{S}_5(u)$

Patch #4

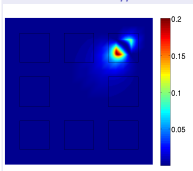
 $\tilde{S}_7(u)$  $\tilde{S}_2(u)$  $\tilde{S}_4(u)$  $\tilde{S}_6(u)$  $\tilde{S}_8(u)$

Non-linear stochastic problem - Anisotropic case

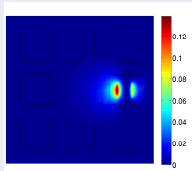
Probabilistic outputs - Quantities of interest

- Sensitivity indices $\tilde{S}_i(u) = \frac{\mathbb{V}(\mathbb{E}(u|\xi_i))}{\max_{x \in \Omega} \mathbb{V}(u)}$ quantifying the relative importance and the **impact** of each random variable ξ_i on the variance $\mathbb{V}(u)$ of multiscale solution u

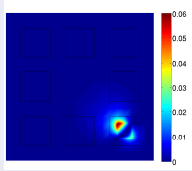
Patch #5

 $\tilde{S}_9(u)$

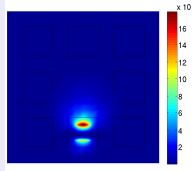
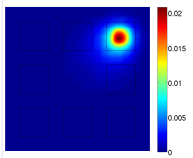
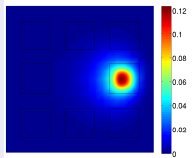
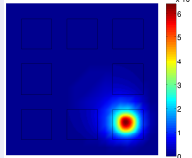
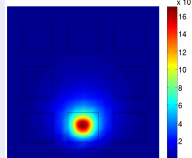
Patch #6

 $\tilde{S}_{11}(u)$

Patch #7

 $\tilde{S}_{13}(u)$

Patch #8

 $\tilde{S}_{15}(u)$  $\tilde{S}_{10}(u)$  $\tilde{S}_{12}(u)$  $\tilde{S}_{14}(u)$  $\tilde{S}_{16}(u)$

Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method with patches
- 3 Computational aspects
- 4 Numerical illustrations
- 5 Conclusions and outlooks**
- 6 Bibliography

Conclusions and outlooks

Conclusions

- A multiscale numerical method dedicated to **localized uncertainties** and possible **localized non-linearities**
 - Domain decomposition method with patches
 - Global-local iterative algorithm with proven convergence properties
- **Analysis** of the algorithm
 - **Consistency** analysis
 - **Convergence** analysis
 - **Robustness** analysis with respect to approximations

On-going works

- Influence of the distance between the patches on the approximation basis
- Improved sampling-based methods for solving non-linear local problems

Future works

- Model reduction/Tensor approximation methods for solving global and local problems to handle high-dimensional problems
- Special treatment of patches with variable positions/orientations
- Extension to linear and non-linear dynamics problems

Thanks for your attention !

Merci de votre attention !

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