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Abstract
We consider a general multivariate affine stochastic recursion and the associated Markov chain on $\mathbb{R}^d$. We assume a natural geometric condition which implies existence of an unbounded stationary solution and we show that the large values of the associated stationary process follow extreme value properties of classical type, with a non trivial extremal index. The proof is based on a spectral gap property for the action of the corresponding Markov operator on spaces of regular functions with slow growth, and on the clustering properties of large values in the recursion.

Keywords : Spectral gap, Extreme value, Affine random recursion, Limit theorem, Point process, Cluster index.

1 Introduction

Let $V = \mathbb{R}^d$ be the d-dimensional Euclidean space and let λ be a probability on the affine group A of V. Let (A_n, B_n) be a sequence of A-valued i.i.d. random variables distributed according to λ and let us consider the affine stochastic recursion on V defined by

$$X_n = A_n X_{n-1} + B_n$$

for $n \in \mathbb{N}$. We denote by P the corresponding Markov kernel on V and by $\mathbb{P}$ the product measure $\lambda^{\otimes \mathbb{N}}$ on $A^{\mathbb{N}}$. Our geometric hypothesis (H) on λ implies that P has a unique invariant probability ρ on V and the support of ρ is unbounded. In our situation (see [13]), the quantity $\rho\{|v| > t\}$ is asymptotic $(t \to \infty)$ to $\alpha^{-1} c\, t^{-\alpha}$ with $\alpha > 0$, $c > 0$. Furthermore, the measure ρ is multivariate regularly varying, a basic property for the development of extreme value theory *i.e* for the study of exceptionally large values of $X_k (1 \leq k \leq n)$ for n large (see [28]). In such a situation of weak dependence, spectral gap properties of operators associated to P play also an important role via a multiple mixing condition described in [7] for the case of step functions. We observe that the same idea was used in the proofs of limit theorems for the largest coefficient in the continued fraction expansion of a real number uniformly distributed in the interval $[0, 1]$ (see [25], [31]), as well as in the proofs of limit theorems for $\sum_{k=1}^{n} X_k$ (see [15]). In the context of geometric ergodicity, assuming a density condition on the law of B_n, partial results were obtained in [21]. Here

we go further in this direction replacing geometric ergodicity by condition (H). We observe that, if λ is singular with respect to Haar measure on A, then the operator P on V is not v-geometrically ergodic in general, hence the classical framework of [22] for asymptotics of Markov chains is not convenient in our setting. However condition (H) implies that the operator P has a spectral gap property in the spaces of Hölder functions with polynomial growth considered below, a fact which allow us to deduce convergence with exponential speed on Hölder functions. A typical example of this situation occurs if the support of λ is finite and generates a dense subsemigroup of the affine group A.

If $\mathbb{Z}_+$ is the set of non negative integers, we denote by $\mathbb{P}_\rho$ the Markov probability on $V^{\mathbb{Z}_+}$ defined by the kernel P and the initial probability ρ. In this paper we establish spectral gap properties for the action of P on Hölder functions and we deduce fundamental extreme value statements for the point processes defined by the $\mathbb{P}_\rho$-stationary sequence $(X_k)_{k\geq 0}$. Our results are based on the fact that the general conditions of multiple mixing and anticlustering used in extreme value theory of stationary processes (see [7]) are valid for affine stochastic recursions, under condition (H). We observe that, in the context of Lipchitz functions, the above mixing property is a consequence of the spectral gap properties studied below ; it turns out that the use of advanced point process theory allows us to extend this mixing property to the classical context of continuous functions. We note that, if $d > 1$, the set of probabilities λ on the affine group A, which satisfy condition (H), is open in a natural weak topology ; hence in this sense, hypothesis (H) is generic. Then, our framework allow us to develop extreme value theory for a large class of natural examples, including the so-called GARCH process as a very special case (see [10]).

In order to sketch our results, we recall that Fréchet's law Φ_α^a with positive parameters α, a is the probability on $\mathbb{R}_+$ defined by the distribution function $\Phi_\alpha^a([0,t]) = exp - at^{-\alpha}$. We denote by μ the projection of λ on the linear group $G = GL(V)$. Also, we consider the associated stochastic linear recursion $Y_n = A_n Y_{n-1}$, we denote by Q the corresponding Markov kernel on $V \setminus \{0\}$ and by $\mathbb{Q} = \mu^{\otimes \mathbb{N}}$ the product measure on $G^{\mathbb{N}}$; we write $S_n = A_n \cdots A_1$ for the product of random matrices $A_k (1 \leq k \leq n)$. Extending previous work of H. Kesten (see [19]), a basic result proved in [13] under condition (H) is that for some $\alpha > 0$, the probability ρ is α-homogeneous at infinity, hence ρ has an asymptotic tail measure $\Lambda \neq 0$ which is a α-homogeneous Q-invariant Radon measure on $V \setminus \{0\}$. The multivariate regular variation of ρ is a direct consequence of this fact. Also, it follows that, if $B_t \subset V$ is the ball of radius $t > 0$ centered at $0 \in V$ and $B_t' = V \setminus B_t$, then we have $\Lambda(B_t') = \alpha^{-1} c t^{-\alpha}$ with $c > 0$. In particular, $\Lambda(B_t')$ is finite and the projection of ρ on $\mathbb{R}_+$, given by the norm map $v \to |v|$ has the same asymptotic tail as Φ_α^c. If $u_n = (cn)^{1/\alpha}$, it follows that the mean number of exceedances of u_n by $|X_k|$ $(1 \leq k \leq n)$ converges to one. It will appear below that u_n is an estimate of $sup\{|X_k| \, ; \, 1 \leq k \leq n\}$.

Then, one of our main results is the convergence in law of the normalized maximum of the sequence $|X_1|, |X_2|, \ldots, |X_n|$ towards Fréchet's law Φ_α^θ with $\theta \in]0,1[$. A closely related point process result is the weak convergence (see [28]) of the time exceedances process

$$N_n^t = \sum_{k=1}^{n} \varepsilon_{n^{-1}k} 1_{\{|X_k|>u_n\}}$$

towards a compound Poisson process with intensity θ and cluster probabilities depending on the renewal point process $\pi_v^\omega = \sum_{i=0}^{\infty} \varepsilon_{S_i(\omega)v}$ and on the Q-invariant measure Λ. The significance of the relation $\theta < 1$ is that, in our situation, values of the sequence $(|X_k|)_{0\leq k\leq n}$ larger than u_n, appear in localized clusters with asymptotic expected cardinality $\theta^{-1} > 1$. This reflects the local dependence of large values in the sequence $(X_k)_{k\geq 0}$ and is in contrast with the well known situation of positive i.i.d. random variables with tail also given by Φ_α^c, where the property $\theta = 1$ is satisfied. If Euclidean norm is replaced by another norm, the value of θ in the new setting is changed but the condition $\theta \in]0,1[$ remains valid. For affine stochastic recursions in dimension one, if A_n, B_n are positive and condition (H) is satisfied, convergence to Fréchet's law and $\theta \in]0,1[$ was proved in [17]. We observe that our result is the natural multivariate extension of this fact. Also, if $d \geq 1$, assuming technical conditions on the random walk X_n and density for the law of B_n with respect to Lebesgue measure, the two above convergences were shown in [21]. Here our proofs use the tools of point processes theory and a remarkable formula (see [2]) for the Laplace functional of a cluster process $C = \sum_{j>0} \varepsilon_{Z_j}$ on $V \setminus \{0\}$, depending only on μ, Λ, which describes locally the large values of $(X_n)_{n\geq 0}$. As a consequence of Fréchet's law and in the spirit of [26], we obtain a logarithm law for affine random walk.

To go further, we consider the linear random walk $Y_n^v = S_n(\omega)v$ on $V \setminus \{0\}$, we observe that condition (H) implies $\lim_{n\to\infty} S_n(\omega)v = 0$, $\mathbb{Q} - a.e$ and we denote by $\mathbb{Q}_\Lambda$ the Markov measure on $(V \setminus \{0\})^{\mathbb{Z}_+}$ defined by the kernel Q and the Q-invariant initial measure Λ. We show below the weak convergence to a limit process N of the sequence of space-time exceedances processes

$$N_n = \sum_{i=1}^{n} \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$$

on $[0,1] \times (V \setminus \{0\})$. In restriction to $[0,1] \times B'_\delta$, with $\delta > 0$, N can be expressed in terms of C and of a Poisson component on $[0,1]$ with intensity $\theta\delta^{-\alpha}$; the expression of C involves the renewal point process π_v^ω and $\mathbb{Q}_\Lambda$. Using the framework and the results of ([2], [3], [7]), we describe a few consequences of this convergence. In particular we consider also, as in ([7], [8]), the convergence of the normalized partial sums $\sum_{i=1}^{n} X_i$ towards stable laws, if $0 < \alpha < 2$, in the framework of extreme value theory. Also, as observed in [7], this convergence is closely connected to the convergence of the sequence of space exceedances point processes on V

$$N_n^s = \sum_{i=1}^{n} \varepsilon_{u_n^{-1}X_i},$$

towards a certain infinitely divisible point process N^s. Here the Laplace functional of N^s can be expressed in terms of Λ and π_v^ω.

In these studies we follow closely the approaches previously developed in ([2], [3], [7]) in the context of extreme value theory for general stationary processes, in particular we make

use of the concepts of tail and cluster processes introduced in [2]. This allow us to prove explicit extreme values properties for affine random walks, under condition (H), and to recover, in a natural setting, the characteristic functions of the above α-stable laws, as described in [15] if $d = 1$ and in ([5], [11]) if $d > 1$, completing thereby the results of ([1], [2], [7]). For self containment reasons we have developed anew a few arguments of ([2], [3]) in our situation. We refer to [4] for information on products of random matrices and to ([14], [16]) for short surveys of the above results.

2 The tail process and the cluster process

In this paper, we will always assume that λ satisfies condition (H) explained below.

2.1 *Homogeneity at infinity of the stationary measure*

We recall condition (H) from [13], for the probability λ on the affine group of V. A semigroup T of $GL(V) = G$ is said to satisfy i-p if

a) T has no invariant finite family of subspaces

b) T contains an element with a dominant eigenvalue which is real and unique.

Condition i-p implies that the action of T on the projective space of V is proximal ; heuristically speaking this means that, T contracts asymptotically two arbitrary given directions to a single one, hence the situation could be compared to a 1-dimensional one. Condition i-p for T is valid if and only if it is valid for the group which is the Zariski closure of T. Hence it is valid in particular if T is Zariski dense in G (see [27]) ; also it is valid for T if and only if it is valid for T^{-1}. Below we will denote by T the closed subsemigroup generated by $supp(\mu)$, the support of μ.

For $g \in G$ we write $\gamma(g) = sup(|g|, |g^{-1}|)$ and we assume $\int log\gamma(g)d\mu(g) < \infty$. For $s \geq 0$ we write $logk(s) = \lim_{n\to\infty} \frac{1}{n} log \int |g|^s d\mu^n(g)$ where μ^n denotes the n^{th} convolution power of μ and we write $L(\mu)$ for the dominant Lyapunov exponent of the product $S_n(\omega) = A_n \cdots A_1$ of random matrices $A_k (1 \leq k \leq n)$ i.e $L(\mu) = \lim_{n\to\infty} \frac{1}{n} \int log|g| d\mu^n(g) = k'(0)$. We denote by $r(g)$ the spectral radius of $g \in G$. We say that T is non arithmetic if $r(T)$ contains two elements with irrational ratio. Condition (H) is the following :

1) supp(λ) has no fixed point in V.

2) There exists $\alpha > 0$ such that $k(\alpha) = \lim_{n\to\infty} \mathbb{E}(|S_n|^\alpha)^{1/n} = 1$.

3) There exists $\varepsilon > 0$ with $\mathbb{E}(|A|^\alpha \gamma^\varepsilon(A) + |B|^{\alpha+\varepsilon}) < \infty$.

4) If $d > 1$, T satisfies i-p and if $d = 1$, T is non arithmetic.

The above conditions imply in particular that $L(\mu) < 0$, $k(s)$ is analytic, $k(s) < 1$ for $s \in]0, \alpha[$ and there exists a unique stationary probability ρ for λ acting by convolution on V ; the support of ρ is unbounded. Property 1 guarantees that ρ has no atom and says that the action of $supp(\lambda)$ is not conjugate to a linear action. Property 2 is responsible for the

α-homogeneity at infinity of ρ described below ; if $k(s)$ is finite on $[0,\infty[$ and there exists $g \in T$ with $r(g) > 1$, then Property 2 is satisfied. Also if $d > 1$, condition i-p is basic for renewal theory of the random walk $S_n(\omega)v$ and it implies that T is non arithmetic.
In the appendix we will show that condition (H) is open in the weak topology of probabilities on the affine group, defined by convergence of moments and of values on continuous compactly supported functions.
Below, we use the decomposition of $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_{>0}$ in polar coordinates, where $\mathbb{S}^{d-1}$ is the unit sphere of V. We consider also the Radon measure ℓ^α on $\mathbb{R}_{>0}$ ($\alpha > 0$) given by $\ell^\alpha(dt) = t^{-\alpha-1}dt$. We recall (see [19]) that, if $(A_n, B_n)_{n\in\mathbb{N}}$ is an i.i.d sequence of A-valued random variables with law λ and $L_\mu < 0$, then ρ is the law of the $\mathbb{P} - a.e$ convergent series $X = \sum_0^\infty A_1 \cdots A_k B_{k+1}$. We observe that a family $h_t (t \in \mathbb{R}^*)$ of automorphisms of the group A is given by $h_t(a, b) = (a, tb)$. Then it follows that the stationary probability for $h_t(\lambda)$ is $t.\rho$, where $t.\rho$ denotes the push forward of ρ under the dilation $v \to tv$.

Theorem 2.1 *(see [13], Theorem C)*
Assume that λ satisfies condition (H). Then the operator P has a unique stationary probability ρ, the support of ρ is unbounded and we have the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t\to 0_+} t^{-\alpha}(t.\rho) = \Lambda = c(\sigma^\alpha \otimes \ell^\alpha)$$

where $c > 0$ and σ^α is a probability on $\mathbb{S}^{d-1}$. Furthermore Λ is a Q-invariant Radon measure on $V \setminus \{0\}$.

We observe that, for $d > 1$, if $supp(\lambda)$ is compact, generates a Zariski dense subgroup of A (see [24]), $L(\mu) < 0$, and T is unbounded, then condition (H) is satisfied. For $d = 1$, if $supp(\lambda)$ is compact, the hypothesis of ([17], Theorem 1.1) is equivalent to condition (H).
The existence of Λ stated in the theorem implies multivariate regular variation of ρ. If the convergence stated in the theorem is valid we say that ρ is homogeneous at infinity ; below we will make essential use of this property.
Under condition (H), Λ gives zero mass to any submanifold and σ^α has positive dimension. We observe that, if the sequence $(A_n, B_n)_{n\in\mathbb{N}}$ is replaced by $(A_n, tB_n)_{n\in\mathbb{N}}$ with $t \in \mathbb{R}^*$, then the asymptotic tail measure is replaced by $t.\Lambda$, in particular the constant c is replaced by $|t|^\alpha c$. We observe that, as shown in [13], the Q-invariant Radon measure Λ is extremal or can be decomposed in two extremal measures. Hence, if the action of T on $\mathbb{S}^{d-1}$ has a unique minimal subset, then Λ is symmetric and the shift invariant measure $\mathbb{Q}_\Lambda$ on $(V \setminus \{0\})^{\mathbb{Z}_+}$ is ergodic. Otherwise $\mathbb{Q}_\Lambda$ decomposes into two ergodic measures. Hence Λ depends only of μ, up to one or two coefficients.
The following is a classical consequence of vague convergence.

Corollary 2.2 *Let f be a non negative Λ-integrable Borel function on $V \setminus \{0\}$ which has a Λ-negligible discontinuity set. Then we have $\lim_{t\to 0_+} t^{-\alpha}(t.\rho)(f) = \Lambda(f)$.*

2.2 *The tail process*

We denote $\Omega = G^{\mathbb{N}}$, $\widehat{\Omega} = G^{\mathbb{Z}}$, and we endow Ω (resp $\widehat{\Omega}$) with the product probability $\mathbb{Q} = \mu^{\otimes\mathbb{N}}$ (resp $\widehat{\mathbb{Q}} = \mu^{\otimes\mathbb{Z}}$).
We define the G-valued cocycle $S_n(\omega)$ where $\omega = (A_k)_{k\in\mathbb{Z}} \in \widehat{\Omega}$, $n \in \mathbb{Z}$ by :

$S_n(\omega) = A_n \cdots A_1$ for $n > 0$, $S_n(\omega) = A_n^{-1} \cdots A_{-1}^{-1}$ for $n < 0$, $S_0(\omega) = Id$

We consider also the random walk $Y_n^v = S_n(\omega)v$ on $V \setminus \{0\}$, starting from $v \neq 0$ and we denote by $\mathbb{Q}_\Lambda$ (resp $\widehat{\mathbb{Q}}_\Lambda$ the Markov measure on $(V \setminus \{0\})^{\mathbb{Z}_+}$ (resp $(V \setminus \{0\})^{\mathbb{Z}}$) for the random walk Y_n^v with initial measure Λ. These measures are invariant under the shift $\widetilde{\tau}$ on $(V \setminus \{0\})^{\mathbb{Z}_+}$ (resp $(V \setminus \{0\})^{\mathbb{Z}}$). If we denote by σ the shift on $\widehat{\Omega}$ and by $\widetilde{\sigma}$ the extended shift on $\widetilde{\Omega} \times (V \setminus \{0\})$ defined by $\widetilde{\sigma}(\omega, v) = (\sigma\omega, A_0 v)$, then $\widehat{\mathbb{Q}} \otimes \Lambda$ is $\widetilde{\sigma}$-invariant and $\widehat{\mathbb{Q}}_\Lambda$ is simply the projection of $\widehat{\mathbb{Q}} \otimes \Lambda$ on $(V \setminus \{0\})^{\mathbb{Z}}$, under the map $(\omega, v) \to (S_k(\omega)v)_{k\in\mathbb{Z}}$.
The normalized restriction of Λ to B_1' is denoted Λ_1, hence $\Lambda_1(B_t') = t^{-\alpha}$ if $t > 1$ and we write $\mathbb{Q}_{\Lambda_1} = c^{-1}\alpha(1_{B_1'} \circ \pi)\mathbb{Q}_\Lambda$, $\widehat{\mathbb{Q}}_{\Lambda_1} = c^{-1}\alpha(1_{B_1'} \circ \pi)\widehat{\mathbb{Q}}_\Lambda$, where π denotes the projection on $V \setminus \{0\}$. We note that the probability $\mathbb{Q}_{\Lambda_1}$ (resp $\widehat{\mathbb{Q}}_{\Lambda_1}$) extends to $V^{\mathbb{Z}_+}$ (resp $V^{\mathbb{Z}}$) and its extension will be still denoted $\mathbb{Q}_{\Lambda_1}$ (resp $\widehat{\mathbb{Q}}_{\Lambda_1}$).
We consider the probability ρ, the shift τ on $V^{\mathbb{Z}_+}$ (resp $V^{\mathbb{Z}}$) the shift-invariant Markov measure $\mathbb{P}_\rho$ (resp $\widehat{\mathbb{P}}_\rho$) on $V^{\mathbb{Z}_+}$ (resp $V^{\mathbb{Z}}$), where ρ is the law of X_0 and $\mathbb{P}_\rho$ is the projection of $\widehat{\mathbb{P}}_\rho$ on $V^{\mathbb{Z}_+}$. Since $\rho(\{0\}) = 0$, we can replace V by $V \setminus \{0\}$ when working under $\mathbb{P}_\rho$. For $0 < j \leq i$ we write $S_j^i = A_i \cdots A_j$ and $S_{i+1}^i = I$. Expectation with respect to $\mathbb{P}$ or $\mathbb{Q}$, $\widehat{\mathbb{Q}}$ will be simply denoted by the symbol $\mathbb{E}$. If expectation is taken with respect to a Markov measure with initial measure ν we will write $\mathbb{E}_\nu$. For a family $Y_j (j \in \mathbb{Z})$ of V-valued random variables and $k, \ell \in \mathbb{Z}\ U\{-\infty, \infty\}$ we denote $M_k^\ell(Y) = \sup\{|Y_j|\ ;\ k \leq j \leq \ell\}$. We observe that, if $t > 0$, condition (H) implies $\rho\{|x| > t\} > 0$, hence as in [2], we can consider the new process $(Y_i^t)_{i\in\mathbb{Z}}$ deduced from $t^{-1}(X_i)_{i\in\mathbb{Z}}$ by conditioning on the set $\{|X_0| > t\}$, for t large, under $\widehat{\mathbb{P}}_\rho$. We recall (see [28]) that a sequence of point processes is said to converge weakly to another point process if there is weak convergence of the corresponding finite dimensional distributions. The following is the detailed form in our case of the general result for multivariate jointly regularly varying stationary processes in [2].

Proposition 2.3 *a) The family of point process $(Y_i^t)_{i\in\mathbb{Z}}$ converges weakly $(t \to \infty)$ to the point process $(Y_i)_{i\in\mathbb{Z}}$ on V given by $Y_i = S_i Y_0$ where Y_0 has law Λ_1 and $(Y_i)_{i\in\mathbb{Z}}$ has law $\widehat{\mathbb{Q}}_{\Lambda_1}$.*

b) We have $\mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\} = \widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) \leq 1\}$ and the process Y_j satisfies $\lim_{|j|\to\infty} |Y_j| = 0$, $\widehat{\mathbb{Q}}_{\Lambda_1} - a.e.$
In particular we have

$$\lim_{n\to\infty} \lim_{t\to\infty} \mathbb{P}_\rho\{\sup_{1\leq k\leq n} |X_k| \leq t / |X_0| > t\} = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \leq 1\} := \theta \in]0, 1[$$

Proof : a) We observe that, since for any $i \geq 0$ $X_i = S_i X_0 + \sum_{j=1}^{i} S_{j+1}^i B_j$ and $\lim_{t\to\infty} \frac{1}{t} \sum_{j=1}^{p} S_{j+1}^i B_j =$

0, $\mathbb{P}_\rho - a.e$, the random vectors $(t^{-1}X_i)_{0\le i\le p+q}$ and $(t^{-1}S_iX_0)_{0\le i\le p+q}$ have the same asymptotic behaviour in $\mathbb{P}_\rho$-law, conditionally on $|X_0| > t$. Also by stationarity of $\widehat{\mathbb{P}}_\rho$, for f continuous and bounded on V^{p+q+1} we have
$\mathbb{E}_\rho\{f(t^{-1}X_{-q},\cdots,t^{-1}X_p)/|X_0|>t\} = \mathbb{E}_\rho\{f(t^{-1}X_0,t^{-1}X_1,\cdots,t^{-1}X_{p+q})\ /\ |X_q|>t\}$.
From above, using Corollary 2.2, $\Lambda\{|x|=1\}=0$, and the formula $\Lambda\{|x|>1\}=\alpha^{-1}c$ we see that the right hand side converges to :

$\alpha c^{-1}\int \mathbb{E}\{f(x,S_1x,\cdots,S_{p+q}x)1_{\{|S_qx|>1\}}d\Lambda(x)$,

Hence, using the definition of $\widehat{\mathbb{Q}}_\Lambda$, we get the weak convergence of the process $(Y_i^t)_{i\in\mathbb{Z}}$ to $(Y_i)_{i\in\mathbb{Z}}$ as stated in a). Since for any $x\in V$ we have $\lim_{i\to\infty} S_ix = 0\ \mathbb{Q}-a.e$, the formula $Y_i = S_iY_0$ gives $\lim_{i\to\infty} Y_i = 0$, $\mathbb{Q}_{\Lambda_1}-a.e$. If $\widehat{\mathbb{Q}}_{\Lambda_1}\{\limsup_{i\to-\infty}|Y_i|>0\}\neq 0$ then, for some $\varepsilon,\varepsilon'>0$ and a sequence $i=i_k\to-\infty$ we have $\widehat{\mathbb{Q}}_{\Lambda_1}\{|Y_i|>\varepsilon\}>\varepsilon'$; since $\widehat{\mathbb{Q}}_\Lambda$ is $\widetilde{\tau}$-invariant we get $\lim_{i\to-\infty}\mathbb{Q}_\Lambda\{|Y_{-i}|>1,|Y_0|>\varepsilon\}>\varepsilon'$. Since $\Lambda(B_\varepsilon)<\infty$ and $\lim_{n\to\infty}S_nx=0\ \mathbb{Q}-a.e$, this gives the required contradiction.

b) In view of a) and Corollary 2.2, since the discontinuity sets of the functions $1_{]0,1]}(M_1^n(Y))$ and $1_{[1,\infty[}(Y_0)$ on V^n are $\mathbb{Q}_{\Lambda_1}$-negligible, we have $\lim_{t\to\infty}\mathbb{P}_\rho\{\sup_{1\le k\le n} t^{-1}|X_k|\le 1/t^{-1}|X_0|>1\} = \mathbb{Q}_{\Lambda_1}\{M_1^n(Y)\le 1\}$.
Hence $\theta = \lim_{n\to\infty}\lim_{t\to\infty}\mathbb{P}_\rho\{\sup_{1\le k\le n}|X_k|\le t/X_0>t\} = \mathbb{Q}_{\Lambda_1}\{\sup_{k\ge1}|Y_k|\le 1\}$.
We write $\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y)\le 1\} = 1-\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y)>1\}$ and we define the random time T by $T = inf\{k\ge1\,;\,|Y_{-k}|>1\}$ if there exists $k\ge1$ with $|Y_{-k}|>1$; if such a k do not exist we take $T=\infty$. We have by definition of T :

$\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y)>1\} = \sum_{k=1}^{\infty}\widehat{\mathbb{Q}}_{\Lambda_1}\{T=k\}$

$\widehat{\mathbb{Q}}_{\Lambda_1}\{T=k\} = \widehat{\mathbb{Q}}_{\Lambda_1}\{|Y_{-1}|\le1,\,|Y_{-2}|\le1,\cdots,|Y_{-k+1}|\le1\ ;\ |Y_{-k}|>1\}$,

Using stationarity of $\widehat{\mathbb{Q}}_\Lambda$, the definition of $\widehat{\mathbb{Q}}_{\Lambda_1}$ and a) we get
$\widehat{\mathbb{Q}}_{\Lambda_1}\{T=k\} = \mathbb{Q}_{\Lambda_1}\{|Y_1|\le1,\cdots,|Y_{k-1}|\le1\ ;\ |Y_k|>1\}$, hence
$\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y)>1\} = \sum_{k=1}^{\infty}\mathbb{Q}_{\Lambda_1}\{|Y_1|\le1,\cdots,|Y_{k-1}|\le1\ ;\ |Y_k|>1\} = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y)>1\}$.
The formula $\widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y)\le1\} = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y)\le1\}$ follows.
The formula $\theta = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y)\le1\}$ and the form of $Y_i(i\ge0)$ given in a) imply $\theta = \mathbb{E}(\int 1_{\{\sup_{i\ge1}|S_ix|\le1\}}d\Lambda_1(x))\le1$. The condition $\theta=1$ would imply for any $i\ge1$: $|S_ix|\le1$ $\mathbb{Q}\otimes\Lambda_1-a.e$, hence $supp(S_i\Lambda_1)\subset\{x\in V\ ;|x|\le1\}$. This would contradict the fact that $supp\Lambda_1$ is unbounded, hence we have $\theta<1$. The inequality $\theta>0$ is obtained in Proposition 2.5 below. For a direct proof using standard arguments in ergodic theory see [6] and appendix. □
We note that the process $(Y_j)_{j\in\mathbb{Z}}$ is not stationary. However this process can be viewed as a simplified version of the stationary process $(X_j)_{j\in\mathbb{Z}}$; for example the property $\lim_{|j|\to\infty}|Y_j|=0$

$\widehat{\mathbb{Q}}_{\Lambda_1} - a.e$ is an analogue of the weak convergence of $X_j(|j| \to \infty)$ to the probability ρ.

2.3 *Anticlustering property*

We are going to show that the set of large values of $X_k(1 \leq k \leq n)$ consists of localized elementary clusters with a few values. An important sufficient condition for localization (see [7]) is proved in Proposition 2.4 below and will allow us to show the existence of a cluster process as defined in[2]. It is called anticlustering and is used in section 4 to decompose the set of values of $X_k(1 \leq k \leq n)$ into successive quasi-independent blocks. For $k \leq \ell$ in $\mathbb{Z}$ we write

$M_k^\ell = \sup_{k \leq i \leq \ell} |X_i| \quad , \quad R_k^\ell = \sum_{i=k}^{\ell} \widehat{\mathbb{P}}_\rho\{|X_i| > u_n / |X_0| > u_n\} = \mathbb{E}_\rho\{\sum_k^\ell 1_{]u_n,\infty[}(|X_i|)/|X_0| > u_n\}$, where $u_n = (cn)^{1/\alpha}$. For $k > 0$, we write also $M_k = M_1^k$.

We observe that $M_k^\ell \leq \sum_k^\ell |X_i|$. Let r_n be any sequence of integers with $r_n = o(n)$, $\lim_{n\to\infty} r_n = \infty$. Then we have $\mathbb{P}_\rho\{M_1^{r_n} > u_n\} \leq r_n \mathbb{P}_\rho\{|X_0| > u_n\}$, hence the homogeneity of ρ at infinity gives $\lim_{n\to\infty} \mathbb{P}_\rho\{M_1^{r_n} > u_n\} = 0$. The condition $r_n = o(n)$ allows us to localize the influence of one large value of $X_k(1 \leq k \leq n)$. It follows that the event $\{M_1^{r_n} > u_n\}$ can be considered as "rare". On the other hand, the last part of Proposition 2.4 below shows that values of $|X_k|(1 \leq k \leq r_n)$ conditionally larger than u_n are taken with finite asymptotic average multiplicity. The homogeneity at infinity of ρ and the arbitrariness of r_n allow us to restrict the study to the sequence u_n instead of $tu_n(t > 0)$.

The following is based on the homogeneity at infinity of ρ, the inequality $0 < k(s) < 1$ if $0 < s < \alpha$, and it will imply the finiteness of such expectations, in the limit.

Proposition 2.4 *Assume $r_n \leq [n^s]$ with $0 < s < 1$, $\lim_{n\to\infty} r_n = \infty$. Then $\lim_{m\to\infty} \lim_{n\to\infty} R_m^{r_n} = 0$. In particular $\lim_{m\to\infty} \lim_{n\to\infty} \widehat{\mathbb{P}}_\rho\{sup(M_{-r_n}^{-m},\ M_m^{r_n}) > u_n / |X_0| > u_n\} = 0$, hence the random walk X_n satisfies anticlustering. For θ_n defined by $\theta_n^{-1} = \mathbb{E}_\rho\{\sum_1^{r_n} 1_{[u_n,\infty[}(|X_i|)/M_{r_n} > u_n\}$ we have*

$\liminf_{n\to\infty} \theta_n > 0$, *and* $\theta_n \leq 1$.

Proof : We observe that :

$\widehat{\mathbb{P}}_\rho\{M_m^{r_n} > u_n / |X_0| > u_n\} \leq R_m^{r_n}$, $\widehat{\mathbb{P}}_\rho\{M_{-r_n}^{-m} > u_n / |X_0| > u_n\} \leq R_{-r_n}^{-m} = R_m^{r_n}$

where we have used stationarity of X_k in the last equality. Hence it suffices to show $\lim_{m\to\infty} \lim_{n\to\infty} R_m^{r_n} = 0$. For $i \geq 0$ we have $X_i = S_i X_0 + \sum_{j=1}^{i} S_{j+1}^i B_j$ where S_i, X_0 are independent, as well as X_0, $\sum_{j=1}^{i} |S_{j+1}^i B_j|$. We write $I_n^i = \widehat{\mathbb{P}}_\rho\{|X_i| > u_n / |X_0| > u_n\}$,

$J_n^i = \widehat{\mathbb{P}}_\rho\{|S_i X_0| > 2^{-1} u_n / |X_0| > u_n\}$, $K_n^i = \widehat{\mathbb{P}}_\rho\{\sum_{j=1}^{i} |S_{j+1}^i B_j| > 2^{-1} u_n / |X_0| > u_n\}$,

hence $R_m^{r_n} = \sum_{i=m}^{r_n} I_n^i \leq \sum_{i=m}^{r_n} J_n^i + \sum_{i=m}^{r_n} K_n^i$.

We are going to show $\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=m}^{r_n} J_n^i = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=m}^{r_n} K_n^i = 0$.

We apply Chebyshev's inequality to the χ-moments of X_n with $\chi \in]0,\alpha[$. We have :

$J_n^i \leq (2u_n^{-1})^\chi \mathbb{E}_\rho(|S_i X_0|^\chi/|X_0| > u_n\} \leq (2u_n^{-1})^\chi \mathbb{E}(|S_i|^\chi)\mathbb{E}_\rho(|X_0|^\chi/|X_0| > u_n)$,

where independence of S_i and X_0 have been used in the last formula. Since the law of X_0 is α-homogeneous at infinity we have :

$\lim_{x\to\infty} x^{-\chi}\mathbb{E}_\rho(|X_0|^\chi/|X_0| > x) = \alpha(\alpha-\chi)^{-1}$, $\limsup_{n\to\infty} J_n^i \leq 2^\chi \mathbb{E}(|S_i|^\chi)\alpha(\alpha-\chi)^{-1}$.

Also, using independence of X_0 and $\sum_{j=1}^{i}|S_{j+1}^i B_j|$:

$$K_n^i = \mathbb{P}\{\sum_{j=1}^{i}|S_{j+1}^i B_j| > 2^{-1}u_n\} \leq (2u_n^{-1})^\chi \mathbb{E}(\sum_{j=1}^{i}|S_{j+1}^i B_j|)^\chi \leq (2u_n^{-1})^\chi \mathbb{E}(\sum_{j=1}^{\infty}|S_{j+1}B_j|)^\chi.$$

It follows $\limsup_{n\to\infty}(\sum_{i=m}^{r_n} J_n^i) \leq 2^\chi\alpha(\alpha-\chi)^{-1}\mathbb{E}(\sum_{i=m}^{\infty}|S_i|^\chi)$.

From [13] we know that, since $0 < \chi < \alpha$, we have $k(\chi) < 1$, hence $\mathbb{E}(|S_i|^\chi)$ decreases exponentially fast to zero ; hence the series $\mathbb{E}(\sum_{i=1}^{\infty}|S_i|^\chi)$ converges and $\lim_{m\to\infty}\mathbb{E}(\sum_{i=m}^{\infty}|S_i|^\chi) = 0$,

$\lim_{m\to\infty}\lim_{n\to\infty}(\sum_{i=m}^{r_n} J_n^i) = 0$.

From above we know that $R_0 = \sum_1^\infty |S_{j+1}B_j|$ has finite χ-moment if $\chi < \alpha$. Then by Chebyshev's inequality : $\sum_{i=m}^{r_n} K_n^i \leq (2u_n^{-1})^\chi r_n \mathbb{E}(R_0^\chi)$.

Since $0 < s < 1$, we can choose $\chi \in]0,\alpha[$ such that $\alpha^{-1}\chi > s$, hence $\lim_{n\to\infty} r_n u_n^{-\chi} = 0$. Then, for any fixed m : $\lim_{n\to\infty}\sum_{i=m}^{r_n} K_n^i = 0$. Hence $\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=m}^{r_n} I_n^i = 0$ and the first result follows.

For the last assertion we follow below a general argument in [2].

From above we have using stationarity :

$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}_\rho\{M_{m+1}^{r_n} > u_n/|X_1| > u_n\} = 0$.

By definition of M_{r_n} : $\mathbb{P}_\rho\{M_{r_n} > u_n\} \geq \sum_{k=0}^{m^{-1}r_n-1}\mathbb{P}_\rho\{|X_{km+1}| > u_n, M_{(k+1)m+1} \leq u_n\}$,

hence using stationarity :

$\mathbb{P}_\rho\{M_{r_n} > u_n\} \geq m^{-1}r_n\mathbb{P}_\rho\{|X_1| > u_n, M_{m+1}^{r_n} \leq u_n\}$.

By definition of θ_n this can be rewritten as :

$\theta_n \geq m^{-1}(1 - \mathbb{P}_\rho\{M_{m+1}^{r_n} > u_n/|X_1| > u_n\})$.

Then for n and m large, since from above the right hand side is close to m^{-1}, we have $\theta_n \geq (2m)^{-1}$, hence $\liminf_{n\to\infty}\theta_n > 0$. By definition of θ_n^{-1} we have $\theta_n^{-1} \geq 1$, hence $\theta_n \leq 1$ □

2.4 *The cluster process*

In general, for a stationary V-valued point process the properties of anticlustering and positivity of the extremal index θ for a sequence $r_n = o(n)$ with $\lim_{n\to\infty} r_n = \infty$, stated in Proposition 2.4, imply the existence of the cluster process (see [2]). For self containment reasons we give in Proposition 2.5 below a proof of this fact, using arguments of [2] ; this gives us also the convergence of θ_n defined in Proposition 2.4 to θ. For later use we include also in the statement the formula of ([2], Theorem 4.3) giving the Laplace functional for the cluster process restricted to B_1'. We recall that the Laplace functional of a random measure $\nu = (\nu_x)_{x\in E}$ on a locally compact separable metric space E endowed with a probability m is given by

$$\psi_\nu(f) = \int exp - \nu_x(f) dm(x)$$

where f is continuous and compactly supported. We recall that weak convergence of a sequence of point processes is equivalent to convergence of their Laplace functionals.
We denote by r_n a sequence as above and we consider the sequence of point processes $C_n = \sum_{i=1}^{r_n} \varepsilon_{u_n^{-1}X_i}$, on $V \setminus \{0\}$ under $\mathbb{P}_\rho$ and conditionally on $M_{r_n} = M_1^{r_n} > u_n$. Using the tail process $(Y_n)_{n\in\mathbb{Z}}$ defined in Proposition 2.3 above, we show that C_n converges weakly to the point process C ; C is a basic quantity for the asymptotics of X_n and is called the cluster process of X_n. As shown in Proposition 2.5 below, the law of C depends only of μ, Λ.
We denote by π_v^ω the renewal point process of the random walk $S_n(\omega)v$ on $V \setminus \{0\}$, given by $\pi_v^\omega = \sum_0^\infty \varepsilon_{S_i(\omega)v}$. For v fixed, the mean measure of the point process π_v^ω is the potential measure $\sum_0^\infty Q^i(v,.)$ of the Markov kernel Q ; if $L(\mu) < 0$ the asymptotics ($|v| \to \infty$) of this Radon measure are described in [13]. The formula below for the Laplace functional of C involves the renewal point process π_v^ω of the linear random walk $S_n(\omega)v$.

Proposition 2.5 *Under $\mathbb{P}_\rho$, the sequence of point processes C_n converges weakly to a point process $C = \sum_1^\infty \varepsilon_{Z_j}$. The law of the point process C is equal to the $\widehat{\mathbb{Q}}_{\Lambda_1}-$ law of the point process $\sum_{j\in\mathbb{Z}} \varepsilon_{S_j x}$ conditional on $\sup_{i\le -1} |S_j x| \le 1$. In particular we have*

$$\widehat{\mathbb{Q}}_{\Lambda_1}\{\lim_{i\to\infty} |Z_i| = 0\} = 1,\ \widehat{\mathbb{Q}}_{\Lambda_1}\{\sup_{i\ge 0} |Z_i| \ge 1\} = 1.$$

Furthermore the sequence θ_n defined in Proposition 2.4 converges to the positive number $\theta = \mathbb{Q}_{\Lambda_1}\{M_1^\infty(Y) \le 1\}$ and we have $\theta^{-1} = \mathbb{E}_{\Lambda_1}(\sum_{j>0} 1_{B_1'(Z_j)}) < \infty$.
If $supp(f) \subset B_1'$ the Laplace functional of C on f is given by

$$1 - \theta^{-1}\mathbb{E}_{\Lambda_1}[exp(f(v) - 1)exp - \pi_v^\omega(f)].$$

Proof : We recall that convergence of Laplace functionals implies weak convergence of the corresponding point processes. Let f be a non negative and continuous function on $V \setminus \{0\}$ which is compactly supported, hence $f(x) = 0$ if $|x| \le \delta$ with $\delta > 0$.

We write for $k \leq \ell$ with $k, \ell \in \mathbb{Z} \cup \{\pm\infty\}$, $M_k^\ell(Y) = \sup_{l \leq j \leq \ell} |Y_j|$ with $Y_j = S_j Y_0$. For k, ℓ, f as above we write $C_k^\ell = exp - \sum_k^\ell f(u_n^{-1} X_j)$, $C_k^\ell(Y) = exp - \sum_k^\ell f(Y_j)$ and we observe that $C_k^\ell \leq 1$. We fix $m > 0$ and we take n so large that the sequence r_n of the above proposition satisfies $r_n > 2m + 1$. When convenient we write $r_n = r$, hence $\mathbb{E}_\rho\{C_1^r; M_1^r > u_n\} = \sum_1^r \mathbb{E}_\rho\{C_1^r; M_1^{j-1} \leq u_n < X_j\}$. We observe that, for $r - m \geq j > m + 1$, we have $C_1^r = C_{j-m}^{j+m}$ except if $sup(M_1^{j-m-1}, M_{j+m+1}^r) > u_n \delta$. We are going to compare $\mathbb{E}_\rho\{C_1^r; M_1^r > u_n\}$ and $(r - 2m)\mathbb{E}_\rho\{C_{-m}^m; M_{-m-1}^{-1} \leq u_n < |X_0|\}$ using those $j's$ which satisfy $m + 1 < j \leq r - m$ and we denote by $\Delta_{n,m}$ their difference.

If we write

$\Delta_{n,m}(j) = \mathbb{E}_\rho\{C_1^r; M_1^{j-1} \leq u_n < |X_j|\} - \mathbb{E}_\rho\{C_{j-m}^{j+m}; M_{j-m-1}^{j-1} \leq u_n < |X_j|\}$

then we have, since $C_k^\ell \leq 1$

$|\Delta_{n,m}| \leq \sum_{m+1}^{r-m} |\Delta_{n,m}(j)| + 2m\mathbb{P}_\rho\{|X_0| > u_n\}$.

Using stationarity of X_n and the above observation we have

$|\Delta_{n,m}(j)| \leq \widehat{\mathbb{P}}\{sup(M_{-r}^{-m-1}, M_{m+1}^r) > u_n\delta; |X_0| > u_n\}$,

hence, using stationarity and the formula $\theta_n = (r_n \mathbb{P}_\rho\{|X_0| > u_n\})^{-1} \mathbb{P}_\rho\{|M_1^r| > u_n\}$

$|\theta_n \mathbb{E}_\rho(C_1^r / M_1^r > u_n) - r^{-1}(r - 2m)\mathbb{E}_\rho(C_{-m}^m; M_{-m-1}^{-1} \leq u_n / |X_0| > u_n)| \leq$

$\widehat{\mathbb{P}}_\rho\{sup(M_{-r}^{-m-1}, M_{m+1}^r) > u_n\delta / |X_0| > u_n\} + 2r^{-1}m$.

Using Proposition 2.3, we see that the discontinuity set of the function $1_{]-\infty,-1]}(M_{-m-1}^{-1}(Y))$ is $\widehat{\mathbb{Q}}_{\Lambda_1}$-negligible hence, using again Proposition 2.3,

$\lim_{n\to\infty} \mathbb{E}_\rho(C_{-m}^m; M_{-m-1}^{-1} \leq u_n / |X_0| > u_n) = \mathbb{E}_{\Lambda_1}(C_{-m}^m(Y); M_{-m-1}^{-1}(Y) \leq 1)$.

Also $\lim_{n\to\infty} r_n^{-1}(r_n - 2m) = 1$ since $\lim_{n\to\infty} r_n = \infty$. We observe that by definition of θ_n and $C_1^r \leq 1$ we have $\theta_n \mathbb{E}_\rho(C_1^r / M_1^r > u_n) \leq \theta_n \leq 1$, hence we can consider convergent subsequences θ_{n_k} with $\lim_{k\to\infty} \theta_{n_k} \in [0, 1]$. The anticlustering property of X_n implies that the limiting values ($n \to \infty$) of $\widehat{\mathbb{P}}_\rho\{sup(M_{-r}^{-m-1}, M_{m+1}^r) > u_n / |X_0| > u_n\}$ are bounded by $\varepsilon_m > 0$ with $\lim_{m\to\infty} \varepsilon_m = 0$. Then the above inequality implies with $r = r_{n_k}$

$\lim_{k\to\infty} |\theta_{n_k} \mathbb{E}_\rho(C_1^r / M_1^r > u_{n_k}) - \mathbb{E}_{\Lambda_1}(C_{-m}^m(Y); M_{-m-1}^{-1}(Y) \leq 1)| \leq \varepsilon_m$.

Since $\lim_{m\to\infty} \mathbb{E}_{\Lambda_1}\{C_{-m}^m(Y); M_{-m-1}^{-1}(Y) \leq 1\} = \mathbb{E}_{\Lambda_1}\{exp - \sum_{-\infty}^{\infty} f(Y_j); M_{-\infty}^{-1}(Y) \leq 1\} := I$, we have $\lim_{k\to\infty} \theta_{n_k} \mathbb{E}_\rho\{C_1^r / M_1^r > u_{n_k}\} = I$, hence the limit of $\theta_n \mathbb{E}_\rho\{C_1^r / M_1^r > u_n\}$ exists and is equal to I. In particular with $f = 0$ and using Proposition 2.3 we get $\lim_{n\to\infty} \theta_n = \widehat{\mathbb{Q}}_{\Lambda_1}\{M_{-\infty}^{-1}(Y) \leq 1\} = \theta$. From above and Proposition 2.4 : $\theta = \lim_{n\to\infty} \theta_n = \liminf_{n\to\infty} \theta_n > 0$

Then we get $\lim_{n\to\infty} \mathbb{E}_\rho\{C_1^{r_n} / M_1^{r_n} > u_n\} = \theta^{-1} I = \widehat{\mathbb{E}}_{\Lambda_1}(exp - \sum_{-\infty}^{\infty} f(Y_j) / M_{-\infty}^{-1}(Y) \leq 1)$ hence the first assertion, using Proposition 2.3. The expression of $(Z_j)_{j\in\mathbb{N}}$ in terms of $(Y_n)_{n\in\mathbb{Z}}$

and the relation $\lim_{|n|\to\infty} Y_n = 0$, $\widehat{\mathbb{Q}}_{\Lambda_1} - a.e$ stated in Proposition 2.3 gives

$$\widehat{\mathbb{Q}}_{\Lambda_1}\{\lim_{i\to\infty} Z_i = 0\} = 1.$$

Since the discontinuity set of $1_{B_1'}$ is Λ_1-negligible, using the weak convergence of C_n to C, the continuous mapping theorem (see [28]) and the convergence of θ_n^{-1} to θ^{-1}, we get the formula $\theta^{-1} = \mathbb{E}_{\Lambda_1}(\sum_1^\infty 1_{B_1'}(Z_j)$. The last formula is proved in ([2], Theorem 4.1). □

3 A spectral gap property and multiple mixing

We denote $X_k^x (k \in \mathbb{N})$ the affine random walk on V governed by λ, starting from $x \in V$ and we write $P\varphi(x) = \int \varphi(hx) d\lambda(h) = \mathbb{E}(\varphi(X_1^x))$.
In this section we use a spectral gap property for a family of operators associated to the process $X_k (1 \le k \le n)$, in order to show the quasi-independence of its successive blocks of length r_n, where r_n is defined in subsection 2.3.

3.1 *Spectral gap property*

It was proved in ([11], Theorem 1) that, given a probability λ on A which satisfies condition (H), the corresponding convolution operator P on V satisfies a "Doeblin-Fortet" inequality (see [18]) for suitable Banach spaces $\mathcal{C}_\chi$ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$ defined below. In particular, it will be essential here to use that the operator P on $\mathcal{H}_{\chi,\varepsilon,\kappa}$ is the direct sum of a 1-dimensional projection π and a contraction U where π and U commute, hence we give also a short proof of this fact below. In order to obtain the relevant multiple mixing property, we show a global Doeblin-Fortet inequality for a family of operators closely related to P. For $\chi, \kappa \ge 0$, we consider the weights ω, η on V defined by $\omega(x) = (1+|x|)^{-\chi}, \eta(x) = (1+|x|)^{-\kappa}$. The space $\mathcal{C}_\chi$ is the space of continuous function φ on V such that $\varphi(x)\omega(x)$ is bounded and we write $|\varphi|_\chi = \sup_{x\in V} |\varphi(x)|\omega(x)$.
For $\varepsilon \in]0, 1]$ we write :

$$[\varphi]_{\varepsilon,\kappa} = \sup_{x\neq y} |x-y|^{-\varepsilon}\eta(x)\eta(y)|\varphi(x) - \varphi(y)|, \quad \|\varphi\| = |\varphi|_\chi + [\varphi]_{\varepsilon,\kappa},$$

and we denote by $\mathcal{H}_{\chi,\varepsilon,\kappa}$ the space of functions φ on V such that $\|\varphi\| < \infty$. We observe that $\mathcal{C}_\chi$ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$ are Banach spaces with respect to the norms $|.|_\chi$ and $\|.\|$ defined above. Also $\mathcal{H}_{\chi,\varepsilon,\kappa} \subset \mathcal{C}_\chi$ with compact injection if $\kappa + \varepsilon < \chi$. We observe that the operator P acts continuously on $\mathcal{C}_\chi$ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$. For a Lipchitz function f on V with non negative real part we define the Fourier-Laplace operator P^f by $P^f\varphi(x) = P(\varphi exp(-f))$. In [11], spectral gap properties for Fourier operators were studied for $f(v) = i < x, v >, x \in V$. Here the calculations are analogous but f will be Lipchitz and bounded. We observe that for functions $f_k (1 \le k \le n)$ and φ as above we have :

$$P^{f_1}P^{f_2}\dots P^{f_n}\varphi(x) = \mathbb{E}\{\varphi(X_n^x) exp - \sum_{k=1}^n f_k(X_k^x)\}$$

Also we note that, for f bounded, with $k(f) = \sup_{x\neq y} |x-y|^{-1}|f(x)-f(y)|$

$$|x-y|^{-\varepsilon}|f(x)-f(y)| \leq \underset{x\neq y}{Inf}(2|f|_\infty |x-y|^{-\varepsilon}, k(f)|x-y|^{1-\varepsilon}) \leq 2|f|_\infty + k(f) := k_1(f),$$

For u, v with non negative real parts we have $|exp(-u) - exp(-v)| \leq |u-v|$. In particular, for f as above, $|exp - f(x) - exp - f(y)| \leq k_1(f)|x-y|^\varepsilon$.
It follows that multiplication by $exp(-f)$ acts continuously on $\mathcal{C}_\chi$, $\mathcal{H}_{\chi,\varepsilon,\kappa}$, hence P^f is a bounded operator on $\mathcal{C}_\chi$ and $\mathcal{H}_{\chi,\varepsilon,\kappa}$. For $m, \gamma > 0$ we denote by $O(m,\gamma)$ the set of operators P^f such that $|f|_\infty \leq m$ and $k(f) \leq \gamma$, hence $k_1(f) \leq 2m + \gamma$. For $p \in \mathbb{N}$ let $O^p(m,\gamma)$ be the set of products of p elements in $O(m,\gamma)$ and $\widehat{O}(m,\gamma) = \underset{p>0}{\cup} O^p(m,\gamma)$. We will endow $\widehat{O}(m,\gamma)$ with the natural norm from $End(\mathcal{H}_{\chi,\varepsilon,\kappa})$. Then we have the

Theorem 3.1 *With the above notations and $0 \leq \chi < 2\kappa < 2\kappa + \varepsilon < \alpha$, there exists $C(m,\gamma) \geq 1$ such that for any $Q \in O(m,\gamma)$ the norm $\|Q\|$ of Q on $\mathcal{H}_{\chi,\varepsilon,\kappa}$ is bounded by $C(m,\gamma)$. Furthermore there exists $r \in [0,1[$, $p \in \mathbb{N}$, $D > 0$ such that for any $Q \in O^p(m,\gamma)$, $\varphi \in \mathcal{H}_{\chi,\varepsilon,\kappa}$:*

$$\|Q\varphi\| \leq r\|\varphi\| + D|\varphi|_\chi.$$

In particular $\widehat{O}(m,\gamma)$ is a bounded subset of $End(\mathcal{H}_{\chi,\varepsilon,\kappa})$ and $C(m,\gamma), r, D$ depend only of m, γ.

The proof depends on the two lemmas given below, and of calculations analogous to those of [11] for Fourier operators.

Lemma 3.2 *$\widehat{O}(m,\gamma)$ is a bounded subset of $End(\mathcal{C}_\chi)$.*

Proof : Since $Re(f) \geq 0$ we have for $Q \in O^\ell(m,\gamma)$ with $\ell \in \mathbb{N}$, $\varphi \in \mathbb{C}_\chi$: $|Q\varphi|_\chi \leq |P^\ell|\varphi||_\chi$, hence it suffices to show that the set $\{P^\ell; \ell \in \mathbb{N}\}$ is bounded in $End(\mathcal{C}_\chi)$. We have for $\varphi \geq 0$, with $A = P^\ell$:

$\omega(x)A\varphi(x) = \omega(x)\mathbb{E}(\varphi(X_\ell^x)) \leq |\varphi|_\chi \mathbb{E}[\omega(x)\omega^{-1}(X_\ell^x)]$.
If $\chi \leq 1$, using independence and the expression of X_ℓ^x we get

$$\omega(x)A\varphi(x) \leq |\varphi|_\chi(1 + \mathbb{E}(|S_\ell|^\chi) + \sum_1^\ell \mathbb{E}(|S_{k+1}^\ell|^\chi \mid \mathbb{E}(|B_k|^\chi)),$$

hence $\sup_{x\in V} \omega(x)A\varphi(x) \leq |\varphi|_\chi(1 + \sup_{\ell\geq 1} \mathbb{E}(|S_\ell|^\chi) + \mathbb{E}(|B_1|^\chi) \sum_1^\infty \mathbb{E}(|S_\ell|^\chi)$.
Since $\chi < \alpha$, we have $\lim_{\ell\to\infty} \mathbb{E}(|S_\ell|^\chi)^{1/\ell} = k(\chi) < 1$, hence $\sup_{x\in V} \omega(x)|A\varphi(x)|$ is bounded by $C_\chi|\varphi|_\chi$ with $C_\chi < \infty$.
If $\chi > 1$, we use Minkowski inequality in $\mathbb{L}^\chi$ and write :

$$\omega(x)|A\varphi(x)| \leq |\varphi|_\chi(1 + \mathbb{E}(|S_\ell|^\chi)^{1/\chi} + \sum_1^\ell \mathbb{E}\{|S_{k+1}^\ell|^\chi)^{1/\chi}\mathbb{E}(|B_k|^\chi)^{1/\chi}\}$$

As above we get
$\sup_{x\in V} \omega(x)|A\varphi(x)| \leq C_\chi|\varphi|_\chi$ with $C_\chi < \infty$. □

Lemma 3.3 *a) For $\beta \in [0, \alpha[$ we have $\sup_n \mathbb{E}(|X_n^0|^\beta) < \infty$.*

b) For $\beta_1, \beta > 0$ and $\beta + \beta_1 < \alpha$, we have $\lim_{n\to\infty} (\mathbb{E}(|S_n|^{\beta_1}|X_n^0|^\beta))^{1/n} < 1$.

c) If $\chi + \varepsilon < \alpha$ the quantity $\widetilde{C}_n = \mathbb{E}\{\sum_1^n |S_i|^\varepsilon (1 + |S_n| + |X_n^0|)^\chi\}$ is bounded. Furthermore, if $2\kappa + \varepsilon < \alpha$, $\widetilde{D}_n = \mathbb{E}\{|S_n|^\varepsilon (1 + |S_n| + |X_n^0|)^{2\kappa}\}$ satisfies $\lim_{n\to\infty} (\widetilde{D}_n)^{1/n} < 1$.

Proof : a) We write $|X_n^0|^\beta = |\sum_1^n S_{k+1}^n B_k|^\beta$. If $\beta \leq 1$ we get :

$\mathbb{E}(|X_n^0|^\beta) \leq \sum_1^n \mathbb{E}(|S_{k+1}^n|^\beta)\mathbb{E}(|B_k|^\beta) = \mathbb{E}(|B_1|^\beta) \sum_0^{n-1} \mathbb{E}(|S_j|^\beta)$

Since $\lim_{j\to\infty} (\mathbb{E}(|S_j|^\beta))^{1/j} < 1$ if $\beta < \alpha$ we get $\sup_{n\geq 0} \mathbb{E}(|X_n^0|^\beta) \leq \mathbb{E}(|B_1|^\beta) \sum_0^\infty \mathbb{E}(|S_j|^\beta) < \infty$.

If $\beta > 1$, we use Minkowski's inequality in $\mathbb{L}^\beta$ as in the proof of Lemma 1.

b) Using Hölder's inequality we have

$\mathbb{E}(|S_n|^{\beta_1}|X_n^0|^\beta) \leq \mathbb{E}(|S_n|^{\beta+\beta_1})^{\beta_1/\beta+\beta_1} \mathbb{E}(|X_n^0|^{\beta+\beta_1})^{\beta/\beta+\beta_1}$,

hence the result follows from a) and the fact that $\lim_{n\to\infty} \mathbb{E}(|S_n|^{\beta+\beta_1})^{1/n} < 1$ since $\beta + \beta_1 < \alpha$.

c) The assertions follows from easy estimations as in b) and the conditions $\chi + \varepsilon < \alpha$, $2\kappa + \varepsilon < \alpha$. □

Proof of Theorem 3.1 We start with a basic observation. For $n > 0$ we have $X_n^x = h_n \cdots h_1 x = S_n x + \sum_1^n S_{k+1}^n B_k$, hence $|X_n^x - X_n^y| = |S_n(x-y)| \leq |S_n||x-y|$. It follows for $k(f) \leq \gamma$, x and y in V :

$$|f(X_n^x) - f(X_n^y)| \leq \gamma |S_n||x-y|.$$

We write $A = T_1 T_2 \cdots T_n$ with $T_i = P^{f_i} \in O(m, \gamma)$ $1 \leq i \leq n$. We have using Markov property,

$A\varphi(x) - A\varphi(y) = I_n(x,y) + J_n(x,y)$ with

$I_n(x,y) = \mathbb{E}\{[exp - \sum_1^n f_i(X_i^x) - exp - \sum_1^n f_i(X_i^y)]\varphi(X_n^x)\}$

$J_n(x,y) = \mathbb{E}\{(exp - \sum_1^n f_i(X_i^y))(\varphi(X_n^x) - \varphi(X_n^y))\}$

Since $Re(f) \geq 0$ we have :

$|exp - \sum_1^n f_i(X_i^x) - exp - \sum_1^n f_i(X_i^y)| \leq \sum_1^n |f_i(X_i^x) - f_i(X_i^y)| \leq (2m + \gamma) \sum_1^n |X_i^x - X_i^y|^\varepsilon$.

The basic observation gives :

$I_n(x,y) \leq (2m+\gamma)|\varphi|_\chi |x-y|^\varepsilon C_n(x)$ with $C_n(x) = \mathbb{E}\{(\sum_1^n |S_i|^\varepsilon (1 + |X_n^x|)^\chi\}$

$J_n(x,y) \leq \mathbb{E}\{|\varphi(X_n^x) - \varphi(X_n^y)|\} \leq [\varphi]_{\varepsilon,\kappa} |x-y|^\varepsilon D_n(x,y)$,

with $D_n(x,y) = \mathbb{E}\{|S_n|^\varepsilon (1 + |X_n^x|)^\kappa (1 + |X_n^y|)^\kappa\}$.

Using symmetry of $|A\varphi(x) - A\varphi(y)|$, $\chi \leq 2\kappa$ and $|X_n^x| \leq |S_n||x| + |X_n^0|$, we get $[A\varphi]_{\varepsilon,\kappa} \leq (2m+\gamma)|\varphi|_\chi \widetilde{C}_n + [\varphi]_{\varepsilon,\kappa} \widetilde{D}_n$ where $\widetilde{C}_n, \widetilde{D}_n$ are as in Lemma 3.3.

Using Lemma 3.3 we can choose $p \in \mathbb{N}$ such that $r = \widetilde{D}_p < 1$, hence for $A \in O^p(m,\gamma)$, $[A\varphi]_{\varepsilon,\kappa} \leq k_1(f)\widetilde{C}_p|\varphi|_\chi + r[\varphi]_{\varepsilon,\kappa}$.
Using Lemma 3.2 we see that there exists $C_\chi \geq 1$ such that $|A\varphi|_\chi \leq C_\chi|\varphi|_\chi$ for $A \in \widehat{O}(m,\gamma)$, $\varphi \in \mathcal{C}_\chi$. Then for $A \in O^p(m,\gamma)$, $\varphi \in \mathcal{H}_{\chi,\varepsilon,\kappa}$ and p as above :

$\|A\varphi\| \leq r\|\varphi\| + (C_\chi + 2m + \gamma)\widetilde{C}_p|\varphi|_\chi = r\|\varphi\| + D|\varphi|_\chi$ with $D > 0$.

For the last assertion, assume $A \in O^n(m,\gamma)$ and write $n = pn_1 + n_0$ with $n_1 \in \mathbb{N}$, $0 \leq n_0 < p$. We have $A = Q_1 \cdots Q_{n_1} R_1 \cdots R_{n_0}$ with $Q_i \in O^p(m,\gamma)$ $(1 \leq i \leq p)$ and $R_j \in O(m,\gamma)$ $(0 \leq j \leq n_0)$, hence $\|R_j\| \leq C_\chi(m,\gamma)$. Finally we get

$\|A\varphi\| \leq (C_\chi(m,\gamma)^{n_0}\left[r^{n_1}\|\varphi\| + D|\varphi|_\chi(r^{n_1-1} + C_\chi\sum_0^{n_1-2} r^k)\right]$,

$\|A\| \leq C_\chi(m,\gamma)^p\left[1 + D(1 + C_\chi(1-r)^{-1})\right] := C(m,\gamma)$, which gives the result. □

For $\chi \in]0,\alpha[$ we consider the function W^χ on V defined by $W^\chi(x) = |x|^\chi$. In Proposition 3.4 below we show that, due to the inequality $0 < k(\chi) < 1$ for $\chi \in]0,\alpha[$, P satisfies a drift condition (see [22]) with respect to W^χ. The same inequality implies also a spectral gap property in the Banach space $\mathcal{H}_{\chi,\varepsilon,\kappa}$ considered in Proposition 3.4 below. For reader's convenience we recall the Doeblin-Fortet spectral gap theorem (see [18]).
Let $(B,|.|)$ be a Banach space, $(L,\|.\|)$ another Banach space with a continuous injection $L \to B$. Let P be a bounded operator on B, which preserves L and satisfies the following conditions

1) The sequence of operator norms $|P^n|$ in is bounded.
2) The injection $L \to B$ is compact.
3) There exists an integer k and $r \in [0,1[$, $D > 0$ such that for any v in L :
$$\|P^k v\| \leq r\|v\| + D|v|$$
4) If $v_n \in L$ is a sequence and $v \in B$ are such that $\|v_n\| \leq 1$ and $\lim_{n\to\infty} |v - v_n| = 0$, there $v \in L$ and $\|v\| \leq 1$

Then in restriction to L, P is the commuting direct sum of a finite dimensional operator π with unimodular spectral values and a bounded operator U with spectral radius $r(U) < 1$.
We observe that, frequently the norm $\|.\|$ on L is given as a sum of a semi-norm $[.]$ and the norm $|.|$; then the inequality in condition 3 can be replaced by
$$[P^k v] \leq r[v] + D|v|$$
such an inequality is called Doeblin-Fortet's inequality.
Our substitute for the strong mixing property (see [29]) uses regularity of functions and is the following.

Proposition 3.4 *For any $\beta \in]0,1]$ there exists $\ell \in \mathbb{N}$ and $b \geq 0$ such that $P^\ell W^\chi \leq \beta W^\chi + b$ for $n \geq \ell$. In particular the sequence of norms $|P^n|_\chi$ is bounded. Furthermore, if $0 < \kappa + \varepsilon < \chi < 2\kappa < 2\kappa + \varepsilon < \alpha$, the injection of $\mathcal{H}_{\chi,\varepsilon,\kappa}$ into $\mathcal{C}_\chi$ is compact and on $\mathcal{H}_{\chi,\epsilon,\kappa}$, the Markov operator P satisfies the direct sum decomposition*
$$P = \rho \otimes 1 + U$$
where $r(U) < 1$ and $U(\rho \otimes 1) = (\rho \otimes 1)U = 0$

If $\alpha = 1$ and $0 < \varepsilon < \chi < 1$, $\kappa = 0$, the same result is valid.

Proof : We verify successively the four above conditions. First we observe that for any $x \in V$,

$$|X_n - X_n^0| \leq |S_n||x|,\ |X_n^x| \leq |X_n^0| + |S_n||x|.$$

If $\chi \leq 1$, it follows

$$\mathbb{E}(|X_n^x|^\chi) \leq \mathbb{E}(|X_n^0|^\chi) + \mathbb{E}(|S_n|^\chi)|x|^\chi.$$

Using the expression of X_n^0 and independence we get $\mathbb{E}(|X_n^0|^\chi) \leq \mathbb{E}(|B_1|^\chi) \sum_0^\infty \mathbb{E}|S_k|^\chi)$. Since $\chi < \alpha$, we have $\mathbb{E}(|X_n^0|^\chi) \leq b < \infty$. On the other hand we have $\lim_{n\to\infty} (\mathbb{E}(|S_n|^\chi))^{1/n} = k(\chi) < 1$, hence for some $\varepsilon > 0$ $k(\chi) + \varepsilon < 1$, and for $n \geq \ell$, $|S_n|^\chi \leq \beta' \leq (k(\chi) + \varepsilon)^n$. It follows, for $n \geq \ell$:

$$P^n W^\chi(x) = \mathbb{E}(|X_n^x|^\chi) \leq \beta' W^\chi(x) + b$$

If $\chi > 1$we use Minkowski inequality, hence :

$$\mathbb{E}(|X_n^x|^\chi) \leq 2^\chi(\mathbb{E}(|X_n^0|^\chi) + \mathbb{E}(|S_n|^\chi|x|^\chi)$$

As above, using $k(\chi) + \varepsilon < 1$ and $n \geq \ell$ we get

$\mathbb{E}(|X_n^x|^\chi) \leq 2^\chi b + 2^\chi(k(\chi) + \varepsilon)^n|x|^\chi,\ P^n W^\chi \leq \beta'' W^\chi + b'$ with $\beta'' < 1$, $b' < \infty$.

We take $\beta = \beta'$ or β'' depending on $\chi \leq 1$ or $\chi > 1$. This allow us now to show that $|P^n|_\chi$ is bounded. We observe that $|\varphi(x)| \leq (1 + W(x))^\chi |\varphi|_\chi$, hence the positivity of P and $P1 = 1$ implies for $n \in \mathbb{N}$,

$$|P^n\varphi|(x) \leq |\varphi|_\chi P^n(2^\chi + 2^\chi W^\chi(x)) = |\varphi|_\chi(2^\chi + 2^\chi P^n W^\chi(x)).$$

From above we get

$$|P^n\varphi|(x) \leq |\varphi|_\chi[2^\chi + 2^\chi(b + \beta W^\chi(x))].$$

Then the definition of $|P^n|_\chi$ gives $|P^n|_\chi \leq 2^\chi(1 + b + \beta)$, hence the boundedness of $|P^n|_\chi$.

In order to show that if $\kappa + \varepsilon < \chi$, the injection of $B_0 = \mathcal{H}_{\kappa,\varepsilon,\chi}$ in $B = \mathcal{C}_\chi$ is compact, we use Ascoli argument and consider a large ball B_t with $t > 0$. We consider $\varphi_n \in \mathcal{H}_{\kappa,\varepsilon,\chi}$ with $\|\varphi_n|| < 1$. The definition on $\|\varphi_n\|$ implies for any $x, y \in B_t$

$$|\varphi_n(x)| \leq (1 + t)^\chi, |\varphi_n(x) - \varphi_n(y)| \leq (1 + t)^{2\kappa}|x - y|^\varepsilon$$

Hence, the restrictions of φ_n to B_t are equicontinuous and we can find a convergent subsequence φ_{n_k}. Using the diagonal procedure and a sequence t_i with $\lim_{i\to\infty} |t_i| = \infty$, we get a convergent subsequence $\varphi_{n_j} \in \mathcal{H}_{\kappa,\varepsilon,\chi}$ with limit a continuous function φ on V. From above we have $|\varphi_{n_j}(x) - \varphi_{n_j}(0)| \leq (1 + |x|)^\kappa|x|^\varepsilon$. hence for some $A, B > 0$, since $\kappa + \varepsilon < \chi$

$$|\varphi_{n_j}(x) - \varphi_{n_j}(0)| \leq (1 + |x|)^{\kappa+\varepsilon}, |\varphi(x)| \leq A + B(1 + |x|)^\chi.$$

It follows that $\varphi \in \mathcal{C}_\chi$. The above inequalities for φ_{n_j} imply

$$|(\varphi_{n_j}(x) - \varphi_{n_j}(0)| - (\varphi(x) - \varphi(0))| \leq 2(1 + |x||^{\kappa+\varepsilon}.$$

Then the convergence of φ_{n_j} to φ, implies with $\varepsilon_{n_j} = |\varphi_{n_j}(0) - \varphi(0)|$,

$|\varphi_{n_j}(x) - \varphi(x)| \leq \varepsilon_{n_j} + 2(1+|x|)^{\kappa+\varepsilon}$, $(1+|x|)^{-\chi}|\varphi_{n_j}(x) - \varphi(x)| \leq \varepsilon_{n_j} + 2(1+|x|)^{\kappa+\varepsilon-\chi}$ with $\lim_{j\to\infty} \varepsilon_{n_j} = 0$. Also for t sufficiently large, and $|x| \geq t$, since $\kappa + \varepsilon < \chi$ we have $(1+|x|)^{\kappa+\varepsilon-\chi} \leq \varepsilon_{n_j}$. Furthermore, the uniform convergence of φ_{n_j} to, φ on B_t implies $\lim_{j\to\infty} (sup\{|\varphi_{n_j}(x) - \varphi(x)| \ ; \ |x| \leq t\}) = 0$. The convergence of $|\varphi_{n_j} - \varphi|_\chi$ to zero follows.
The convergence of $\varphi_{n_j}(x)$ to $\varphi(x)$ for any $x \in V$ and the defintion of $\|\varphi_{n_j}\|$, implies $\|\varphi\| \leq \lim_{j\to\infty} \|\varphi_{n_j}\| \leq 1$, hece $\varphi \in L$, hence condition 4 is satisfied.

With $f = 0$ in Theorem 3.1 we have $P^f = P$. In particular there exists $k > 0$ such that $\|P^k\varphi\| \leq r\|\varphi\| + D|\varphi|_\chi$ if $\varphi \in \mathcal{H}_{\chi,\varepsilon,\kappa}$. Hence from [18], we know that the above conditions imply that P is the direct sum of a finite rank operator and a bounded operator U which satisfies $r(U) < 1$. Now it suffices to show that the equation $P\varphi = z\varphi$ with $|z| = 1$, $\varphi \in \mathcal{H}_{\chi,\varepsilon,\kappa}$ implies that φ is constant and $z = 1$. From the convergence in law of X_n^x to ρ we know that for any $x \in V$, the sequence of measures $P^n(x,.)$ converges weakly to ρ. Also we have $|\varphi| \in \mathcal{H}_{\chi,\varepsilon,\kappa}$ and the sequence $n^{-1}\sum_1^n P^k|\varphi|$ converges to $\rho(|\varphi|)$. Since $|\varphi(x)| = |z^n\varphi(x)| \leq P^n(x,|\varphi|)$ we get $|\varphi(x)| \leq \rho(|\varphi|)$, hence $|\varphi|$ is bounded. Since $z^n\varphi(x) = \mathbb{E}(\varphi(X_n^x))$ and X_n^x converges in law to ρ, we get $\lim_{n\to\infty} z^n\varphi(x) = \rho(\varphi)$. This implies $z = 1$ and $\varphi(x) = \rho(\varphi)$ for any $x \in V$.
For the last assertion, in view of the above, we have only to verify the contraction condition. We write $[\varphi]_\varepsilon = \sup_{x\neq y} |x-y|^{-\varepsilon}|\varphi(x) - \varphi(y)|$. Then we have

$$\mathbb{E}(|\varphi(X_n^x) - \varphi(X_n^y)|) \leq [\varphi]_\varepsilon |X_n^x - X_n^y|^\varepsilon \leq [\varphi]_\varepsilon |x-y|^\varepsilon \mathbb{E}(|S_n|^\varepsilon).$$

Since $\varepsilon < \alpha$, we have $0 < k(\varepsilon) < r < 1$ for some r, hence $[P^n\varphi]_\varepsilon \leq r[\varphi]_\varepsilon$ for n large. □

3.2 *A mixing property with speed for the system* $(V^{\mathbb{Z}_+}, \tau, \mathbb{P}_\rho)$.

In general, if the law of B_n has no density with respect to Lebesgue measure, the operator P on $\mathbb{L}^2(\rho)$ don't satisfy spectral gap properties hence the stationary process is not strongly mixing in the sense of [29], but Proposition 3.4 above shows that it is still ergodic. Then, using Theorem 3.1 and Proposition 3.4, it is shown below that the system $(V^{\mathbb{Z}_+}, \tau, \mathbb{P}_\rho)$ satisfies a multiple mixing condition with respect to Lipchitz functions. For a very general framework covering mixing conditions with respect to regular functions, see [9]. For a study of extreme value properties for random walks on some classes of homogeneous spaces, using $\mathbb{L}^2$-spectral gap methods, we refer to [20]. Since, using Proposition 2.4, the stationary process $(X_n)_{n\in\mathbb{N}}$ satisfies also anticlustering, we see below that extreme value theory can be developed for $(X_n)_{n\in\mathbb{N}}$ following the arguments of ([2] , [3]) which were developed under conditions $\mathcal{A}(u_n)$, $\mathcal{A}'(u_n)$, using continuous functions.
However it turns out that the mixing property $\mathcal{A}(u_n)$ of [2] for continuous functions can be proved, as a consequence of the corresponding convergences involving Lipchitz functions

and point process theory.
Let f be a bounded continuous function with non negative real part on $[0,1]\times(V\setminus\{0\})$. Let r_n be an integer valued sequence with $\lim_{n\to\infty} r_n=\infty$, $r_n=o(n)$ and $k_n=[r_n^{-1}n]$. For $0\le i\le n,\ 0\le j\le n$, $x\in V\setminus\{0\}$, $\omega\in V^{\mathbb{Z}_+}$ we write :

$\overline{f}_n^j(x)=f(n^{-1}j,u_n^{-1}x), f_{i,n}(\omega)=\overline{f}_n^i(X_i),\ f_{i,n}^j(\omega)=\overline{f}_n^j(X_i)$.

In view of heavy notations, in some formulae we will write $r_n=r$, $k_n=k$, $\ell_n=\ell$. For f Lipchitz we denote by $k(f)$ the Lipchitz constant of f, and assume $supp(f)\subset[0,1]\times B'_\delta$ with $\delta>0$. We consider below the quantity $\mathbb{E}_\rho(exp-\sum_{i=1}^n f_{i,n})$ which is the Laplace functional of the point process $\sum_{i=1}^n \varepsilon_{u_n^{-1}X_i}$. For its analysis we use the classical Bernstein method of gaps, *i.e* we decompose the interval $[1,n]$ into large subintervals separated by smaller but still large ones.

Proposition 3.5 *Let f be a compactly supported Lipchitz function on $[0,1]\times(V\setminus\{0\})$ with $Ref\ge 0$. Assume that the sequence $r_n\in\mathbb{N}$ satisfies $r_n=o(n)$, $\lim_{n\to\infty}(logn)^{-1}r_n=\infty$ and write $|f|_\infty=m, k(f)=\gamma$, $supp(f)\subset[0,1]\times B'_\delta, \delta>0$. Then, with the above notations there exists $C(\delta,m,\gamma)<\infty$ such that,*

$$I_n(f):=|\mathbb{E}_\rho\{exp(-\sum_1^n f_{i,n})\}-\prod_{j=1}^{k_n}\mathbb{E}_\rho\{exp(-\sum_{(j-1)r_n+1}^{jr_n} f_{i,n}^{jr_n})\}|\le C(\delta,m,\gamma)sup(r_n^{-1},n^{-1}r_n).$$

In particular with $r_n=[n^{1/2}]$ we get $sup(n^{-1}r_n,r_n^{-1})\le 2n^{-1/2}$

Proof : We write $[0,n]=[0,k_nr_n]\cup]k_nr_n,n]$, we decompose the interval $[0,k_nr_n]$ into k_n intervals $J_j=[jr_n,(j+1)r_n[$ and we distinguish in J_j the subinterval of length ℓ_n $J'_j=[(j+1)r_n-\ell_n,(j+1)r_n[$; the large integer ℓ_n will be specified below.

We write for f fixed, $I(n)=|\mathbb{E}_\rho(exp-\sum_1^n f_{i,n})-\prod_{j=1}^k\mathbb{E}_\rho(exp-\sum_{i=(j-1)r+1}^{jr} f_{i,n}^{jr}|$.

Then the triangular inequality gives $I(n)\le I_1(n)+I_2(n)+I_3(n)+I_4(n)$ with

$$I_1(n)=|\mathbb{E}_\rho(exp-\sum_1^n f_{i,n})-\mathbb{E}_\rho(exp-\sum_1^{kr} f_{i,n})|$$

$$I_2(n)=|\mathbb{E}_\rho(exp-\sum_1^{kr} f_{i,n})-\mathbb{E}_\rho(exp-\sum_{j=1}^k\sum_{i=(j-1)r+1}^{jr-\ell} f_{i,n})|$$

$$I_3(n)=|\mathbb{E}_\rho(exp-\sum_{j=1}^k\sum_{i=(j-1)r+1}^{jr-\ell} f_{i,n})-\prod_{j=1}^k\mathbb{E}_\rho(exp-\sum_{i=1}^{r-\ell} f_{i,n}^{jr})|$$

$$I_4(n)=|\prod_{j=1}^k\mathbb{E}_\rho(exp-\sum_{i=1}^{r-\ell} f_{i,n}^{jr})-\prod_{j=1}^k\mathbb{E}_\rho(exp-\sum_{i=1}^{r} f_{i,n}^{jr})|$$

where stationarity of $\mathbb{P}_\rho$ has been used in the expressions of $I_3(n), I_4(n)$. The quantities $I_1,\ I_2,\ I_4$ are boundary terms ; their estimation below is based only on the fact that r_n (resp ℓ_n) is small with respect to n (resp r_n), the form of u_n, and f has non negative real part. On the other hand estimation of I_3 depends on Theorem 3.1 and Proposition 3.4.

Using the inequality $|exp(-x) - exp(-y)| \leq |x-y|$ for x, y with non negative real parts we get $I_1(n) \leq \sum_{kr+1}^{n} \mathbb{E}_\rho(f_{i,n})$. Let $\delta > 0$ be as above such that $f(t,x) = 0$ for $t \in [0,1]$, $|x| < \delta$, and observe that $n - kr < r$. Then the above bound for $I_1(n)$ gives :

$I_1(n) \leq r_n |f|_\infty \mathbb{P}_\rho\{u_n^{-1}|X_1| \geq \delta\}$.

Since $\lim_{n\to\infty} n^{-1} r_n = 0$, the definition of u_n and Theorem 2.1 gives $\lim_{n\to\infty} I_1(n) = 0$. Also $I_1(n)$ is bounded by $n^{-1} r_n$, up to a coefficient depending only on m, δ. For $I_2(n)$, a similar argument involving each interval J_j and the subinterval J_j' gives :

$I_2(n) \leq k_n \ell_n |f|_\infty \mathbb{P}_\rho\{u_n^{-1}|X_1| \geq \delta\}$.

Using $k_n r_n \leq n$ we get $\lim_{n\to\infty} n^{-1} k_n \ell_n \leq \lim_{n\to\infty} r_n^{-1} \ell_n$ i.e $\lim_{n\to\infty} I_2(n) = 0$ if $\lim_{n\to\infty} r_n^{-1}\ell_n = 0$.

Also we can bound $I_2(n)$ by $r_n^{-1}\ell_n$, up to a coefficient depending only on m, δ.

For $I_4(n)$, we use the inequality $|\prod_1^n z_j - \prod_1^n w_j| \leq \sum_1^n |z_j - w_j|$ if $|z_j|$ and $|w_j|$ are less than 1. Hence :

$I_4(n) \leq \sum_{j=1}^{k} |\mathbb{E}_\rho(exp - \sum_1^{r-\ell} f_{i,n}^{jr}) - \mathbb{E}_\rho(exp - \sum_1^{r} f_{i,n}^{jr})| \leq |f|_\infty k_n \ell_n \mathbb{P}_\rho\{|X_1| > \delta u_n\}$

As above we get $\lim_{n\to\infty} I_4(n) = 0$ if $\lim_{n\to\infty} r_n^{-1}\ell_n = 0$, and a bound for $I_4(n)$ of the same form as for $I_2(n)$.

The estimation of $I_3(n)$ is more delicate and depends on Lemma 3.6 below. We begin with the inequality : $I_3(n) \leq D(n) + I_5(n) + I_3(n - r_n)$ where

$D(n) = |\mathbb{E}_\rho(exp - \sum_{j=1}^{k} \sum_{(j-1)r+1}^{jr-\ell} f_{i,n}) - \mathbb{E}_\rho(exp - \sum_{i=1}^{r-\ell} f_{i,n}) \mathbb{E}_\rho(exp - \sum_{j=2}^{k} \sum_{(j-1)r+1}^{jr-\ell} f_{i,n})|,$

$I_5(n) = |(\mathbb{E}_\rho(exp - \sum_1^{r-\ell} f_{i,n}) \mathbb{E}_\rho(exp - \sum_{j=2}^{k} \sum_{(j-1)r+1}^{jr-\ell} f_{i,n}) - \mathbb{E}_\rho(exp - \sum_1^{r-\ell} f_{i,n}^r) \mathbb{E}_\rho(exp - \sum_{j=2}^{k} \sum_{(j-1)r+1}^{jr-\ell} f_{i,n})|,$

$I_3(n-r) = |\mathbb{E}_\rho(exp - \sum_{j=2}^{k} \sum_{(j-r)r+1}^{jr-\ell} f_{i,n}) - \prod_{j=2}^{k} \mathbb{E}_\rho(exp - \sum_1^{r-\ell} f_{i,n}^{jr})|.$

Using as above the inequality $|exp(-x) - exp(-y)| \leq |x-y|$, and $Re(f) \geq 0$ we get :

$I_5(n) \leq |\mathbb{E}_\rho(\sum_1^{r-\ell} f_{i,n}| - \mathbb{E}_\rho(\sum_1^{r-\ell} f_{i,n}^r)|.$

Since f is Lipchitz we have, for t', t'' in $[0,1]$, $x \in V \setminus \{0\}$: $|f(t',x) - f(t'',x)| \leq k(f)|t' - t''|$. Since $|n^{-1}i - n^{-1}r_n| \leq n^{-1} r_n$ we have

$I_5(n) \leq (r_n - \ell_n) n^{-1} r_n k(f) \mathbb{P}_\rho\{u_n^{-1}|X_1| \geq \delta\} \leq r_n^2 n^{-1} k(f) \mathbb{P}_\rho\{|X_1| \geq \delta u_n\}$.

Using Theorem 2.1 we get $I_5(n) \leq C n^{-2} r_n^2$ with a constant C depending on $k(f)$ and δ.

In order to estimate $D(n)$ we consider the family of operators $P_{i,n}$ on the space $\mathcal{H}_{\chi,\varepsilon,\kappa}$ with $\chi, \varepsilon, \kappa$ as in Proposition 3.4, defined by $P_{i,n}\varphi(x) = \mathbb{E}((exp - f_{i,n}(\omega))\varphi(X_i^x))$ and the function $\psi_n(\omega)$ defined by $\psi_n(\omega) = \mathbb{E}\left(exp - \sum_{j=2}^{k} \sum_{i=(j-1)r+1}^{jr-\ell} f_{i,n}^{i+r} / X_r^x\right)$. Since, $u_n \geq 1$, for n large with $m = |f|_\infty$, $\gamma = k(f)$, the functions $f_{i,n}$ satisfy $|f_{i,n}|_\infty \leq m$, $k(f_{i,n}) \leq \gamma$, hence

the operators $P_{i,n}$ belong to $O(m,\gamma) \subset End\mathcal{H}_{\chi,\varepsilon,\kappa}$. With the above notations, the products of operators $P_{i,n}$ belong to $\widehat{O}(m,\gamma)$. Also, using Proposition 3.4 we know that on $\mathcal{H}_{\chi,\varepsilon,\kappa}$ we can write $P = \rho \otimes 1 + U$ where U has spectral radius $r(U)$ less then 1 and U commutes with the projection $\rho \otimes 1$. We note also that for f as above and $\psi \in \mathcal{H}_{\chi,\varepsilon,\kappa}$ we have :

$$|\rho(P^f\psi)| \leq \rho(P|\psi|) = \rho(|\psi|) \leq \|\psi\|$$

Then Lemma 3.6 below implies the convergence of $D(n)$ to zero with speed.

Now, in order to prove the proposition, we are left to show $\lim_{n\to\infty} I_3(n) = 0$. We iterate k_n times the inequality : $I_3(n) \leq D(n) + I_5(n) + I_3(n - r_n)$. We get, using Lemma 3.6 :
$I_3(n) \leq I_3(n - r_n) + C'(f)(n^{-2}r_n^2 + r_1^{\ell_n}(U)) \leq C'(f)(k_n r_1^{\ell_n}(U) + n^{-1}r_n)$,
with $C'(f) \geq 1$, depending on m, γ. Since $r_n = o(n)$, it remains to choose ℓ_n such that $\ell_n = o(r_n)$ with $\lim_{n\to\infty} k_n r_1^{\ell_n}(U) = 0$. These conditions can be written as

$$\lim_{n\to\infty} r_n^{-1}\ell_n = 0, \ \lim_{n\to\infty} r_n^{-1} n r_1^{\ell_n}(U) = 0.$$

The choice of ℓ_n with the above properties is possible since :

$$r_1(U) < 1, \ \lim_{n\to\infty} n^{-1}r_n = 0 \text{ and } \lim_{n\to\infty} (logn)^{-1}r_n = \infty.$$

One can take $\ell_n < r_n$ with $(logn)^{-1}\ell_n = \infty$. The above estimations of I_1, I_2, I_3, I_4, I_5 give bounds by $sup(n^{-1}r_n, r_n^{-1})$, up to a coefficient depending on δ, m, γ only. □

Lemma 3.6 *There exists positive numbers $C_1(U)$, $r_1(U) \in]r(U), 1[$ and $C(f)$ depending only of m, γ such that, for $n \in \mathbb{N}$ and $\ell_n < r_n$, $D(n)$, as above :*

$$D(n) = |\rho(P_{1,n} \cdots P_{r_n-\ell_n,n} U^{\ell_n}\psi_n)| \leq C_1(U)C(f)(r_1(U))^{\ell_n}.$$

Proof : We observe that Markov's property implies $\mathbb{E}(e^{-f(X_1^x)}g(\omega)) = P^f(\mathbb{E}(g(\omega)))$ where f is as above, $g(\omega)$ is a function depending only of ω throught the random variables $X_k^x (k \geq 2)$ and $\mathbb{E}(g(\omega))$ is a function of x. We apply this property to $\mathcal{H}_{\chi,\varepsilon,\kappa}$ with $f = f_{i,n}$ $(1 \leq i \leq r - \ell)$ or $f = 0$, $g = \psi_n$ as above, hence writing $P^\ell = \rho \otimes 1 + U^\ell$ and

$$D(n) = |\rho(P_{1,n} \cdots P_{r-\ell,n} P^\ell \psi_n| - \rho(P_{1,n} \cdots P_{r-\ell,n} 1)\rho(\psi_n)| = |\rho(P_{1,n} \cdots P_{r-\ell,n} U^\ell \psi_n)|,$$

Proposition 3.4 implies the existence of $C_1(U) < \infty, r_1(U) \in]r(U), 1[$ with

$$\|U^\ell \psi_n\| \leq C_1(U) r_1^\ell(U) \|\psi_n\|.$$

On the other hand, since ψ_n is of the form $\psi_n = A1$ with $A \in \widehat{O}(m,\gamma)$, we have, using Theorem 3.1, $\|\psi_n\| \leq C(f)$ with $C(f)$ depending on m, γ. It follows $D(n) \leq C_1(U)C(f)(r_1(U))^{\ell_n}$. □

4 Asymptotics of exceedances processes

4.1 *Statements of results*

Let E be a complete separable metric space which is locally compact, $M_+(E)$ the space of positive Radon measures on E, $M_p(E)$ its subspace of point measures, $\mathcal{C}_+^c(E)$ (resp $\mathcal{L}_+^c(E)$) the space of non negative and compactly supported continuous (resp Lipchitz) functions. Then it is well known that the vague topology on $M_+(E)$ is given by a metric

and then $M_+(E)$ becomes a complete separable metric space. Furthermore this metric is constructed (see [28] Lemma 3.11, Proposition 3.17) using a countable family $(h_i)_{i\in I}$ of functions in $\mathcal{L}^c_+(E)$ and $M_p(E)$ is a closed subset of $M_+(E)$. It follows that, in various situations with respect to weak convergence of random measures, $\mathcal{C}^c_+(E)$ can be replaced by $\mathcal{L}^c_+(E)$.

Below, assuming condition (H), we describe the asymptotics of the space-time exceedances process $N_n = \sum_1^n \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$ under the probability $\mathbb{P}_\rho$ and we state a few corollaries. The results are formally analogous to results for stationary processes proved in ([2], [3]) under general conditions. Here however, corresponding conditions have been proved in sections 2, 3 for the affine random walk X_n ; hence the results described below are new for affine random walks.

It is convenient to express the Laplace formula below in terms of the renewal point process $\pi_v^\omega = \sum_0^\infty \varepsilon_{S_n(\omega)v}$ of the linear random walk $S_n(\omega)v$.

We denote by $\sum_{i\geq 0} \varepsilon_{T_i^\delta}$ the homogeneous point Poisson process on $[0,1]$ with intensity $p(\delta) = \theta\delta^{-\alpha}$ and by $\sum_{j>0} \varepsilon_{Z_{ij}}$ $(i \geq 0)$ an i.i.d collection of copies of the cluster process $C = \sum_{j>0} \varepsilon_{Z_j}$ described in Proposition 2.5, independent of $\sum_{i\geq 0} \varepsilon_{T_i^\delta}$. Since we have $|X_n^x - X_n^y| \leq |S_n||x-y|$ and $\lim_{n\to\infty} |S_n| = 0$, $\mathbb{P} - a.e$ it is possible to replace $\mathbb{P}_\rho$ by $\mathbb{P}$ and X_n by X_n^x with x fixed, in the statements. We give the corresponding proof for the logarithm law only.

Theorem 4.1 *The sequence of normalized space-time point processes* $N_n = \sum_{i=1}^n \varepsilon_{(n^{-1}i, u_n^{-1}X_i)}$ *on the space* $[0,1] \times (V \setminus \{0\})$ *converges weakly to a point process* N. *For any* $\delta > 0$, *the law of the restriction of* N *to* $[0,1] \times B'_\delta$ *is the same as the law of the point process on* $[0,1] \times B'_\delta$ *given by :*

$$\sum_{i\geq 0} \sum_{j>0} \varepsilon_{(T_i^\delta, \delta Z_{ij})} \ 1_{\{|Z_{ij}|>1\}}.$$

If η *denotes the law of* N *and* $f \in \mathcal{C}^c_+([0,1] \times B'_\delta)$, *then* $log\psi_\eta(f)$ *is equal to*

$$\theta\delta^{-\alpha} \int_0^1 \mathbb{E}_{\Lambda_1}(1 - exp - \sum_{j>0} f(t, \delta Z_j))dt = c^{-1} \int_0^1 \mathbb{E}_\Lambda[(exp\ f_t(v) - 1)exp - \pi_v^\omega(f_t)]dt$$

where $f_t(x) = f(t,x)$

Assuming the mixing and anticlustering conditions for continuous functions, this statement was proved in [3]. Here we will use Propositions 2.5, 3.4 and point process theory.

Now as a consequence of Theorem 4.1, the mixing property stated in Proposition 3.5 for Lipchitz functions can be extended to compactly supported continuous functions. Then, in particular, the mixing condition $\mathcal{A}(u_n)$ of [2] is valid here and the basic conditions of extreme value theory (see [8]) are satisfied in our context.

Corollary 4.2 *With the notation of Proposition 3.5, assume f is a continuous compactly supported function on $[0,1] \times (V \setminus \{0\})$. Then we have the convergence $\lim_{n\to\infty} I_n(f) = 0$.*

Since the space exceedances process $N_n^s = \sum_{i=1}^{n} \varepsilon_{u_n^{-1} X_i}$ is the projection of N_n on $V \setminus \{0\}$ we have the

Corollary 4.3 *The normalized space exceedance process N_n^s converges weakly to a point process N^s. The law of the restriction of N^s to B'_δ is the same as the law of the point process*

$$\overline{Q}^\delta = \sum_{i=0}^{T^\delta} \sum_{j=1}^{\infty} \varepsilon_{\delta Z_{ij}} 1_{\{|Zij|>1\}}$$

where T^δ is a Poisson random variable with mean $p(\delta) = \theta\delta^{-\alpha}$.
The Laplace functional of N^s is given by

$$exp\ c^{-1}\mathbb{E}_\Lambda[(exp\ f(v) - 1)exp - \pi_v^\omega(f)].$$

Assuming the mixing and anticlustering conditions for continuous functions, this statement was proved in [2], using the formula for Laplace functionals in Proposition 2.5.

We consider the $\mathbb{N}$-valued random variable $\zeta = \pi_v^\omega(B'_1)$ and we write $\theta_k = \mathbb{Q}_{\Lambda_1}\{\zeta = k\}$ for $k \geq 1$; in particular we have $\theta_1 = \theta$, $\theta_k \geq \theta_{k+1}$.

Corollary 4.4 *The sequence of normalized time exceedances process $N_n^t = \sum_{i=1}^{n} \varepsilon_{n^{-1}i} 1_{\{|X_i|>u_n\}}$ converges weakly $(n \to \infty)$ to the homogeneous compound Poisson process N^t on $[0,1]$ with intensity θ, and cluster probabilities $\nu_k (k \geq 1)$ where $\nu_k = \theta^{-1}(\theta_k - \theta_{k+1})$.*

Under special hypotheses, including density of the law of B_n with respect to Lebesgue measure, this statement was proved in [21].
Fréchet's law for $M_n^x = sup\{|X_k^x|; 1 \leq k \leq n\}$ is a simple consequence of Corollary 4.4 as follows.

Corollary 4.5 *For any $x \in V$ and $t > 0$ we have the convergence in law of $u_n^{-1} M_n^x$ to Fréchet's law Φ_α^θ,*

$$\lim_{n\to\infty} \mathbb{P}\{u_n^{-1} M_n^x < t\} = exp - \theta t^{-\alpha} = \Phi_\alpha^\theta([0,t])$$

with $\theta = \mathbb{Q}_{\Lambda_1}\{\sup_{n\geq 1} |S_n(\omega)v| \leq 1\}$. Furthermore the normalized law of the entrance time τ_t^x of $|X_n^x|$ in $[t, \infty[$ converges to the exponential law with parameter $c\theta$, i.e

$$\lim_{t\to\infty} \mathbb{P}\{t^{-\alpha}\tau_t^x > k\} = exp - c\theta k.$$

It was observed in [26] that Sullivan's logarithm law for excursions of geodesics around the cusps of hyperbolic manifolds (see [30]), in the case of the modular surface, is a consequence of Fréchet's law for the continuous fraction expansion of a real number uniformly distributed in $[0,1](see[25])$. For more detailed extreme value properties in the context of pointwise convergence, we refer to ([10], p 168-179). Here, in this vein, we have the following logarithm law.

Corollary 4.6 *For any $x \in V$, we have the $\mathbb{P}-a.e$ convergence*

$$\limsup_{n\to\infty} \frac{log|X_n^x|}{logn} = \lim_{n\to\infty} \frac{logM_n^x}{logn} = \frac{1}{\alpha}.$$

We observe that a logarithm law and a modified Fréchet law have been obtained in [20] for random walks on some homogeneous spaces of arithmetic character, using $\mathbb{L}^2$-spectral gap methods.

4.2 *Proofs of point process convergences*

The proof of Theorem 4.1 will follow of three lemmas.
We denote by $(X_{k,j})_{k\in\mathbb{N}}$ an i.i.d sequence of copies of the process $(X_j)_{j\in\mathbb{N}}$ and we write

$$\widetilde{N}_{k,n} = \sum_{j=1}^{r_n} \delta_{(n^{-1}kr_n, u_n^{-1}X_{k,j})}, \widetilde{N}_n = \sum_{k=1}^{k_n} \widetilde{N}_{k,n},$$

where $r_n,\ k_n$ are as in section 2. For $k_n > 0$ we denote by $\mathbb{E}_\rho^{(k_n)}$ the expectation corresponding to the product probability of k_n copies of $\mathbb{P}_\rho$.
If f is a non negative and compactly supported Lipchitz function on $[0,1]\times V\setminus\{0\}$, we have, using independence :

$$\mathbb{E}_\rho^{(k_n)}(exp - \widetilde{N}_n(f)) = \prod_{k=1}^{k_n} \mathbb{E}_\rho(exp - \sum_{j=1}^{r_n} f(n^{-1}k, u_n^{-1}X_{k,j})).$$

This relation and the multiple mixing property in Proposition 3.5 show that, on functions f as above, the asymptotic behaviour of the Laplace functionals of N_n under $\mathbb{E}_\rho$, and $\widetilde{N}_n$ under $\mathbb{E}_\rho^{(k_n)}$, are the same. We begin by considering the convergence of $\mathbb{E}_\rho^{(k_n)}(exp-\widetilde{N}_n(f))$.
Lemma 4.7 below is a general statement giving the weak convergence of a sequence of random measures, using only the convergence of the values of the Laplace functionals on Lipchitz functions.
Lemmas 4.8, 4.9 are reformulations of part of the proof of Theorem 2.3 in [3], which was considered in a general setting.

Lemma 4.7 *Let E be a separable metric space endowed with a probability m and assume E to be locally compact. Let ν_n be a sequence of random measures on E and, for f non negative Lipchitz and compactly supported, assume that the sequence of Laplace functionals $\psi_{\nu_n}(f)$ converges to $\psi(f)$ and $\psi(sf)$ is continuous at $s=0$, then the sequence ν_n converges*

weakly. A random measure $\nu = (\nu_x)_{x\in E}$ *on* (E,m) *is well defined by the values of its Laplace functionals* $\psi_\nu(f) = \int exp - \nu_x(f)dm(x)$ *with* f *as above.*

Proof : We begin by the last assertion and we use the family of Lipchitz functions $(h_i)_{i\in I}$ considered in the above subsection. If the random measures ν, ν' satisfy $\psi_\nu(f) = \psi_{\nu'}(f)$ for any $f \in \mathcal{L}^c_+(E)$ and $\lambda_1, \lambda_2, \cdots, \lambda_p$ are non negative numbers then we have $\psi_\nu(\sum_{i=1}^{i=p} \lambda_i h_i) = \psi_{\nu'}(\sum_{i=1}^{i=p} \lambda_i h_i)$. It follows that the random vectors $(\nu(h_1), \cdots, \nu(h_p))$ and $(\nu'(h_1), \cdots, \nu'(h_p))$ have the same Laplace transforms, hence the same laws. Furthermore, for rational numbers $r_j < r'_j$ the finite intersections of sets of the form $\{\mu \in M_+(E), \mu(h_i) \in]r_j, r'_j[)\}$ form a countable basic B of open subsets in $M_+(E)$ stable under finite intersection, hence a π-system (see [28]). Then from above, ν, ν' are equal on B ; since the σ-field generated by B coincide with the Borel field, one has $\nu = \nu'$.
We observe that, if a sequence of random measures ν_n is such that for any $f \in \mathcal{L}_+(E)$ the sequence of real random variables $\nu_n(f)$ is tight, then the sequence ν_n itself is tight. This follows for a corresponding result in [28] for $f \in \mathcal{C}^c_+(E)$ since any such f is dominated by an element of $\mathcal{L}^c_+(E)$.
Assuming the convergence of $\psi_{\nu_n}(f)$ to $\psi_\nu(f)$ for any $f \in \mathcal{L}^c_+(E)$ and the continuity at $s = 0$ of $\psi_\nu(sf)$, we get that $\psi_\nu(sf)$ is the Laplace transform of the real random variable $\nu(f)$, and the convergence of the sequence $\nu_n(f)$ to $\nu(f)$ for any $f \in \mathcal{L}^c_+(E)$. From above and the continuity hypothesis of $\psi(sf)$ at $s = 0$, we get that the sequence ν_n is tight. If ν_{n_i} is a subsequence converging weakly to the random measure ν we have $\lim_{n\to\infty} \psi_{\nu_n}(f) = \lim_{j\to\infty} \psi_{\nu_{n_j}}(f)$ for any $f \in \mathcal{L}^c_+(E)$. Since such a limit is independent of the subsequence, we get from above that two possible weak limits of random measures are equal. Hence the sequence ν_n converges weakly to ν. □

Lemma 4.8 *Let* f *be a non negative and compactly supported continuous function on* $[0,1] \times B'_\delta$ *and let* $\sum_{j>0} \varepsilon_{Z_j}$ *be the cluster process for the affine random walk* $(X_k)_{k\in\mathbb{N}}$*. Then :*

a) $\lim_{n\to\infty} [log\mathbb{E}^{(k_n)}_\rho(exp - \widetilde{N}_n(f)) + \sum_1^{k_n}(1 - \mathbb{E}_\rho(exp - \widetilde{N}_{k,n}(f)))] = 0.$

b) $\lim_{n\to\infty} \sum_1^{k_n}(1 - \mathbb{E}_\rho(exp - \widetilde{N}_{k,n}(f))) = \theta\delta^{-\alpha} \int_0^1 \mathbb{E}_{\Lambda_1}(1 - exp - \sum_{j>0} f(t, \delta Z_j))dt.$

Lemma 4.9 *Let* $\sum_{i\geq 0} \varepsilon_{T^\delta_i}$ *be a homogeneous Poisson process of intensity* $p(\delta) > 0$ *on* $[0,1]$*, which is independent of the sequence of cluster processes* $\sum_{j>0}^{\infty} \varepsilon_{Z_{ij}}$*.*
Then for any non negative and compactly supported continuous function f *on* $[0,1] \times B'_\delta$*, the Laplace functional of the point process* $Q^\delta = \sum_{i\geq 0} \sum_{j>0} \varepsilon_{(T^\delta_i, \delta Z_{ij})} 1_{\{|Z_{ij}|>1\}}$ *restricted to* $[0,1] \times B'_\delta$ *is equal to :*

$$\psi^\delta(f) = exp - p(\delta) \int_0^1 \mathbb{E}_{\Lambda_1}(1 - exp - \sum_1^\infty f(t, \delta Z_j))dt$$

Proof of Theorem 4.1 Let f be a compactly supported Lipchitz function on $[0,1] \times B'_\delta$.
Using Proposition 3.5, lemmas 4.8 implies that, on non-negative compactly supported Lipchitz functions on $[0,1] \times B'_\delta$, the Laplace functionals of N_n and $\widetilde{N}_n$ have the same limit, namely

$$\psi^\delta(f) = exp - p(\delta) \int_0^1 \mathbb{E}_{\Lambda_1}(1 - exp - \sum_1^\infty f(t, \delta Z_j))dt.$$

We observe that, for fixed f as above, the function $s \to \psi^\delta(sf)$ is continuous at $s = 0$. Since the function $s \to \psi_n(sf) = \mathbb{E}_\rho(exp - sN_n(f))$ is the Laplace transform of the non negative random variable $N_n(f)$, the continuity theorem for Laplace transforms implies that the sequence $N_n(f)$ converges in law to some random variable. Since this is valid for any f as above, Lemma 4.7 implies that the sequence of point processes N_n itself is tight. Since moreover the sequence of Laplace functionals $\psi_n(f)$ converges to $\psi^\delta(f)$, Lemma 4.7 implies that there exists a unique point process N on $[0,1] \times (V \setminus \{0\})$ such that the sequence N_n converges weakly to N. As stated in Lemmas 4.8, 4.9 the restriction of N to $[0,1] \times B'_\delta$ is given by the point process formula in the theorem. Lemma 4.9 implies that the Laplace functional of N on the function $f \in \mathcal{C}^c_+([0,1] \times B'_\delta$ is equal to

$$\psi^\delta(f) = exp - \theta\delta^{-\alpha} \int_0^1 \mathbb{E}_{\Lambda_1}[1 - exp - \sum_{j>0} f(t, \delta Z_j)]dt.$$

The first part of the formula giving the Laplace functional of N on f follows. The second part is a consequence of the last formula in Proposition 2.5 applied to the function $v \to f(t, \delta v)$ and of the α-homogeneity of Λ. □

Proof of Theorem 4.2 The first term $\mathbb{E}_\rho(exp - \sum_1^n f_{i,n})$ in $I_n(f)$ is the value of the Laplace functional of N_n on the continuous function f. Hence Theorem 4.1 implies its convergence to the Laplace functional of N on f. The same remark is valid for the second term in $I_n(f)$, if N_n is replaced by $\widetilde{N}_n$; the limit of $\widetilde{N}_n$ is also N, using Lemma 4.7 and Proposition 3.5. Then for any f in $\mathcal{C}^c_+([0,1] \times (V \setminus \{0\})$ we have $\lim_{n\to\infty} I_n(f) = \lim_{n\to\infty} |\mathbb{E}_\rho[exp - N_n(f) - \mathbb{E}^{(k_n)}_\rho(exp - \widetilde{N}_n(f)]| = 0$ □

Proof of Corollary 4.3 The point process N^s_n is the projection of N_n on $V \setminus \{0\}$. Since $[0,1]$ is compact and the projection is continuous, the continuous mapping theorem implies the required convergence, using the first part of Theorem 4.1. The formula for the Laplace functional of N^s is a direct consequence of the second part in Theorem 4.1 applied to a function independent of t. □

Proof of Corollary 4.4 For $\varphi \in C^+_c([0,1])$ we have $N^t_n(\varphi) = N_n(\varphi 1_{B'_1})$. Since the discontinuity set of $1_{B'_1}$ is Λ-negligible, Theorem 4.1 gives the convergence of $N^t_n(\varphi)$ to $N^t(\varphi)$. With $f = \varphi \otimes 1_{B'_1}$, the formula for the Laplace functional $\psi_\eta(f)$ of N gives the

Laplace functional $\psi_{\eta^t}(\varphi)$ of N^t in the logarithmic form

$$log\ \psi_{\eta^t}(\varphi) = -\theta \int_0^1 \mathbb{E}_{\Lambda_1}[1-\varphi(t)\gamma]dt.$$

The expression of the generating function of the random variable $\gamma = \sum_{j\geq 1} 1_{B_1'}(Z_j)$ follows from the last formula in Proposition 2.5 :

$$\sum_1^\infty e^{-sk}\nu_k = 1-(e^s-1)\theta^{-1}\mathbb{E}_{\Lambda_1}[exp - s\ \pi_v^\omega(B_1')].$$

Hence $\nu_k = \theta^{-1}(\theta_k - \theta_{k+1})$

In view of Theorem 4.1, the point process N^t can be written as $N^t = \sum_{k\geq 0} \gamma_k \varepsilon_{T_k^1}$, where the random variables γ_k are i.i.d with the same law as γ, hence N^t coincides with the compound Poisson process described in the statement. □

Proof of Corollary 4.5 Replacing u_n by δu_n ($\delta > 0$) in Corollary 4.4, we see that the point process on $[0,1]$ given by $N_{n,\delta}^t = \sum_{k=1}^n \varepsilon_{n^{-1}k} 1_{\{|X_k|>\delta u_n\}}$ converges to $N_\delta^t = \sum_{k\geq 0} \gamma_k \varepsilon_{T_k^\delta}$ where $\sum_{k\geq 0} \varepsilon_{T_k^\delta}$ is the Poisson process on $[0,1]$ with intensity $\theta\ \delta^{-\alpha}$ and the γ_k are i.i.d random variables as in the proof of Corollary 4.4. It follows that for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbb{P}_\rho\{N_{n,\delta}^t(1) = 0\} = exp - \theta\ \delta^{-\alpha}.$$

Since $\mathbb{P}_\rho\{N_{n,\delta}^t(1) = 0\} = \mathbb{P}_\rho\{M_n \leq u_n\delta\}$, the convergence of $u_n^{-1}M_n$ to Fréchet's law follows.

If M_n is replaced by $M_n^x (x \in V)$, the same proof as the one given below for the logarithm law remains valid. The last assertion in the corollary is a direct consequence of Fréchet's law. □

4.3 *Proofs of logarithm's law*

The proof of the logarithm's law is based on Fréchet's law and depends on two lemmas as follows.

Lemma 4.10 *We have* $\mathbb{P}_\rho - a.e$ *:*

$$\limsup_{n\to\infty} \frac{log|X_n|}{logn} \leq \limsup_{n\to\infty} \frac{logM_n}{logn} \leq \frac{1}{\alpha}.$$

Proof : Let $\varepsilon > 0$, $A_n(\varepsilon) = \{|X_n| \geq n^{1/\alpha+\varepsilon}\} \subset V^{\mathbb{Z}_+}$, $A_n'(\varepsilon) = V^{\mathbb{Z}_+} \setminus A_n(\varepsilon)$. Stationarity of X_n implies $\mathbb{P}_\rho(A_n(\varepsilon)) = \mathbb{P}_\rho\{|X_0| > n^{1/\alpha+\varepsilon}\}$. Since $\lim_{n\to\infty} n^{1+\alpha\varepsilon}\ell^\alpha(n^{1/\alpha+\varepsilon}, \infty) = 1$, we have $\sum_1^\infty \mathbb{P}_\rho\{A_n(\varepsilon)\} < \infty$. Then Borel-Cantelli's lemma implies that $\mathbb{P}_\rho\{\bigcup_1^\infty \bigcap_{j\geq n} A_j'(\varepsilon)\} = 1$ i.e $\mathbb{P}_\rho - a.e$ there exists $n_0(\omega)$ such that for $n \geq n_0(\omega)$, $|X_n(\omega)| \leq n^{1/\alpha+\varepsilon}$.

Then we deduce that $\mathbb{P}_\rho - a.e$: $\limsup_{n\to\infty} \frac{logM_n}{logn} \leq \frac{1}{\alpha} + \varepsilon$. Since ε is arbitrary we get : $\limsup_{n\to\infty} \frac{logM_n}{logn} \leq \frac{1}{\alpha}$. □

Lemma 4.11 *We have* $\mathbb{P}_\rho - a.e : \limsup_{n\to\infty} \frac{log|X_n|}{logn} \geq \frac{1}{\alpha}$.

Proof : Let $\varepsilon \in]0, 1/\alpha[$, $B(\varepsilon) = \left\{ \limsup_{n\to\infty} \frac{log|X_n|}{logn} \leq \frac{1}{\alpha} - \varepsilon \right\}$, $B_n(\varepsilon) = \left\{ \sup_{j\geq n} \frac{log|X_j|}{logj} \leq \frac{1}{\alpha} - \frac{\varepsilon}{2} \right\}$.

The sequence $B_n(\varepsilon)$ is increasing and $B(\varepsilon) \subset \overset{\infty}{\underset{2}{\cup}} B_n(\varepsilon)$. We are going to show $\mathbb{P}_\rho\{B_n(\varepsilon)\} = 0$.

For $p \geq n \geq 2$, $p \in \mathbb{N}$, we define $B_{n,p}(\varepsilon) = \{ \sup_{n\leq j\leq p} |X_j| \leq p^{1/\alpha - \varepsilon/2} \}$, hence $B_n(\varepsilon) \subset B_{n,p}(\varepsilon)$. Using stationarity we get $\mathbb{P}_\rho\{B_{n,p}(\varepsilon)\} = \mathbb{P}_\rho\{p^{-1/\alpha-\varepsilon/2} M_{p-n+1} \leq p^{-\varepsilon/2}\}$. Also, using Corollary 4.3, we have $\limsup_{n\to\infty t\geq 0} |\mathbb{P}_\rho\{n^{-1/\alpha} M_n \leq t\} - e^{-c\theta t^{-\alpha}}| = 0$ which implies $\lim_{p\to\infty} \mathbb{P}_\rho\{B_{n,p}(\varepsilon)\} = 0$.

Since the function $p \to B_{n,p}(\varepsilon)$ is decreasing and $B_n(\varepsilon) = \underset{p\geq 2}{\cap} B_{n,p}(\varepsilon)$ we have for $n \geq 2$:

$\mathbb{P}_\rho\{B_n(\varepsilon)\} = \lim_{p\to\infty} \mathbb{P}_\rho\{B_{n,p}(\varepsilon)\} = 0$, i.e $\mathbb{P}_\rho\{B(\varepsilon)\} = 0$.

We see that $\mathbb{P}_\rho - a.e$, $\limsup_{n\to\infty} \frac{log|X_n|}{logn} \geq \frac{1}{\alpha} - \varepsilon$, and, since ε is arbitrary we conclude : $\limsup_{n\to\infty} \frac{log|X_n|}{logn} \geq \frac{1}{\alpha}$. □

Proof of Corollary 4.6 From Lemmas 4.10, 4.11 we have $\mathbb{P}_\rho - a.e$,

$\limsup_{n\to\infty} \frac{logM_n}{logn} = \limsup_{n\to\infty} \frac{log|X_n|}{logn} = \frac{1}{\alpha}$.

Hence the definition of M_n implies $\lim_{n\to\infty} \frac{logM_n}{logn} = \frac{1}{\alpha}$.

Hence, for a set of $\rho \otimes \mathbb{P}$- probability 1 in $V \times A^{\mathbb{N}}$ we have

$\frac{1}{\alpha} = \limsup_{n\to\infty} \frac{log|X_n|(\omega)}{logn} = \lim_{n\to\infty} \frac{logM_n}{logn}$,

hence for a subsequence $n_k(\omega)$, $\frac{1}{\alpha} = \lim_{k\to\infty} \frac{log|X_{n_k}(\omega)|}{logn_k}$.

On the other hand we have for any $x \in V$: $|X_n - X_n^x| \leq |S_n||X_0 - x|$ and $\mathbb{P} - a.e$: $\lim_{n\to\infty} |S_n| = 0$.

Also $|log|X_n| - log|X_n^x|| \leq |S_n||X_0 - x| sup(|X_n|^{-1}, |X_n^x|^{-1})$, hence for any $x \in V$, $\mathbb{P}_\rho - a.e$:

$\lim_{n\to\infty} |log|X_n| - log|X_n^x|| = 0$.

It follows $\lim_{k\to\infty} \frac{log|X_{n_k}|}{logn_k} = \frac{1}{\alpha}$, and $\mathbb{P} - a.e$, $\limsup_{n\to\infty} \frac{log|X_n^x|}{logn} \geq \frac{1}{\alpha}$.

A similar argument shows that $\mathbb{P} - a.e, \limsup_{n\to\infty} \frac{log|X_n^x|}{logn} \leq \frac{1}{\alpha}$.

Furthermore, for any $n \geq 1$, $x \in V$:

$|M_n^x - M_n| \leq \sup_{<}\{|S_k|; 1 \leq k \leq n\}|x - X_0|$

where the sequence on the right is $\mathbb{P}-a.e$ bounded. Then, an argument as above shows that, $\mathbb{P}-a.e: \lim_{n\to\infty} \frac{logM_n}{logn} = \frac{1}{\alpha}$. $\square$

5 Convergence to stable laws

The convergence to stable laws of the normalized sums $\sum_{i=1}^{n} X_i$ under hypothesis (H) was shown in ([11], [15]) where explicit formulae for the corresponding characteristic functions were given. It was observed there that these formulae involved the asymptotic tail Λ of ρ, as well as the renewal point process $\pi_v^\omega = \sum_0^\infty \varepsilon_{S_i(\omega)v}$. A similar situation ocurred in the dynamical context of [12], where the limiting law was expressed in terms of an induced transformation. We observe that the connection between stable laws for $\sum_{i=1}^{n} X_i$, where $(X_i)_{i\in\mathbb{N}}$ is a stationary process, and point process theory had been already developed in [7] in the context of sample autocorrelation functions. For a recent analysis of the involved properties in this setting see [23]. Here we give new proofs of the results given in ([11], [15]), following the point process approach. In particular we get also a direct proof of the convergence for the related space point process $N^s = \sum_{i=1}^{n} \varepsilon_{u_n^{-1}X_i}$, via the analysis of Laplace functionals.

5.1 *On the space exceedances process*

We give here a direct proof of the convergence of N_n^s and we deduce the convergence of the characteristic function for the random variable $N_n^s(f)$, for f compactly supported.

Theorem 5.1 *Let f be a complex valued compactly supported Lipchitz function on $V\setminus\{0\}$ which satisfies $Re(f)\geq 0$. Then we have*

$$\lim_{n\to\infty} \mathbb{E}_\rho(exp - N_n^s(f)) = exp\ c^{-1}\mathbb{E}_\Lambda[(exp\ f(v)-1)exp - \pi_v^\omega(f)].$$

The proof depends on two lemmas where notations explained above are used. For $i\leq j$ we write $C_n(i,j) = exp - \sum_{k=i}^{j} f(u_n^{-1}X_k) - 1$, and we note the equality

$$C_n(1,r_n) = \sum_{i=1}^{r_n} [C_n(i,r_n) - C_n(i+1,r_n)]$$

where $C_n(r_n+1,r_n)=0$ and r_n is a sequence as in Proposition 2.5. We note also that

$$|C_n(i,j)|\leq 2,\ |C_n(i,j) - C_n(i+1,j)|\leq 2.$$

We are going to compare $C_n(1,r_n)$ to $C_{n,k}(1,r_n)$ where

$$C_{n,k}(1,r_n) = \sum_{i=1}^{r_n} [C_n(i,i+k) - C_n(i+1,i+k)],$$

we write $\Delta_{n,k}$ for their difference, $\varepsilon_n = r_n\mathbb{P}_\rho\{|X|>u_n\}$ and $supp(f)\subset B'_\delta$ with $\delta>0$.

Then we have the

Lemma 5.2 $\lim_{k\to\infty} \limsup_{n\to\infty} \varepsilon_n^{-1}\mathbb{E}_\rho(|\Delta_{n,k}|) = 0.$

Proof : We can assume that $r_n > k$. We observe that

$$1 + C_n(i, r_n) = (1 + C_n(i+1, r_n))exp - f(u_n^{-1}X_i).$$

Also $C_n(i+1, r_n) - C_n(i+1, i+k) = (exp - \sum_{j=i+1}^{i+k} f(X_j))(exp - \sum_{j=i+k+1}^{r_n} f(X_j) - 1)$.
Hence we have

$$\Delta_{n,k} = \sum_{i=1}^{r_n} (exp - f(u_n^{-1}X_i) - 1)[C_n(i+1, r_n) - C_n(i+1, i+k)] = \Delta'_{n,k} + \Delta''_{n,k}$$

with $\Delta'_{n,k}$ (resp $\Delta''_{n,k}$) is the above sum with index i restricted to $[1, r_n - k]$ (resp $]r_n - k, r_n]$). As observed above the expression under Σ is bounded by 4 and vanishes unless $|X_i| > \delta u_n$ for some $i \in [1, r_n - k]$ and $M_{k+i+1}^{r_n} > \delta u_n$. Then we get using stationary,

$$\mathbb{E}_\rho(|\Delta'_{n,k}|) \le 4r_n\mathbb{P}_\rho\{|X_0| > \delta u_n, M_{k+1}^{r_n} > \delta u_n\}.$$

Since the process $(X_k)_{k\in\mathbb{Z}_+}$ satisfies anticlustering, it follows $\lim_{k\to\infty} \limsup_{n\to\infty} \varepsilon_n^{-1}\mathbb{E}_\rho(|\Delta'_{n,k}|) = 0$. Also, stationarity implies

$$\mathbb{E}_\rho(|\Delta''_{n,k}|) \le 4 \sum_{i=r_n-k+1}^{r_n} \mathbb{P}_\rho\{|X_i| > \delta u_n\} = 4k\mathbb{P}_\rho\{|X| > \delta u_n\}.$$

Since ρ is homogeneous at infinity and $\lim_{n\to\infty} r_n^{-1}k = 0$, we get $\lim_{k\to\infty} \limsup_{n\to\infty} \varepsilon_n^{-1}\mathbb{E}_\rho(|\Delta''_{n,k}|) = 0$, hence the required assertion. □

Lemma 5.3 *We have the following convergences.*

1) For any $k \ge 1$ $\lim_{n\to\infty} k_n\mathbb{E}_\rho[C_{n,k}(1, r_n)] = c^{-1}\mathbb{E}_\Lambda[(expf(v) - 1)exp - \sum_{i=0}^{k} f(S_i v)]$

2) $\lim_{k\to\infty} \limsup_{n\to\infty} k_n\mathbb{E}_\rho(|\Delta_{n,k}|) = 0$

Proof : 1) Using stationarity we have

$$k_n\mathbb{E}_\rho[C_{n,k}(1, r_n)] = k_n r_n\mathbb{E}_\rho[C_n(1, k+1) - C_n(2, k+1)] =$$
$$k_n r_n\mathbb{E}_\rho[exp - \sum_{j=0}^{k} f(u_n^{-1}X_j) - exp - \sum_{j=1}^{k} f(u_n^{-1}X_j)].$$

The function $f^{(k)}$ on $(V \setminus \{0\})^{k+1}$ given by

$$f^{(k)}(x_0, x_1 \cdots, x_k) = exp - \sum_{j=0}^{k} f(x_j) - exp - \sum_{j=1}^{k} f(x_j) = (exp\ f(x_0) - 1)exp - \sum_{j=0}^{k} f(x_j)$$

is bounded, uniformly continuous on $(B'_\delta)^{k+1}$ and $\lim_{n\to\infty} n^{-1}k_n r_n = 1$. Hence, the homogeneity at infinity of ρ, the conditional convergence of $u_n^{-1}X_j$ to $S_j X_0$ and the definition of Λ imply

$$\lim_{n\to\infty} k_n\mathbb{E}_\rho[C_{n,k}(1, r_n)] = c^{-1}\mathbb{E}_\Lambda[(exp\ f(v) - 1)exp - \sum_{j=0}^{k} f(S_j v)].$$

2) We have the equality, $k_n\mathbb{E}_\rho(|\Delta_{n,k}|) = k_n r_n \mathbb{P}_\rho\{|X| > u_n\}\varepsilon_n^{-1}\mathbb{E}_\rho(|\Delta_{n,k}|)$.

Then, using Lemma 5.2, the relation $\lim_{n\to\infty} n^{-1}k_n r_n = 1$ and the homogeneity at infinity of ρ, assertion 2 follows. □

Proof of Theorem 5.1 With r_n as in Proposition 5.3 above, the multiple mixing property gives

$$\lim_{n\to\infty}[\mathbb{E}_\rho(exp - N_n^s(f)) - (\mathbb{E}_\rho(1 + C_n(1, r_n)))^{k_n}] = 0,$$

hence it suffices to study the sequence $(1+\mathbb{E}_\rho(C_n(1, r_n)))^{k_n}$. Since $Ref \geq 0$ and $supp(f) \subset B'_\delta$ we have

$$\mathbb{E}_\rho(|C_n(1, r_n)|) \leq \mathbb{E}_\rho(|1-exp-\sum_{i=1}^{r_n} f(u_n^{-1}X_i)|) \leq \mathbb{E}_\rho(\sum_{i=1}^{r_n} |f(u_n^{-1}X_i)|) \leq r_n|f|_\infty \mathbb{P}_\rho\{|X_0| > \delta u_n\}.$$

The last inequality implies the $\mathbb{L}^1$-convergence to zero of $\sum_{i=1}^{r_n} f(u_n^{-1}X_i)$. Then the first in equality gives $\lim_{n\to\infty} \mathbb{E}_\rho(|C_n(1, r_n)|) = 0$.

It follows that the behaviour of the sequence $[1+\mathbb{E}_\rho(C_n(1, r_n))]^{k_n}$ for n large is determined by the behaviour of $k_n\mathbb{E}_\rho(C_n(1, r_n))$. We have for $k \geq 1$,

$$k_n\mathbb{E}_\rho(C_n(1, r_n)) = k_n\mathbb{E}_\rho(C_{n,k}(1, r_n)) + k_n\mathbb{E}_\rho(\Delta_{n,k}).$$

Since $supp(f) \subset B'_\delta$ and $\lim_{j\to\infty} |S_j v| = 0\ \mathbb{Q} - a.e$, the series $\sum_{j=1}^{\infty} f(S_j v)$ converges $\mathbb{Q} - a.e$.

Since $Ref \geq 0$, it follows $\lim_{k\to\infty} \mathbb{E}(exp - \sum_{j=1}^{k} f(S_j v)) = exp - \pi_v^\omega(f)$.

Then dominated convergence and Lemma 5.3 imply

$$\lim_{k\to\infty}\ \lim_{n\to\infty} k_n\mathbb{E}_\rho(C_{n,k}(1, r_n)) = \mathbb{E}_\Lambda[(exp\ f(v) - 1)exp - \pi_v^\omega(f)].$$

This equality and the second assertion in Lemma 5.3 give the result. □

Corollary 5.4 *Let $m > 0$, $\delta > 0$, $\gamma \geq 0$, and let f be a $\mathbb{R}^m$-valued continuous function on $V \setminus \{0\}$ which satisfies the conditions*

1) f is locally Lipchitz

2) $f(v) = 0$ for $|v| < \delta$

3) $\sup_{v\in V} |v|^{-\gamma}|f(v)| = c_\gamma < \infty$

Then we have , for any $u \in \mathbb{R}^m$,

$\lim_{n\to\infty} \mathbb{E}_\rho(exp - i < u, N_n^s(f) >) = exp\ c^{-1}\mathbb{E}_\Lambda[(exp\ i < u, f(v) > -1)exp - i < u, \pi_v^\omega(f) >]$

Proof : We consider the random variable $Y_n = N_n^s(f)$. For $a \geq 1$, let θ_a be the function from $V \setminus \{0\}$ to $[0, 1]$ defined by

$$\theta_a(v) = 1 \text{ for } |v| \leq a,\ \theta_a(v) = a + 1 - |v| \text{ for } |v| \in [a, a + 1],\ \theta_a(v) = 0 \text{ for } |v| \geq a + 1.$$

Then θ_a is Lipchitz, hence $f\theta_a$ is Lipchitz and compactly supported. Then the theorem gives,
$\lim_{n\to\infty} \mathbb{E}_\rho(exp\ i < u, N_n^s(f\theta_a) >) = exp\ c^{-1}\mathbb{E}_\Lambda[(exp - i < u, f\theta_a(v) > -1)exp\ \pi_v^\omega(i < u, f\theta_a(v) >)] = \Phi_a(u)$
Since $\Lambda(B'_\delta) < \infty$, the function $u \to \Phi_a(u)$ is continuous on $\mathbb{R}^m$. It follows that the sequence of random variables $Y_n^a = N_n^s(f\theta_a)$ converges in law to the random variable Y_∞^a which has characteristic function Φ_a. On the other hand we have $\lim_{a\to\infty} \theta_a = 1$, hence by dominated convergence we get,

$$\lim_{a\to\infty} \Phi_a(u) = exp\ c^{-1}\mathbb{E}_\Lambda[(exp - i < u, f(v) > -1)exp\ i < u, \pi_v^\omega(f) >] = \Phi(u).$$

We recall that, for v fixed, the series $\sum_{j=0}^{\infty} f(S_j v)$ converges $\mathbb{Q} - a.e$ to a finite sum, hence the function $u \to \Phi(u)$ is continuous. In other words, Y_∞^a converges in law $(a \to \infty)$ to the random variable Y with characteristic function Φ.
Also for, $z, z' \in \mathbb{C}$ with $Rez \leq 0$, $Rez' \leq 0$ we have $|exp\ z - exp\ z'| \leq |z - z'|$. If we choose $\beta \in]0,1[$, $\gamma > 0$ such that $\beta\gamma \in]0, \alpha[$, then we have for any $\varepsilon > 0$,

$$\mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta}\mathbb{E}_\rho[(\sum_{j=1}^{n} (f(u_n^{-1}X_j)1_{\{|X_j|>au_n\}})^\beta],$$

$$\mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta}c_\gamma^\beta n\mathbb{E}_\rho[|u_n^{-1}X|^{\beta\gamma}1_{\{|X|>au_n\})}].$$

Using Corollary 2.2, with $W(x) = |x|$ it follows $\limsup_{n\to\infty} \mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta}c_\gamma^\beta\Lambda(W^{\beta\gamma}1_{\{|W|>a\}})$.
Since $0 < \beta\gamma < \alpha$, we get $\lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{P}_\rho\{|Y_n^a - Y_n| > \varepsilon\} = 0$
Since $\varepsilon > 0$ is arbitrary, the convergence in law of Y_n to the random variable Y follows, hence the corollary. □
In order to prepare the study of limits for the sums $T_n = \sum_{j=1}^{n} X_j$ if $0 < \alpha < 2$, we write for $a > 0 : \psi_a(v) = v(1 - \varphi_a(v)$ where
$\varphi_a(v) = 1$ if $|v| \leq a$, $\varphi_a(v) = 2 - a^{-1}|v|$ if $a \leq |v| \leq 2\varepsilon$, $\varphi_a(v) = 0$ if $|v| > 2a$.
Hence $0 \leq \varphi_a| \leq 1_{[0,2a]}$ and $k(\varphi_a) \leq a^{-1}$. Then a consequence of Corollary 5.4 with $m = d$, $\gamma = 1$ is the following

Corollary 5.5 *The sequence of V-valued random variables $N_n^s(\psi_a)$ converges in law to the random variable with characteristic function*
$exp\ c^{-1}\mathbb{E}_\Lambda[(exp - i < u, \psi_a(v) > -1)exp\ i < u, \pi_v^\omega(\psi_a) >]$.

5.2 *Convergence to stable laws for $T_n = \sum_{i=1}^{n} X_i$*

In this subsection we write $\psi(v) = v$ and we study the convergence of $N_n^s(\psi) = u_n^{-1}\ T_n$ towards a stable law, extending the weak convergence of N_n studied in the above subsection. We need here the last part of the spectral gap result in Proposition 3.4 for the operator P.

We have the

Theorem 5.6 *Let $0 < \alpha < 2$. Then there exists a sequence d_n in V such that the sequence of random variables $n^{-1/\alpha}(T_n - d_n)$ converges in law to a non degenerate stable law.*
If $0 < \alpha < 1$, we have $d_n = 0$.
If $1 < \alpha < 2$, we have $d_n = n\mathbb{E}_\rho(X)$
If $\alpha = 1$, we have $d_n = n\ \mathbb{E}_\rho[X\varphi_1(X)]$.

Explicit expressions for the characteristic functions of the limits are given in the proofs. Non degeneracy of the limit laws are proved in [11] and [15]. For the proofs, we follow the approach of [7] and we need two lemmas corresponding to the cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$.
In the proofs below we use the normalization $u_n = (cn)^{1/\alpha}$ instead of $n^{1/\alpha}$ as in the theorem.

Lemma 5.7 *Assume $0 < \alpha < 1$. Then for any $u \in V$ and with the notation of Corollary 5.5, $T^a = N^s(\psi_a)$ converges in law $(a \to 0)$ to T with characteristic function given by*

$$exp\ c^{-1}\mathbb{E}_\Lambda[(exp - i < u, v > -1)exp\ i < u, \sum_{j=0}^{\infty} S_j v >] = \Phi(u).$$

Also, for any $\delta > 0$ we have

$$\lim_{a\to 0}\ \limsup_{n\to\infty} \mathbb{P}_\rho\left\{ |\sum_{j=1}^{n} u_n^{-1} X_j \varphi_a(u_n^{-1}X_j)| > \delta \right\} = 0.$$

Proof : Using dominated convergence, the first part follows from Corollary 5.5.
On the other hand, Markov inequality gives

$$\mathbb{P}_\rho\left\{ |\sum_{j=1}^{n} u_n^{-1} X_j \varphi_a(u^{-1}X_j)| > \delta \right\} \leq \delta^{-1} u_n^{-1} \mathbb{E}_\rho(|X| 1_{\{|X|<2au_n\}})$$

The homogeneity at infinity of ρ and Karamata's lemma (see [28] p.26) gives that the right hand side is equivalent to

$$\delta^{-1} n^{1-\alpha^{-1}} \alpha(1-\alpha)^{-1}(2au_n)\mathbb{P}_\rho\{|X| > 2au_n\},$$

i.e to $\delta^{-1}\ a^{1-\alpha}$ up to a coefficient independent of n. Hence the result since $1 - \alpha > 0$. □

Lemma 5.8 *Assume $1 \leq \alpha < 2$ and write $\overline{\psi}_a(v) = v\varphi_a(v)$. Then we have the convergence*

$$\lim_{a\to 0}\ \limsup_{n\to\infty} \mathbb{E}_\rho(|N_n^s(\overline{\psi}_a) - \mathbb{E}_\rho(N_n^s(\overline{\psi}_a)|^2) = 0.$$

Proof : It suffices to show that for any $u \in \mathbb{S}^{d-1}$:

$$\lim_{a\to 0}\ \limsup_{n\to\infty} \mathbb{E}_\rho(| < u, N_n^s(\overline{\psi}_a) > -\mathbb{E}_\rho(< u, N_n^s(\overline{\psi}_a) > |^2) = 0.$$

We write $f_{a,n}(v) = \overline{\psi}_a(u_n^{-1}v)$, $\overline{\psi}_{a,n} = f_{a,n} - \rho(f_{a,n})$.
Hence $|f_{a,n}(v)| \leq u_n^{-1}|v| 1_{\{|v|\leq 2u_n\}}$, $[f_{a,n}] \leq 3u_n^{-1}$. We have the equality

$$\mathbb{E}_\rho(| < u, N_n(\overline{\psi}_a) > -\mathbb{E}_\rho(< u, N_n(\overline{\psi}_a) >)|^2) = A_{n,a} + 2B_{n,a}$$

with

$A_{n,a} = n\mathbb{E}_\rho(|< u, \psi_{a,n}(X_0) >|^2),\ B_{n,a} = \sum_{j=1}^{n}(n-j)\mathbb{E}_\rho(< u, \overline{\psi}_{a,n}(X_0) >< u, \overline{\psi}_{a,n}(X_j))$.

Now the proof splits into two parts a) and b) corresponding to the studies of $A_{n,a}, B_{n,a}$.

a) We have, using the above estimation of $f_{a,n}$,

$$n\mathbb{E}_\rho(|< u, \overline{\psi}_{a,n}(X_0) >|^2) \leq n\mathbb{E}_\rho(|f_{a,n}(X_0)|^2) \leq n\mathbb{E}_\rho(u_n^{-2}|X_0|^2 1_{\{|X_0|<2au_n\}}).$$

Then Karamata's lemma implies that, for n large, the right hand side is equivalent to $n^{1-2\alpha^{-1}}(2au_n)^2((2a)^\alpha n)^{-1}, i.e$ to $a^{2-\alpha}$. Hence, since $\alpha \in]0,2[$, we get $\lim_{a\to 0}\ \limsup_{n\to\infty} A_{n,a} = 0$ uniformly in $u \in \mathbb{S}^{d-1}$.

b) Markov property for the process $(X_i)_{i\geq 0}$ implies for $i \geq 1$,

$\mathbb{E}_\rho(< u, \overline{\psi}_{a,n}(X_0) >< u, \overline{\psi}_{a,n}(X_i) >) = \mathbb{E}_\rho(< u, \overline{\psi}_{a,n}(X_0) >< u, P^i\overline{\psi}_{a,n}(X_0) >)$.

First we consider the case $\alpha \in]1,2[$ and we apply Proposition 3.4 to P acting on the Banach space $\mathcal{H} = \mathcal{H}_{\chi,\varepsilon,\kappa}$ with $\chi \in]1,\alpha[$, $\varepsilon = 1$ and κ choosen according to Proposition 3.4. We observe that for $h \in \mathcal{H}$ we have $|h(v)| \leq \|h\|(1+|v|)$. Since $\overline{\psi}_{a,n} \in \mathcal{H}$, we have

$\mathbb{E}_\rho(< u, \overline{\psi}_{a,n}(X_0) >< u, P^i\overline{\psi}_{a,n}(X_0) >) = \mathbb{E}_\rho(< u, f_{a,n}(X_0) >< u, U^i f_{a,n}(X_0) >)$,

where we have used the decomposition $P^i = \rho\otimes 1 + U^i$. Schwarz inequality allow us to bound the right hand side by the square root of $\mathbb{E}_\rho(|f_{a,n}(X_0)|^2)\mathbb{E}_\rho(|U^i f_{a,n}(X_0)|^2 1_{\{|X_0|<2an\}})$.

Since $|U^i f_{a,n}(v)| \leq (1+|v|)\|U^i\|\ \|f_{a,n}\|$, the quantity $\mathbb{E}_\rho(< u, f_{a,n}(X_0) >< u, U^i f_{a,n}(X_0) >)$ is bounded by $\|U^i\|\ \|f_{a,n}\|[u_n^{-2}\mathbb{E}_\rho(|X_0|^2 1_{\{|X_0|<2au_n\}})]^{1/2}\ [\mathbb{E}_\rho(1+|X_0|)^2 1_{\{|X_0|<2au_n\}})]^{1/2}$.

Then Karamata's lemma implies that, up to a coefficient independent of n, the above expression is bounded by $\|U^i\|\ \|f_{a,n}\|[n^{-1/2}a^{1-\alpha/2}]\ [1 + n^{\alpha^{-1}-1/2}a^{1-\alpha/2}]$.

Since $\|f_{a,n}\| \leq nu_n^{-1}$, it follows that $B_{n,a}$, uniformly in $u \in \mathbb{S}^{d-1}$ and up to a coefficient, is bounded by

$n(\sum_{i=0}^{\infty}\|U^i\|)n^{-1}[n^{1/2-\alpha^{-1}}a^{1-\alpha/2} + a^{2-\alpha}] = \sum_{i=0}^{\infty}\|U^i\|\ [a^{2-\alpha} + a^{1-\alpha/2}n^{1/2-\alpha^{-1}}]$.

Since $r(U) < 1$ we have $\sum_{i=0}^{\infty}\|U^i\| < \infty$, hence $\limsup_{n\to\infty} B_{n,a}$ is bounded by $a^{2-\alpha}$, up to a coefficient independent of n. Since $1 < \alpha < 2$, the lemma follows of the two above convergences.

If $\alpha = 1$, we need to use the Banach space $\mathcal{H}' = \mathcal{H}_{\chi,\varepsilon,\kappa}$ with $0 < \varepsilon < \chi < 1$, $\kappa = 0$, considered in Proposition 3.4. We use also the inequality $\|f\|_{a,n} \leq c_1 a^{1-\chi}n^{-\varepsilon}$ with $c_1 > 0$, shown below. We note that for $h \in \mathcal{H}'$, we have $|h(v)| \leq \|h\|(1+|v|^\varepsilon)$ in particular and up to a constant independent of n, a we have

$$|U^i f_{a,n}(v)| \leq \|U^i\|\ \|f_{a,n}\|(1+|v|^\varepsilon) \leq \|U^i\|(a^{1-\chi}n^{-\varepsilon})(1+|v|^\varepsilon).$$

Hence we can bound $\mathbb{E}_\rho(< u, f_{a,n}(X_0) >< u, U^i f_{a,n}(X_0) >)$ by

$2c_1\|U^i\|(a^{1-\chi}n^{-\varepsilon})[n^{-2}\mathbb{E}_\rho(|X_0|^2)1_{\{|X_0|<2na\}}]^{1/2}[\mathbb{E}_\rho(1+|X_0|^{2\varepsilon})1_{\{|X_0|<2na\}}]^{1/2}$,

which, if $\varepsilon > 1/2$, can be estimated using Karamata's lemma by

$2c_1\|U^i\|(a^{1-\chi}n^{-\varepsilon})(an^{-1})^{1/2}(na)^{\varepsilon-1/2} = c_2 a^{1-\chi+\varepsilon}n^{-1}$.

It follows that $B_{n,a}$ can be estimated by $2c_2(\sum_{i=0}^{\infty}\|U^i\|)a^{1-\chi+\varepsilon}$. Since, using Proposition 3.4 we have $r_1(U)<1$, it follows if $1/2<\varepsilon<\chi<1$, $\lim_{a\to 0}\limsup_{n\to\infty} B_{n,a}=0$. $\square$

Proof of Theorem 5.6 For $\alpha\in]0,1[$ the proof follows of Lemma 5.7. We observe that dominated convergence implies the continuity of Φ at zero, hence Φ is a characteristic function. From Lemma 5.7 we know that if $Y_n=N_n^s(1)$, $Y_n^a=N_n^s(1-\varphi_a)$,

1) For any $a>0$, Y_n^a converges in law $(n\to\infty)$ to T^a

2) T^a converges in law $(a\to 0)$ to T

3) For any $\varepsilon>0$, we have $\lim_{a\to 0}\limsup_{n\to\infty}\mathbb{P}\{|Y_n-Y_n^a)>\varepsilon\}=0$.

It follows that the sequence $Y_n=\sum_{i=1}^{n}u_n^{-1}X_i$ converges in law $(n\to\infty)$ to the random variable T with characteristic function Φ.

For $1<\alpha<2$, we write

$Y_n=N_n^s(\psi)-\mathbb{E}_\rho(N_n^s(\psi))=(u_n^{-1}\sum_{j=1}^{n}X_j)-\mathbb{E}_\rho(X), Y_n^a=N_n^s(\psi_a)-\mathbb{E}_\rho(N_n^s(\psi_a))$,

so that $Y_n^a-Y_n=N_n^s(\overline{\psi}_{a,n})-\mathbb{E}_\rho(N_n^s(\overline{\psi}_{a,n}))$. Then for any $\varepsilon>0$ Lemma 5.8 gives,

$$\lim_{a\to 0}\limsup_{n\to\infty}\mathbb{P}_\rho\{|Y_n^a-Y_n|>\varepsilon\}=0$$

Furthermore, the sequence $N_n^s(\psi_a)$ converges in law $(n\to\infty)$ to T^a and $\mathbb{E}_\rho(N_n^s(\psi_a))=n\mathbb{E}_\rho[u_n^{-1}X(1-\varphi_a)(u_n^{-1}X)]$ converges to the value $b(a)$ of Λ on the function $v\to v(1-\varphi_a(v))$, as follows from $\alpha>1$ and the homogeneity at infinity of ρ. Hence the sequence Y_n^a converges in law $(n\to\infty)$ to $T^a-b(a)=Y^a$. Finally Y^a converges in law $(a\to 0)$ to the random variable T with characteristic function Φ defined by,

$\Phi(u)=exp\ c^{-1}\mathbb{E}_\Lambda[(exp-i<u,v>-1+i<u,v>)exp\ i<u,\sum_{j=0}^{\infty}S_jv>]$

$+i\mathbb{E}_\Lambda[<u,v>(exp\ i<u,\sum_{j=0}^{\infty}S_jv>-1)]$.

This follows of Theorem 5.1, of dominated convergence $(a\to 0)$ and of the following inequalities

$|exp-i<u,\psi_a(v)>-1+i<u,\psi_a(v)>|\leq Inf(2+|u|\ |v|,4|v|^2|u|^2)$,

$|<u,\psi_a(v)>\mathbb{E}(exp\ i<u,\sum_{j=1}^{\infty}\psi_a(S_jv)>-1)|\leq Inf(|u|\ |v|,2|u|^2|v|^2\sum_{j=1}^{\infty}\mathbb{E}(|S_j|))$,

where $\alpha>1$ gives $\sum_{j=1}^{\infty}\mathbb{E}(|S_j|)<\infty$. Continuity of Φ at zero follows also from the above inequalities.

As in [7], we deduce the convergence in law of the sequence Y_n to T.

If $\alpha=1$, we write $Y_n=\sum_{j=1}^{n}n^{-1}X_j-\mathbb{E}_\rho(\varphi_1(n^{-1}X))$ and

$$Y_n^a=\sum_{j=1}^{n}n^{-1}X_i(1-\varphi_a(n^{-1}X_j))-b_n(a)=N_n^s(\psi_a)-b_n(a)$$

where $b_n(a) = \mathbb{E}_\rho[X(\varphi_1 - \varphi_a)(n^{-1}X))]$. With the new notations the above inequalities are still valid. The homogeneity at infinity of ρ gives now

$\lim_{n\to\infty} b_n(a) = c^{-1}\mathbb{E}_\Lambda(v(\varphi_1 - \varphi_a)) = b(a)$.

It follows that the sequence Y_n^a converges in law ($n \to \infty$) to the random variable T_1^a with characteristic function,

$$exp\ c^{-1}\mathbb{E}_\Lambda[(exp - i < u, \psi_a(v) > -1)exp\ i(< u, \pi_v^\omega(\psi_a) > -i < u, b(a) >)]$$

We insert the expression $i < u, v > (\varphi_1 - \varphi_a(v))$ with the adequate sign in each of the above factors inside the expectation $\mathbb{E}_\Lambda$. Then dominated convergence ($a \to 0$) shows that $T_1^a - b(a)$ converges in law to the random variable T with characteristic function

$$\Phi(u) = exp\ c^{-1}\mathbb{E}_\Lambda[A(u,v) + B(u,v)]$$

with $A(u,v) = (exp - i < u, v > -1 + i < u, v >)\varphi_1(v)$,

$B(u,v) = i < u, v > \varphi_1(v)(exp\ i < u, \sum_{j=1}^{\infty} S_j v > -1)$.

As in ([11], [15]), the stability of the limiting laws follow from the formula for $\Phi(u)$. If $0 < \alpha < 2$, $\alpha \neq 1$ the formula for $\Phi(u)$ shows that for any $n \in \mathbb{N}$ we have $\Phi^n(u) = \Phi(n^{1/\alpha}u)$, hence T has a stable law of index α.

If $\alpha = 1$, we have with $\gamma_n = c\ \mathbb{E}_\Lambda[v(\varphi_1(n^{-1}v) - \varphi_1(v)]$, $\Phi^n(u) = \Phi(n\ u)\ exp - in < u, \gamma_n >$. This implies that T follows a stable law with index 1. □

6 Appendix

6.1 *On positivity of the extremal index*

We give a direct proof of the positivity of $\mathbb{Q}_{\Lambda_1}\{\sup_{n>0} |S_n(\omega)v| < 1\}$ following [6].

Proposition 6.1 *Let (Y, τ, m) be a dynamical system where m is a τ-invariant probability and let f be a measurable function on Y. If $\lim_{n\to\infty} \sum_0^{n-1} f(\tau^k y) = -\infty \quad m - a.e$, then there exists $c < 0$ and a subset $Y_1 \subset Y$ of positive measure such that for every $y \in Y_1,\ n \geq 1$,*

$$T_n(y) = \sum_0^{n-1} f(\tau^k y) \leq c$$

Proof : Let $M_n(y) = \sup_{0\leq k\leq n} T_k(y)$, hence

$$M_{n+1}(y) = \sup[f(y), f(y) + M_n(\tau y)] = f(y) + M_n^+(\tau y).$$

Since $\lim_{n\to\infty} T_n(y) = -\infty\ m - a.e$, the function $M_\infty(y) = \lim_{n\to\infty} M_n(y)$ is finite $m - a.e$, hence

$$f(y) = M_\infty^+(y) - M_\infty^-(y) - M_\infty^+(\tau y).$$

If $M_\infty^- = 0\ m - a.e$, then $f(y) = M_\infty^+(y) - M_\infty^+(\tau y)$ is a coboundary. By considering the return times to a set on which M_∞^+ is bounded from below, we get a contradiction

with the above coboundary equation. Hence we have, for some $c < 0$, $M_\infty(y) \leq c$ on a set Y_1 of positive measure, i.e $M_\infty(y) \leq c < 0$ on $Y_1 \subset X$ with $m(Y_1) > 0$. Hence $T_n(y) \leq M_n(y) \leq M_\infty(y) \leq c$ on Y_1 for any $n \geq 1$. □

Using the notations of section 2 we consider the action of T on the unit sphere $(g, x) \to \frac{gx}{|gx|} = g.x$, we assume that T is strongly irreducible and we denote by ν a μ-stationary probability on $\mathbb{S}^{d-1}$. We denote by $\mathcal{U}(supp(\nu))$ the set of non void open subsets of $supp(\nu)$, hence for any $U \in \mathcal{U}(supp\nu))$ we have $\nu(U) > 0$.

Corollary 6.2 *With the above notations, we assume that the semigroup T is strongly irreducible and $L_\mu = \int log|gx| d\mu(g) d\nu(x) < 0$. Then, there exists $\varepsilon > 0$, $\Omega_1 \subset \Omega$ with $\mathbb{Q}(\Omega_1) > 0$ and a map $\omega \to U_\varepsilon(\omega)$ from Ω_1 to $\mathcal{U}(supp(\nu))$ such that*

$$\sup\{|S_n(\omega)x| < 1 \ ; \ n > 0\} < 1 - \varepsilon$$

for any $x \in U_\varepsilon(\omega)$, $\omega \in \Omega_1$.
In particular, we have $\mathbb{Q}_{\Lambda_1}\{\sup |S_n(\omega)v| < 1 \ ; \ n > 0\} > 0$

Proof : We denote $Y = \Omega \times supp(\nu)$ and we write $y = (\omega, x) \in Y$. We consider the dynamical system (Y, τ, m) defined by $\tau(\omega, x) = (\sigma\omega, \ A_1(\omega).x)$. The hypothesis implies with $f(y) = log|A_1(\omega)x|$, $m(f) < 0$ $T_n(y) = log|S_n(\omega)x| : \lim_{n\to\infty} T_n(y) = -\infty$, $m - a.e.$
Then Proposition 6.1 implies the existence of a set $Y_1 \subset \Omega \times supp(\nu)$ of positive $\mathbb{Q} \otimes \nu$ measure and $\varepsilon' > 0$ such that for $n > 0$, $|S_n(\omega)v| < 1 - \varepsilon'$. Hence there exists $\Omega_1 \subset \Omega$ with $\mathbb{Q}(\Omega_1) > 0$ such that, for $\omega \in \Omega_1$, there exists $S_\omega \subset \mathbb{S}^{d-1}$ with $\nu(S_\omega) > 0$ and $\sup\{|S_n(\omega)v| \ ; \ n > 0\} < 1 - \varepsilon'$ for $v \in S_\omega$. Since T is strongly irreducible, ν gives measure zero to any proper subspace (see [13]), hence S_ω contains $v_1^\omega, \cdots, v_d^\omega$ which are linearly independent, for any $\omega \in \Omega_1$. Then, for any $v = \sum_{i=1}^{d} \lambda_i v_i^\omega$ with $\sum_{i=1}^{d} |\lambda_i| < (1 - \varepsilon'/2)^{-1}$ we have

$$|S_n(\omega)v| \leq \sum_{i=1}^{d} |\lambda_i| \ |S_i(\omega)v_i^\omega| \leq (1 - \varepsilon'/2)^{-1}(1 - \varepsilon') < 1 - \varepsilon'/2,$$

for $n > 0$. But the set

$$U_\omega = supp(\nu) \cap \{v, \sum_{i=1}^{d} |\lambda_i| < (1 - \varepsilon'/2)^{-1}\}$$

is an open non void subset of $supp(\nu)$ since $(1 - \varepsilon'/2)^{-1} > 1$ and $v_i^\omega \in U_\omega$, for $i = 1, \cdots, d$. Hence, for $\omega \in \Omega_1$ and $x \in U_\omega$, $\varepsilon = \varepsilon'/2$, we have $\sup\{|S_n(\omega)x| \ ; \ n > 0\} < 1 - \varepsilon$. From [13], we know that the set $supp(\sigma^\alpha)$ is also the support of a μ-stationary measure ν and $\Lambda = \sigma^\alpha \otimes \ell^\alpha$ with $c > 0$. Since for any $U \in \mathcal{U}(supp(\nu))$ the set $\{v = tx \ ; \ 1 < t < (1 - \varepsilon)^{-1}, \ x \in U\}$ has positive Λ_1-measure, we get by definition of $\mathbb{Q}_{\Lambda_1}$

$$\mathbb{Q}_{\Lambda_1}\{\sup |S_n(\omega)v| < 1 \ ; \ n > 0\} > 0$$

□

6.2 *Condition (H) is open if $d > 1$*

We denote by T_μ the closed subsemigroup of G generated by $supp(\mu)$, where μ is a probability on G. We consider weak topologies for probability measures on G and on A. We denote by $M^1(G)$ $(resp\ M^1(A))$ the set of probabilities on $G(resp\ A)$. We denote by $\mathcal{W}(A)$ the weak topology on $M^1(A)$ defined by the convergence on continuous compactly supported functions as well as of the moments $\int(\gamma^k(g) + |b|^k(h))d\lambda(h)$ for any $k \in \mathbb{N}$.

Theorem 6.3 *If $d > 1$, condition (H) is open in the weak topology $\mathcal{W}(A)$ on $M^1(A)$.*

We will need the Proposition

Proposition 6.4 *Condition i-p is open for the weak topology on $M^1(G)$.*

Proof : Assume $\mu \in M^1(G)$, satisfies i-p and let $\mu_n \in M^1(G)$ be a sequence which converges weakly to μ. Then $supp(\mu_n)$ and T_{μ_n} are closed subsets of G which converges to $supp(\mu)$ and T_μ respectively. If γ is a proximal element of T_μ, then by perturbation theory there exists a neighbourhood of γ in G which consists of proximal elements. Hence there exists $\gamma_n \in T_{\mu_n}$ which is also proximal.
On the other hand T_{μ_n} is irreducible for large n. Otherwise there exists a proper subspace $W^n \subset V$ with $T_{\mu_n}(W^n) = W^n$. Let $W \subset V$ be the limit of a subsequence of W^n. Then, clearly $T_\mu(W) = W$, which contradicts the irreducibility of T_μ.
In order to show the strong irreducibility of T_{μ_n} for n large, we show the irreducibility of $Zc_0(T_{\mu_n})$, the connected component of the Zariski closure $Zc(T_{\mu_n})$ of T_{μ_n} (see [24]). Since T_{μ_n} is irreducible, the Lie group $Zc_0(T_{\mu_n})$ is reductive and has finite index in $Zc(T_{\mu_n})$. We decompose V as the direct sum of its isotypic components $V_i^{(n)}(1 \leq i \leq p_n)$ under the action of $Zc_0(T_{\mu_n})$: $V = \overset{p_n}{\underset{i=1}{\oplus}} V_i^{(n)}$. Since $Zc_0(T_{\mu_n})$ has finite index in $Zc(T_{\mu_n})$ we can assume, by taking a suitable power, that $\gamma_n \in Zc_0(T_{\mu_n})$. The uniqueness of the above decomposition of V and the relation $\gamma_n v = \lambda_n v$, $v = \overset{p_n}{\underset{i=1}{\sum}} v_i$, $v_i \in V_i^{(n)}$, with λ_n a simple dominant eigenvalue of γ_n implies $\gamma_n v_i = \lambda_n v_i$; hence the proximality of γ_n implies that v belongs to a unique $V_i^{(n)}$, to $V_1^{(n)}$ say. Also the irreducibility of T_{μ_n} implies that T_{μ_n} permutes the subspaces $V_i^{(n)}(1 \leq i \leq p_n)$. Since $V_1^{(n)}$ is isotypic and γ_n is proximal, the subspace $V_1^{(n)}$ is T_{μ_n}-irreducible. The same is valid for any $V_i^{(n)} = g(V_1^{(n)})$ since $g\gamma_n g^{-1}$ is also proximal, for $g \in T_{\mu_n}$. Assume $Zc_0(T_{\mu_n})$ is not irreducible for n large ; then it follows that $p_n \in]1, d]$ and $r_n = dim\ V_1^{(n)} \in [1, d[$. It follows that we can assume $p_n = p$ and $r_n = r$ for n large. Hence, taking convergent subsequences of $V_i^{(n)}(1 \leq i \leq p)$ we obtain proper subspaces $V_i(1 \leq i \leq p)$ which are permuted by T_μ ; the irreducibility of T_μ implies that their sum is V, hence we have $V = \overset{p}{\underset{1}{\oplus}} V_i$, which contradicts the strong irreducibility of T_μ. Hence T_{μ_n} satisfies condition $i - p$ for n large. □

Proof of Theorem 6.3
Let $\lambda_n \in M^1(A)$ be a sequence which converges to $\lambda \in M^1(A)$ in the weak topology $\mathcal{W}(A)$ and let us denote by μ_n the projection of λ_n on G. We verify the stability of conditions 1, 2 in (H), since condition 3 follows of the definition of $\mathcal{W}(A)$ and condition 4 is a direct consequence of Proposition 6.4.

1) Assume that $supp(\lambda_n)$ has a fixed point $x_n \in V$ for n large. Since the closed subset $supp(\lambda_n)$ converges to $supp(\lambda)$, we can find a convergent subsequence of x_n to a point x in $(V)U(\mathbb{S}^{d-1})$, endowed with the visual topology, such that x is $supp(\lambda)$-invariant. If $x \in V$ we have a contradiction since $supp(\lambda)$ has no fixed point in V. If $x \in \mathbb{S}^{d-1}$, we have also a contradiction since the projective action of $supp(\mu)$ has no fixed point.

2) Using Lemma 6.4, since finiteness of moments for μ_n is valid, we get that for μ_n and for any $s \geq 0$, the corresponding operator P^s has a spectral gap on the relevant Hölder space on $\mathbb{S}^{d-1}$ (see [13]). The moment condition implies that perturbation theory is valid for the operators P^s. Hence the spectral radius $k(s)$ varies continuously. In particular, since we have $k(s) > 1$ for μ and $s > \alpha$, and $L(\mu) < 0$ the same is valid for μ_n with n large. Hence there exists $\alpha_n > 0$ close to α such that $k(\alpha_n) = 1$. □

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