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► **To cite this version:**

Yves Guivarc'H, Emile Le Page. On spectral gap properties and extreme value theory for multivariate affine stochastic recursions. 2017. hal-01307535v3

HAL Id: hal-01307535

<https://hal.science/hal-01307535v3>

Preprint submitted on 13 Dec 2017

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On spectral gap properties and extreme value theory for multivariate affine stochastic recursions

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Abstract

We consider a general multivariate affine stochastic recursion and the associated Markov chain on \mathbb{R}^d . We assume a natural geometric condition which implies existence of an unbounded stationary solution and we show that the large values of the associated stationary process follow extreme value properties of classical type, with a non trivial extremal index. We develop some explicit consequences such as convergence to Fréchet's law or to an exponential law, as well as convergence to a stable law. The proof is based on a spectral gap property for the action of associated positive operators on spaces of regular functions with slow growth, and on the clustering properties of large values in the recursion.

Keywords : Spectral gap, Extreme value, Affine random recursion, Point process, Extremal index, Excursion, Ruin, Laplace functional.

1 Introduction

Let $V = \mathbb{R}^d$ be the d -dimensional Euclidean space and let λ be a probability on the affine group H of V , μ the projection of λ on the linear group $G = GL(V)$. Let (A_n, B_n) be a sequence of H -valued i.i.d. random variables distributed according to λ and let us consider the affine stochastic recursion on V defined by

$$X_n = A_n X_{n-1} + B_n$$

for $n \in \mathbb{N}$. We denote by P the corresponding Markov kernel on V and by \mathbb{P} the product measure $\lambda^{\otimes \mathbb{N}}$ on $H^{\mathbb{N}}$. Our geometric hypothesis (c-e) on λ involves contraction and expansion properties and is robust for the relevant weak topology if $d > 1$; it implies that P has a unique invariant probability ρ on V , with unbounded support. In our situation (see [13]), the quantity $\rho\{|v| > t\}$ is asymptotic ($t \rightarrow \infty$) to $\alpha^{-1} c t^{-\alpha}$ with $\alpha > 0$, $c > 0$. More precisely the measure ρ is multivariate regularly varying, a basic property for the development of extreme value theory i.e. for the study of exceptionally large values of X_k ($1 \leq k \leq n$) for n large (see [30]), our main goal in this paper. One non trivial aspect of hypothesis (c-e) is that it implies unboundness of the subsemigroup of G generated by $\text{supp}(\mu)$, hence multiplicity of large values of $|X_n|$. Also, from a heuristic point of view, hypothesis (c-e) allows us, to reduce the d -dimensional linear situation to a 1-dimensional setting.

1. Corresponding author

In such a situation of weak dependance, spectral gap properties of operators associated to P play an important role via a multiple mixing condition described in [7] for step functions. We observe that the same idea has been used in various closely related situations : limit theorems for the largest coefficient in the continued fraction expansion of a real number (see [27], [37]), limit theorems for the ergodic sums $\sum_{k=1}^n X_k$ along the above stochastic recursion (see [15]), geodesic excursions on the modular surface (see [17], [28]), "shrinking target" problem (see [26], [32]). In our setting, spectral gap properties of operators associated to P allow us to study the path behaviour of the Markov chain defined by P . In the context of geometric ergodicity (see [31]) for the Markov operator P acting on measurable functions, assuming a density condition on the law of B_n , partial results were obtained in [22]. However simple examples show (see below) that, in general, P has no spectral gap in $\mathbb{L}^2(\rho)$. Here we go further in this direction replacing geometric ergodicity by condition (c-e). Condition (c-e) implies that the operator P has a spectral gap property in the spaces of Hölder functions with polynomial growth considered below, a fact which allow us to deduce convergence with exponential speed on Hölder functions. A typical example of this situation occurs if the support of λ is finite and generates a dense subsemigroup of the affine group H .

If \mathbb{Z}_+ is the set of non negative integers, we denote by \mathbb{P}_ρ the Markov probability on $V^{\mathbb{Z}_+}$ defined by the kernel P and the initial probability ρ . In this paper we establish spectral gap properties for the action of P on Hölder functions and we deduce fundamental extreme value statements for the point processes defined by the \mathbb{P}_ρ -stationary sequence $(X_k)_{k \geq 0}$. Our results are based on the fact that the general conditions of multiple mixing and anticlustering used in extreme value theory of stationary processes (see [2], [7]) are valid for affine stochastic recursions, under condition (c-e). We observe that, in the context of Lipschitz functions, the above mixing property is a consequence of the spectral gap properties studied below ; it turns out that the use of advanced point process theory allows us to extend this mixing property to the setting of compactly supported continuous functions considered in [2]. Using these basic results, we give proofs of a few extreme value properties, following closely [2].

Then, our framework allow us to develop extreme value theory for a large class of natural examples in collective risk theory, including the so-called GARCH process as a very special case (see [9]). In this context, some of our results have natural interpretations as asymptotics of ruin probabilities or ruin times [6].

In order to sketch our results, we recall that Fréchet's law Φ_α^a with positive parameters α, a is the max-stable probability on \mathbb{R}_+ defined by the distribution function $\Phi_\alpha^a([0, t]) = \exp(-at^{-\alpha})$. Also, we consider the associated stochastic linear recursion $Y_n = A_n Y_{n-1}$, we denote by Q the corresponding Markov kernel on $V \setminus \{0\}$ and by $\mathbb{Q} = \mu^{\otimes \mathbb{N}}$ the product measure on $\Omega = G^{\mathbb{N}}$; we write $S_n = A_n \cdots A_1$ for the product of random matrices $A_k (1 \leq k \leq n)$. Extending previous work of H. Kesten (see [20]), a basic result proved in [13] under condition (c-e) is that for some $\alpha > 0$, the probability ρ is α -homogeneous at infinity,

hence ρ has an asymptotic tail measure $\Lambda \neq 0$ which is a α -homogeneous Q -invariant Radon measure on $V \setminus \{0\}$. The multivariate regular variation of ρ is a direct consequence of this fact. Also, it follows that, if $U_t \subset V$ is the closed ball of radius $t > 0$ centered at $0 \in V$ and $U'_t = V \setminus U_t$, then we have $\Lambda(U'_t) = \alpha^{-1}ct^{-\alpha}$ with $c > 0$. In particular, $\Lambda(U'_t)$ is finite and the projection of ρ on \mathbb{R}_+ , given by the norm map $v \rightarrow |v|$ has the same asymptotic tail as Φ_α^c . We write $\Lambda_0 = c^{-1}\alpha\Lambda$ hence $\Lambda_0(U'_1) = 1$ and we denote by Λ_1 the restriction of Λ_0 to U'_1 . If $u_n > 0$ satisfies $(u_n)^\alpha = \alpha^{-1}cn$, it follows that the mean number of exceedances of u_n by $|X_k|$ ($1 \leq k \leq n$) converges to one. It will appear below that u_n is an estimate of $\sup\{|X_k|; 1 \leq k \leq n\}$ and that normalization by u_n reduces extreme values for the process X_n to excursions at infinity for the linear random walk $S_n(\omega)v$ and the measures Λ_0, Λ_1 .

Then, one of our main results is the convergence in law of the u_n -normalized maximum of the sequence $|X_1|, |X_2|, \dots, |X_n|$ towards Fréchet's law Φ_α^θ with $\theta \in]0, 1[$. A closely related point process result is the weak convergence of the time exceedances process

$$N_n^t = \sum_{k=1}^n \varepsilon_{n^{-1}k} 1_{\{|X_k| > u_n\}}$$

towards a compound Poisson process with intensity θ and cluster probabilities depending on the occupation measure $\pi_v^\omega = \sum_{i=0}^{\infty} \varepsilon_{S_i(\omega)v}$ of the associated linear random walk $S_i(\omega)v$ and on Λ_1 . The expression ε_x denotes the Dirac mass at x . The significance of the relation $\theta < 1$ is that, in our situation, values of the sequence $(|X_k|)_{0 \leq k \leq n}$ larger than u_n , appear in localized clusters with asymptotic expected cardinality $\theta^{-1} > 1$. This reflects the local dependance of large values in the sequence $(X_k)_{k \geq 0}$ and is in contrast with the well known situation of positive i.i.d. random variables with tail also given by Φ_α^c , where the same convergences with $\theta = 1$ is satisfied. If Euclidean norm is replaced by another norm, the value of the extremal index θ is changed but the condition $\theta \in]0, 1[$ remains valid. For affine stochastic recursions in dimension one, if A_n, B_n are positive and condition (c-e) is satisfied, convergence to Fréchet's law and $\theta \in]0, 1[$ was proved in [18], using [20]. We observe that our result is the natural multivariate extension of this fact. Here, our proofs use the tools of point processes theory and a remarkable formula (see [2]) for the Laplace functional of a cluster point process $C = \sum_{j \in \mathbb{Z}} \varepsilon_{Z_j}$ on $V \setminus \{0\}$, which describes in small time the large values of $(X_n)_{n \geq 0}$. As a consequence of Fréchet's law and in the spirit of [28], we obtain a logarithm law for affine random walk.

To go further, we consider the linear random walk $S_n(\omega)v$ on $V \setminus \{0\}$, we observe that condition (c-e) implies $\lim_{n \rightarrow \infty} S_n(\omega)v = 0$, \mathbb{Q} -a.e. and we denote by \mathbb{Q}_{Λ_0} (resp. \mathbb{Q}_{Λ_1}) the Markov measure on $(V \setminus \{0\})^{\mathbb{Z}_+}$ defined by the kernel Q and the initial measure Λ_0 (resp. Λ_1). We show below the weak convergence to a limit process N of the sequence of space-time exceedances processes

$$N_n = \sum_{i=1}^n \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$$

on $[0, 1] \times (V \setminus \{0\})$. In restriction to $[0, 1] \times U'_\delta$, with $\delta > 0$, N can be expressed in terms of

C and of a Poisson component on $[0, 1]$ with intensity $\theta\delta^{-\alpha}$; the expression of the Laplace functional of C involves the occupation measure π_v^ω and \mathbb{Q}_{Λ_1} . Using the framework and the results of ([2], [3], [7]), we describe a few consequences of this convergence. In particular we consider also, as in ([7], [8]), the convergence of the normalized partial sums $\sum_{i=1}^n X_i$ towards stable laws, if $0 < \alpha < 2$, in the framework of extreme value theory. Also, as observed in [7], this convergence is closely connected to the convergence of the sequence of space exceedances point processes on V

$$N_n^s = \sum_{i=1}^n \varepsilon_{u_n^{-1} X_i},$$

towards a certain infinitely divisible point process N^s . Another consequence of this convergence is the description of the asymptotics of the normalized hitting time of any dilated Borel subset of U'_1 with positive Λ -measure, and negligible boundary : weak convergence to a non trivial exponential law is valid. We observe that convergence of normalized hitting times to an exponential law is a well studied property in the context of the "shrinking target" problem. Here we are concerned with a special "random dynamical system" (see [32]) associated to the affine action on \mathbb{R}^d and the hitting times of shrinking neighbourhoods of the sphere at infinity of \mathbb{R}^d considered as a weakly attractive target. The point process approach and the use of spectral gap properties allow us to deal with this geometrically more complex situation which involves a non trivial extremal index.

In these studies we follow closely the approaches previously developed in ([2], [7]) in the context of extreme value theory for general stationary processes, in particular we make full use of the concepts of tail and cluster processes introduced in [2]. This allow us to recover, in a natural setting, the characteristic functions of the above α -stable laws, as described in [15] if $d = 1$ and in [10] if $d > 1$, completing thereby the results of ([1], [2], [7]). Furthermore, in view of our results we observe that, with respect to the Q -invariant measure Λ_0 , the potential theory of the associated linear random walk enters essentially in the description of the extreme value asymptotics for the affine random walk X_n , through excursions, occupation measures and capacities.

For self containment reasons we have developed anew a few arguments of ([2]) in our situation. However we have modified somewhat the scheme of ([2], [33]) for the construction of the tail process, introducing a shift invariant measure $\widehat{\mathbb{Q}}_{\Lambda_0}$ on $(V \setminus \{0\})^{\mathbb{Z}}$ which governs the excursions at infinity of the associated linear random walk $S_n(\omega)v$. This lead us to an essentially self-contained presentation for the extremal index and the point process asymptotics. It is the use of its restriction $\widehat{\mathbb{Q}}_{\Lambda_1}$, which allows one to express geometrically the point processes N, N^s, N^t , as in ([2], [3]).

Here is the structure of our paper.

In section 2, we recall a basic result of [13] and we define two processes of probabilistic significance. This allow us to connect the extreme values of the affine random walk to the excursions of its associated linear random walk, as in [2]. In view of the results in section

4, it is essential to deal with \mathbb{Z} -indexed processes, hence to introduce the shift invariant measure $\widehat{\mathbb{Q}}_{\Lambda_0}$.

In section 3 we develop the spectral gap properties for affine random walks. In particular we deduce the multiple mixing property which plays an essential role in section 4.

In section 4 we describe the new results and give the corresponding proofs.

Section 5 describe briefly new proofs of the convergence to stable laws, in the context of point processes using the approach of [2], [3], [7].

In section 6 we show that our basic hypothesis (c-e) is "generic" in the weak topology of measures on the affine group H .

We refer to ([14], [16]) for surveys of the above results. After submission of this paper the authors became aware of the content of the book [5], where closely related results are proved by different methods. We thank the referees for substancial and useful comments. We thank also J.P. Conze for suggesting the use of dynamical methods in the above context.

2 The tail process and the cluster process

2.1 Homogeneity at infinity of the stationary measure

We recall condition (c-e) from [13], for the probability λ on the affine group of V .

A semigroup T of $GL(V) = G$ is said to satisfy i-p if

- a) T has no invariant finite family of proper subspaces
- b) T contains an element with a dominant eigenvalue which is real and unique.

Condition i-p implies that the action of T on the projective space of V is proximal ; heuristically speaking this means that, T contracts asymptotically two arbitrary given directions to a single one, hence the situation could be compared to a 1-dimensional one. Condition i-p for T is valid if and only if it is valid for the group which is the Zariski closure of T , (see [25]). We recall that a Zariski closed subset of an algebraic group like G or H is a subset defined as the set of zeros of a family of polynomials in the coordinates. The corresponding topology is much weaker than the usual locally compact topology . In particular the Zariski closure of a subsemigroup is a subgroup which is closed in the usual topology and which has a finite number of connected components. Hence condition i-p is valid if T is Zariski dense in G (see [29]) ; also it is valid for T if it is valid for T^{-1} . Below we will denote by T the closed subsemigroup generated by $\text{supp}(\mu)$, the support of μ .

For $g \in G$ we write $\gamma(g) = \sup(|g|, |g^{-1}|)$ and we assume $\int \log \gamma(g) d\mu(g) < \infty$. For $s \geq 0$ we write $k(s) = \lim_{n \rightarrow \infty} \left(\int |g|^s d\mu^n(g) \right)^{1/n}$ where μ^n denotes the n^{th} convolution power of μ and we write $L(\mu)$ for the dominant Lyapunov exponent of the product $S_n(\omega) = A_n \cdots A_1$ of random matrices $A_k (1 \leq k \leq n)$ i.e. $L(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log |g| d\mu^n(g) = k'(0)$. We denote by $r(g)$ the spectral radius of $g \in G$. We say that T is non arithmetic if $r(T)$ contains two elements with irrational ratio. Condition (c-e) is the following :

- 1) $\text{supp}(\lambda)$ has no fixed point in V .
- 2) There exists $\alpha > 0$ such that $k(\alpha) = \lim_{n \rightarrow \infty} (\mathbb{E}|S_n|^\alpha)^{1/n} = 1$.
- 3) There exists $\varepsilon > 0$ with $\mathbb{E}(|A|^{\alpha-\varepsilon} + |B|^{\alpha+\varepsilon}) < \infty$.
- 4) If $d > 1$, T satisfies i-p and if $d = 1$, T is non arithmetic.

The above conditions imply in particular that $L(\mu) < 0$, $k(s)$ is analytic, $k(s) < 1$ for $s \in]0, \alpha[$ and there exists a unique stationary probability ρ for λ acting by convolution on V ; the support of ρ is unbounded. Property 1 guarantees that ρ has no atom and says that the action of $\text{supp}(\lambda)$ is not conjugate to a linear action. Properties 2,3 imply that T is unbounded and are responsible for the α -homogeneity at infinity of ρ described below; if $k(s)$ is finite on $[0, \infty[$ and there exists $g \in T$ with $r(g) > 1$, then Property 2 is satisfied. Also if $d > 1$, condition i-p is basic for renewal theory of the linear random walk $S_n(\omega)v$ and it implies that T is non arithmetic.

In the appendix we will show that if $d > 1$ condition (c-e) is open in the weak topology of probabilities on the affine group, defined by convergence of moments and of values on continuous compactly supported functions.

Below, we use the decomposition of $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_{>0}$ in polar coordinates, where \mathbb{S}^{d-1} is the unit sphere of V . We consider also the Radon measure ℓ^α on $\mathbb{R}_{>0}$ ($\alpha > 0$) given by $\ell^\alpha(dt) = t^{-\alpha-1}dt$. We recall (see [20]) that, if $(A_n, B_n)_{n \in \mathbb{N}}$ is an i.i.d. sequence of H -valued random variables with law λ and $L(\mu) < 0$, then ρ is the law of the \mathbb{P} -a.e. convergent series $X = \sum_0^\infty A_1 \cdots A_k B_{k+1}$. The following is basic for our analysis.

Theorem 2.1 (see [13], Theorem C)

Assume that λ satisfies condition (c-e). Then the operator P has a unique stationary probability ρ , the support of ρ is unbounded and we have the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho) = \Lambda = c(\sigma^\alpha \otimes \ell^\alpha)$$

where $c > 0$ and σ^α is a probability on \mathbb{S}^{d-1} . Furthermore Λ is a Q -invariant Radon measure on $V \setminus \{0\}$ with unbounded support.

We observe that, for $d > 1$, if $\text{supp}(\lambda)$ is compact and has no fixed point in V , $L(\mu) < 0$, T is Zariski dense in G and is unbounded, then condition (c-e) is satisfied. For $d = 1$, if $\text{supp}(\lambda)$ is compact, the hypothesis of ([18], Theorem 1.1) is equivalent to condition (c-e). The existence of Λ stated in the theorem implies multivariate regular variation of ρ . Since the convergence stated in the theorem is valid we say that ρ is homogeneous at infinity; below we will make essential use of this property. We note that the fact that $\text{supp}(\Lambda)$ is unbounded follows from condition 2 above, hence of the unboundness of T .

Under condition (c-e), Λ gives zero mass to any proper affine subspace and σ^α has positive dimension. We observe that, if the sequence $(A_n, B_n)_{n \in \mathbb{N}}$ is replaced by $(A_n, tB_n)_{n \in \mathbb{N}}$ with $t \in \mathbb{R}^*$, then the asymptotic tail measure is replaced by $t.\Lambda$, in particular the constant c is

replaced by $|t|^\alpha c$. In subsection 2.2 below, in view of normalisation, it is natural to replace Λ by $\Lambda_0 = c^{-1}\alpha\Lambda$; this takes into account the magnitude of B_n and implies $\Lambda_0(U'_1) = 1$. Also, as shown in [13], the Q -invariant Radon measure Λ_0 is extremal or can be decomposed in two extremal measures. Hence, if the action of T on \mathbb{S}^{d-1} has a unique minimal subset, then Λ_0 is symmetric, $\text{supp}(\sigma^\alpha)$ is equal to this minimal subset and the shift invariant Markov measure \mathbb{Q}_{Λ_0} on $(V \setminus \{0\})^{\mathbb{Z}_+}$ is ergodic. Otherwise \mathbb{Q}_{Λ_0} decomposes into at most two ergodic measures. Hence Λ_0 depends only of μ , possibly up to one positive coefficient. The following is a consequence of vague convergence.

Corollary 2.2 *Let f be a bounded Borel function on $V \setminus \{0\}$ which has a Λ -negligible discontinuity set and such that $\text{supp}(f)$ is bounded away from zero. Then we have*

$$\lim_{t \rightarrow 0_+} t^{-\alpha} (t.\rho)(f) = \Lambda(f).$$

2.2 The tail process

We denote $\Omega = G^{\mathbb{N}}$, $\widehat{\Omega} = G^{\mathbb{Z}}$, and we endow Ω with the product probability $\mathbb{Q} = \mu^{\otimes \mathbb{N}}$. We define the G -valued cocycle $S_n(\omega)$ where $\omega = (A_k)_{k \in \mathbb{Z}} \in \widehat{\Omega}$, $n \in \mathbb{Z}$ by :

$$S_n(\widehat{\omega}) = A_n \cdots A_1 \text{ for } n > 0, S_n(\widehat{\omega}) = A_{n+1}^{-1} \cdots A_0^{-1} \text{ for } n < 0, S_0(\widehat{\omega}) = Id$$

We consider also the random walk $S_n(\omega)v$ on $V \setminus \{0\}$, starting from $v \neq 0$, $\omega \in \Omega$ and we denote by \mathbb{Q}_{Λ_0} the Markov measure on $(V \setminus \{0\})^{\mathbb{Z}_+}$ for the random walk $S_n(\omega)v$ with initial measure Λ_0 . This measure is invariant under the shift on $(V \setminus \{0\})^{\mathbb{Z}_+}$. We recall that the shift s (resp. \widehat{s}) on the product space $\mathbb{D}^{\mathbb{Z}_+}$ (resp. $\mathbb{D}^{\mathbb{Z}}$) is the map defined on $\omega = (\omega_k)_{k \in \mathbb{Z}_+}$ (resp. $\widehat{\omega} = (\widehat{\omega}_k)_{k \in \mathbb{Z}}$) by $s(\omega)_k = \omega_{k+1}$ (resp. $\widehat{s}(\widehat{\omega})_k = \widehat{\omega}_{k+1}$). We recall below Proposition 4 of [12] which extends the construction of the natural extension for a non invertible transformation (see also [34]); this construction is standard for finite invariant measures.

Let Δ (resp. $\widehat{\Delta}$) be the shift on Ω (resp. $\widehat{\Omega}$), σ (resp. $\widehat{\sigma}$) the map of $\Omega \times (V \setminus \{0\}) = E$ into itself (resp. $\widehat{\Omega} \times (V \setminus \{0\}) = \widehat{E}$) defined by

$$\sigma(\omega, v) = (\Delta\omega, A_0(\omega)v) \text{ (resp. } \widehat{\sigma}(\widehat{\omega}, v) = (\widehat{\Delta}\widehat{\omega}, A_0(\widehat{\omega})v).$$

We denote by τ (resp. $\widehat{\tau}$) the shift on $(V \setminus \{0\})^{\mathbb{Z}_+}$ (resp. $(V \setminus \{0\})^{\mathbb{Z}}$) and we note that $\mathbb{Q} \otimes \Lambda_0$ is σ -invariant we observe that the map $(\omega, v) \rightarrow (S_k(\omega)v)_{k \in \mathbb{Z}_+}$ defines the dynamical system $((V \setminus \{0\})^{\mathbb{Z}_+}, \tau, \mathbb{Q}_{\Lambda_0})$ as a measure preserving factor of $(E, \sigma, \mathbb{Q} \otimes \Lambda_0)$. The construction of a natural extension has been detailed in [12] and we state the result.

Proposition 2.3 (see [12])

There exists a unique $\widehat{\sigma}$ -invariant measure $\widehat{\mathbb{Q}} \otimes \widehat{\Lambda}$ on \widehat{E} with projection $\mathbb{Q} \otimes \Lambda$ en E .

Since the map defined by $p(\omega, v) = (S_k(\omega)v)_{k \in \mathbb{Z}_+}$ (resp. $\widehat{p}(\widehat{\omega}, v) = (S_k(\widehat{\omega})v)_{k \in \mathbb{Z}}$) commutes with σ, τ (resp. $\widehat{\sigma}, \widehat{\tau}$) and $\widehat{\mathbb{Q}} \otimes \widehat{\Lambda}_0$ is $\widehat{\sigma}$ -invariant with projection $\mathbb{Q} \otimes \Lambda_0$ on $\Omega \times (V \setminus \{0\})$ it follows, that the push-forward measure $\widehat{p}(\widehat{\mathbb{Q}} \otimes \widehat{\Lambda}_0) = \widehat{\mathbb{Q}}_{\Lambda_0}$ on $(V \setminus \{0\})^{\mathbb{Z}}$, is $\widehat{\tau}$ -invariant and

has projection $\mathbb{Q}_{\Lambda_0} = p(\mathbb{Q} \otimes \Lambda_0)$ on $(V \setminus \{0\})^{\mathbb{Z}_+}$. The definition of $\widehat{\mathbb{Q}}_{\Lambda_0}$ is used below in the construction of the tail process (see [2]) of the affine random walk $(X_k)_{k \in \mathbb{Z}}$.

We define the probability $\widehat{\mathbb{Q}}_{\Lambda_1}$ by $\widehat{\mathbb{Q}}_{\Lambda_1} = (1_{U'_1} \circ \pi) \widehat{\mathbb{Q}}_{\Lambda_0}$ where π denotes the projection of $(V \setminus \{0\})^{\mathbb{Z}}$ on $V \setminus \{0\}$. The restriction of Λ_0 to U'_1 is denoted Λ_1 , hence $\Lambda_1(U'_t) = t^{-\alpha}$ if $t > 1$ and we denote by \mathbb{Q}_{Λ_1} the Markov measure defined by Q and the initial measure Λ_1 . We note that the probability \mathbb{Q}_{Λ_1} (resp $\widehat{\mathbb{Q}}_{\Lambda_1}$) extends to $V^{\mathbb{Z}_+}$ (resp $V^{\mathbb{Z}}$) and its extension will be still denoted \mathbb{Q}_{Λ_1} (resp $\widehat{\mathbb{Q}}_{\Lambda_1}$).

In order to illustrate the above construction, we take $d = 1$, hence $V \setminus \{0\} = \mathbb{R}^*$, $\Lambda = c|x|^{-\alpha-1}dx$ where $\alpha > 0$ and μ satisfies $\int \log|a|d\mu(a) < 0$. Since $\int |a|^\alpha d\mu(a) = 1$, we can denote by μ_α the new probability defined by $d\mu_\alpha(a) = |a|^{-\alpha}d\mu^*(a)$ and μ^* is the push forward of μ by the map $a \rightarrow a^{-1}$; it follows $\int \log|a|d\mu_\alpha(a) < 0$. We denote by Q^* the Markov kernel defined by convolution with μ_α on \mathbb{R}^* , hence $Q^*\Lambda = \Lambda$. Then it is easy to verify that $\widehat{\mathbb{Q}}_{\Lambda_0} = \mathbb{Q}_\alpha^* \otimes \Lambda_0 \otimes \mathbb{Q}$ where $\mathbb{Q}_\alpha^* = \mu_\alpha^{\otimes(-\mathbb{N})}$ and \widehat{E} is identified with $G^{-\mathbb{N}} \times V \times G^{\mathbb{N}}$. Also we denote by $\mathbb{Q}_v^* \otimes \varepsilon_v$ (resp. $\varepsilon_v \otimes \mathbb{Q}_v$) the Markov probability on $(V \setminus \{0\})^{\otimes(-\mathbb{Z}_+)}$ (resp. $(V \setminus \{0\})^{\otimes \mathbb{Z}_+}$) associated to Q^* (resp. Q) and the initial measure ε_v . Then we have $\widehat{\mathbb{Q}}_{\Lambda_1} = \int \mathbb{Q}_v^* \otimes \varepsilon_v \otimes \mathbb{Q}_v d\Lambda_1(v)$. If $d > 1$, such formulae remain valid, with Q^* equal to the adjoint operator to Q in $\mathbb{L}^2(\Lambda)$.

We consider the probability ρ , the shift-invariant Markov measure \mathbb{P}_ρ (resp $\widehat{\mathbb{P}}_\rho$) on $V^{\mathbb{Z}_+}$ (resp $V^{\mathbb{Z}}$), where ρ is the law of X_0 and \mathbb{P}_ρ is the projection of $\widehat{\mathbb{P}}_\rho$ on $V^{\mathbb{Z}_+}$. Since $\rho(\{0\}) = 0$, we can replace V by $V \setminus \{0\}$ when working under \mathbb{P}_ρ . For $0 < j \leq i$ we write $S_j^i = A_i \cdots A_j$ and $S_i^{i+1} = I$. Expectation with respect to \mathbb{P} or \mathbb{Q} , will be simply denoted by the symbol \mathbb{E} . If expectation is taken with respect to a Markov measure with initial measure ν , we will write \mathbb{E}_ν . For a family $Y_j (j \in \mathbb{Z})$ of V -valued random variables and k, ℓ in $\mathbb{Z} \cup \{-\infty, \infty\}$, we denote $M_k^\ell(Y) = \sup\{|Y_j| ; k \leq j \leq \ell\}$. We observe that, if $t > 0$, condition (c-e) implies $\rho\{|x| > t\} > 0$, hence as in [2], we can consider the new process $(Y_i^t)_{i \in \mathbb{Z}}$ deduced from $t^{-1}(X_i)_{i \in \mathbb{Z}}$ by conditioning on the set $\{|X_0| > t\}$, for t large, under $\widehat{\mathbb{P}}_\rho$. We recall (see [30]) that a sequence of point processes on a separable locally compact space E is said to converge weakly to another point process if there is weak convergence of the corresponding finite dimensional distributions. The following is a detailed form in our case of the general result for multivariate jointly regularly varying stationary processes in [2]. The tail process appears here to be closely related to the stationary process (with infinite measure) $\widehat{\mathbb{Q}}_{\Lambda_0}$, which plays therefore an important geometric role.

Proposition 2.4 *a) The family of finite dimensional distributions of the point process $(Y_i^t)_{i \in \mathbb{Z}}$ converges weakly ($t \rightarrow \infty$) to those of the point process $(Y_i)_{i \in \mathbb{Z}}$ on V given by $Y_i = S_i Y_0$ where $(Y_i)_{i \in \mathbb{Z}}$ has law $\widehat{\mathbb{Q}}_{\Lambda_1}$.*

b) We have $\mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\} = \widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) \leq 1\}$,

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_\rho\left\{ \sup_{1 \leq k \leq n} |X_k| \leq t / |X_0| > t \right\} = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\} := \theta$$

c) In particular $\theta \in]0, 1[$

Proof a) We observe that, since for any $i \geq 0$, $X_i = S_i X_0 + \sum_{j=1}^i S_i^{j+1} B_j$ and $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{j=1}^i S_i^{j+1} B_j = 0$, \mathbb{P}_ρ -a.e, the random vectors $(t^{-1}X_i)_{0 \leq i \leq p+q}$ and $(t^{-1}S_i X_0)_{0 \leq i \leq p+q}$ have the same asymptotic behaviour in \mathbb{P}_ρ -law, conditionally on $|X_0| > t$. Also by stationarity of $\widehat{\mathbb{P}}_\rho$, for f continuous and bounded on V^{p+q+1} we have $\mathbb{E}_\rho\{f(t^{-1}X_{-q}, \dots, t^{-1}X_p) / |X_0| > t\} = \mathbb{E}_\rho\{f(t^{-1}X_0, t^{-1}X_1, \dots, t^{-1}X_{p+q}) / |X_q| > t\}$. From above, using Corollary 2.2, $\Lambda\{|x| = 1\} = 0$ and the formula $\Lambda\{|x| > 1\} = \alpha^{-1}c$, we see that the right hand side converges to :

$$c^{-1}\alpha \int \mathbb{E}\{f(x, S_1 x, \dots, S_{p+q} x) 1_{\{|S_q x| > 1\}}\} d\Lambda(x) = \widehat{\mathbb{Q}}_{\Lambda_0}(f 1_{\{S_q^{-1}(U'_1)\}})$$

We observe that Corollary 2.2 can be used here for fixed $\omega \in \Omega$ since the condition $|X_q| > t$ reduces the transformed expression under \mathbb{E}_ρ to a bounded function supported in $S_q^{-1}(U'_1)$. Hence, using stationarity of $\widehat{\mathbb{Q}}_\Lambda$, we get the weak convergence of the process $(Y_i^t)_{i \in \mathbb{Z}}$ to $(Y_i)_{i \in \mathbb{Z}}$ as stated in a).

b) In view of a) and Corollary 2.2, since the discontinuity sets of the functions $1_{]0,1[}(M_1^n(Y))$ and $1_{]1,\infty[}(Y_0)$ on V^n are \mathbb{Q}_{Λ_1} -negligible, we have $\lim_{t \rightarrow \infty} \mathbb{P}_\rho\{\sup_{1 \leq k \leq n} t^{-1}|X_k| \leq 1/t^{-1}|X_0| > 1\} = \mathbb{Q}_{\Lambda_1}\{M_1^n(Y) \leq 1\}$.

Hence $\theta = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_\rho\{\sup_{1 \leq k \leq n} |X_k| \leq t/X_0 > t\} = \mathbb{Q}_{\Lambda_1}\{\sup_{k \geq 1} |Y_k| \leq 1\}$.

We write $\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) \leq 1\} = 1 - \widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) > 1\}$ and we define the random time T by $T = \inf\{k \geq 1; |Y_{-k}| > 1\}$ if there exists $k \geq 1$ with $|Y_{-k}| > 1$; if such a k does not exist we take $T = \infty$. We have by definition of T :

$$\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) > 1\} = \sum_{k=1}^{\infty} \widehat{\mathbb{Q}}_{\Lambda_1}\{T = k\},$$

$$\widehat{\mathbb{Q}}_{\Lambda_1}\{T = k\} = \widehat{\mathbb{Q}}_{\Lambda_1}\{|Y_{-1}| \leq 1, |Y_{-2}| \leq 1, \dots, |Y_{-k+1}| \leq 1; |Y_{-k}| > 1\},$$

Using $\widehat{\tau}$ -invariance of $\widehat{\mathbb{Q}}_\Lambda$, the definition of $\widehat{\mathbb{Q}}_{\Lambda_1}$ and a), we get

$$\widehat{\mathbb{Q}}_{\Lambda_1}\{T = k\} = \mathbb{Q}_{\Lambda_1}\{|Y_1| \leq 1, \dots, |Y_{k-1}| \leq 1; |Y_k| > 1\},$$

$$\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) > 1\} = \sum_{k=1}^{\infty} \mathbb{Q}_{\Lambda_1}\{|Y_1| \leq 1, \dots, |Y_{k-1}| \leq 1; |Y_k| > 1\} = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) > 1\}.$$

The formula $\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) \leq 1\} = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\}$ follows.

c) The formula $\theta = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\}$ and the form of $Y_i (i \geq 0)$ given in a) imply $\theta = \mathbb{E}(\int_{i \geq 1} 1_{\{\sup |S_i x| \leq 1\}} d\Lambda_1(x)) \leq 1$. The condition $\theta = 1$ would imply for any $i \geq 1$: $|S_i x| \leq 1$,

$\mathbb{Q} \otimes \Lambda_1$ -a.e., hence $\{0\} \not\subset \text{supp}(S_i \Lambda_1) \subset U_1$. This would contradict the fact that $\text{supp}(\Lambda_1)$ is unbounded, hence we have $\theta < 1$. The condition $\theta = 0$ implies $\mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\} = 0$ and, since $Y_j = S_j Y_0$, we have $|S_j y| > 1$ \mathbb{Q}_{Λ_1} -a.e for some $j > 1$. Since Λ is Q -invariant, the corresponding hitting law of $\text{supp}(\Lambda_1)$ is absolutely continuous with respect to Λ_1 . Then, using Markov property we see that $S_k(\omega)y$ returns to $\text{supp}(\Lambda_1)$ infinitely often \mathbb{Q}_{Λ_1} -a.e. Since condition (c-e) implies $\lim_{j \rightarrow \infty} |S_j y| = 0$, \mathbb{Q} -a.e. for any $y \neq 0$, this is a contradiction. \square

2.3 Anticlustering property

We will show in section 2.4 that, asymptotically, the set of large values of X_k consists of a sequence of localized elementary clusters. An important sufficient condition for localization (see [7]) is proved in Proposition 2.5 below and will allow us to show the existence of a cluster process, following [2]. It is called anticlustering and is used in section 4 to decompose the set of values of X_k ($1 \leq k \leq n$) into successive quasi-independent blocks. For $k \leq \ell$ in \mathbb{Z} we write

$$M_k^\ell = \sup_{k \leq i \leq \ell} |X_i|, \quad R_k^\ell = \sum_{i=k}^{\ell} \widehat{\mathbb{P}}_\rho\{|X_i| > u_n/|X_0| > u_n\},$$

where $u_n = (\alpha^{-1}cn)^{1/\alpha}$. For $k > 0$, we write also $M_k = M_1^k$.

We observe that $M_k^\ell \leq \sum_{k \leq i \leq \ell} |X_i|$. Let r_n be any sequence of integers with $r_n = o(n)$, $\lim_{n \rightarrow \infty} r_n = \infty$. Then we have $\mathbb{P}_\rho\{M_1^{r_n} > u_n\} \leq r_n \mathbb{P}_\rho\{|X_0| > u_n\}$, hence the homogeneity of ρ at infinity gives $\lim_{n \rightarrow \infty} \mathbb{P}_\rho\{M_1^{r_n} > u_n\} = 0$. The condition $r_n = o(n)$ allows us to localize the influence of one large value of X_k ($1 \leq k \leq n$). It follows that the event $\{M_1^{r_n} > u_n\}$ can be considered as "rare". The homogeneity at infinity of ρ and the arbitrariness of r_n allow us to restrict the study to the sequence u_n instead of tu_n ($t > 0$).

The following is based on the homogeneity at infinity of ρ , the inequality $0 < k(s) < 1$ if $0 < s < \alpha$.

Proposition 2.5 *Assume $r_n \leq [n^s]$ with $0 < s < 1$, $\lim_{n \rightarrow \infty} r_n = \infty$. Then $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} R_m^{r_n} = 0$. In particular $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \widehat{\mathbb{P}}_\rho\{\sup(M_{-r_n}^-, M_m^{r_n}) > u_n/|X_0| > u_n\} = 0$, hence the random walk X_n satisfies anticlustering. Furthermore we have $\widehat{\mathbb{Q}}_{\Lambda_1}\{\lim_{|t| \rightarrow \infty} |Y_t| = 0\} = 1$.*

Proof We observe that :

$\widehat{\mathbb{P}}_\rho\{M_m^{r_n} > u_n/|X_0| > u_n\} \leq R_m^{r_n}$, $\widehat{\mathbb{P}}_\rho\{M_{-r_n}^- > u_n/|X_0| > u_n\} \leq R_{-r_n}^- = R_m^{r_n}$ where we have used stationarity of X_k in the last equality. Hence it suffices to show $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} R_m^{r_n} = 0$. For $i \geq 0$ we have $X_i = S_i X_0 + \sum_{j=1}^i S_i^{j+1} B_j$ where S_i, X_0 are independent, as well as $X_0, \sum_{j=1}^i |S_i^{j+1} B_j|$. We write $I_n^i = \widehat{\mathbb{P}}_\rho\{|X_i| > u_n/|X_0| > u_n\}$,

$$J_n^i = \widehat{\mathbb{P}}_\rho\{|S_i X_0| > 2^{-1}u_n/|X_0| > u_n\}, \quad K_n^i = \widehat{\mathbb{P}}_\rho\{\sum_{j=1}^i |S_i^{j+1} B_j| > 2^{-1}u_n/|X_0| > u_n\},$$

$$\text{hence } R_m^{r_n} = \sum_{i=m}^{r_n} I_n^i \leq \sum_{i=m}^{r_n} J_n^i + \sum_{i=m}^{r_n} K_n^i.$$

We are going to show $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=m}^{r_n} J_n^i = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=m}^{r_n} K_n^i = 0$.

We apply Chebyshev's inequality to the χ -moments of X_n with $\chi \in]0, \alpha[$. We have :

$$J_n^i \leq (2u_n^{-1})^\chi \mathbb{E}_\rho(|S_i X_0|^\chi / |X_0| > u_n) \leq (2u_n^{-1})^\chi \mathbb{E}(|S_i|^\chi) \mathbb{E}_\rho(|X_0|^\chi / |X_0| > u_n),$$

where independance of S_i and X_0 has been used in the last formula. Since the law of X_0 is α -homogeneous at infinity we have :

$$\lim_{x \rightarrow \infty} x^{-\chi} \mathbb{E}_\rho(|X_0|^\chi / |X_0| > x) = \alpha(\alpha - \chi)^{-1}, \quad \limsup_{n \rightarrow \infty} J_n^i \leq 2^\chi (\mathbb{E}|S_i|^\chi) \alpha(\alpha - \chi)^{-1}.$$

It follows $\limsup_{n \rightarrow \infty} (\sum_{i=m}^{r_n} J_n^i) \leq 2^\chi \alpha(\alpha - \chi)^{-1} \mathbb{E}(\sum_{i=m}^{\infty} |S_i|^\chi)$, hence $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=m}^{r_n} J_n^i = 0$, since $\lim_{i \rightarrow \infty} (\mathbb{E}|S_i|^\chi)^{1/i} = k(\chi) < 1$.

Also, using independence of X_0 and $\sum_{j=1}^i |S_i^{j+1} B_j|$:

$$K_n^i = \mathbb{P}_\rho\left\{ \sum_{j=1}^i |S_i^{j+1} B_j| > 2^{-1} u_n \right\} \leq (2u_n^{-1})^\chi \mathbb{E}\left(\sum_{j=1}^i |S_i^{j+1} B_j|\right)^\chi \leq (2u_n^{-1})^\chi \mathbb{E}\left(\sum_{j=1}^{\infty} |S_{j-1} B_j|\right)^\chi.$$

From above using independance of S_{j-1} and B_j we know that $R_0 = \sum_1^\infty |S_{j-1} B_j|$ has finite χ -moment if $\chi \leq 1$. Then by Chebyshev's inequality : $\sum_{i=m}^{r_n} K_n^i \leq (2u_n^{-1})^\chi r_n \mathbb{E}(R_0^\chi)$.

Also if $\chi \in [1, \alpha[$ we can use Minkowski's inequality and independance in order to show the finiteness of $\mathbb{E}(|R_0|^\chi)$. Since $0 < s < 1$, we can choose $\chi \in]0, \alpha[$ such that $\alpha^{-1} \chi > s$, hence

$$\lim_{n \rightarrow \infty} r_n u_n^{-\chi} = 0. \quad \text{Then, for any fixed } m : \lim_{n \rightarrow \infty} \sum_{i=m}^{r_n} K_n^i = 0.$$

From [13] we know that, since $0 < \chi < \alpha$, we have $k(\chi) < 1$, hence $\mathbb{E}(|S_i|^\chi)$ decreases exponentially fast to zero ; then the series $\mathbb{E}(\sum_{i=1}^{\infty} |S_i|^\chi)$ converges and

$$\lim_{m \rightarrow \infty} \mathbb{E}\left(\sum_{i=m}^{\infty} |S_i|^\chi\right) = 0, \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \left(\sum_{i=m}^{r_n} J_n^i\right) = 0.$$

Hence $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=m}^{r_n} I_n^i = 0$ and the result follows.

The anticlustering property $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \widehat{\mathbb{P}}_\rho\{\sup(M_{-r_n}^{-m}, M_m^{r_n}) > u_n / |X_0| > u_n\} = 0$ shows that, given $\varepsilon > 0$, $u > 0$ and using Proposition 2.4, there exists $m \in \mathbb{N}$ such that for $r \geq m$ we have $\widehat{\mathbb{Q}}_{\Lambda_1}\{\sup_{m \leq |j| \leq r} |Y_j| \geq u\} \leq \varepsilon$. Hence $\widehat{\mathbb{Q}}_{\Lambda_1}\{\lim_{|j| \rightarrow \infty} |Y_j| = 0\} = 1$.

□

2.4 The cluster process

In general, for a stationary V -valued point process with an associated tail process, the properties of anticlustering and positivity of the extremal index θ for a sequence $r_n = o(n)$ with $\lim_{n \rightarrow \infty} r_n = \infty$, imply the existence of the cluster process (see [2]). For self containment reasons we give in Proposition 2.6 below a proof of this fact, using arguments of [2] ; this gives us also the convergence of θ_n defined by $\theta_n^{-1} = \mathbb{E}_\rho\left\{\sum_1^{r_n} 1_{[u_n, \infty[}(|X_k|)/M_{r_n} > u_n\right\}$ to θ defined in Proposition 2.4. We note that the condition $\sum_1^{r_n} 1_{[u_n, \infty[}(X_k) > 0$ implies $M_{r_n} > u_n$, hence we have $\theta_n^{-1} = r_n (\mathbb{P}_\rho\{|X_0| > u_n\}) (\mathbb{P}_\rho\{M_{r_n} > u_n\})^{-1}$. For later use we include also in the statement the formula of ([2], Theorem 4.3) giving the Laplace functional for the cluster process restricted to U'_1 . We recall that the Laplace functional of a random measure

η on a locally compact separable metric space E , where the space $M_+(E)$ of positive Radon measures on E is endowed with a probability m , is given by

$$\psi_\eta(f) = \int \exp(-\nu(f)) dm(\nu)$$

with f continuous non negative and $\text{supp}(f)$ compact. We recall also that weak convergence of a sequence of point processes is equivalent to convergence of their Laplace functionals.

We denote by r_n a sequence as above and we consider the sequence of point processes $C_n = \sum_{i=1}^{r_n} \varepsilon_{u_n^{-1}X_i}$, on $E = V \setminus \{0\}$ under \mathbb{P}_ρ and conditionally on $M_{r_n} = M_1^{r_n} > u_n$. In

particular $\theta_n^{-1} = \mathbb{E}_\rho(C_n(U'_1))$. Using the tail process $(Y_n)_{n \in \mathbb{Z}}$ defined in Proposition 2.4 above, we show that C_n converges weakly to the point process C ; C is a basic quantity for the asymptotics of X_n and is called the cluster process of X_n . As shown in Proposition 2.6 below, the law of C can be expressed in terms of $\widehat{\mathbb{Q}}_{\Lambda_1}$ and depends only of μ, Λ_1 . By definition, C selects the large values of the process X_n and describes the local multiplicity of large values in a typical cluster for the process X_n . We denote $\widehat{F} = (V \setminus \{0\})^{\mathbb{Z}}$, $\widehat{F}_- = \{v \in \widehat{F}; \sup_{k \leq -1} |v_k| \leq 1\}$. Since $\lim_{|i| \rightarrow \infty} Y_i = 0$, Proposition 2.5 implies $\widehat{\mathbb{Q}}_{\Lambda_1}(\widehat{F}_-) > 0$, hence we can

define the conditional probability $\overline{\mathbb{Q}}_{\Lambda_1}$ on \widehat{F}_- by $\overline{\mathbb{Q}}_{\Lambda_1} = (\widehat{\mathbb{Q}}_{\Lambda_1}(\widehat{F}_-))^{-1} (1_{\widehat{F}_-} \widehat{\mathbb{Q}}_{\Lambda_1})$. Proposition 2.6 below implies that the law of C is $\overline{\mathbb{Q}}_{\Lambda_1}$, hence we can define a natural version of C as $C = \sum_{j \in \mathbb{Z}} \varepsilon_{Z_j}$ where $Z_j(v)$ is defined as the j -projection of $v = (v_k)_{k \in \mathbb{Z}} \in \widehat{F}_-$, where \widehat{F}_- is endowed with the probability $\overline{\mathbb{Q}}_{\Lambda_1}$.

We denote by π_v^ω the occupation measure of the random walk $S_n(\omega)v$ on $V \setminus \{0\}$, given by $\pi_v^\omega = \sum_0^\infty \varepsilon_{S_i(\omega)v}$. For v fixed, the mean measure of π_v^ω is the potential measure $\sum_0^\infty Q^i(v, \cdot)$ of the Markov kernel Q ; if $L(\mu) < 0$ the asymptotics ($|v| \rightarrow \infty$) of this Radon measure are described in [13]. The formula below for the Laplace functional of C involves the occupation measure π_v^ω of the linear random walk $S_n(\omega)v$ and Λ_1 ; it plays an essential role below.

With the above notations we have the.

Proposition 2.6 *Under \mathbb{P}_ρ , the sequence of point processes C_n converges weakly to a point process C . The law of the point process C is equal to the $\widehat{\mathbb{Q}}_{\Lambda_1}$ - law of the point process $\sum_{j \in \mathbb{Z}} \varepsilon_{S_j x}$ conditional on $\sup_{j \leq -1} |S_j x| \leq 1$. In particular we have for $C = \sum_{j \in \mathbb{Z}} \varepsilon_{Z_j}$ with Z_j as above*

$$\overline{\mathbb{Q}}_{\Lambda_1} \{ \lim_{|i| \rightarrow \infty} |Z_i| = 0 \} = 1, \quad \overline{\mathbb{Q}}_{\Lambda_1} \{ \sup_{i \geq 1} |Z_i| \geq 1 \} = 1.$$

Furthermore the sequence θ_n defined above converges to the positive number θ and :

$$\sup_{n \in \mathbb{N}} \mathbb{E}_\rho \left\{ \left(\sum_1^{r_n} 1_{[u_n, \infty[} |X_i| \right)^2 / M_{r_n} > u_n \right\} < \infty, \quad \theta^{-1} = \overline{\mathbb{E}}_{\Lambda_1} \left(\sum_{j \in \mathbb{Z}} 1_{U'_1}(Z_j) \right) < \infty.$$

If $\text{supp}(f) \subset U'_1$, the Laplace functional of C on f is given by

$$1 - \theta^{-1} \overline{\mathbb{E}}_{\Lambda_1} [(\exp f(v) - 1) \exp(-\pi_v^\omega(f))].$$

Proof Let f be a non negative and continuous function on $V \setminus \{0\}$ which is compactly supported, hence $f(x) = 0$ if $|x| \leq \delta$ with $\delta > 0$.

We write for $k \leq \ell$ with $k, \ell \in \mathbb{Z} \cup \{\pm\infty\}$, $M_k^\ell(Y) = \sup_{k \leq j \leq \ell} |Y_j|$ with $Y_j = S_j Y_0$. For k, ℓ, f as above we write $C_k^\ell = \exp(-\sum_k^\ell f(u_n^{-1} X_j))$, $C_k^\ell(Y) = \exp(-\sum_k^\ell f(Y_j))$ and we observe that $C_k^\ell \leq 1$. We fix $m > 0$ and we take n so large that the sequence r_n of the above proposition satisfies $r_n > 2m + 1$. When convenient we write $r_n = r$, hence $\mathbb{E}_\rho(C_1^r; M_1^r > u_n) = \sum_1^r \mathbb{E}_\rho(C_1^r; M_1^{j-1} \leq u_n < X_j)$. We observe that, for $r - m \geq j > m + 1$, we have $C_1^r = C_{j-m}^{j+m}$ except if $\sup(M_1^{j-m-1}, M_{j+m+1}^r) > u_n \delta$. We are going to compare $\mathbb{E}_\rho(C_1^r; M_1^r > u_n)$ and $(r - 2m) \widehat{\mathbb{E}}_\rho(C_{-m}^m; M_{-m-1}^{-1} \leq u_n < |X_0|)$ using those j 's which satisfy $m + 1 < j \leq r - m$ and we denote by $\Delta_{n,m}$ their difference.

If we write

$$\Delta_{n,m}(j) = \mathbb{E}_\rho(C_1^r; M_1^{j-1} \leq u_n < |X_j|) - \mathbb{E}_\rho(C_{j-m}^{j+m}; M_{j-m-1}^{j-1} \leq u_n < |X_j|),$$

then we have using stationarity and $C_k^\ell \leq 1$:

$$|\Delta_{n,m}| \leq \sum_{m+1}^{r-m} |\Delta_{n,m}(j)| + 2m \mathbb{P}_\rho\{|X_0| > u_n\}.$$

Using stationarity of X_n with respect to $\widehat{\mathbb{P}}_\rho$ and the above observation we have

$$|\Delta_{n,m}(j)| \leq \widehat{\mathbb{P}}_\rho\{\sup(M_{-r}^{-m-1}, M_{m+1}^r) > u_n \delta; |X_0| > u_n\}.$$

Also using the formula $\theta_n = (r_n \mathbb{P}_\rho\{|X_0| > u_n\})^{-1} \mathbb{P}_\rho\{|M_1^r| > u_n\}$, we have $\theta_n \mathbb{E}_\rho(C_1^r/M_1^r > u_n) = r_n^{-1} \mathbb{E}_\rho(C_1^r; M_1^r > u_n) \mathbb{P}_\rho\{|X_0| > u_n\}^{-1}$. Then the use of stationarity and the above estimations for $\Delta_{n,m}(j)$ and $\Delta_{n,m}$ give the basic relation,

$$\begin{aligned} & |\theta_n \mathbb{E}_\rho(C_1^r/M_1^r > u_n) - r^{-1}(r - 2m) \widehat{\mathbb{E}}_\rho(C_{-m}^m; M_{-m-1}^{-1} \leq u_n / |X_0| > u_n)| \leq \\ & \widehat{\mathbb{P}}_\rho\{\sup(M_{-r}^{-m-1}, M_{m+1}^r) > u_n \delta / |X_0| > u_n\} + 2m r_n^{-1}. \end{aligned}$$

Using Proposition 2.4, we see that the discontinuity set of the function $1_{]0,1]}(M_{-m-1}^{-1}(Y))$ is $\widehat{\mathbb{Q}}_{\Lambda_1}$ -negligible, hence using again Proposition 2.4,

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}}_\rho(C_{-m}^m; M_{-m-1}^{-1} \leq u_n / |X_0| > u_n) = \widehat{\mathbb{E}}_{\Lambda_1}(C_{-m}^m(Y); M_{-m-1}^{-1}(Y) \leq 1).$$

Also $\lim_{n \rightarrow \infty} r_n^{-1}(r_n - 2m) = 1$ since $\lim_{n \rightarrow \infty} r_n = \infty$. We observe that, by definition of θ_n and $C_1^r \leq 1$, we have $\theta_n \mathbb{E}_\rho(C_1^r/M_1^r > u_n) \leq \theta_n \leq 1$. The anticlustering property of X_n implies that the limiting values ($n \rightarrow \infty$) of $\widehat{\mathbb{P}}_\rho\{\sup(M_{-r}^{-m-1}, M_{m+1}^r) > \delta u_n / |X_0| > u_n\}$ are bounded by $\varepsilon_m > 0$ with $\lim_{m \rightarrow \infty} \varepsilon_m = 0$. Then the above inequality implies

$$\limsup_{n \rightarrow \infty} |\theta_n \widehat{\mathbb{E}}_\rho(C_1^r/M_1^r > u_n) - \widehat{\mathbb{E}}_{\Lambda_1}(C_{-m}^m(Y); M_{-m-1}^{-1}(Y) \leq 1)| \leq \varepsilon_m + 2m r_n^{-1}.$$

Since $\lim_{m \rightarrow \infty} \widehat{\mathbb{E}}_{\Lambda_1}(C_{-m}^m(Y); M_{-m-1}^{-1}(Y) \leq 1) = \widehat{\mathbb{E}}_{\Lambda_1}(\exp(-\sum_{-\infty}^\infty f(Y_j)); M_{-\infty}^{-1}(Y) \leq 1) := I$, we have $\lim_{n \rightarrow \infty} \theta_n \mathbb{E}_\rho(C_1^r/M_1^r > u_n) = I$. In particular with $f = 0$ and using Proposition 2.4, we get $\lim_{n \rightarrow \infty} \theta_n = \widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) \leq 1\} = \theta > 0$.

Then we get $\lim_{n \rightarrow \infty} \mathbb{E}_\rho(C_1^{r_n}/M_1^{r_n} > u_n) = \theta^{-1} I = \widehat{\mathbb{E}}_{\Lambda_1}(\exp(-\sum_{-\infty}^\infty f(Y_j))/M_{-\infty}^{-1}(Y) \leq 1)$ hence the first assertion, using Proposition 2.4. The expression of $(Z_j)_{j \in \mathbb{N}}$ in terms of \widehat{F}_- and $\overline{\mathbb{Q}}_{\Lambda_1}$

explained above and the relation $\lim_{|n| \rightarrow \infty} Y_n = 0$, $\widehat{\mathbb{Q}}_{\Lambda_1}$ -a.e. stated in Proposition 2.5 gives

$$\widehat{\mathbb{Q}}_{\Lambda_1} \left\{ \lim_{i \rightarrow \infty} Z_i = 0 \right\} = 1.$$

Since the discontinuity set of $1_{U'_1}$ is Λ_1 -negligible, using the weak convergence of C_n to C , the continuous mapping theorem (see [30]) and the convergence of θ_n^{-1} to θ^{-1} , we get the formula $\theta^{-1} \geq \overline{\mathbb{E}}_{\Lambda_1}(\sum_{j \in \mathbb{Z}} 1_{U'_1}(Z_j))$. To go further we write $W_n = \sum_1^{r_n} 1_{[u_n, \infty[}(X_i)$ and we observe that the stated formula for $\lim_{n \rightarrow \infty} \theta_n^{-1}$ is a consequence of the uniform boundedness

of $\mathbb{E}_\rho\{W_n^2/M_{r_n} > u_n\}$. We have

$$\mathbb{E}_\rho\{W_n^2/M_{r_n} > u_n\} = \theta_n^{-1} + 2 \sum_{1 \leq i < j \leq r_n} \mathbb{P}_\rho\{|X_i| > u_n, |X_j| > u_n/M_{r_n} > u_n\},$$

hence using the convergence of θ_n^{-1} to θ^{-1} , we see that it suffices to bound the second term in the above formula. Writing $j - i = p$, stationarity gives

$$\mathbb{P}_\rho\{|X_i| > u_n, |X_j| > u_n/M_{r_n} > u_n\} = \mathbb{P}_\rho\{|X_0| > u_n, |X_p| > u_n\}(\mathbb{P}_\rho\{M_{r_n} > u_n\})^{-1}.$$

We know already that $\theta^{-1} = \lim_{n \rightarrow \infty} (r_n \mathbb{P}_\rho\{|X_1| > u_n\})(\mathbb{P}_\rho\{M_{r_n} > u_n\})^{-1}$ and we will now use a calculation similar to the one in the proof of Proposition 2.5.

In particular we have :

$$\mathbb{P}_\rho\{|X_0| > u_n, |X_p| > u_n\} \leq \mathbb{P}_\rho\{|X_0| > u_n, |S_p X_0| > 2^{-1}u_n\} + \mathbb{P}_\rho\{|X_0| > u_n, \sum_1^p |S_p^{j+1} B_j| > 2^{-1}u_n\}.$$

With $0 < \chi < \alpha$ we have using independance :

$$\mathbb{P}_\rho\{|X_0| > u_n, |S_p X_0| > 2^{-1}u_n\} \leq 2^\chi u_n^{-\chi} \mathbb{E}(|S_p|^\chi) \mathbb{E}_\rho(|X_0|^\chi 1_{\{|X_0| > u_n\}}).$$

Since the law of X_0 is homogeneous at infinity, the right hand side is bounded by

$$C(\chi) \mathbb{P}_\rho\{|X_0| > u_n\} \mathbb{E}(|S_p|^\chi).$$

On the other hand we have

$$\begin{aligned} \mathbb{P}_\rho\{|X_0| > u_n, |S_p| > 2^{-1}u_n\} &= \mathbb{P}_\rho\{|X_0| > u_n\} \mathbb{P}_\rho\{|S_p| > 2^{-1}u_n\} \leq \\ &(2u_n^{-1})^\chi (\mathbb{E}|S_p|^\chi) (\mathbb{P}_\rho\{|X_0| > u_n\}) \leq C'(\chi) u_n^{-\chi} \mathbb{P}_\rho\{|X_0| > u_n\} \end{aligned}$$

It follows that $\mathbb{P}_\rho\{|X_p| > u_n / |X_0| > u_n\}$ is bounded by $C''(\chi)(u_n^{-\chi} + \mathbb{E}(|S_p|^\chi))$, hence

$\sum_{1 \leq i < j \leq r_n} \mathbb{P}_\rho\{|X_i| > u_n, |X_j| > u_n/M_{r_n} > u_n\}$ is bounded by

$$D(\chi) r_n^2 \left(\sum_1^\infty (\mathbb{E}|S_p|^\chi + u_n^{-\chi}) (\mathbb{P}_\rho\{|X_0| > u_n\}) (\mathbb{P}_\rho\{M_{r_n} > u_n\})^{-1} \leq D'(\chi) r_n \left(\sum_1^\infty \mathbb{E}(|S_p|^\chi), \right.$$

since $r_n (\mathbb{P}_\rho\{|X_0| > u_n\}) (\mathbb{P}_\rho\{M_{r_n} > u_n\})^{-1}$ converges to θ^{-1} .

Then the uniform boundedness of $\mathbb{E}_\rho(W_n^2/M_{r_n} > u_n)$ will follow if $r_n u_n^{-\chi}$ is bounded. In view of the form of r_n, u_n , this amounts to the boundedness of $n^s n^{-\chi/\alpha}$ with $s < 1$. Hence it suffices to choose χ with $\alpha s \leq \chi < \alpha$ in order to get the result.

The last formula is proved in ([2], Theorem 4.1). A different proof is sketched in section 5.

□

3 A spectral gap property and multiple mixing

We denote $X_k^x (k \in \mathbb{N})$ the affine random walk on V governed by λ , starting from $x \in V$ and we write $P\varphi(x) = \int \varphi(hx)d\lambda(h) = \mathbb{E}(\varphi(X_1^x))$.

In this section we use a spectral gap property for a family of operators associated to the process $X_k (1 \leq k \leq n)$, in order to show the quasi-independence of its successive blocks of length r_n , where r_n is defined in subsection 2.3.

3.1 Spectral gap property

It was proved in ([10], Theorem 1) that, given a probability λ on H which satisfies condition (c-e), the corresponding convolution operator P on V satisfies a "Doebelin-Fortet" inequality (see [19]) for suitable Banach spaces \mathcal{C}_χ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$ defined below. In particular, it will be essential here to use that the operator P on $\mathcal{H}_{\chi,\varepsilon,\kappa}$ is the direct sum of a 1-dimensional projection π and a contraction U where π and U commute, hence we give also a short proof of this fact below. In order to obtain the relevant multiple mixing property, we show a global Doebelin-Fortet inequality for a family of operators closely related to P . For $\chi, \kappa \geq 0$, we consider the weights ω, η on V defined by $\omega(x) = (1+|x|)^{-\chi}, \eta(x) = (1+|x|)^{-\kappa}$. The space \mathcal{C}_χ is the space of continuous functions φ on V such that $\varphi(x)\omega(x)$ are bounded and we write $|\varphi|_\chi = \sup_{x \in V} |\varphi(x)|\omega(x)$.

For $\varepsilon \in]0, 1]$ we write :

$$[\varphi]_{\varepsilon,\kappa} = \sup_{x \neq y} |x - y|^{-\varepsilon} \eta(x)\eta(y) |\varphi(x) - \varphi(y)|, \quad \|\varphi\| = |\varphi|_\chi + [\varphi]_{\varepsilon,\kappa},$$

and we denote by $\mathcal{H}_{\chi,\varepsilon,\kappa}$ the space of functions φ on V such that $\|\varphi\| < \infty$. We observe that \mathcal{C}_χ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$ are Banach spaces with respect to the norms $|\cdot|_\chi$ and $\|\cdot\|$ defined above. Also $\mathcal{H}_{\chi,\varepsilon,\kappa} \subset \mathcal{C}_\chi$ with compact injection if $\kappa + \varepsilon < \chi$. We observe that the operator P acts continuously on \mathcal{C}_χ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$. For a Lipschitz function f on V with non negative real part we define the Fourier-Laplace operator P^f by $P^f \varphi(x) = P(\varphi \exp(-f))$. In [10], spectral gap properties for Fourier operators were studied for $f(v) = i \langle x, v \rangle, x \in V$. Here the calculations are analogous but f will be Lipschitz and bounded. We observe that for functions $f_k (1 \leq k \leq n)$ and φ as above we have :

$$P^{f_1} P^{f_2} \dots P^{f_n} \varphi(x) = \mathbb{E}\{\varphi(X_n^x) \exp(-\sum_{k=1}^n f_k(X_k^x))\}$$

Also we note that, for f bounded, with $k(f) = \sup_{x \neq y} |x - y|^{-1} |f(x) - f(y)| < \infty$

$$|x - y|^{-\varepsilon} |f(x) - f(y)| \leq \inf_{x \neq y} (2|f|_\infty |x - y|^{-\varepsilon}, k(f) |x - y|^{1-\varepsilon}) \leq 2|f|_\infty + k(f) := k_1(f),$$

For u, v with non negative real parts we have $|\exp(-u) - \exp(-v)| \leq |u - v|$. In particular, for f as above, $|\exp(-f(x)) - \exp(-f(y))| \leq k_1(f) |x - y|^\varepsilon$.

It follows that multiplication by $\exp(-f)$ acts continuously on $\mathcal{C}_\chi, \mathcal{H}_{\chi,\varepsilon,\kappa}$, hence P^f is a bounded operator on \mathcal{C}_χ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$. For $m, \gamma > 0$ we denote by $O(m, \gamma)$ the set of operators P^f such that $|f|_\infty \leq m$ and $k(f) \leq \gamma$, hence $k_1(f) \leq 2m + \gamma$. For $p \in \mathbb{N}$ let $O^p(m, \gamma)$ be

the set of products of p elements in $O(m, \gamma)$ and $\widehat{O}(m, \gamma) = \bigcup_{p>0} O^p(m, \gamma)$. We will endow $\widehat{O}(m, \gamma)$ with the natural norm from $End(\mathcal{H}_{\chi, \varepsilon, \kappa})$. Then we have the

Theorem 3.1 *With the above notations and $0 \leq \chi < 2\kappa < 2\kappa + \varepsilon < \alpha$, there exists $C(m, \gamma) \geq 1$ such that for any $Q \in O(m, \gamma)$ the norm $\|Q\|$ of Q on $\mathcal{H}_{\chi, \varepsilon, \kappa}$ is bounded by $C(m, \gamma)$. Furthermore there exists $r \in [0, 1[$, $p \in \mathbb{N}$, $D > 0$ such that for any $Q \in O^p(m, \gamma)$, $\varphi \in \mathcal{H}_{\chi, \varepsilon, \kappa}$:*

$$\|Q\varphi\| \leq r\|\varphi\| + D|\varphi|_\chi.$$

In particular $\widehat{O}(m, \gamma)$ is a bounded subset of $End(\mathcal{H}_{\chi, \varepsilon, \kappa})$ and $C(m, \gamma), r, D$ depend only of m, γ .

The proof depends on the two lemmas given below, and of calculations analogous to those of [10] for Fourier operators.

Lemma 3.2 *$\widehat{O}(m, \gamma)$ is a bounded subset of $End(\mathcal{C}_\chi)$.*

Proof Since $Re(f) \geq 0$ we have for $Q \in O^\ell(m, \gamma)$ with $\ell \in \mathbb{N}$, $\varphi \in \mathcal{C}_\chi$: $|Q\varphi|_\chi \leq |P^\ell|\varphi|_\chi$, hence it suffices to show that the set $\{P^\ell; \ell \in \mathbb{N}\}$ is bounded in $End(\mathcal{C}_\chi)$. We have for $\varphi \geq 0$, with $M = P^\ell$:

$$\omega(x)M\varphi(x) = \omega(x)\mathbb{E}(\varphi(X_\ell^x)) \leq |\varphi|_\chi \mathbb{E}[\omega(x)\omega^{-1}(X_\ell^x)].$$

If $\chi \leq 1$, using independance and the expression of X_ℓ^x we get

$$\omega(x)M\varphi(x) \leq |\varphi|_\chi (1 + \mathbb{E}|S_\ell|^\chi + \sum_1^\ell (\mathbb{E}|S_\ell^{k+1}|^\chi) (\mathbb{E}|B_k|^\chi),$$

hence $\sup_{x \in V} \omega(x)M\varphi(x) \leq |\varphi|_\chi (1 + \sup_{\ell \geq 1} \mathbb{E}|S_\ell|^\chi + (\mathbb{E}|B_1|^\chi) (\sum_1^\infty \mathbb{E}|S_\ell|^\chi)$.

Since $\chi < \alpha$, we have $\lim_{\ell \rightarrow \infty} (\mathbb{E}|S_\ell|^\chi)^{1/\ell} = k(\chi) < 1$, hence $\sup_{x \in V} \omega(x)|M\varphi(x)|$ is bounded by

$C_\chi |\varphi|_\chi$ with $C_\chi < \infty$.

If $\chi > 1$, we use Minkowski's inequality in \mathbb{L}^χ and write :

$$\omega(x)|M\varphi(x)| \leq |\varphi|_\chi (1 + (\mathbb{E}|S_\ell|^\chi)^{1/\chi} + \sum_1^\ell \mathbb{E}(|S_\ell^{k+1}|^\chi)^{1/\chi} (\mathbb{E}|B_k|^\chi)^{1/\chi})$$

As above we get

$$\sup_{x \in V} \omega(x)|M\varphi(x)| \leq C_\chi |\varphi|_\chi \text{ with } C_\chi < \infty. \quad \square$$

Lemma 3.3 *a) For $\beta \in [0, \alpha[$ we have $\sup_n \mathbb{E}|X_n^0|^\beta < \infty$.*

b) For $\beta_1, \beta > 0$ and $\beta + \beta_1 < \alpha$, we have $\lim_{n \rightarrow \infty} (\mathbb{E}(|S_n|^{\beta_1} |X_n^0|^\beta))^{1/n} < 1$.

c) If $\chi + \varepsilon < \alpha$ the quantity $\tilde{C}_n = \mathbb{E}(\sum_1^n |S_i|^\varepsilon (1 + |S_n| + |X_n^0|)^\chi)$ is bounded. Furthermore, if $2\kappa + \varepsilon < \alpha$, then $\tilde{D}_n = \mathbb{E}(|S_n|^\varepsilon (1 + |S_n| + |X_n^0|)^{2\kappa})$ satisfies $\lim_{n \rightarrow \infty} (\tilde{D}_n)^{1/n} < 1$.

Proof a) We write $|X_n^0|^\beta = |\sum_1^n S_n^{k+1} B_k|^\beta$. If $\beta \leq 1$ we get :

$$\mathbb{E}(|X_n^0|^\beta) \leq \sum_1^n (\mathbb{E}|S_n^{k+1}|^\beta) (\mathbb{E}|B_k|^\beta) = (\mathbb{E}|B_1|^\beta) (\sum_0^{n-1} \mathbb{E}|S_j|^\beta)$$

Since $\lim_{j \rightarrow \infty} (\mathbb{E}|S_j|^\beta)^{1/j} < 1$ if $\beta < \alpha$ we get $\sup_{n \geq 0} \mathbb{E}|X_n^0|^\beta \leq (\mathbb{E}|B_1|^\beta) (\sum_0^\infty \mathbb{E}|S_j|^\beta) < \infty$.

If $\beta > 1$, we use Minkowski's inequality in \mathbb{L}^β as in the proof of Lemma 3.2.

b) Using Hölder's inequality we have

$$\mathbb{E}(|S_n|^{\beta_1} |X_n^0|^\beta) \leq (\mathbb{E}|S_n|^{(\beta+\beta_1)\beta_1/\beta+\beta_1}) \mathbb{E}|X_n^0|^{(\beta+\beta_1)\beta/\beta+\beta_1},$$

hence the result follows from a) and the fact that $\lim_{n \rightarrow \infty} (\mathbb{E}|S_n|^{\beta+\beta_1})^{1/n} < 1$ since $\beta + \beta_1 < \alpha$.

c) The assertions follows from easy estimations as in b) and the conditions $\chi + \varepsilon < \alpha$, $2\kappa + \varepsilon < \alpha$. \square

Proof of Theorem 3.1 We start with a basic observation. For $n > 0$ we have $X_n^x = h_n \cdots h_1 x = S_n x + \sum_1^n S_n^{k+1} B_k$, hence $|X_n^x - X_n^y| = |S_n(x - y)| \leq |S_n||x - y|$. It follows for $k(f) \leq \gamma$, x and y in V :

$$|f(X_n^x) - f(X_n^y)| \leq \gamma |S_n| |x - y|.$$

We write $M = T_1 T_2 \cdots T_n$ with $T_i = P^{f_i} \in O(m, \gamma)$ $1 \leq i \leq n$. We have using Markov property,

$$M\varphi(x) - M\varphi(y) = I_n(x, y) + J_n(x, y) \text{ with}$$

$$I_n(x, y) = \mathbb{E}([\exp(-\sum_1^n f_i(X_i^x)) - \exp(-\sum_1^n f_i(X_i^y))] \varphi(X_n^x))$$

$$J_n(x, y) = \mathbb{E}((\exp(-\sum_1^n f_i(X_i^y))) (\varphi(X_n^x) - \varphi(X_n^y)))$$

Since $Re(f) \geq 0$ we have :

$$|\exp(-\sum_1^n f_i(X_i^x)) - \exp(-\sum_1^n f_i(X_i^y))| \leq \sum_1^n |f_i(X_i^x) - f_i(X_i^y)| \leq (2m + \gamma) \sum_1^n |X_i^x - X_i^y|^\varepsilon.$$

The basic observation gives :

$$I_n(x, y) \leq (2m + \gamma) |\varphi|_\chi |x - y|^\varepsilon C_n(x) \text{ with } C_n(x) = \mathbb{E}(\sum_1^n |S_i|^\varepsilon (1 + |X_n^x|)^\chi)$$

$$J_n(x, y) \leq \mathbb{E}|\varphi(X_n^x) - \varphi(X_n^y)| \leq [\varphi]_{\varepsilon, \kappa} |x - y|^\varepsilon D_n(x, y),$$

$$\text{with } D_n(x, y) = \mathbb{E}(|S_n|^\varepsilon (1 + |X_n^x|)^\kappa (1 + |X_n^y|)^\kappa).$$

Using symmetry of $|M\varphi(x) - M\varphi(y)|$, $\chi \leq 2\kappa$ and $|X_n^x| \leq |S_n||x| + |X_n^0|$, we get $[M\varphi]_{\varepsilon, \kappa} \leq (2m + \gamma) |\varphi|_\chi \tilde{C}_n + [\varphi]_{\varepsilon, \kappa} \tilde{D}_n$ where \tilde{C}_n, \tilde{D}_n are as in Lemma 3.3.

Using Lemma 3.3 we can choose $p \in \mathbb{N}$ such that $r = \tilde{D}_p < 1$, hence for $M \in O^p(m, \gamma)$,

$$[M\varphi]_{\varepsilon, \kappa} \leq k_1(f) \tilde{C}_p |\varphi|_\chi + r [\varphi]_{\varepsilon, \kappa}.$$

Using Lemma 3.2 we see that there exists $C_\chi \geq 1$ such that $|M\varphi|_\chi \leq C_\chi |\varphi|_\chi$ for $M \in \hat{O}(m, \gamma)$, $\varphi \in \mathcal{C}_\chi$. Then for $M \in O^p(m, \gamma)$, $\varphi \in \mathcal{H}_{\chi, \varepsilon, \kappa}$ and p as above :

$$\|M\varphi\| \leq r \|\varphi\| + (C_\chi + 2m + \gamma) \tilde{C}_p |\varphi|_\chi = r \|\varphi\| + D |\varphi|_\chi \text{ with } D > 0.$$

For the last assertion, assume $M \in O^n(m, \gamma)$ and write $n = pn_1 + n_0$ with $n_1 \in \mathbb{N}$, $0 \leq n_0 < p$. We have $M = Q_1 \cdots Q_{n_1} R_1 \cdots R_{n_0}$ with $Q_i \in O^p(m, \gamma)$ ($1 \leq i \leq p$) and $R_j \in O(m, \gamma)$ ($0 \leq j \leq n_0$), hence $\|R_j\| \leq C_\chi(m, \gamma)$. Finally we get

$$\begin{aligned} \|M\varphi\| &\leq C_\chi(m, \gamma)^{n_0} \left[r^{n_1} \|\varphi\| + D|\varphi|_\chi (r^{n_1-1} + C_\chi \sum_0^{n_1-2} r^k) \right], \\ \|M\| &\leq C_\chi(m, \gamma)^p [1 + D(1 + C_\chi(1 - r)^{-1})] := C(m, \gamma), \text{ which gives the result. } \quad \square \end{aligned}$$

For $\chi \in]0, \alpha[$ we consider the function W^χ on V defined by $W^\chi(x) = |x|^\chi$. In Proposition 3.4 below we show that, due to the inequality $0 < k(\chi) < 1$ for $\chi \in]0, \alpha[$, P satisfies a drift condition (see [23]) with respect to W^χ . The same inequality implies also a spectral gap property in the Banach space $\mathcal{H}_{\chi, \varepsilon, \kappa}$ considered in Proposition 3.4 below. For reader's convenience we recall the Doeblin-Fortet spectral gap theorem (see [19]).

Let $(F, |\cdot|)$ be a Banach space, $(L, \|\cdot\|)$ another Banach space with a continuous injection $L \rightarrow F$. Let P be a bounded operator on F , which preserves L and satisfies the following conditions

- 1) The sequence of operator norms $|P^n|$ in is bounded.
- 2) The injection $L \rightarrow F$ is compact.
- 3) There exists an integer k and $r \in [0, 1[$, $D > 0$ such that for any v in L :

$$\|P^k v\| \leq r\|v\| + D|v|$$

- 4) If $v_n \in L$ is a sequence and $v \in F$ are such that $\|v_n\| \leq 1$ and $\lim_{n \rightarrow \infty} |v - v_n| = 0$,

there $v \in L$ and $\|v\| \leq 1$

Then in restriction to L , P is the commuting direct sum of a finite dimensional operator π with unimodular spectral values and a bounded operator U with spectral radius $r(U) < 1$. We observe that, frequently the norm $\|\cdot\|$ on L is given as a sum of a semi-norm $|\cdot|$ and the norm $|\cdot|$; then the inequality in condition 3 can be replaced by

$$|P^k v| \leq r|v| + D|v|$$

such an inequality is called Doeblin-Fortet's inequality.

Our substitute for the strong mixing property (see [30]) uses regularity of functions and is the following.

Proposition 3.4 *For any $\beta \in]0, 1]$ there exists $\ell \in \mathbb{N}$ and $b \geq 0$ such that $P^\ell W^\chi \leq \beta W^\chi + b$ for $n \geq \ell$. In particular the sequence of norms $|P^n|_\chi$ is bounded. Furthermore, if $0 < \kappa + \varepsilon < \chi < 2\kappa < 2\kappa + \varepsilon < \alpha$, the injection of $\mathcal{H}_{\chi, \varepsilon, \kappa}$ into \mathcal{C}_χ is compact and on $\mathcal{H}_{\chi, \varepsilon, \kappa}$, the Markov operator P satisfies the direct sum decomposition*

$$P = \rho \otimes 1 + U$$

where $r(U) < 1$ and $U(\rho \otimes 1) = (\rho \otimes 1)U = 0$

If $\alpha = 1$ and $0 < \varepsilon < \chi < 1$, $\kappa = 0$, the same result is valid.

Proof We verify successively the four above conditions. First we observe that for any $x \in V$,

$$|X_n^x - X_n^0| \leq |S_n||x|, \quad |X_n^x| \leq |X_n^0| + |S_n||x|.$$

If $\chi \leq 1$, it follows

$$\mathbb{E}|X_n^x|^\chi \leq \mathbb{E}|X_n^0|^\chi + (\mathbb{E}|S_n|^\chi)|x|^\chi.$$

Using the expression of X_n^0 and independence we get $\mathbb{E}|X_n^0|^\chi \leq (\mathbb{E}|B_1|^\chi)(\sum_0^\infty \mathbb{E}|S_k|^\chi)$. Since $\chi < \alpha$, we have $\mathbb{E}(|X_n^0|^\chi) \leq b < \infty$. On the other hand we have $\lim_{n \rightarrow \infty} (\mathbb{E}(|S_n|^\chi))^{1/n} = k(\chi) < 1$, hence for some $\varepsilon > 0$ $k(\chi) + \varepsilon < 1$, and for $n \geq \ell$, $|S_n|^\chi \leq \beta' \leq (k(\chi) + \varepsilon)^n$. It follows, for $n \geq \ell$:

$$P^n W^\chi(x) = \mathbb{E}|X_n^x|^\chi \leq \beta' W^\chi(x) + b$$

If $\chi > 1$ we use Minkowski's inequality, hence :

$$\mathbb{E}|X_n^x|^\chi \leq 2^\chi (\mathbb{E}|X_n^0|^\chi + \mathbb{E}|S_n|^\chi |x|^\chi)$$

As above, using $k(\chi) + \varepsilon < 1$ and $n \geq \ell$ we get

$$\mathbb{E}|X_n^x|^\chi \leq 2^\chi b + 2^\chi (k(\chi) + \varepsilon)^n |x|^\chi, \quad P^n W^\chi \leq \beta'' W^\chi + b' \text{ with } \beta'' < 1, b' < \infty.$$

We take $\beta = \beta'$ or β'' depending on $\chi \leq 1$ or $\chi > 1$. This allow us now to show that $|P^n|_\chi$ is bounded. We observe that $|\varphi(x)| \leq (1 + W(x))^\chi |\varphi|_\chi$, hence the positivity of P and $P1 = 1$ implies for $n \in \mathbb{N}$,

$$|P^n \varphi|(x) \leq |\varphi|_\chi P^n(2^\chi + 2^\chi W^\chi(x)) = |\varphi|_\chi (2^\chi + 2^\chi P^n W^\chi(x)).$$

From above we get

$$|P^n \varphi|(x) \leq |\varphi|_\chi [2^\chi + 2^\chi (b + \beta W^\chi(x))].$$

Then the definition of $|P^n|_\chi$ gives $|P^n|_\chi \leq 2^\chi (1 + b + \beta)$, hence the boundedness of $|P^n|_\chi$.

In order to show that if $\kappa + \varepsilon < \chi$, the injection of $\mathcal{H}_{\kappa, \varepsilon, \chi}$ in $F = \mathcal{C}_\chi$ is compact, we use Ascoli's argument and consider a large ball U_t with $t > 0$. We consider $\varphi_n \in \mathcal{H}_{\kappa, \varepsilon, \chi}$ with $\|\varphi_n\| < 1$. The definition on $\|\varphi_n\|$ implies for any $x, y \in U_t$

$$|\varphi_n(x)| \leq (1 + t)^\chi, |\varphi_n(x) - \varphi_n(y)| \leq (1 + t)^{2\kappa} |x - y|^\varepsilon$$

Hence, the restrictions of φ_n to U_t are equicontinuous and we can find a convergent subsequence φ_{n_k} . Using the diagonal procedure and a sequence t_i with $\lim_{i \rightarrow \infty} |t_i| = \infty$, we get a convergent subsequence $\varphi_{n_j} \in \mathcal{H}_{\kappa, \varepsilon, \chi}$ with limit a continuous function φ on V . From above we have $|\varphi_{n_j}(x) - \varphi_{n_j}(0)| \leq (1 + |x|)^\kappa |x|^\varepsilon$. hence for some $A, B > 0$, since $\kappa + \varepsilon < \chi$

$$|\varphi_{n_j}(x) - \varphi_{n_j}(0)| \leq (1 + |x|)^{\kappa + \varepsilon}, |\varphi(x)| \leq A + B(1 + |x|)^\chi.$$

It follows that $\varphi \in \mathcal{C}_\chi$. The above inequalities for φ_{n_j} imply

$$|(\varphi_{n_j}(x) - \varphi_{n_j}(0)) - (\varphi(x) - \varphi(0))| \leq 2(1 + |x|)^{\kappa + \varepsilon}.$$

Then the convergence of φ_{n_j} to φ , implies with $\varepsilon_{n_j} = |\varphi_{n_j}(0) - \varphi(0)|$,

$$|\varphi_{n_j}(x) - \varphi(x)| \leq \varepsilon_{n_j} + 2(1 + |x|)^{\kappa + \varepsilon}, \quad (1 + |x|)^{-\chi} |\varphi_{n_j}(x) - \varphi(x)| \leq \varepsilon_{n_j} + 2(1 + |x|)^{\kappa + \varepsilon - \chi}$$

with $\lim_{j \rightarrow \infty} \varepsilon_{n_j} = 0$. Also for t sufficiently large, and $|x| \geq t$, since $\kappa + \varepsilon < \chi$ we have $(1 + |x|)^{\kappa + \varepsilon - \chi} \leq \varepsilon_{n_j}$. Furthermore, the uniform convergence of φ_{n_j} to φ on U_t implies $\lim_{j \rightarrow \infty} (\sup\{|\varphi_{n_j}(x) - \varphi(x)| ; |x| \leq t\}) = 0$. The convergence of $|\varphi_{n_j} - \varphi|_\chi$ to zero follows.

The convergence of $\varphi_{n_j}(x)$ to $\varphi(x)$ for any $x \in V$ and the definition of $\|\varphi_{n_j}\|$, implies $\|\varphi\| \leq \lim_{j \rightarrow \infty} \|\varphi_{n_j}\| \leq 1$, hence $\varphi \in L$, and condition 4 is satisfied.

With $f = 0$ in Theorem 3.1 we have $P^f = P$. In particular there exists $k > 0$ such that $\|P^k \varphi\| \leq r\|\varphi\| + D|\varphi|_\chi$ if $\varphi \in \mathcal{H}_{\chi, \varepsilon, \kappa}$. Hence from [19], we know that the above conditions imply that P is the direct sum of a finite rank operator and a bounded operator U which satisfies $r(U) < 1$. Now it suffices to show that the equation $P\varphi = z\varphi$ with $|z| = 1$, $\varphi \in \mathcal{H}_{\chi, \varepsilon, \kappa}$ implies that φ is constant and $z = 1$. From the convergence in law of X_n^x to ρ we know that for any $x \in V$, the sequence of measures $P^n(x, \cdot)$ converges weakly to ρ . Also we have $|\varphi| \in \mathcal{H}_{\chi, \varepsilon, \kappa}$ and the sequence $n^{-1} \sum_1^n P^k |\varphi|$ converges to $\rho(|\varphi|)$. Since $|\varphi(x)| = |z^n \varphi(x)| \leq P^n(x, |\varphi|)$ we get $|\varphi(x)| \leq \rho(|\varphi|)$, hence $|\varphi|$ is bounded. Since $z^n \varphi(x) = \mathbb{E}(\varphi(X_n^x))$ and X_n^x converges in law to ρ , we get $\lim_{n \rightarrow \infty} z^n \varphi(x) = \rho(\varphi)$. This implies $z = 1$ and $\varphi(x) = \rho(\varphi)$ for any $x \in V$.

For the last assertion, in view of the above, we have only to verify the contraction condition. We write $[\varphi]_\varepsilon = \sup_{x \neq y} |x - y|^{-\varepsilon} |\varphi(x) - \varphi(y)|$. Then we have

$$\mathbb{E}(|\varphi(X_n^x) - \varphi(X_n^y)|) \leq [\varphi]_\varepsilon |X_n^x - X_n^y|^\varepsilon \leq [\varphi]_\varepsilon |x - y|^\varepsilon \mathbb{E}(|S_n|^\varepsilon).$$

Since $\varepsilon < \alpha$, we have $0 < k(\varepsilon) < r < 1$ for some r , hence $[P^n \varphi]_\varepsilon \leq r[\varphi]_\varepsilon$ for n large. \square

3.2 A mixing property with speed for the system $(V^{\mathbb{Z}_+}, \tau, \mathbb{P}_\rho)$.

In general, if the law of B_n has no density with respect to Lebesgue measure, the operator P on $\mathbb{L}^2(\rho)$ doesn't satisfy spectral gap properties, hence the stationary process X_n is not strongly mixing in the sense of [31] but Proposition 3.4 above shows that it is still ergodic. A simple example is as follows. Let $V = \mathbb{R}$ and let P be the operator defined on $\mathbb{L}^2(\rho)$ by the formula $P\varphi(x) = \frac{1}{2}[\varphi(\frac{x}{2}) + \varphi(\frac{x+1}{2})]$. Then P preserves $[0, 1]$, ρ is uniform measure on $[0, 1]$ and the adjoint P^* in $\mathbb{L}^2(\rho)$ of P can be identified with the map $x \rightarrow \{2x\}$ on $[0, 1]$ endowed with Lebesgue measure. Then the spectrum of P^* in $\mathbb{L}^2(\rho)$ is contained in $\{|z| = 1\}$ (and is in fact absolutely continuous). Hence P^* has no spectral gap in $\mathbb{L}^2(\rho)$; by duality this is true also of P .

Then, using Theorem 3.1 and Proposition 3.4, it is shown below that the system $(V^{\mathbb{Z}_+}, \tau, \mathbb{P}_\rho)$ satisfies a multiple mixing condition with respect to Lipschitz functions. For a study of extreme value properties for random walks on some classes of homogeneous spaces, using \mathbb{L}^2 -spectral gap methods, we refer to [21]. Since, using Proposition 2.4, the stationary process $(X_n)_{n \in \mathbb{N}}$ satisfies also anticlustering, we see below that extreme value theory can be developed for $(X_n)_{n \in \mathbb{N}}$ following the arguments of ([2], [3]) which were developed under mixing conditions involving continuous functions.

However it turns out that the mixing property $\mathcal{A}'(u_n)$ of [3] for continuous functions can be proved, as a consequence of the corresponding convergences involving Lipschitz functions

and point process theory. This condition plays an essential role in the study of space-time convergence (see [3]).

Let f be a bounded continuous function with non negative real part on $[0, 1] \times (V \setminus \{0\})$. Let r_n be an integer valued sequence with $\lim_{n \rightarrow \infty} r_n = \infty$, $r_n = o(n)$ and $k_n = \lceil r_n^{-1} n \rceil$. For $0 \leq i \leq n$, $0 \leq j \leq n$, $x \in V \setminus \{0\}$, $\omega \in V^{\mathbb{Z}^+}$ we write :

$$\bar{f}_n^j(x) = f(n^{-1}j, u_n^{-1}x), f_{i,n}(\omega) = \bar{f}_n^i(X_i), f_{i,n}^j(\omega) = \bar{f}_n^j(X_i).$$

In view of heavy notations, in some formulae we will write $r_n = r$, $k_n = k$, $\ell_n = \ell$. For f Lipschitz we denote by $k(f)$ the Lipschitz constant of f , and assume $\text{supp}(f) \subset [0, 1] \times U'_\delta$ with $\delta > 0$. We consider below the quantity $\mathbb{E}_\rho(\exp(-\sum_{i=1}^n f_{i,n}))$ which is the Laplace functional of the point process $\sum_{i=1}^n \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$. For its analysis we use the classical Bernstein method of gaps, i.e. we decompose the interval $[1, n]$ into large subintervals separated by smaller but still large ones.

Proposition 3.5 *Let f be a compactly supported Lipschitz function on $[0, 1] \times (V \setminus \{0\})$ with $\text{Ref} \geq 0$. Assume that the sequence $r_n \in \mathbb{N}$ satisfies $r_n = o(n)$, $\lim_{n \rightarrow \infty} (\log n)^{-1} r_n = \infty$ and write $|f|_\infty = m$, $k(f) = \gamma$, $\text{supp}(f) \subset [0, 1] \times U'_\delta$, $\delta > 0$. Then, with the above notations there exists $C(\delta, m, \gamma) < \infty$ such that,*

$$I_n(f) := |\mathbb{E}_\rho\{\exp(-\sum_{i=1}^n f_{i,n})\} - \prod_{j=1}^{k_n} \mathbb{E}_\rho\{\exp(-\sum_{i=(j-1)r_n+1}^{jr_n} f_{i,n}^{jr_n})\}| \leq C(\delta, m, \gamma) \sup(r_n^{-1}, n^{-1}r_n).$$

In particular with $r_n = \lceil n^{1/2} \rceil$ we get $\sup(n^{-1}r_n, r_n^{-1}) \leq 2n^{-1/2}$

Proof We write $[0, n] = [0, k_n r_n] \cup]k_n r_n, n]$, we decompose the interval $[0, k_n r_n]$ into k_n intervals $J_j = [jr_n, (j+1)r_n[$ and we distinguish in J_j the subinterval of length ℓ_n $J'_j = [(j+1)r_n - \ell_n, (j+1)r_n[$; the large integer ℓ_n will be specified below.

We write for f fixed, $I(n) = |\mathbb{E}_\rho(\exp(-\sum_1^n f_{i,n})) - \prod_{j=1}^k \mathbb{E}_\rho(\exp(-\sum_{i=(j-1)r+1}^{jr} f_{i,n}^{jr}))|$.

Then the triangular inequality gives $I(n) \leq I_1(n) + I_2(n) + I_3(n) + I_4(n)$ with

$$\begin{aligned} I_1(n) &= |\mathbb{E}_\rho(\exp(-\sum_1^n f_{i,n})) - \mathbb{E}_\rho(\exp(-\sum_1^{kr} f_{i,n}))| \\ I_2(n) &= |\mathbb{E}_\rho(\exp(-\sum_1^{kr} f_{i,n})) - \mathbb{E}_\rho(\exp(-\sum_{j=1}^k \sum_{i=(j-1)r+1}^{jr-\ell} f_{i,n}))| \\ I_3(n) &= |\mathbb{E}_\rho(\exp(-\sum_{j=1}^k \sum_{i=(j-1)r+1}^{jr-\ell} f_{i,n})) - \prod_{j=1}^k \mathbb{E}_\rho(\exp(-\sum_{i=1}^{r-\ell} f_{i,n}^{jr}))| \\ I_4(n) &= |\prod_{j=1}^k \mathbb{E}_\rho(\exp(-\sum_{i=1}^{r-\ell} f_{i,n}^{jr})) - \prod_{j=1}^k \mathbb{E}_\rho(\exp(-\sum_{i=1}^r f_{i,n}^{jr}))| \end{aligned}$$

where stationarity of \mathbb{P}_ρ has been used in the expressions of $I_3(n), I_4(n)$. The quantities I_1, I_2, I_4 are boundary terms; their estimation below is based only on the fact that r_n (resp. ℓ_n) is small with respect to n (resp. r_n), the form of u_n , and f has non negative real part. On the other hand, estimation of I_3 depends on Theorem 3.1 and Proposition 3.4.

Using the inequality $|\exp(-x) - \exp(-y)| \leq |x - y|$ for x, y with non negative real parts we get $I_1(n) \leq \sum_{kr+1}^n \mathbb{E}_\rho(f_{i,n})$. Let $\delta > 0$ be as above such that $f(t, x) = 0$ for $t \in [0, 1]$, $|x| < \delta$, and observe that $n - kr < r$. Then the above bound for $I_1(n)$ gives :

$$I_1(n) \leq r_n |f|_\infty \mathbb{P}_\rho\{u_n^{-1} |X_1| \geq \delta\}.$$

Since $\lim_{n \rightarrow \infty} n^{-1} r_n = 0$, the definition of u_n and Theorem 2.1 give $\lim_{n \rightarrow \infty} I_1(n) = 0$. Also $I_1(n)$ is bounded by $n^{-1} r_n$, up to a coefficient depending only on m, δ . For $I_2(n)$, a similar argument involving each interval J_j and the subinterval J'_j gives :

$$I_2(n) \leq k_n \ell_n |f|_\infty \mathbb{P}_\rho\{u_n^{-1} |X_1| \geq \delta\}.$$

Using $k_n r_n \leq n$ we get $\lim_{n \rightarrow \infty} n^{-1} k_n \ell_n \leq \lim_{n \rightarrow \infty} r_n^{-1} \ell_n$, i.e. $\lim_{n \rightarrow \infty} I_2(n) = 0$ if $\lim_{n \rightarrow \infty} r_n^{-1} \ell_n = 0$.

Also we can bound $I_2(n)$ by $r_n^{-1} \ell_n$, up to a coefficient depending only on m, δ .

For $I_4(n)$, we use the inequality $|\prod_1^n z_j - \prod_1^n w_j| \leq \sum_1^n |z_j - w_j|$ if $|z_j|$ and $|w_j|$ are less than 1.

Hence :

$$I_4(n) \leq \sum_{j=1}^k |\mathbb{E}_\rho(\exp(-\sum_1^{r-\ell} f_{i,n}^{jr})) - \mathbb{E}_\rho(\exp(-\sum_1^r f_{i,n}^{jr}))| \leq |f|_\infty k_n \ell_n \mathbb{P}_\rho\{|X_1| > \delta u_n\}$$

As above we get $\lim_{n \rightarrow \infty} I_4(n) = 0$ if $\lim_{n \rightarrow \infty} r_n^{-1} \ell_n = 0$, and a bound for $I_4(n)$ of the same form as for $I_2(n)$.

The estimation of $I_3(n)$ is more delicate and depends on Lemma 3.6 below. We begin with the inequality : $I_3(n) \leq D(n) + I_5(n) + I_3(n - r_n)$ where

$$D(n) = |\mathbb{E}_\rho(\exp(-\sum_{j=1}^k \sum_{(j-1)r+1}^{jr-\ell} f_{i,n})) - \mathbb{E}_\rho(\exp(-\sum_{i=1}^{r-\ell} f_{i,n})) \mathbb{E}_\rho(\exp(-\sum_{j=2}^k \sum_{(j-1)r+1}^{jr-\ell} f_{i,n}))|,$$

$$I_5(n) = |\mathbb{E}_\rho(\exp(-\sum_1^{r-\ell} f_{i,n})) \mathbb{E}_\rho(\exp(-\sum_{j=2}^k \sum_{(j-1)r+1}^{jr-\ell} f_{i,n})) - \mathbb{E}_\rho(\exp(-\sum_1^{r-\ell} f_{i,n}^r))$$

$$\mathbb{E}_\rho(\exp(-\sum_{j=2}^k \sum_{(j-1)r+1}^{jr-\ell} f_{i,n}))|,$$

$$I_3(n - r) = |\mathbb{E}_\rho(\exp(-\sum_{j=2}^k \sum_{(j-r)r+1}^{jr-\ell} f_{i,n})) - \prod_{j=2}^k \mathbb{E}_\rho(\exp(-\sum_1^{r-\ell} f_{i,n}^{jr}))|.$$

Using as above the inequality $|\exp(-x) - \exp(-y)| \leq |x - y|$, and $Re(f) \geq 0$ we get :

$$I_5(n) \leq |\mathbb{E}_\rho(\sum_1^{r-\ell} f_{i,n}) - \mathbb{E}_\rho(\sum_1^{r-\ell} f_{i,n}^r)|.$$

Since f is Lipschitz we have, for t', t'' in $[0, 1]$, $x \in V \setminus \{0\}$: $|f(t', x) - f(t'', x)| \leq k(f) |t' - t''|$. Since $|n^{-1} i - n^{-1} r_n| \leq n^{-1} r_n$ we have

$$I_5(n) \leq (r_n - \ell_n) n^{-1} r_n k(f) \mathbb{P}_\rho\{u_n^{-1} |X_1| \geq \delta\} \leq r_n^2 n^{-1} k(f) \mathbb{P}_\rho\{|X_1| \geq \delta u_n\}.$$

Using Theorem 2.1 we get $I_5(n) \leq C n^{-2} r_n^2$ with a constant C depending on $k(f)$ and δ .

In order to estimate $D(n)$ we consider the family of operators $P_{i,n}$ on the space $\mathcal{H}_{\chi, \varepsilon, \kappa}$ with $\chi, \varepsilon, \kappa$ as in Proposition 3.4, defined by $P_{i,n} \varphi(x) = \mathbb{E}((\exp(-f_{i,n}(\omega))) \varphi(X_i^x))$ and the function ψ_n defined by $\psi_n(y) = \mathbb{E}\{\exp(-\sum_{j=2}^k \sum_{i=(j-1)r+1}^{jr-\ell} f_{i,n}^{i+r}) / X_r^x = y\}$. Since, $u_n \geq 1$, for n large with $m = |f|_\infty$, $\gamma = k(f)$, the functions $f_{i,n}$ satisfy $|f_{i,n}|_\infty \leq m$, $k(f_{i,n}) \leq \gamma$, hence

the operators $P_{i,n}$ belong to $O(m, \gamma) \subset \text{End}\mathcal{H}_{\chi, \varepsilon, \kappa}$. With the above notations, the products of operators $P_{i,n}$ belong to $\widehat{O}(m, \gamma)$. Also, using Proposition 3.4 we know that on $\mathcal{H}_{\chi, \varepsilon, \kappa}$ we can write $P = \rho \otimes 1 + U$ where U has spectral radius $r(U)$ less than 1 and U commutes with the projection $\rho \otimes 1$. We note also that for f as above and $\psi \in \mathcal{H}_{\chi, \varepsilon, \kappa}$ we have :

$$|\rho(P^f \psi)| \leq \rho(P|\psi|) = \rho(|\psi|) \leq \|\psi\|$$

Then Lemma 3.6 below implies the convergence of $D(n)$ to zero with speed.

Now, in order to prove the proposition, we are left to show $\lim_{n \rightarrow \infty} I_3(n) = 0$. We iterate k_n times the inequality : $I_3(n) \leq D(n) + I_5(n) + I_3(n - r_n)$. We get, using Lemma 3.6 : $I_3(n) \leq I_3(n - r_n) + C'(f)(n^{-2}r_n^2 + r_1^{\ell_n}(U)) \leq C'(f)(k_n r_1^{\ell_n}(U) + n^{-1}r_n)$, with $C'(f) \geq 1$, depending on m, γ . Since $r_n = o(n)$, it remains to choose ℓ_n such that $\ell_n = o(r_n)$ with $\lim_{n \rightarrow \infty} k_n r_1^{\ell_n}(U) = 0$. These conditions can be written as

$$\lim_{n \rightarrow \infty} r_n^{-1} \ell_n = 0, \quad \lim_{n \rightarrow \infty} r_n^{-1} n r_1^{\ell_n}(U) = 0.$$

The choice of ℓ_n with the above properties is possible since :

$$r_1(U) < 1, \quad \lim_{n \rightarrow \infty} n^{-1} r_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (\log n)^{-1} r_n = \infty.$$

One can take $\ell_n < r_n$ with $(\log n)^{-1} \ell_n = \infty$. The above estimations of I_1, I_2, I_3, I_4, I_5 give bounds by $\sup(n^{-1} r_n, r_n^{-1})$, up to a coefficient depending on δ, m, γ only. \square

Lemma 3.6 *There exist positive numbers $C_1(U), r_1(U) \in]r(U), 1[$ and $C(f)$ depending only of m, γ such that, for $n \in \mathbb{N}$ and $\ell_n < r_n$, $D(n)$, as above :*

$$D(n) = |\rho(P_{1,n} \cdots P_{r-\ell_n, n} U^{\ell_n} \psi_n)| \leq C_1(U) C(f) (r_1(U))^{\ell_n}.$$

Proof We observe that Markov's property implies $\mathbb{E}(e^{-f(X_1^x)} g(\omega)) = P^f(\mathbb{E}(g(\omega)))$ where f is as above, $g(\omega)$ is a function depending on ω through the random variables $X_k^x (k \geq 1)$ and $\mathbb{E}(g(\omega))$ is a function of x . We apply this property to $\mathcal{H}_{\chi, \varepsilon, \kappa}$ with $f = f_{i,n}$ ($1 \leq i \leq r - \ell$) or $f = 0, g = \psi_n$ as above, hence writing $P^\ell = \rho \otimes 1 + U^\ell$ and

$$D(n) = |\rho(P_{1,n} \cdots P_{r-\ell, n} P^\ell \psi_n) - \rho(P_{1,n} \cdots P_{r-\ell, n} 1) \rho(\psi_n)| = |\rho(P_{1,n} \cdots P_{r-\ell, n} U^\ell \psi_n)|,$$

Proposition 3.4 implies the existence of $C_1(U) < \infty, r_1(U) \in]r(U), 1[$ with

$$\|U^\ell \psi_n\| \leq C_1(U) r_1^\ell(U) \|\psi_n\|.$$

On the other hand, since ψ_n is of the form $\psi_n = M1$ with $M \in \widehat{O}(m, \gamma)$ we have, using Theorem 3.1, $\|\psi_n\| \leq C(f)$ with $C(f)$ depending on m, γ . It follows $D(n) \leq C_1(U) C(f) (r_1(U))^{\ell_n}$. \square

4 Asymptotics of exceedances processes

4.1 Statements of results

Let E be a complete separable metric space which is locally compact, $M_+(E)$ the space of positive Radon measures on E , $M_p(E)$ its subspace of point measures, $\mathcal{C}_+^c(E)$ (resp $\mathcal{L}_+^c(E)$) the space of non negative and compactly supported continuous (resp Lipschitz)

functions. Then it is well known that the vague topology on $M_+(E)$ is given by a metric and with respect to this metric, $M_+(E)$ is a complete separable metric space. Furthermore this metric is constructed (see [30] Lemma 3.11, Proposition 3.17) using a countable family $(h_i)_{i \in I}$ of functions in $\mathcal{L}_+^c(E)$ and $M_p(E)$ is a closed subset of $M_+(E)$. It follows that, in various situations with respect to weak convergence of random measures, $\mathcal{C}_+^c(E)$ can be replaced by $\mathcal{L}_+^c(E)$.

Below, assuming condition (c-e), we describe the asymptotics of the space-time exceedances process $N_n = \sum_{i=1}^n \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$ under the probability \mathbb{P}_ρ and we state a few corollaries. The results are formally analogous to results for stationary processes proved in ([2], [3]) under general conditions. Here however, corresponding conditions have been proved in sections 2, 3 for the affine random walk hence the results described below are new for affine random walks but the scheme of the proofs is given in [2], [3].

It is convenient to express the Laplace formulae below in terms of the occupation measure $\pi_v^\omega = \sum_0^\infty \varepsilon_{S_n(\omega)v}$ of the linear random walk $S_n(\omega)v$ on $V \setminus \{0\}$. We observe that these formulae depend only of the linear part of the affine random walk ; this is a consequence of the choice of the normalization by u_n .

We denote by $\sum_{i \geq 0} \varepsilon_{T_i^\delta}$ the homogeneous point Poisson process on $[0, 1]$ with intensity $p(\delta) = \theta \delta^{-\alpha}$ and by $\sum_{j \in \mathbb{Z}} \varepsilon_{Z_{ij}}$ ($i \geq 0$) an i.i.d. collection of copies of the cluster process $C = \sum_{j \in \mathbb{Z}} \varepsilon_{Z_j}$ described in Proposition 2.6, independant of $\sum_{i \geq 0} \varepsilon_{T_i^\delta}$. Since we have $|X_n^x - X_n^y| \leq |S_n||x - y|$ and $\lim_{n \rightarrow \infty} |S_n| = 0$, \mathbb{P} -a.e. it is possible to replace \mathbb{P}_ρ by \mathbb{P} and X_n by X_n^x with x fixed, in the statements. We give the corresponding proof for the logarithm law only.

Theorem 4.1 *The sequence of normalized space-time point processes $N_n = \sum_{i=1}^n \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$ on the space $[0, 1] \times (V \setminus \{0\})$ converges weakly to a point process N . For any $\delta > 0$, the law of the restriction of N to $[0, 1] \times U'_\delta$ is the same as the law of the point process on $[0, 1] \times U'_\delta$ given by :*

$$\sum_{i \geq 0} \sum_{j \in \mathbb{Z}} \varepsilon_{(T_i^\delta, \delta Z_{ij})} \mathbf{1}_{\{|Z_{ij}| > 1\}}.$$

If η denotes the law of N , $f \in \mathcal{C}_+^c([0, 1] \times U'_\delta)$ and ψ_η is the Laplace functional of η , then $-\log \psi_\eta(f)$ is equal to

$$\theta \delta^{-\alpha} \int_0^1 \widehat{\mathbb{E}}_{\Lambda_1} (1 - \exp(- \sum_{j \in \mathbb{Z}} f(t, \delta Z_j))) dt = \int_0^1 \mathbb{E}_{\Lambda_0} (\exp f_t(v) - 1) \exp(-\pi_v^\omega(f_t)) dt$$

where $f_t(x) = f(t, x)$

Assuming the mixing and anticlustering conditions for compactly supported continuous functions, this statement was proved in [3]. Here we will use Propositions 2.6, 3.4 and point

process theory. We observe that, due to the normalization by u_n the law of N depends only of μ, Λ_0 .

Now as a consequence of Theorem 4.1, the mixing property stated in Proposition 3.5 for Lipschitz functions can be extended to compactly supported continuous functions. Then, in particular, the mixing conditions $\mathcal{A}(u_n)$ and $\mathcal{A}'(u_n)$ of ([2], [3]) are valid here and the basic conditions of extreme value theory (see [8]) are satisfied in our context.

Corollary 4.2 *With the notation of Proposition 3.5, assume f is a continuous compactly supported function on $[0, 1] \times (V \setminus \{0\})$. Then we have the convergence $\lim_{n \rightarrow \infty} I_n(f) = 0$.*

Since the space exceedances process $N_n^s = \sum_{i=1}^n \varepsilon_{u_n^{-1} X_i}$ is the projection of N_n on $V \setminus \{0\}$ we have the

Corollary 4.3 *The normalized space exceedances process N_n^s converges weakly to a point process N^s . The law of the restriction of N^s to U_δ^s is the same as the law of the point process*

$$\overline{Q}^\delta = \sum_{i=0}^{T^\delta} \sum_{j \in \mathbb{Z}} \varepsilon_{\delta Z_{ij}} 1_{\{|Z_{ij}| > 1\}}$$

where T^δ is a Poisson random variable with mean $p(\delta) = \theta \delta^{-\alpha}$, independant of Z_{ij} for $i \geq 0, j \in \mathbb{Z}$.

The Laplace functional of N^s is given in logarithmic form by

$$-\mathbb{E}_{\Lambda_0}[(\exp f(v) - 1) \exp(-\pi_v^\omega(f))].$$

Assuming the mixing and anticlustering conditions for continuous functions, this statement was proved in [2], using the formula for Laplace functionals in Proposition 2.6.

We consider the \mathbb{N} -valued random variable $\zeta = \pi_v^\omega(U_1^s)$ and we write $\zeta_k = \mathbb{Q}_{\Lambda_1}\{\zeta = k\}$ for $k \geq 1$; in particular we have $\zeta_1 = \theta, \zeta_k \geq \zeta_{k+1}$.

Corollary 4.4 *The sequence of normalized time exceedances process $N_n^t = \sum_{i=1}^n \varepsilon_{n^{-1} i} 1_{\{|X_i| > u_n\}}$ converges weakly ($n \rightarrow \infty$) to the homogeneous compound Poisson process N^t on $[0, 1]$ with intensity θ , and cluster probabilities $\nu_k (k \geq 1)$ where $\nu_k = \theta^{-1}(\zeta_k - \zeta_{k+1})$.*

Under special hypotheses, including density of the law of B_n with respect to Lebesgue measure, this statement was proved in [22].

Fréchet's law for $M_n^x = \sup\{|X_k^x|; 1 \leq k \leq n\}$ is a simple consequence of Corollary 4.4 as follows.

Corollary 4.5 For any $x \in V$ and $t > 0$ we have the convergence in law of $u_n^{-1}M_n^x$ to Fréchet's law Φ_α^θ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}\{u_n^{-1}M_n^x < t\} = \exp(-\theta t^{-\alpha}) = \Phi_\alpha^\theta([0, t])$$

with $\theta = \mathbb{Q}_{\Lambda_1}\{\sup_{n \geq 1} |S_n(\omega)v| \leq 1\}$. Furthermore the law of the normalized hitting time $t^{-\alpha}\tau_t^x$ of U_t^1 by the process $|X_n^x|$ converges to the exponential law with parameter $c\theta$, i.e.

$$\lim_{t \rightarrow \infty} \mathbb{P}\{t^{-\alpha}\tau_t^x > u\} = \exp(-c\theta u).$$

For $d = 1$, another proofs of Laplace formulae in 4.3, 4.5 were given in [5] (see 3.1.1, 3.2.1)

It was observed in [28] that Sullivan's logarithm law for excursions of geodesics around the cusps of hyperbolic manifolds (see [36]), in the case of the modular surface, is a consequence of Fréchet's law for the continuous fraction expansion of a real number uniformly distributed in $[0, 1]$ (see [27]). Here, in this vein, we have the following logarithm law.

e

Corollary 4.6 For any $x \in V$, we have the \mathbb{P} -a.e. convergence

$$\limsup_{n \rightarrow \infty} \frac{\log |X_n^x|}{\log n} = \frac{1}{\alpha} = \limsup_{n \rightarrow \infty} \frac{\log M_n^x}{\log n}.$$

If x is random, we observe that a logarithm law and a modified Fréchet law have been obtained in [21] for random walks on some homogeneous spaces of arithmetic character, using \mathbb{L}^2 -spectral gap methods.

Given a Borel subset A of U_1^1 and a real number $t > 1$, we can also consider the hitting time τ_{tA}^x of the dilated set tA under by the process X_n^x (see [35] p. 290). In the context of collective risk theory this hitting time can be interpreted as a ruin time associated to the entrance of X_n^x in the set tA (see [6]). We observe that, in contrast to the associated linear random walk, the event of ruin (in infinite time) occurs here with probability 1, due to the finiteness of the stationary measure ρ . However, in law, the ruin scenario is the same : excursion at infinity of the associated linear random walk.

Then the convergence to the point process N^s stated in Corollary 4.3 gives the

Corollary 4.7 Let A be a Borel subset of V such that $\bar{A} \subset U_1^1$, $\Lambda(A) > 0$ $\Lambda(\partial A) = 0$. Then for any $x \in V$, the normalized hitting time $t^{-\alpha}\tau_{tA}^x$ of the set tA converges in law to the exponential law with parameter $c\theta(A)$ with

$$\theta(A) = (\Lambda_0(A))^{-1} \mathbb{Q}_{\Lambda_0}\{\sum_1^\infty 1_A(S_i y) = 0, y \in A\} \in]0, \theta].$$

We observe that, in the notation of [35], the quantity $\mathbb{Q}\{\sum_1^\infty 1_A(S_i y) = 0\}$ is, for $y \in A$, the escape probability from A for the linear random walk $S_i y$ on $V \setminus \{0\}$. Hence the number $\gamma(A) = \mathbb{Q}_{\Lambda_0}\{\sum_1^\infty 1_A(S_i y) = 0, y \in A\} \in]0, \Lambda_0(A)]$ is the corresponding capacity with respect

to the Q -invariant measure Λ_0 . The positivity of $\theta(A)$ is a simple consequence of the formula for $\theta(A)$ and of the dynamical argument used in the proof of Proposition 2.4 for the inequality $\theta > 0$.

The inequality $\tau_{tA}^x \geq \tau_t^x$ implies $\theta(A) \leq \theta < 1$. We observe that, in the context of statistics of hitting times for hyperbolic dynamical systems, convergence to an exponential law is also valid, but since almost every point is repulsive, the corresponding condition $\theta = 1$ is then generically satisfied (see for example [26], [32]). Here the property $\theta(A) \in]0, 1[$ is a consequence of the contraction-expansion property, which follows from the unboundness of the semigroup generated by $\text{supp}(\mu)$. Heuristically speaking, the sphere at infinity of V is weakly attractive for the affine random walk.

4.2 Proofs of point process convergences

The proof of Theorem 4.1 will follow of three lemmas.

We denote by $(X_{k,j})_{k \in \mathbb{N}}$ an i.i.d. sequence of copies of the process $(X_j)_{j \in \mathbb{N}}$ and we write

$$\tilde{N}_{k,n} = \sum_{j=1+(k-1)r_n}^{kr_n} \varepsilon_{(n^{-1}kr_n, u_n^{-1}X_{k,j})}, \tilde{N}_n = \sum_{k=1}^{k_n} \tilde{N}_{k,n},$$

where r_n, k_n are as in section 2.

For $k_n > 0$ we denote by $\mathbb{E}_\rho^{(k_n)}$ the expectation corresponding to the product probability of k_n copies of \mathbb{P}_ρ .

If f is a non negative and compactly supported Lipschitz function on $[0, 1] \times V \setminus \{0\}$, we have, using independance :

$$\mathbb{E}_\rho^{(k_n)}(\exp(-\tilde{N}_n(f))) = \prod_{k=1}^{k_n} \mathbb{E}_\rho(\exp(-\sum_{j=1+(k-1)r_n}^{kr_n} f(n^{-1}kr_n, u_n^{-1}X_{k,j}))).$$

This relation and the multiple mixing property in Proposition 3.5 show that, on functions f as above, the asymptotic behaviour of the Laplace functionals of N_n under \mathbb{E}_ρ , and \tilde{N}_n under $\mathbb{E}_\rho^{(k_n)}$, are the same. We begin by considering the convergence of $\mathbb{E}_\rho^{(k_n)}(\exp(-\tilde{N}_n(f)))$. Lemma 4.8 below is a general statement giving the weak convergence of a sequence of random measures, using only the convergence of the values of the Laplace functionals on Lipschitz functions. Lemmas 4.9, 4.10 are reformulations of parts of the proof of Theorem 2.3 in [3], which was considered in a general setting.

Lemma 4.8 *Let E be a locally compact separable metric space. Let η_n be a sequence of random measures on E and, for f non negative Lipschitz and compactly supported, assume that the sequence of Laplace functionals $\psi_{\eta_n}(f)$ converges to $\psi(f)$ and $\psi(sf)$ is continuous at $s = 0$; then the sequence η_n converges weakly. A random measure η on E , is well defined by the values of its Laplace functional on Lipschitz functions.*

Proof We begin by the last assertion and we use the family of Lipschitz functions $(h_i)_{i \in I}$ considered in the above subsection. If the random measures η, η' satisfy $\psi_\eta(f) = \psi_{\eta'}(f)$ for any $f \in \mathcal{L}_+^c(E)$ and $\lambda_1, \lambda_2, \dots, \lambda_p$ are non negative numbers then we have $\psi_\eta(\sum_{i=1}^{i=p} \lambda_i h_i) =$

$\psi_{\eta'}(\sum_{i=1}^{i=p} \lambda_i h_i)$. It follows that the random vectors $(\eta(h_1), \dots, \eta(h_p))$ and $(\eta'(h_1), \dots, \eta'(h_p))$ have the same Laplace transforms, hence the same laws. Furthermore, for rational numbers $r_j < r'_j$ the finite intersections of sets of the form $\{\mu \in M_+(E), \mu(h_i) \in]r_j, r'_j[)\}$ define a countable basic \mathcal{U} of open subsets in $M_+(E)$ stable under finite intersection, hence a π -system (see [30]). Then from above, η, η' are equal on \mathcal{U} ; since the sigma-field generated by \mathcal{U} coincides with the Borel sigma-field, one has $\eta = \eta'$.

We observe that, if a sequence of random measures η_n is such that for any $f \in \mathcal{L}_+(E)$ the sequence of real random variables $\eta_n(f)$ is tight, then the sequence η_n itself is tight. This follows for a corresponding result in [30] for $f \in \mathcal{C}_+^c(E)$ since any such f is dominated by an element of $\mathcal{L}_+^c(E)$.

Assuming the convergence of $\psi_{\eta_n}(f)$ to $\psi_\eta(f)$ for any $f \in \mathcal{L}_+^c(E)$ and the continuity at $s = 0$ of $\psi_\eta(sf)$, we get that $\psi_\eta(sf)$ is the Laplace transform of the real random variable $\eta(f)$, hence the convergence of the sequence $\eta_n(f)$ to $\eta(f)$ for any $f \in \mathcal{L}_+^c(E)$. From above and the continuity hypothesis of $\psi(sf)$ at $s = 0$, we get that the sequence η_n is tight. If η_{n_i} is a subsequence converging weakly to the random measure η we have the convergence of the corresponding Laplace functionals for any $f \in \mathcal{L}_+^c(E)$. Since such a limit is independant of the subsequence, we get from above that two possible weak limits of random measures are equal. Hence the sequence η_n converges weakly to η . \square

Lemma 4.9 *Let f be a non negative and compactly supported continuous function on $[0, 1] \times U'_\delta$ and let $\sum_{j \in \mathbb{Z}} \varepsilon_{Z_j}$ be the cluster process for the affine random walk $(X_k)_{k \in \mathbb{N}}$. Then :*

- a) $\lim_{n \rightarrow \infty} [\log \mathbb{E}_\rho^{(k_n)}(\exp(-\tilde{N}_n(f))) + \sum_{k=1}^{k_n} (1 - \mathbb{E}_\rho(\exp(-\tilde{N}_{k,n}(f))))] = 0.$
- b) $\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} (1 - \mathbb{E}_\rho(\exp(-\tilde{N}_{k,n}(f)))) = \theta \delta^{-\alpha} \int_0^1 \widehat{\mathbb{E}}_{\Lambda_1}(1 - \exp(-\sum_{j \in \mathbb{Z}} f(t, \delta Z_j))) dt.$

Lemma 4.10 *Let $\sum_{i \geq 0} \varepsilon_{T_i^\delta}$ be a homogeneous Poisson process of intensity $p(\delta) > 0$ on $[0, 1]$, which is independant of the sequence of cluster processes $\sum_{j \in \mathbb{Z}} \varepsilon_{Z_{ij}}$.*

Then for any non negative and compactly supported continuous function f on $[0, 1] \times U'_\delta$, the Laplace functional of the point process $Q^\delta = \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} \varepsilon_{(T_i^\delta, \delta Z_{ij})} 1_{\{|Z_{ij}| > 1\}}$ restricted to $[0, 1] \times U'_\delta$ satisfies :

$$\log \psi^\delta(f) = -p(\delta) \int_0^1 \widehat{\mathbb{E}}_{\Lambda_1}(1 - \exp(-\sum_{j \in \mathbb{Z}} f(t, \delta Z_j))) dt$$

Proof of Theorem 4.1 Let f be a non negative and compactly supported Lipschitz function on $[0, 1] \times U'_\delta$. Using Proposition 3.5, Lemma 4.9 implies that, on such functions the Laplace functionals of N_n and \tilde{N}_n have the same limit, namely

$$\psi^\delta(f) = \exp[-p(\delta) \int_0^1 \widehat{\mathbb{E}}_{\Lambda_1}(1 - \exp(-\sum_{j \in \mathbb{Z}} f(t, \delta Z_j))) dt].$$

We observe that, for fixed f as above, the function $s \rightarrow \psi^\delta(sf)$ is continuous at $s = 0$. Since the function $s \rightarrow \psi_n(sf) = \mathbb{E}_\rho(\exp(-sN_n(f)))$ is the Laplace transform of the non negative random variable $N_n(f)$, the continuity theorem for Laplace transforms implies that the sequence $N_n(f)$ converges in law to some random variable. Since the sequence of Laplace functionals $\psi_n(f)$ converges to $\psi^\delta(f)$, Lemma 4.8 implies that there exists a unique point process N on $[0, 1] \times (V \setminus \{0\})$ such that the sequence N_n converges weakly to N . As stated in Lemma 4.10, for f as above the restriction of N to $[0, 1] \times U'_\delta$ is given by the point process formula in the theorem. A density argument shows that the Laplace functional of N on the function $f \in \mathcal{C}_+^c([0, 1] \times U'_\delta)$ is equal to

$$\psi^\delta(f) = \exp[-p(\delta) \int_0^1 \widehat{\mathbb{E}}_{\Lambda_1} [1 - \exp(-\sum_{j \in \mathbb{Z}} f(t, \delta Z_j))] dt].$$

The point process formula for N follows, as well as the first part of the formula giving the Laplace functional of N . The second part is a consequence of the last formula in Proposition 2.6 applied to the function $v \rightarrow f(t, \delta v)$ and of the α -homogeneity of Λ_0 . \square

Proof of Corollary 4.2 The first term $\mathbb{E}_\rho(\exp(-\sum_1^n f_{i,n}))$ in $I_n(f)$ is the value of the Laplace functional of N_n on the continuous function f . Hence the weak convergence in Theorem 4.1 implies its convergence to the Laplace functional of N on f . The same remark is valid for the second term in $I_n(f)$, if N_n is replaced by \tilde{N}_n : the limit of \tilde{N}_n is also N , using Lemma 4.8 and Proposition 3.5. Then for any f in $\mathcal{C}_+^c([0, 1] \times (V \setminus \{0\}))$ we have :
 $\lim_{n \rightarrow \infty} I_n(f) = \lim_{n \rightarrow \infty} |\mathbb{E}_\rho(\exp(-N_n(f))) - \mathbb{E}_\rho^{(k_n)}(\exp(-\tilde{N}_n(f)))| = 0 \square$

Proof of Corollary 4.3 The point process N_n^s is the projection of N_n on $V \setminus \{0\}$. Since $[0, 1]$ is compact and the projection is continuous, the continuous mapping theorem implies the required convergence, using the first part of Theorem 4.1. The formula for the Laplace functional of N^s is a direct consequence of the second part in Theorem 4.1 applied to a function independent of t . \square

Proof of Corollary 4.4 For $\varphi \in \mathcal{C}_+^c([0, 1])$ we have $N_n^t(\varphi) = N_n(\varphi \otimes 1_{U'_1})$. Since the discontinuity set of $1_{U'_1}$ is Λ -negligible, Theorem 4.1 gives the convergence of $N_n^t(\varphi)$ to $N^t(\varphi)$. With $f = \varphi \otimes 1_{U'_1}$, the formula for the Laplace functional $\psi_\eta(f)$ of N gives the Laplace functional $\psi_{\eta^t}(\varphi)$ of N^t in the logarithmic form

$$\log \psi_{\eta^t}(\varphi) = -\theta \int_0^1 \widehat{\mathbb{E}}_{\Lambda_1} [1 - \exp - \varphi(x)\gamma] dt.$$

The expression of the generating function of the random variable $\gamma = \sum_{j \in \mathbb{Z}} 1_{U'_1}(Z_j)$ follows from the last formula in Proposition 2.6 :

$$\sum_1^\infty e^{-sk} \nu_k = 1 - (e^s - 1)\theta^{-1} \widehat{\mathbb{E}}_{\Lambda_1} [\exp(-s \pi_v^\omega(U'_1))].$$

Hence $\nu_k = \theta^{-1}(\zeta_k - \zeta_{k+1})$

In view of Theorem 4.1, the point process N^t can be written as $N^t = \sum_{k \geq 0} \gamma_k \varepsilon_{T_k^1}$, where the random variables γ_k are i.i.d. with the same law as γ , hence N^t coincides with the

compound Poisson process described in the statement. \square

Proof of Corollary 4.5 Replacing u_n by δu_n ($\delta > 0$) in Corollary 4.4, we see that the point process on $[0, 1]$ given by $N_{n,\delta}^t = \sum_{k=1}^n \varepsilon_{n^{-1}k} 1_{\{|X_k| > \delta u_n\}}$ converges to $N_\delta^t = \sum_{k \geq 0} \gamma_k \varepsilon_{T_k^\delta}$ where $\sum_{k \geq 0} \varepsilon_{T_k^\delta}$ is the Poisson process on $[0, 1]$ with intensity $\theta \delta^{-\alpha}$ and the γ_k are i.i.d. random variables as in the proof of Corollary 4.4. It follows that for any $\delta > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_\rho \{N_{n,\delta}^t(1) = 0\} = \exp(-\theta \delta^{-\alpha}).$$

Since $\mathbb{P}_\rho \{N_{n,\delta}^t(1) = 0\} = \mathbb{P}_\rho \{M_n \leq u_n \delta\}$, the convergence of $u_n^{-1} M_n$ to Fréchet's law follows.

If M_n is replaced by M_n^x with $x \in V$, the same proof as the one given below for the logarithm law remains valid. The last assertion in the corollary is a direct consequence of Fréchet's law. \square

4.3 Proof of logarithm's law

The proof of logarithm's law is based on Fréchet's law and depends on two lemmas as follows.

Lemma 4.11 *We have \mathbb{P}_ρ -a.e. :*

$\limsup_{n \rightarrow \infty} \frac{\log |X_n|}{\log n} \leq \limsup_{n \rightarrow \infty} \frac{\log M_n}{\log n} \leq \frac{1}{\alpha}$. Furthermore, for any $\varepsilon > 0$, and for n large we have $|X_n| \leq n^{1/\alpha + \varepsilon}$.

Proof Let $\varepsilon > 0$, $\alpha_n(\varepsilon) = \{|X_n| \geq n^{1/\alpha + \varepsilon}\} \subset V^{\mathbb{Z}_+}$, $\alpha'_n(\varepsilon) = V^{\mathbb{Z}_+} \setminus \alpha_n(\varepsilon)$. Stationarity of X_n implies $\mathbb{P}_\rho \{\alpha_n(\varepsilon)\} = \mathbb{P}_\rho \{|X_0| \geq n^{1/\alpha + \varepsilon}\}$. Since $\lim_{n \rightarrow \infty} n^{1+\alpha\varepsilon} \ell^\alpha(n^{1/\alpha + \varepsilon}, \infty) = 1$, with $\ell^\alpha(dt) = t^{-\alpha-1} dt$, Corollary 2.2 gives $\sum_1^\infty \mathbb{P}_\rho \{\alpha_n(\varepsilon)\} < \infty$. Then Borel-Cantelli's lemma implies that $\mathbb{P}_\rho \{\bigcap_{j \geq n} \alpha'_j(\varepsilon)\} = 1$, hence \mathbb{P}_ρ -a.e. there exists $n_0(\omega)$ such that for $n \geq n_0(\omega)$, $|X_n(\omega)| \leq n^{1/\alpha + \varepsilon}$.

Then we deduce that \mathbb{P}_ρ -a.e. : $\limsup_{n \rightarrow \infty} \frac{\log M_n}{\log n} \leq \frac{1}{\alpha} + \varepsilon$. Since ε is arbitrary we get :

$$\limsup_{n \rightarrow \infty} \frac{\log M_n}{\log n} \leq \frac{1}{\alpha}. \quad \square$$

Lemma 4.12 *We have \mathbb{P}_ρ -a.e. : $\limsup_{n \rightarrow \infty} \frac{\log |X_n|}{\log n} \geq \frac{1}{\alpha}$.*

Proof Let $\varepsilon \in]0, 1/\alpha[$, $\beta(\varepsilon) = \left\{ \limsup_{n \rightarrow \infty} \frac{\log |X_n|}{\log n} \leq \frac{1}{\alpha} - \varepsilon \right\}$, $\beta_n(\varepsilon) = \left\{ \sup_{j \geq n} \frac{\log |X_j|}{\log j} \leq \frac{1}{\alpha} - \frac{\varepsilon}{2} \right\}$.

The sequence $\beta_n(\varepsilon)$ is increasing and $\beta(\varepsilon) \subset \bigcup_2^\infty \beta_n(\varepsilon)$. We are going to show $\mathbb{P}_\rho \{\beta_n(\varepsilon)\} = 0$.

For $p \geq n \geq 2$, $p \in \mathbb{N}$, we define $\beta_{n,p}(\varepsilon) = \{ \sup_{n \leq j \leq p} |X_j| \leq p^{1/\alpha - \varepsilon/2} \}$, hence $\beta_n(\varepsilon) \subset \beta_{n,p}(\varepsilon)$.

Using stationarity we get $\mathbb{P}_\rho\{\beta_{n,p}(\varepsilon)\} \leq \mathbb{P}_\rho\{p^{-1/\alpha}M_{p-n+1} \leq p^{-\varepsilon/2}\}$. Also, using Corollary 4.5, we have $\lim_{n \rightarrow \infty} (\sup_{t > 0} |\mathbb{P}_\rho\{n^{-1/\alpha}M_n \leq t\} - e^{-c\theta t^{-\alpha}}|) = 0$ which gives $\lim_{p \rightarrow \infty} \mathbb{P}_\rho\{\beta_{n,p}(\varepsilon)\} = 0$.

Since $\beta_n(\varepsilon) = \bigcap_{p \geq n} \beta_{n,p}(\varepsilon)$ we have for $n \geq 2$:

$$\mathbb{P}_\rho\{\beta_n(\varepsilon)\} \leq \lim_{p \rightarrow \infty} \mathbb{P}_\rho\{\beta_{n,p}(\varepsilon)\} = 0, \text{ i.e. } \mathbb{P}_\rho\{\beta(\varepsilon)\} = 0.$$

We see that \mathbb{P}_ρ -a.e., $\limsup_{n \rightarrow \infty} \frac{\log|X_n|}{\log n} \geq \frac{1}{\alpha} - \varepsilon$, and, since ε is arbitrary we conclude

$$\limsup_{n \rightarrow \infty} \frac{\log|X_n|}{\log n} \geq \frac{1}{\alpha}. \quad \square$$

Proof of Corollary 4.6 From Lemmas 4.11, 4.12 we have \mathbb{P}_ρ -a.e.,

$$\limsup_{n \rightarrow \infty} \frac{\log M_n}{\log n} \geq \limsup_{n \rightarrow \infty} \frac{\log|X_n|}{\log n} = \frac{1}{\alpha}.$$

On the other hand, for n large and $\varepsilon > 0$, Lemma 4.11 gives $\frac{\log M_n}{\log n} \leq \frac{1}{\alpha} + 2\varepsilon$, hence

$$\limsup_{n \rightarrow \infty} \frac{\log M_n}{\log n} = \frac{1}{\alpha}.$$

Then, for a set of $\rho \otimes \mathbb{P}$ -probability 1 in $V \times H^{\mathbb{N}}$ we have

$$\frac{1}{\alpha} = \limsup_{n \rightarrow \infty} \frac{\log|X_n|(\omega)}{\log n} = \limsup_{n \rightarrow \infty} \frac{\log M_n}{\log n},$$

hence for a subsequence $n_k(\omega)$, $\frac{1}{\alpha} = \lim_{k \rightarrow \infty} \frac{\log|X_{n_k}(\omega)|}{\log n_k}$.

On the other hand we have for any $x \in V$: $|X_n - X_n^x| \leq |S_n||X_0 - x|$ and $\lim_{n \rightarrow \infty} |S_n| = 0$, \mathbb{P} -a.e.

Also $|\log|X_n| - \log|X_n^x|| \leq |S_n||X_0 - x| \sup(|X_n|^{-1}, |X_n^x|^{-1})$, hence for any $x \in V$, \mathbb{P}_ρ -a.e. :

$$\lim_{n \rightarrow \infty} |\log|X_n| - \log|X_n^x|| = 0.$$

It follows $\lim_{k \rightarrow \infty} \frac{\log|X_{n_k}^x|}{\log n_k} = \frac{1}{\alpha}$, and \mathbb{P} -a.e., $\limsup_{n \rightarrow \infty} \frac{\log|X_n^x|}{\log n} \geq \frac{1}{\alpha}$.

A similar argument shows that \mathbb{P} -a.e., $\limsup_{n \rightarrow \infty} \frac{\log|X_n^x|}{\log n} \leq \frac{1}{\alpha}$.

Furthermore, for any $n \geq 1$, $x \in V$:

$$|M_n^x - M_n| \leq \sup\{|S_k|; 1 \leq k \leq n\}|x - X_0|$$

where the sequence on the right is \mathbb{P} -a.e. bounded. Then, it follows that, \mathbb{P} -a.e. :

$$\limsup_{n \rightarrow \infty} \frac{\log M_n^x}{\log n} = \frac{1}{\alpha}. \quad \square$$

4.4 Proof of the normalized hitting time convergence

The proof depends on the following lemma.

Lemma 4.13 *For any $\lambda \geq 0$, we have*

$$\log \mathbb{E}(\exp(-\lambda N^s(A))) = - \int (1 - \exp(-\lambda 1_A(y))) \mathbb{E}(\exp(-\lambda \sum_1^\infty 1_A(S_i y))) d\Lambda_1(y).$$

In particular we have

$$\mathbb{Q}_{\Lambda_1}\{N^s(A) = 0\} = \exp(-\mathbb{Q}_{\Lambda_1}\{\sum_1^\infty 1_A(S_i y) = 0, y \in A\}) \text{ and } \mathbb{Q}_{\Lambda_1}\{N^s(\partial A) = 0\} = 1.$$

Proof Corollary 4.3 says that the above logarithmic formula is valid if 1_A is replaced by an arbitrary function f in $C_+^c(U'_1)$. We observe that if $A \subset U'_1$ is compact, then 1_A is the decreasing limit of a sequence in $C_+^c(U'_1)$, hence dominated convergence implies the validity of the formula in this case. Also, if A is bounded, ∂A is compact, hence the formula is valid for ∂A . Taking the limit in the formula at $\lambda = \infty$ we get,

$$\log \mathbb{P}_{\Lambda_0}\{N^s(\partial A) = 0\} = -\mathbb{Q}_{\Lambda_1}\{\sum_1^\infty 1_{\partial A}(S_i y) = 0, y \in \partial A\}.$$

Since $\Lambda_0(\partial A) = 0$ we have \mathbb{Q}_{Λ_1} -a.e., $\sum_1^\infty 1_{\partial A}(S_i y) = \sum_1^\infty 1_A(S_i y)$, hence for A bounded,

$$\log \mathbb{E}(\exp(-\lambda N^s(A))) = - \int (1 - \exp(-\lambda 1_A(y))) \mathbb{E}(\exp(-\lambda \sum_1^\infty 1_A(S_i y))) d\Lambda_1(y).$$

For A unbounded we proceed by exhaustion in U'_1 with the sets $U_t \cap U'_1$. Using again the formula for general A as above, we get

$$\mathbb{Q}_{\Lambda_0}\{N^s(\partial A) = 0\} = \exp(-\mathbb{Q}_{\Lambda_1}\{\sum_1^\infty 1_{\partial A}(S_i y), y \in \partial A\}).$$

Since $\Lambda_1(\partial A) = 0$, we have $\mathbb{Q}_{\Lambda_1}\{N^s(\partial A) = 0\} = 1$. \square

Proof of Corollary 4.7 We have by definition, with $n(t) = [zt^\alpha]$:

$$\mathbb{P}_\rho\{t^{-\alpha} \tau_{tA}^x > z\} = \mathbb{P}_\rho\{N_{n(t)}^s(tA) = 0\}$$

and we know the weak convergence of N_n^s to N^s . We observe that, for a sequence $a_n > 0$ converging to $a > 0$, and for any function φ on U'_1 with N^s -negligible discontinuities, the convergence of $N'_n(\varphi) = \int \varphi(a_n v) dN^s(v)$ to $\int \varphi(av) dN^s(v)$ is valid.

Since $\Lambda_0(\partial A) = 0$ and $\lim_{t \rightarrow \infty} tn(t)^{-1/\alpha} = z^{-1/\alpha}$:

$$\lim_{t \rightarrow \infty} \mathbb{P}_\rho\{N_{n(t)}^s(tA) = 0\} = \mathbb{Q}_{\Lambda_1}\{N^s(z^{-1/\alpha} A) = 0\}.$$

Using Lemma 4.13 and the expression of the Laplace transform of $N^s(A)$, we get

$$\lim_{t \rightarrow \infty} \mathbb{P}_\rho\{t^{-\alpha} \tau_{tA}^x > z\} = \exp(-cz\theta(A)).$$

In this formula, we can replace \mathbb{P}_ρ by \mathbb{P} , since for any $x \in V$, the point process N_n^s converges also weakly to N^s under \mathbb{P} . This follows from the use of ε -Hölder functions for $\varepsilon < \inf(1, \alpha)$, Lemma 4.8 and the inequality

$$|\varphi(S_n x) - \varphi(S_n y)| \leq [\varphi]_\varepsilon |S_n|^\varepsilon |x - y|^\varepsilon,$$

since $\sum_1^\infty |S_n|^\varepsilon < \infty$, \mathbb{Q} -a.e. and $\int |y|^\varepsilon d\rho(y) < \infty$.

The proof of positivity for $\theta(A)$ is the same as in Proposition 2.4 with A instead of $\text{supp}(\Lambda_1)$ and $\theta(A)$ instead of θ . \square

5 Convergence to stable laws

The convergence to stable laws of the normalized sums $\sum_{i=1}^n X_i$ under (c-e) was shown in ([10], [15]) where explicit formulae for the corresponding characteristic functions were given and non degeneracy of the limiting laws was proved. It was observed that these formulae involved the asymptotic tail Λ of ρ , as well as the occupation measure $\pi_v^\omega = \sum_0^\infty \varepsilon_{S_i(\omega)v}$. A similar situation occurred in the dynamical context of [11], where the limiting law was expressed in terms of an induced transformation. We observe that the connection with convergence to stable laws for $\sum_{i=1}^n X_i$, where $(X_i)_{i \in \mathbb{N}}$ is a stationary process, and point process theory had been already developed in [8]. For an analysis of the involved properties in this setting see [24]. For another approach, not using spectral gap properties, see chapter 4 of the recent book [5]. Here we give new proofs of the results given in ([10], [15]), following and completing the point process approach of [7] in the case of affine stochastic recursions. In particular we get also a direct proof of the convergence for the related space point process $N^s = \sum_{i=1}^n \varepsilon_{u_n^{-1}X_i}$, via a detailed analysis of Laplace functionals and without use of the cluster process.

5.1 On the space exceedances process

We give here a direct proof of the convergence of N_n^s , already shown in Corollary 4.3 above and we deduce the convergence of the characteristic function for the random variable $N_n^s(f)$, for f compactly supported. We make use of the mixing property in Proposition 3.5 for Lipschitz functions depending only on $v \in V$.

Theorem 5.1 *Let f be a complex valued compactly supported Lipschitz function on $V \setminus \{0\}$ which satisfies $\operatorname{Re}(f) \geq 0$. Then we have*

$$-\log \lim_{n \rightarrow \infty} \mathbb{E}_\rho \exp(-N_n^s(f)) = \mathbb{E}_{\Lambda_0}[(\exp f(v) - 1) \exp(-\pi_v^\omega(f))].$$

The proof depends on two lemmas where notations for r_n, k_n explained above are used.

For $i \leq j$ we write $C_n(i, j) = \exp(-\sum_{k=i}^j f(u_n^{-1}X_k)) - 1$, and we note the equality

$$C_n(1, r_n) = \sum_{i=1}^{r_n} [C_n(i, r_n) - C_n(i+1, r_n)]$$

where $C_n(r_n+1, r_n) = 0$ and r_n is a sequence as in Proposition 2.6. We note also that

$$|C_n(i, j)| \leq 2, \quad |C_n(i, j) - C_n(i+1, j)| \leq 2.$$

We are going to compare $C_n(1, r_n)$ to $C_{n,k}(1, r_n)$ for k large, where

$$C_{n,k}(1, r_n) = \sum_{i=1}^{r_n} [C_n(i, i+k) - C_n(i+1, i+k)],$$

we write $\Delta_{n,k}$ for their difference, $\varepsilon_n = r_n \mathbb{P}_\rho\{|X| > u_n\}$ and we assume that $\text{supp}(f) \subset U'_\delta$ with $\delta > 0$.

Using anticlustering we will obtain the approximation of the small quantity $C_n(1, r_n)$ by $C_{n,k}(1, r_n)$ up to ε_n . Then taking $k_n = \lfloor nr_n^{-1} \rfloor$ large and using the definition of Λ , we will get the limiting form of $k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)]$, hence of $k_n \mathbb{E}_\rho[C_n(1, r_n)]$. We need the two following lemmas.

Lemma 5.2 $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \varepsilon_n^{-1} \mathbb{E}_\rho(|\Delta_{n,k}|) = 0$.

Proof We observe that

$$1 + C_n(i, r_n) = (1 + C_n(i + 1, r_n)) \exp(-f(u_n^{-1} X_i)).$$

$$\text{Also } C_n(i + 1, r_n) - C_n(i + 1, i + k) = (\exp(-\sum_{j=i+1}^{i+k} f(X_j))) (\exp(-\sum_{j=i+k+1}^{r_n} f(X_j)) - 1).$$

Hence we have

$$\Delta_{n,k} = \sum_{i=1}^{r_n} (\exp(-f(u_n^{-1} X_i)) - 1) [C_n(i + 1, r_n) - C_n(i + 1, i + k)] = \Delta'_{n,k} + \Delta''_{n,k}$$

where $\Delta'_{n,k}$ (resp. $\Delta''_{n,k}$) is the above sum with index i restricted to $[1, r_n - k]$ (resp. $]r_n - k, r_n]$). As observed above, the expression under Σ is bounded by 4 and vanishes unless $|X_i| > \delta u_n$ for some $i \in [1, r_n - k]$ and $M_{k+i+1}^{r_n} > \delta u_n$. Then we get using stationarity,

$$\mathbb{E}_\rho(|\Delta'_{n,k}|) \leq 4r_n \mathbb{P}_\rho\{|X_0| > \delta u_n, M_{k+1}^{r_n} > \delta u_n\}.$$

Since the process $(X_k)_{k \in \mathbb{Z}_+}$ satisfies anticlustering, it follows $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \varepsilon_n^{-1} \mathbb{E}_\rho(|\Delta'_{n,k}|) = 0$. Also, stationarity implies

$$\mathbb{E}_\rho(|\Delta''_{n,k}|) \leq 4 \sum_{i=r_n-k+1}^{r_n} \mathbb{P}_\rho\{|X_i| > \delta u_n\} = 4k \mathbb{P}_\rho\{|X| > \delta u_n\}.$$

Since ρ is homogeneous at infinity and $\lim_{n \rightarrow \infty} r_n^{-1} k = 0$, we get $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \varepsilon_n^{-1} \mathbb{E}_\rho(|\Delta''_{n,k}|) = 0$, hence the required assertion. \square

Lemma 5.3 *We have the following convergences.*

- 1) For any $k \geq 1$ $\lim_{n \rightarrow \infty} k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)] = \mathbb{E}_{\Lambda_0}[(\exp f(v) - 1) \exp(-\sum_{i=0}^k f(S_i v))]$
- 2) $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} k_n \mathbb{E}_\rho(|\Delta_{n,k}|) = 0$

Proof 1) Using stationarity we have

$$\begin{aligned} k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)] &= k_n r_n \mathbb{E}_\rho[C_n(1, k + 1) - C_n(2, k + 1)] = \\ &= k_n r_n \mathbb{E}_\rho[\exp(-\sum_{j=0}^k f(u_n^{-1} X_j)) - \exp(-\sum_{j=1}^k f(u_n^{-1} X_j))]. \end{aligned}$$

The function $f^{(k)}$ on $(V \setminus \{0\})^{k+1}$ given by

$$f^{(k)}(x_0, x_1, \dots, x_k) = \exp(-\sum_{j=0}^k f(x_j)) - \exp(-\sum_{j=1}^k f(x_j)) = -(\exp(f(x_0)) - 1) \exp(-\sum_{j=0}^k f(x_j))$$

is bounded, uniformly continuous on $(U'_\delta)^{k+1}$ and $\lim_{n \rightarrow \infty} n^{-1}k_n r_n = 1$. Hence, Proposition 2.4 implies

$$\lim_{n \rightarrow \infty} k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)] = -\mathbb{E}_{\Lambda_0}[(\exp f(v) - 1)\exp(-\sum_{j=0}^k f(S_j v))].$$

2) We have the equality, $k_n \mathbb{E}_\rho(|\Delta_{n,k}|) = k_n r_n \mathbb{P}_\rho\{|X| > u_n\} \varepsilon_n^{-1} \mathbb{E}_\rho(|\Delta_{n,k}|)$.

Then, using Lemma 5.2, the relation $\lim_{n \rightarrow \infty} n^{-1}k_n r_n = 1$ and the homogeneity at infinity of ρ , assertion 2 follows. \square

Proof of Theorem 5.1 With r_n as in Proposition 4.2 above, the multiple mixing property in Proposition 3.5 for functions depending only of $v \in V$ gives

$$\lim_{n \rightarrow \infty} [\mathbb{E}_\rho(\exp(-N_n^s(f))) - (\mathbb{E}_\rho(1 + C_n(1, r_n)))^{k_n}] = 0,$$

hence it suffices to study the sequence $(1 + \mathbb{E}_\rho(C_n(1, r_n)))^{k_n}$. Since $Ref \geq 0$ and $\text{supp}(f) \subset U'_\delta$ we have :

$$\mathbb{E}_\rho(|C_n(1, r_n)|) \leq \mathbb{E}_\rho(|1 - \exp(-\sum_{i=1}^{r_n} f(u_n^{-1} X_i))|) \leq \mathbb{E}_\rho(\sum_{i=1}^{r_n} |f(u_n^{-1} X_i)|) \leq r_n |f|_\infty \mathbb{P}_\rho\{|X_0| > \delta u_n\}.$$

The last inequality implies the \mathbb{L}^1 -convergence to zero of $\sum_{i=1}^{r_n} f(u_n^{-1} X_i)$. Then the first one gives $\lim_{n \rightarrow \infty} \mathbb{E}_\rho(|C_n(1, r_n)|) = 0$.

It follows that the behaviour of the sequence $[1 + \mathbb{E}_\rho(C_n(1, r_n))]^{k_n}$ for n large is determined by the behaviour of $k_n \mathbb{E}_\rho(C_n(1, r_n))$. We have for $k \geq 1$,

$$k_n \mathbb{E}_\rho(C_n(1, r_n)) = k_n \mathbb{E}_\rho(C_{n,k}(1, r_n)) + k_n \mathbb{E}_\rho(\Delta_{n,k}).$$

Since $\text{supp}(f) \subset U'_\delta$ and $\lim_{j \rightarrow \infty} |S_j v| = 0$ \mathbb{Q} -a.e., the series $\sum_{j=1}^{\infty} f(S_j v)$ converges \mathbb{Q} -a.e.

Since $Ref \geq 0$, it follows $\lim_{k \rightarrow \infty} \mathbb{E}(\exp(-\sum_{j=1}^k f(S_j v))) = \exp(-\pi_v^\omega(f))$.

Then dominated convergence and Lemma 5.3 imply

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} k_n \mathbb{E}_\rho(C_{n,k}(1, r_n)) = -\mathbb{E}_{\Lambda_0}[(\exp f(v) - 1)\exp(-\pi_v^\omega(f))].$$

This equality and the second assertion in Lemma 5.3 give the result. \square

Corollary 5.4 *Let $m > 0$, $\delta > 0$, $\gamma \geq 0$, and let f be a \mathbb{R}^m -valued continuous function on $V \setminus \{0\}$ which satisfies the conditions*

- 1) f is locally Lipschitz
- 2) $f(v) = 0$ for $|v| < \delta$
- 3) $\sup_{v \in V} |v|^{-\gamma} |f(v)| = c_\gamma < \infty$

Then we have , for any $u \in \mathbb{R}^m$,

$$\log \lim_{n \rightarrow \infty} \mathbb{E}_\rho(\exp(-i \langle u, N_n^s(f) \rangle)) = -\mathbb{E}_{\Lambda_0}[(\exp(i \langle u, f(v) \rangle) - 1)\exp(-i \langle u, \pi_v^\omega(f) \rangle)]$$

Proof We consider the \mathbb{R}^m -valued random variable $Y_n = N_n^s(f)$. For $a \geq 1$, let $\theta_a(v)$ be the function from $V \setminus \{0\}$ to $[0, 1]$ defined by

$$\theta_a(v) = 1 \text{ for } |v| \leq a, \theta_a(v) = a + 1 - |v| \text{ for } |v| \in [a, a + 1], \theta_a(v) = 0 \text{ for } |v| \geq a + 1.$$

Then θ_a is Lipschitz, hence $f\theta_a$ is Lipschitz and compactly supported. Then Theorem 5.1 gives, in logarithmic form

$$\lim_{n \rightarrow \infty} \mathbb{E}_\rho(\exp i \langle u, N_n^s(f\theta_a) \rangle) = -\mathbb{E}_{\Lambda_0}[(\exp(-i \langle u, f\theta_a(v) \rangle) - 1)\exp(\pi_v^\omega(i \langle u, f\theta_a(v) \rangle))] = \Phi_a(u)$$

Since $\Lambda(U'_\delta) < \infty$, the function $u \rightarrow \Phi_a(u)$ is continuous on \mathbb{R}^m . It follows that the sequence of random variables $Y_n^a = N_n^s(f\theta_a)$ converges in law to the random variable Y_∞^a which has characteristic function Φ_a . On the other hand we have $\lim_{a \rightarrow \infty} \theta_a = 1$, hence by dominated convergence we get in logarithmic form,

$$\lim_{a \rightarrow \infty} \Phi_a(u) = -\mathbb{E}_{\Lambda_0}[(\exp(-i \langle u, f(v) \rangle) - 1)\exp(i \langle u, \pi_v^\omega(f) \rangle)] = \Phi(u).$$

We recall that, for v fixed, the series $\sum_{j=0}^{\infty} f(S_j v)$ converges \mathbb{Q} -a.e. to a finite sum, hence the function $u \rightarrow \Phi(u)$ is continuous. In other words, Y_∞^a converges in law ($a \rightarrow \infty$) to the random variable Y with characteristic function Φ .

If we choose $\beta \in]0, 1[$, $\gamma > 0$ such that $\beta\gamma \in]0, \alpha[$, then for any $\varepsilon > 0$, Markov's inequality gives

$$\mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta} \mathbb{E}_\rho\left[\sum_{j=1}^n (f(u_n^{-1} X_j) 1_{\{|X_j| > au_n\}})^\beta\right],$$

$$\mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta} c_\gamma^\beta n \mathbb{E}_\rho[|u_n^{-1} X|^{\beta\gamma} 1_{\{|X| > au_n\}}].$$

Using Corollary 2.2, it follows $\limsup_{n \rightarrow \infty} \mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta} c_\gamma^\beta \Lambda(W^{\beta\gamma} 1_{\{W > a\}})$, where

$$W(x) = |x|.$$

Since $0 < \beta\gamma < \alpha$, we get $\lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} = 0$

Since $\varepsilon > 0$ is arbitrary, the convergence in law of Y_n to the random variable Y follows, hence the corollary. \square

In order to prepare the study of limits for the sums $T_n = \sum_{j=1}^n X_j$ if $0 < \alpha < 2$, we write for

$a > 0$ $\psi_a(v) = v(1 - \varphi_a(v))$, where

$$\varphi_a(v) = 1 \text{ if } |v| \leq a, \varphi_a(v) = 2 - a^{-1}|v| \text{ if } a \leq |v| \leq 2a, \varphi_a(v) = 0 \text{ if } |v| > 2a.$$

Hence $0 \leq \varphi_a \leq 1_{[0, 2a]}$ and $k(\varphi_a) \leq a^{-1}$. In particular we will use below the function φ_1 corresponding to $a = 1$. Then a consequence of Corollary 5.4, if $\gamma = 1$ is the following

Corollary 5.5 *The sequence of V -valued random variables $N_n^s(\psi_a)$ converges in law to the random variable with characteristic function which logarithm is*

$$\mathbb{E}_{\Lambda_0}[(\exp(-i \langle u, \psi_a(v) \rangle) - 1)\exp(i \langle u, \pi_v^\omega(\psi_a) \rangle)].$$

5.2 Convergence to stable laws for $T_n = \sum_{i=1}^n X_i$

In this subsection we write $\psi(v) = v$ and we study the convergence of $N_n^s(\psi) = u_n^{-1} T_n$ towards a stable law, relying on the weak convergence of N_n^s studied in the above subsection. We need here the last part of the spectral gap result in Proposition 3.4 for the operator P .

We have the

Theorem 5.6 *Let $0 < \alpha < 2$. Then there exists a sequence d_n in V such that the sequence of random variables $n^{-1/\alpha}(T_n - d_n)$ converges in law to a non degenerate stable law.*

If $0 < \alpha < 1$, we have $d_n = 0$.

If $1 < \alpha < 2$, we have $d_n = n\mathbb{E}_\rho(X)$

If $\alpha = 1$, we have $d_n = n \mathbb{E}_\rho[X\varphi_1(X)]$.

Explicit expressions for the characteristic functions of the limits are given in the proofs. Non degeneracy of the limit laws are proved in [10] and [15]. For the proofs, we follow the approach of [7] and we need two lemmas corresponding to the cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$.

In the proofs below we use the normalization $u_n = (\alpha^{-1}cn)^{1/\alpha}$ instead of $n^{1/\alpha}$ as in the theorem.

Lemma 5.7 *Assume $0 < \alpha < 1$. Then for any $u \in V$ and with the notation of Corollary 5.5, $T^a = N^s(\psi_a)$ converges in law ($a \rightarrow 0$) to T with characteristic function $\Phi(u)$ given by*

$$-\log\Phi(u) = c^{-1}\mathbb{E}_{\Lambda_0}[(\exp(-i \langle u, v \rangle) - 1)\exp(i \langle u, \sum_{j=0}^{\infty} S_j v \rangle)].$$

Also, for any $\delta > 0$ we have

$$\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_\rho \left\{ \left| \sum_{j=1}^n u_n^{-1} X_j \varphi_a(u_n^{-1} X_j) \right| > \delta \right\} = 0.$$

Proof Using dominated convergence, the first part follows from Corollary 5.5.

On the other hand, Markov's inequality gives

$$\mathbb{P}_\rho \left\{ \left| \sum_{j=1}^n u_n^{-1} X_j \varphi_a(u_n^{-1} X_j) \right| > \delta \right\} \leq n\delta^{-1}u_n^{-1}\mathbb{E}_\rho(|X|1_{\{|X| < 2au_n\}})$$

The homogeneity at infinity of ρ and Karamata's lemma (see [30] p.26) gives that the right hand side is equivalent to

$$\delta^{-1}n^{1-\alpha^{-1}}\alpha(1-\alpha)^{-1}(2au_n)\mathbb{P}_\rho\{|X| > 2au_n\},$$

i.e. to $\delta^{-1}a^{1-\alpha}$, up to a coefficient independant of n . Since $1-\alpha > 0$ the result follows. \square

Lemma 5.8 *Assume $1 \leq \alpha < 2$ and write $\bar{\psi}_a(v) = v\varphi_a(v)$ where φ_a is defined in the proof of Corollary 5.4. Then we have the convergence*

$$\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}_\rho(|N_n^s(\bar{\psi}_a) - \mathbb{E}_\rho(N_n^s(\bar{\psi}_a))|^2) = 0.$$

Proof It suffices to show that for any $u \in \mathbb{S}^{d-1}$:

$$\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}_\rho(|\langle u, N_n^s(\bar{\psi}_a) \rangle - \mathbb{E}_\rho(\langle u, N_n^s(\bar{\psi}_a) \rangle)|^2) = 0.$$

We write $f_{a,n}(v) = \bar{\psi}_a(u_n^{-1}v)$, $\bar{\psi}_{a,n} = f_{a,n} - \rho(f_{a,n})$.

Hence $|f_{a,n}(v)| \leq u_n^{-1}|v|1_{\{|v| \leq 2au_n\}}$, $k(f_{a,n}) \leq 3u_n^{-1}$. We have the equality

$$\mathbb{E}_\rho(|\langle u, N_n(\bar{\psi}_a) \rangle - \mathbb{E}_\rho(\langle u, N_n(\bar{\psi}_a) \rangle)|^2) = A_{n,a} + 2B_{n,a}$$

with

$$A_{n,a} = n\mathbb{E}_\rho(|\langle u, \bar{\psi}_{a,n}(X_0) \rangle|^2), \quad B_{n,a} = \sum_{j=1}^n (n-j)\mathbb{E}_\rho(\langle u, \bar{\psi}_{a,n}(X_0) \rangle \langle u, \bar{\psi}_{a,n}(X_j) \rangle).$$

Now the proof splits into two parts a) and b) corresponding to the studies of $A_{n,a}$, $B_{n,a}$.

a) We have, using the above estimation of $f_{a,n}$,

$$n\mathbb{E}_\rho(|\langle u, \bar{\psi}_{a,n}(X_0) \rangle|^2) \leq n\mathbb{E}_\rho(|f_{a,n}(X_0)|^2) \leq n\mathbb{E}_\rho(u_n^{-2}|X_0|^2 1_{\{|X_0| < 2au_n\}}).$$

Then Karamata's lemma implies that, for n large, the right hand side is equivalent to $n^{1-2\alpha^{-1}}(2au_n)^2((2a)^\alpha n)^{-1}$, i.e. to $a^{2-\alpha}$. Hence, since $\alpha \in]0, 2[$, we get $\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} A_{n,a} = 0$

uniformly in $u \in \mathbb{S}^{d-1}$.

b) Markov property for the process $(X_i)_{i \geq 0}$ implies for $i \geq 1$,

$$\mathbb{E}_\rho(\langle u, \bar{\psi}_{a,n}(X_0) \rangle \langle u, \bar{\psi}_{a,n}(X_i) \rangle) = \mathbb{E}_\rho(\langle u, \bar{\psi}_{a,n}(X_0) \rangle \langle u, P^i \bar{\psi}_{a,n}(X_0) \rangle).$$

First we consider the case $\alpha \in]1, 2[$ and we apply Proposition 3.4 to P acting on the Banach space $\mathcal{H} = \mathcal{H}_{\chi, \varepsilon, \kappa}$ with $\chi \in]1, \alpha[$, $\varepsilon = 1$ and κ choosen according to Proposition 3.4. We observe that for $h \in \mathcal{H}$ we have $|h(v)| \leq \|h\|(1 + |v|)$. Since $\bar{\psi}_{a,n} \in \mathcal{H}$, we have

$$\mathbb{E}_\rho(\langle u, \bar{\psi}_{a,n}(X_0) \rangle \langle u, P^i \bar{\psi}_{a,n}(X_0) \rangle) = \mathbb{E}_\rho(\langle u, f_{a,n}(X_0) \rangle \langle u, U^i f_{a,n}(X_0) \rangle),$$

where we have used the decomposition $P^i = \rho \otimes 1 + U^i$. The Schwarz inequality allows us to bound the right hand side by the square root of $\mathbb{E}_\rho(|f_{a,n}(X_0)|^2) \mathbb{E}_\rho(|U^i f_{a,n}(X_0)|^2 1_{\{|X_0| < 2an\}})$. Since $|U^i f_{a,n}(v)| \leq (1 + |v|)\|U^i\| \|f_{a,n}\|$, the quantity $\mathbb{E}_\rho(\langle u, f_{a,n}(X_0) \rangle \langle u, U^i f_{a,n}(X_0) \rangle)$ is bounded by $\|U^i\| \|f_{a,n}\| [u_n^{-2} \mathbb{E}_\rho(|X_0|^2 1_{\{|X_0| < 2au_n\}})]^{1/2} [\mathbb{E}_\rho(1 + |X_0|)^2 1_{\{|X_0| < 2au_n\}}]^{1/2}$.

Then Karamata's lemma implies that, up to a coefficient independant of n , the above expression is bounded by $\|U^i\| \|f_{a,n}\| [n^{-1/2} a^{1-\alpha/2}] [1 + n^{\alpha^{-1}-1/2} a^{1-\alpha/2}]$.

Since $\|f_{a,n}\| \leq nu_n^{-1}$, it follows that $B_{n,a}$, uniformly in $u \in \mathbb{S}^{d-1}$ and up to a coefficient, is bounded by

$$n \left(\sum_{i=0}^{\infty} \|U^i\| \right) n^{-1} [n^{1/2-\alpha^{-1}} a^{1-\alpha/2} + a^{2-\alpha}] = \sum_{i=0}^{\infty} \|U^i\| [a^{2-\alpha} + a^{1-\alpha/2} n^{1/2-\alpha^{-1}}].$$

Since $r(U) < 1$ we have $\sum_{i=0}^{\infty} \|U^i\| < \infty$, hence $\limsup_{n \rightarrow \infty} B_{n,a}$ is bounded by $a^{2-\alpha}$, up to a coefficient independant of n . Since $1 < \alpha < 2$, and in view of the two above convergences the lemma follows.

If $\alpha = 1$, we need to use the Banach space $\mathcal{H}' = \mathcal{H}_{\chi, \varepsilon, \kappa}$ with $0 < \varepsilon < \chi < 1$, $\kappa = 0$, considered in Proposition 3.4. We use also the inequality $\|f\|_{a,n} \leq c_1 a^{1-\chi} n^{-\varepsilon}$ with $c_1 > 0$,

shown below. We note that for $h \in \mathcal{H}'$, we have $|h(v)| \leq \|h\|(1 + |v|^\varepsilon)$ in particular and up to a constant independent of n and a , we have

$$|U^i f_{a,n}(v)| \leq \|U^i\| \|f_{a,n}\| (1 + |v|^\varepsilon) \leq \|U^i\| (a^{1-\chi} n^{-\varepsilon}) (1 + |v|^\varepsilon).$$

Hence we can bound $\mathbb{E}_\rho(\langle u, f_{a,n}(X_0) \rangle \langle u, U^i f_{a,n}(X_0) \rangle)$ by

$$2c_1 \|U^i\| (a^{1-\chi} n^{-\varepsilon}) [n^{-2} \mathbb{E}_\rho(|X_0|^2) 1_{\{|X_0| < 2na\}}]^{1/2} [\mathbb{E}_\rho(1 + |X_0|^{2\varepsilon}) 1_{\{|X_0| < 2na\}}]^{1/2},$$

which can be estimated using Karamata's lemma, for $\varepsilon > 1/2$, by

$$2c_2 \|U^i\| (a^{1-\chi} n^{-\varepsilon}) (an^{-1})^{1/2} (na)^{\varepsilon-1/2} = c_2 a^{1-\chi+\varepsilon} n^{-1}.$$

It follows that $B_{n,a}$ can be estimated by $2c_2 \sum_{i=0}^{\infty} \|U^i\| a^{1-\chi+\varepsilon}$. Since using Proposition 3.4 we have $r(U) < 1$, if $1/2 < \varepsilon < \chi < 1$, it follows $\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} B_{n,a} = 0$. \square

Proof of Theorem 5.6 For $\alpha \in]0, 1[$ the proof follows from Lemma 5.7. We observe that dominated convergence implies the continuity of Φ at zero, hence Φ is a characteristic function. From Lemma 5.7 we know that if $Y_n = N_n^s(1)$, $Y_n^a = N_n^s(1 - \varphi_a)$,

- 1) For any $a > 0$, Y_n^a converges in law ($n \rightarrow \infty$) to T^a
- 2) T^a converges in law ($a \rightarrow 0$) to T
- 3) For any $\varepsilon > 0$, we have $\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\{|Y_n - Y_n^a| > \varepsilon\} = 0$.

It follows that the sequence $Y_n = \sum_{i=1}^n u_n^{-1} X_i$ converges in law ($n \rightarrow \infty$) to the random variable T with characteristic function Φ .

For $1 < \alpha < 2$, we write

$$Y_n = N_n^s(\psi) - \mathbb{E}_\rho(N_n^s(\psi)) = u_n^{-1} \sum_{j=1}^n X_j - \mathbb{E}_\rho(X), \quad Y_n^a = N_n^s(\psi_a) - \mathbb{E}_\rho(N_n^s(\psi_a)),$$

so that $Y_n^a - Y_n = N_n^s(\bar{\psi}_{a,n}) - \mathbb{E}_\rho(N_n^s(\bar{\psi}_{a,n}))$. Then, for any $\varepsilon > 0$, Lemma 5.8 gives,

$$\lim_{a \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} = 0$$

Furthermore, the sequence $N_n^s(\psi_a)$ converges in law ($n \rightarrow \infty$) to T^a and $\mathbb{E}_\rho(N_n^s(\psi_a)) = n \mathbb{E}_\rho[u_n^{-1} X (1 - \varphi_a)(u_n^{-1} X)]$ converges to the value $b(a)$ of Λ on the function $v \rightarrow v(1 - \varphi_a(v))$, as follows from $\alpha > 1$ and the homogeneity at infinity of ρ . Hence the sequence Y_n^a converges in law ($n \rightarrow \infty$) to $T^a - b(a) = Y^a$. Finally Y^a converges in law ($a \rightarrow 0$) to the random variable T with characteristic function Φ defined in logarithmic form by

$$\begin{aligned} -\Phi(u) &= \mathbb{E}_{\Lambda_0}[(\exp(-i \langle u, v \rangle) - 1 + i \langle u, v \rangle) \exp(i \langle u, \sum_{j=0}^{\infty} S_j v \rangle)] \\ &+ i \mathbb{E}_{\Lambda_0}[\langle u, v \rangle (\exp(i \langle u, \sum_{j=0}^{\infty} S_j v \rangle) - 1)]. \end{aligned}$$

This follows of Theorem 5.1, of dominated convergence ($a \rightarrow 0$) and of the following inequalities

$$\begin{aligned} |\exp(-i \langle u, \psi_a(v) \rangle) - 1 + i \langle u, \psi_a(v) \rangle| &\leq \inf(2 + |u| |v|, 4|v|^2 |u|^2), \\ |\langle u, \psi_a(v) \rangle \mathbb{E}(\exp(i \langle u, \sum_{j=1}^{\infty} \psi_a(S_j v) \rangle) - 1)| &\leq \inf(|u| |v|, 2|u|^2 |v|^2 \sum_{j=1}^{\infty} \mathbb{E}|S_j|), \end{aligned}$$

where $\alpha > 1$ gives $\sum_{j=1}^{\infty} \mathbb{E}|S_j| < \infty$. Continuity of Φ at zero follows also from the above inequalities.

Then, as in [7], we deduce the convergence in law of the sequence Y_n to T .

If $\alpha = 1$, we write $Y_n = \sum_{j=1}^n n^{-1} X_j - \mathbb{E}_{\rho}(\varphi_1(n^{-1} X))$ and

$$Y_n^a = \sum_{j=1}^n n^{-1} X_j (1 - \varphi_a(n^{-1} X_j)) - b_n(a) = N_n^s(\psi_a) - b_n(a)$$

where $b_n(a) = \mathbb{E}_{\rho}[X(\varphi_1 - \varphi_a)(n^{-1} X)]$. With the new notations, the above inequalities are still valid. The homogeneity at infinity of ρ gives now

$$\lim_{n \rightarrow \infty} b_n(a) = c^{-1} \mathbb{E}_{\Lambda_0}(v(\varphi_1 - \varphi_a)) = b(a).$$

It follows that the sequence Y_n^a converges in law ($n \rightarrow \infty$) to the random variable T_1^a with characteristic function given in logarithmic form by

$$-\mathbb{E}_{\Lambda_0}[(\exp(-i \langle u, \psi_a(v) \rangle) - 1)(\exp(i \langle u, \pi_v^{\omega}(\psi_a) \rangle) - i \langle u, b(a) \rangle)]$$

We insert the expression $i \langle u, v \rangle (\varphi_1 - \varphi_a(v))$ with the adequate sign in each of the above factors inside the expectation \mathbb{E}_{Λ_0} . Then dominated convergence ($a \rightarrow 0$) shows that $T_1^a - b(a)$ converges in law to the random variable T with characteristic function given in logarithmic form by

$$\Phi(u) = -\mathbb{E}_{\Lambda_0}[A(u, v) + B(u, v)]$$

with $A(u, v) = (\exp(-i \langle u, v \rangle) - 1 + i \langle u, v \rangle) \varphi_1(v)$,

$$B(u, v) = i \langle u, v \rangle \varphi_1(v) (\exp(i \langle u, \sum_{j=1}^{\infty} S_j v \rangle) - 1).$$

As in ([10], [15]), the stability of the limiting laws follow from the formula for $\Phi(u)$. If $0 < \alpha < 2$, $\alpha \neq 1$ the formula for $\Phi(u)$ shows that for any $n \in \mathbb{N}$ we have $\Phi^n(u) = \Phi(n^{1/\alpha} u)$, hence T has a stable law of index α .

If $\alpha = 1$, we have with $\gamma_n = c \mathbb{E}_{\Lambda_0}[v(\varphi_1(n^{-1} v) - \varphi_1(v))]$, $\Phi^n(u) = \Phi(n u) \exp(-in \langle u, \gamma_n \rangle)$.

This implies that T follows a stable law with index 1. \square

Remark 1 *The idea used above in the proof of the limiting form of N_n^s can be used in other similar situations, using only anticlustering and the definition of Λ . This is the case for the Laplace functional of the cluster process C stated in Proposition 2.6.*

There we write, as in the proof of Theorem 5.1

$$C'_n(k, \ell) = \exp(-\sum_{k=1}^{\ell} f(u_n^{-1} X_i)) 1_{\{M_{k, \ell} > u_n\}}$$

$$C'_n(1, r_n) = \exp(-\sum_{i=1}^{r_n} f(u_n^{-1} X_i)) 1_{\{M_{r_n} > u_n\}}$$

$$C'_{n, k}(1, r_n) = \exp(-\sum_{i=1}^{r_n} [C'_n(i, i+k) - C'_n(i+1, i+k)])$$

Then the approximation of $C'(1, r_n)$ by $C'_{n, k}(1, r_n)$ up to ε_n is still valid. Furthermore, the definition of Λ_0 shows that for k_n large $k_n \mathbb{E}_{\rho}[C'_{n, k}(1, r_n)]$ is close to

$$I = -\mathbb{E}_{\Lambda_0}([\exp(-\pi_v^\omega(f))(\exp(f(v)))(1_{\{\sup_{i>0} |S_i v| > 1\}}) - 1_{\{\sup_{i \geq 0} |S_i v| > 1\}})]$$

To conclude we write using the notation of 2.4

$$\mathbb{E}_\rho[\exp(-C_n) = \mathbb{E}_\rho\{C'_n(1, r_n)/M_{r_n} > u_n\} = (\frac{n}{r_n} \mathbb{E}_\rho[C'_n(1, r_n)])(\frac{r_n}{n \mathbb{P}_\rho\{M_{r_n} > u_n\}}).$$

From above the first factor converges to I . The second factor is asymptotic to

$$(\theta n \mathbb{P}_\rho\{|X_0| > u_n\})^{-1} \text{ since } \theta_n^{-1} = r_n \mathbb{P}_\rho\{|X_0| > u_n/M_{r_n} > u_n\} \text{ converges to } \theta^{-1}.$$

If $\text{supp}(f) \subset U'_1, \theta^{-1}I$ is the sum of

$$-\theta^{-1} \mathbb{E}_{\Lambda_1}[\exp(-\sum_1^\infty f(S_i v))(1_{\{\sup_{i \geq 1} |S_i v| \leq 1\}})] = 1 \text{ and of the last expression in Proposition 2.6.}$$

6 Appendix : Condition (c-e) is open if $d > 1$

We denote by T_μ the closed subsemigroup of G generated by $\text{supp}(\mu)$, where μ is a probability on G . We consider weak topologies for probability measures on G and on H . We denote by $M^1(G)$ (resp. $M^1(H)$) the set of probabilities on G (resp. H). We denote by $\mathcal{W}(H)$ the weak topology on $M^1(H)$ defined by the convergence on continuous compactly supported functions as well as of the moments $\int(\gamma^k(g) + |b|^k(h))d\lambda(h)$ for any $k \in \mathbb{N}$. An element $\gamma \in G$ is said to be proximal if it has a unique simple dominant real eigenvalue.

Theorem 6.1 *If $d > 1$, condition (c-e) is open in the weak topology $\mathcal{W}(H)$ on $M^1(H)$.*

We will need the Proposition

Proposition 6.2 *Condition i-p is open for the weak topology on $M^1(G)$.*

Proof Assume $\mu \in M^1(G)$, satisfies i-p and let $\mu_n \in M^1(G)$ be a sequence which converges weakly to μ . Then $\text{supp}(\mu_n)$ and T_{μ_n} are closed subsets of G which converges to $\text{supp}(\mu)$ and T_μ respectively. If γ is a proximal element of T_μ , then by perturbation theory there exists a neighbourhood of γ in G which consists of proximal elements. Hence there exists $\gamma_n \in T_{\mu_n}$ which is also proximal.

On the other hand T_{μ_n} is irreducible for large n . Otherwise there exists a proper subspace $W^n \subset V$ with $T_{\mu_n}(W^n) = W^n$. Let $W \subset V$ be the limit of a subsequence of W^n . Then, clearly $T_\mu(W) = W$, which contradicts the irreducibility of T_μ .

In order to show the strong irreducibility of T_{μ_n} for n large, we show the irreducibility of $Z_{c_0}(T_{\mu_n})$, the connected component of the Zariski closure $Zc(T_{\mu_n})$ of T_{μ_n} (see [25]). Since T_{μ_n} is irreducible, the Lie group $Z_{c_0}(T_{\mu_n})$ is reductive and has finite index in $Zc(T_{\mu_n})$.

We decompose V as the direct sum of its isotypic components $V_i^{(n)} (1 \leq i \leq p_n)$ under the action of $Z_{c_0}(T_{\mu_n}) : V = \bigoplus_{i=1}^{p_n} V_i^{(n)}$. Since $Z_{c_0}(T_{\mu_n})$ has finite index in $Zc(T_{\mu_n})$ we can assume, by taking a suitable power, that $\gamma_n \in Z_{c_0}(T_{\mu_n})$. The uniqueness of the above

decomposition of V and the relation $\gamma_n v = \lambda_n v$, $v = \sum_{i=1}^{p_n} v_i$, $v_i \in V_i^{(n)}$, with λ_n a simple dominant eigenvalue of γ_n implies $\gamma_n v_i = \lambda_n v_i$; hence the proximality of γ_n implies that v belongs to a unique $V_i^{(n)}$, to $V_1^{(n)}$ say. Also the irreducibility of T_{μ_n} implies that T_{μ_n} permutes the subspaces $V_i^{(n)}$ ($1 \leq i \leq p_n$). Since $V_1^{(n)}$ is isotypic and γ_n is proximal, the subspace $V_1^{(n)}$ is T_{μ_n} -irreducible. The same is valid for any $V_i^{(n)} = g(V_1^{(n)})$ since $g\gamma_n g^{-1}$ is also proximal, for $g \in T_{\mu_n}$. Assume $Zc_0(T_{\mu_n})$ is not irreducible for n large; then it follows that $p_n \in]1, d]$ and $r_n = \dim V_1^{(n)} \in [1, d[$. It follows that we can assume $p_n = p$ and $r_n = r$ for n large with $p > 1$, $r < d$. Hence, taking convergent subsequences of $V_i^{(n)}$ ($1 \leq i \leq p$) we obtain proper subspaces V_i ($1 \leq i \leq p$) which are permuted by T_μ ; the irreducibility of T_μ implies that their sum is V , hence we have $V = \bigoplus_1^p V_i$, which contradicts the strong irreducibility of T_μ . Hence T_{μ_n} satisfies condition i-p for n large. \square

Proof of Theorem 6.1

Let $\lambda_n \in M^1(H)$ be a sequence which converges to $\lambda \in M^1(H)$ in the weak topology $\mathcal{W}(H)$ and let us denote by μ_n the projection of λ_n on G . We verify the stability of conditions 1, 2 in (c-e) if $d > 1$, since condition 3 follows of the definition of $\mathcal{W}(H)$ and condition 4 is a direct consequence of Proposition 6.2. We denote by \mathbb{S}_∞^{d-1} the sphere at infinity of V and we observe that the group H acts continuously on the compactification $(V) \cup \mathbb{S}_\infty^{d-1}$ endowed with the visual topology.

1) Assume that $\text{supp}(\lambda_n)$ has a fixed point $x_n \in V$ for n large. Since the closed subset $\text{supp}(\lambda_n)$ converges to $\text{supp}(\lambda)$, we can find a convergent subsequence of x_n to a point x in $(V) \cup (\mathbb{S}_\infty^{d-1})$, such that x is $\text{supp}(\lambda)$ -invariant. If $x \in V$ we have a contradiction since $\text{supp}(\lambda)$ has no fixed point in V . If $x \in \mathbb{S}_\infty^{d-1}$, we have also a contradiction since condition i-p implies that the projective action of $\text{supp}(\mu)$ has no fixed point, if $d > 1$.

2) Using Lemma 6.4, since finiteness of moments for μ_n is valid, we get that for μ_n and for any $s \geq 0$, the corresponding operator P^s has a spectral gap on the relevant Hölder space on \mathbb{S}_∞^{d-1} (see [13]). The moment condition implies that perturbation theory is valid for the operators P^s . Hence the spectral radius $k(s)$ varies continuously. In particular, since we have $k(s) > 1$ for μ and $s > \alpha$, and $L(\mu) < 0$, the same is valid for μ_n with n large. Hence there exists $\alpha_n > 0$ close to α such that $k(\alpha_n) = 1$ and $L(\mu_n) < 0$. \square

Acknowledgment

This research, did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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