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The Stable, Center-Stable, Center, Center-Unstable, Unstable Manifolds

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1. INTRODUCTION

Our purpose is to give a proof of the existence and smoothness of the invariant manifolds in the title for a system of ordinary differential equations defined in a neighborhood of a critical point, periodic orbit, or periodic surface. In system (1) below when the center equation (y -equation) is absent, the existence, and to some extent the smoothness, of stable and unstable manifolds is well known (see Theorem 4.1 on page 330 and Theorem 4.2 on page 333 of Coddington and Levinson [1], for example). For the associated perturbed system with no center equation [system (45) below with no y -equation] the existence of an invariant manifold called a periodic surface or integral manifold is well known [2]–[8]. For smoothness of this manifold see [2], p. 480; [3], [8]. Sections 6 and 7 of this paper are in part related to the idea of the classical periodic surface. See the remark at the end of Section 6 and Theorem 5 in Section 7. A discussion of the work of Krylov-Bogoliubov-Mitropolsky [2], [6], Diliberto [3], and Levinson [7] occurs in Hale [4]. The techniques we use in this paper are closely related to those of Krylov-Bogoliubov-Mitropolsky.

For Hamiltonian systems or systems of differential equations with an integral, the existence of two-dimensional subcenter manifolds (invariant manifolds related to pairs of imaginary eigenvalues) is known; see, for example, [9]–[12]. Lykova [13], [14] seems to have been the first to consider systems from the manifold standpoint with a center equation present but without an integral. However, she considered only the two-dimensional center case. Chen [15] has considered a corresponding problem for diffeomorphisms in the one-dimensional and two-dimensional center case. With no restriction on dimensions the center-stable, center, center-unstable

manifolds have occurred only recently in the work of Pliss [16] and Kelley [17], [18]. The theorem of Pliss states (without going into detail) that the stability of the center-stable manifold is completely determined by the stability of the center manifold. Although Pliss only proved the theorem for systems of ordinary differential equations in a neighborhood of a critical point, the same theorem is true for systems in a neighborhood of a periodic orbit or periodic surface. A proof of this extension to the Pliss theorem is given in [18]. In [17] is found an elementary application of the concept of the center manifold to Hamiltonian systems of equations.

Professor J. Hale has brought to our attention many of the references for this paper—in particular, [2], [4], [6], [7], [13], and [14]. The referee has pointed out that Lemma 2 below is a direct consequence of Gronwall's inequality since the right-hand derivative of the absolute value is less than or equal to the absolute value of the derivative. Section 4 of the present paper is due to D. V. Anasov.

2. NOTATION

The norm $|\cdot|$ will represent the Euclidean norm on vectors and the operator norm on matrices, and $\langle \cdot, \cdot \rangle$ will represent the usual scalar product on pairs of vectors. If $F = F(p)$ is a smooth vector valued function of the vector p , then F_p will represent the Jacobian matrix of partial derivatives. D_p^ρ will designate the usual partial differential operator; $D_p^\rho = \partial^{|\rho|} / \partial p_1^{\rho_1} \cdots \partial p_n^{\rho_n}$, where $\rho = (\rho_1, \dots, \rho_n)$ is an n -tuple of nonnegative integers, $|\rho| = \rho_1 + \cdots + \rho_n$, and $n = \dim p$. In the proof of Lemma 3 below, it will be convenient to use the notation $\rho = \rho_1 + \rho_2$, where $\rho_j (j = 1, 2)$ designates an n -tuple rather than a component of an n -tuple. The meaning will be clear from the context.

3. INVARIANT MANIFOLDS

Consider the real, $C^k (1 \leq k < \infty)$ system of ordinary differential equations

$$\begin{aligned} \dot{\theta} &= a + \tilde{\Theta}(\theta, x, y, z), & \dot{x} &= Ax + \tilde{X}(\theta, x, y, z), \\ \dot{y} &= By + \tilde{Y}(\theta, x, y, z), & \dot{z} &= Cz + \tilde{Z}(\theta, x, y, z), \end{aligned} \tag{1}$$

where A, B, C are constant square matrices in real canonical form; A has eigenvalues with negative real parts; B has eigenvalues with zero real parts

($B \equiv 0$ is allowed); C has eigenvalues with positive real parts; θ, x , etc., are vectors; a is a constant vector; $\tilde{\Theta}, \tilde{X}, \tilde{Y}, \tilde{Z}$ are defined and C^k in

$$N_\delta = \{(\theta, x, y, z) | \theta \text{ arbitrary, } |x| + |y| + |z| < \delta\},$$

and have multiple period ω in θ ; $\tilde{\Theta}, \tilde{X}, \tilde{Y}, \tilde{Z}, (\tilde{X}, \tilde{Y}, \tilde{Z})_{(x,y,z)} \equiv 0$ when $(x, y, z) = 0$.

Equation (1) represents a system of ordinary differential equations in a neighborhood of a critical point, periodic orbit, or periodic surface, depending on whether θ is absent from (1), $\dim \theta = 1$, or $\dim \theta > 1$, respectively. In the last two cases the condition $a \neq 0$ would also hold, but for our purposes one need not assume anything about a except that it is constant.

THEOREM 1. *For system (1) with $3 \leq k < \infty$, there exists invariant manifolds*

$$M^+ = \{(\theta, x, y, z) | \theta \text{ arbitrary, } |x| < \delta_1, y = v^+(\theta, x), z = w^+(\theta, x)\},$$

$$M^- = \{(\theta, x, y, z) | \theta \text{ arbitrary, } x = u^-(\theta, z), y = v^-(\theta, z), |z| < \delta_1\},$$

where v^+, w^+, u^-, v^- are real vector-valued functions defined and C^{k-2} in some neighborhood N_{δ_1} for δ_1 sufficiently small; v^+, w^+, u^-, v^- have multiple period ω in θ ; $v^+, w^+, u^-, v^-, (v^+, w^+, u^-, v^-)_{(x,z)} \equiv 0$ when $(x, z) = 0$; M^+, M^- are (locally) unique.

For system (1) with $2 \leq k < \infty$ there exist invariant manifolds

$$M^{*+} = \{(\theta, x, y, z) | \theta \text{ arbitrary, } |x| + |y| < \delta_1, z = w^{*+}(\theta, x, y)\},$$

$$M^* = \{(\theta, x, y, z) | \theta \text{ arbitrary, } x = u^*(\theta, y), |y| < \delta_1, z = w^*(\theta, y)\},$$

$$M^{*-} = \{(\theta, x, y, z) | \theta \text{ arbitrary, } x = u^{*-}(\theta, y, z), |y| + |z| < \delta_1\},$$

where w^{*+}, u^*, w^*, u^{*-} are real vector-valued functions defined and C^{k-1} in some neighborhood N_{δ_1} for δ_1 sufficiently small; w^{*+}, u^*, w^*, u^{*-} have multiple period ω in θ ; $w^{*+}, u^*, w^*, u^{*-}, (w^{*+}, u^*, w^*, u^{*-})_{(x,y,z)} \equiv 0$ when $(x, y, z) = 0$ (M^{*+}, M^*, M^{*-} need not be unique).

The invariant manifolds $M^+, M^{*+}, M^*, M^{*-}, M^-$ are called, respectively, the stable manifold, the center-stable manifold, the center manifold, the center-unstable manifold, and the unstable manifold.

Proof. Introducing the scalar change of variables $(x, y, z) \rightarrow (\lambda x, \lambda y, \lambda z)$ and multiplying $\tilde{\Theta}, \tilde{X}, \tilde{Y}, \tilde{Z}$ by $\phi(|x|^2 + |y|^2 + |z|^2 + K\lambda^2)$ where K is a sufficiently large positive constant and $\phi(r)$ is a C^∞ real-valued function satisfying $\phi(r) \equiv 1$ for $0 \leq r \leq \frac{1}{2}$ and $\phi(r) \equiv 0$ for $1 \leq r < \infty$, we obtain

$$\begin{aligned} \dot{x} &= a + \Theta(\theta, x, y, z, \lambda), & \dot{x} &= Ax + X(\theta, x, y, z, \lambda), \\ \dot{y} &= By + Y(\theta, x, y, z, \lambda), & \dot{z} &= Cz + Z(\theta, x, y, z, \lambda), \end{aligned} \tag{2}$$

where

$$\begin{aligned}\Theta(\theta, x, y, z, \lambda) &= \phi(|x|^2 + |y|^2 + |z|^2 + K\lambda^2)\tilde{\Theta}(\theta, \lambda x, \lambda y, \lambda z), \\ X(\theta, x, y, z, \lambda) &= \phi(|x|^2 + |y|^2 + |z|^2 + K\lambda^2)\lambda^{-1}\tilde{X}(\theta, \lambda x, \lambda y, \lambda z),\end{aligned}$$

etc., and the following conditions hold.

(2i) Θ, X, Y, Z exist and are continuous for all $(\theta, x, y, z, \lambda)$ and for each λ fixed are C^k in (θ, x, y, z) .

(2ii) Θ, X, Y, Z have multiple period ω in θ .

(2iii) $\Theta, X, Y, Z, (X, Y, Z)_{(x,y,z)} \equiv 0$ when $(x, y, z) = 0$.

(2iv) $\Theta, X, Y, Z \equiv 0$ for $|x|^2 + |y|^2 + |z|^2 \geq 1$.

(2v) $D_{(\theta,x,y,z)}^{\rho}(\Theta, X, Y, Z) \rightarrow 0$ uniformly in (θ, x, y, z) as $\lambda \rightarrow 0$ for $0 \leq |\rho| \leq k$.

If $\lambda \neq 0$, then systems (1) and (2) are locally (near $\{(\theta, x, y, z) | \theta$ arbitrary, $(x, y, z) = 0\}$) related by a scalar change of variables. Therefore it is sufficient to prove Theorem 1 for system (2). More precisely, however, we will prove Theorem 2 which will imply Theorem 1.

THEOREM 2. *For system (2) with $3 \leq k < \infty$ there exists invariant manifolds*

$$\begin{aligned}M_{\lambda}^{+} &= \{(\theta, x, y, z) | \theta \text{ arbitrary, } |x| < 1, y = v^{+}(\theta, x, \lambda), \\ &\quad z = w^{+}(\theta, x, \lambda), |\lambda| < \delta\}, \\ M_{\lambda}^{-} &= \{(\theta, x, y, z) | \theta \text{ arbitrary, } x = u^{-}(\theta, z, \lambda), y = v^{-}(\theta, z, \lambda), \\ &\quad |z| < 1, |\lambda| < \delta\},\end{aligned}$$

where the following conditions hold.

(3 i) $v^{+}, w^{+}, u^{-}, v^{-}$ are real vector-valued functions defined and continuous in

$$N_1^{\delta} = \{(\theta, x, z, \lambda) | \theta \text{ arbitrary, } |x| + |z| < 1, |\lambda| < \delta\}$$

for some $\delta > 0$ sufficiently small, and for each λ fixed these functions are C^{k-2} in (θ, x, z) .

(3 ii) $v^{+}, w^{+}, u^{-}, v^{-}$ have multiple period ω in θ .

(3 iii) $v^{+}, w^{+}, u^{-}, v^{-}, (v^{+}, w^{+}, u^{-}, v^{-})_{(x,z)} \equiv 0$ when $(x, z) = 0$.

For system (2) with $2 \leq k < \infty$ there exist invariant manifolds

$$\begin{aligned}M_{\lambda}^{*+} &= \{(\theta, x, y, z) | (\theta, x, y) \text{ arbitrary, } z = w^{*+}(\theta, x, y, \lambda), |\lambda| < \delta\}, \\ M_{\lambda}^{*} &= \{(\theta, x, y, z) | (\theta, y) \text{ arbitrary, } x = u^{*}(\theta, y, \lambda), z = w^{*}(\theta, y, \lambda), |\lambda| < \delta\}, \\ M_{\lambda}^{*-} &= \{(\theta, x, y, z) | (\theta, y, z) \text{ arbitrary, } x = u^{*-}(\theta, y, z, \lambda), |\lambda| < \delta\},\end{aligned}$$

where the following conditions hold.

(3 iv) w^{*+} , u^* , w^* , u^{*-} are real vector-valued functions defined and continuous in

$$N^\delta = \{(\theta, x, y, z, \lambda) | (\theta, x, y, z) \text{ arbitrary, } |\lambda| < \delta\}$$

for some $\delta > 0$ sufficiently small, and for each λ fixed these functions are C^{k-1} in (θ, x, y, z) .

(3 v) w^{*+} , u^* , w^* , u^{*-} have multiple period ω in θ .

(3 vi) w^{*+} , u^* , w^* , u^{*-} , $(w^{*+}, u^*, w^*, u^{*-})_{(x,y,z)} \equiv 0$ when $(x, y, z) = 0$. Moreover, M_λ^+ , M_λ^- are (locally) unique (but M_λ^{*+} , M_λ^* , M_λ^{*-} need not be).

Proof. Let (ψ, ξ, η, ζ) where $\psi = \psi(t) = \psi(t, \theta, x, y, z, \lambda)$, $\xi = \xi(t) = \xi(t, \theta, x, y, z, \lambda)$, etc., represent the unique solution of (2) with initial condition (θ, x, y, z) at $t = 0$. From (2 i, iv) the solution exists and is continuous for all $(t, \theta, x, y, z, \lambda)$ and for each λ fixed is C^k in (t, θ, x, y, z) .

The functions v^+ , w^+ which determine M_λ^+ will now be constructed as the unique solution to the differential-integral system

$$\dot{\theta} = a + \Theta(\theta, x, v^+(\theta, x, \lambda), w^+(\theta, x, \lambda), \lambda), \quad (3a)$$

$$\dot{x} = Ax + X(\theta, x, v^+(\theta, x, \lambda), w^+(\theta, x, \lambda), \lambda),$$

$$v^+(\theta, x, \lambda) = \int_{+\infty}^0 e^{-B\sigma} Y(\psi^+, \xi^+, v^+(\psi^+, \xi^+, \lambda), w^+(\psi^+, \xi^+, \lambda), \lambda) d\sigma, \quad (3b)$$

$$w^+(\theta, x, \lambda) = \int_{+\infty}^0 e^{-C\sigma} Z(\psi^+, \xi^+, v^+(\psi^+, \xi^+, \lambda), w^+(\psi^+, \xi^+, \lambda), \lambda) d\sigma,$$

where (ψ^+, ξ^+) with $\psi^+ = \psi^+(t) = \psi^+(t, \theta, x, v^+, w^+, \lambda)$, $\xi^+ = \xi^+(t) = \xi^+(t, \theta, x, v^+, w^+, \lambda)$ represents the unique solution to (3a) with initial condition (θ, x) at $t = 0$. To explicitly designate the *functional* dependence of the solution of (3a) on v^+ , w^+ , these functions are included in the arguments of ψ^+ , ξ^+ . In (3b) the functions ψ^+ , ξ^+ occurring in the integrand are understood to be $\psi^+(\sigma) = \psi^+(\sigma, \theta, x, v^+, w^+, \lambda)$, $\xi^+(\sigma) = \xi^+(\sigma, \theta, x, v^+, w^+, \lambda)$. Assuming (3) has a unique solution (v^+, w^+) , $v^+ = v^+(\theta, x, \lambda)$, $w^+ = w^+(\theta, x, \lambda)$, which satisfies conditions (3 i-iii), we can easily show that M_λ^+ is an invariant manifold for system (2). Since (3a) is an autonomous system,

$$\begin{aligned} \psi^+(\sigma, \psi^+(t, \theta, x, v^+, w^+, \lambda), \xi^+(t, \theta, x, v^+, w^+, \lambda), \lambda) \\ = \psi^+(t + \sigma, \theta, x, v^+, w^+, \lambda), \end{aligned}$$

$$\begin{aligned} \xi^+(\sigma, \psi^+(t, \theta, x, v^+, w^+, \lambda), \xi^+(t, \theta, x, v^+, w^+, \lambda), v^+, w^+, \lambda) \\ = \xi^+(t + \sigma, \theta, x, v^+, w^+, \lambda). \end{aligned}$$

Replacing (θ, x) in (3b) by $(\psi^+(t), \xi^+(t))$, we have

$$\begin{aligned} v^+(\psi^+(t), \xi^+(t), \lambda) &= \int_{+\infty}^0 e^{-B\sigma} Y(\psi^+(t + \sigma), \dots) d\sigma \\ &= \int_{+\infty}^t e^{-B(\tau-t)} Y(\psi^+(\tau), \dots) d\tau, \\ w^+(\psi^+(t), \xi^+(t), \lambda) &= \int_{+\infty}^0 e^{-C\sigma} Z(\psi^+(t + \sigma), \dots) d\sigma \\ &= \int_{+\infty}^t e^{-C(\tau-t)} Z(\psi^+(\tau), \dots) d\tau. \end{aligned}$$

Let $v^+(t) = v^+(\psi^+(t), \xi^+(t), \lambda)$, $w^+(t) = w^+(\psi^+(t), \xi^+(t), \lambda)$; then a direct calculation shows

$$\begin{aligned} (d/dt)v^+(t) &= Bv^+(t) + Y(\psi^+(t), \xi^+(t), v^+(t), w^+(t), \lambda), \\ (d/dt)w^+(t) &= Cw^+(t) + Z(\psi^+(t), \xi^+(t), v^+(t), w^+(t), \lambda). \end{aligned}$$

Because solutions of (2) are unique,

$$\begin{aligned} \psi(t, \theta, x, v^+(\theta, x, \lambda), w^+(\theta, x, \lambda), \lambda) &= \psi^+(t, \theta, x, v^+, w^+, \lambda), \\ \xi(t, \theta, x, v^+(\theta, x, \lambda), w^+(\theta, x, \lambda), \lambda) &= \xi^+(t, \theta, x, v^+, w^+, \lambda), \\ \eta(t, \theta, x, v^+(\theta, x, \lambda), w^+(\theta, x, \lambda), \lambda) \\ &= v^+(\psi^+(t, \theta, x, v^+, w^+, \lambda), \xi^+(t, \theta, x, v^+, w^+, \lambda), \lambda), \\ \zeta(t, \theta, x, v^+(\theta, x, \lambda), w^+(\theta, x, \lambda), \lambda) \\ &= w^+(\psi^+(t, \theta, x, v^+, w^+, \lambda), \xi^+(t, \theta, x, v^+, w^+, \lambda), \lambda), \end{aligned}$$

and M_λ^+ is an invariant manifold for (2).

To solve (3), inequalities involving the matrices A, B, C are basic.

LEMMA 1. *There exists $\mu > 0$, $\gamma \geq 0$, $\frac{1}{2}\mu > \gamma$, such that, for all x, y, z ,*

$$\begin{aligned} \langle Ax, x \rangle &\leq -2\mu |x|^2, \\ |\langle By, y \rangle| &\leq \gamma |y|^2, \\ \langle Cz, z \rangle &\geq 2\mu |z|^2 \end{aligned} \tag{4}$$

and these inequalities imply

$$\begin{aligned} |e^{At}| &\leq e^{-2\mu t} & (0 \leq t < \infty), \\ |e^{Bt}| &\leq e^{\gamma|t|} & (-\infty < t < \infty), \\ |e^{-Ct}| &\leq e^{-2\mu t} & (0 \leq t < \infty). \end{aligned} \tag{5}$$

This lemma is well known and the proof is omitted. These inequalities are introduced as a lemma because they are used extensively below.

We now develop a useful generalization of an inequality used by Hale [5].

LEMMA 2. *Let a be a nonnegative constant, and let $b(t)$ be a continuous real-valued function defined on a finite or infinite interval I which contains the origin. If $\varphi(t)$ is a C^1 vector or matrix which satisfies*

$$|\dot{\varphi}(t)| \leq a|\varphi(t)| + b(t), \quad t \in I \quad (6)$$

then

$$|\varphi(t)| \leq e^{a|t|} \left\{ |\varphi(0)| + \int_0^t e^{-a|\tau|} b(\tau) d\tau \right\}, \quad t \in I.$$

Proof. From (6) for $t \geq 0$,

$$\begin{aligned} |\varphi(t)| - |\varphi(0)| &\leq |\varphi(t) - \varphi(0)| = \left| \int_0^t \dot{\varphi}(\tau) d\tau \right| \\ &\leq \int_0^t |\dot{\varphi}(\tau)| d\tau \leq \int_0^t \{a|\varphi(\tau)| + b(\tau)\} d\tau. \end{aligned}$$

Therefore,

$$|\varphi(t)| \leq |\varphi(0)| + \int_0^t \{a|\varphi(\tau)| + b(\tau)\} d\tau. \quad (7)$$

Consider the scalar function $\psi(t)$ defined by

$$\dot{\psi}(t) = a\psi(t) + b(t), \quad \psi(0) = |\varphi(0)|.$$

Thus

$$\psi(t) = |\varphi(0)| + \int_0^t \{a\psi(\tau) + b(\tau)\} d\tau. \quad (8)$$

Subtracting (8) from (7), we obtain

$$|\varphi(t)| - \psi(t) \leq \int_0^t a\{|\varphi(\tau)| - \psi(\tau)\} d\tau$$

and it now follows from the Gronwall inequality ([1], problem 1, Chapter 1) that, for $t \geq 0$, $t \in I$,

$$|\varphi(t)| \leq \psi(t) = e^{at} \left\{ |\varphi(0)| + \int_0^t e^{-a\tau} b(\tau) d\tau \right\}.$$

The proof for $t < 0$ is similar. This completes the proof of Lemma 2.

We now proceed to solve (3) by means of a contraction mapping in a Banach space. For l, m positive integers define

$$\mathfrak{X}_m^l = \{\chi = \chi(\theta, x) \text{ satisfying (9i-v)}\}.$$

(9 i) χ is a real vector-valued function defined and C^l for all θ and $|x| < 1$.

(9 ii) $\dim \chi = m$.

(9 iii) χ has multiple period ω in θ .

(9 iv) $\chi, \chi_x \equiv \mathbf{0}$ when $x = 0$.

(9 v) $\|\chi\| = \max_{0 \leq |\rho| \leq l} \sup_{\theta} \sup_{|x| < 1} |D_{(\theta, x)}^\rho \chi(\theta, x)| < \infty$.

With the norm $\|\cdot\|$ given in (9 v), \mathfrak{X}_m^l is a Banach space. Define $\tilde{\mathfrak{X}}_m^l$ to be the closed unit ball in \mathfrak{X}_m^l .

Let $\mathfrak{X}_y^l = \mathfrak{X}_{\dim y}^l$, etc. For $v \in \tilde{\mathfrak{X}}_y^{k-1}$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, $2 \leq k < \infty$, consider the system

$$\begin{aligned} \dot{\theta} &= a + \Theta(\theta, x, v(\theta, x), w(\theta, x), \lambda), \\ \dot{x} &= Ax + X(\theta, x, v(\theta, x), w(\theta, x), \lambda), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \psi^{(v, w)} &= \psi = \psi(t) = \psi(t, \theta, x, v, w, \lambda), \\ \xi^{(v, w)} &= \xi = \xi(t) = \xi(t, \theta, x, v, w, \lambda) \end{aligned} \tag{11}$$

represents the unique solution of (10) with initial condition (θ, x) at $t = 0$.

LEMMA 3. For $0 \leq t < \infty$, all $\theta, |x| < 1$, $v \in \tilde{\mathfrak{X}}_y^{k-1}$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, $|\lambda| < \delta_0$ with $\delta_0 > 0$ chosen sufficiently small, the solution (ψ, ξ) of (10) [given explicitly in (11)] exists and satisfies

$$\begin{aligned} (11 \text{ i}) \quad & |\xi(t)| \leq e^{-\mu t} |x|, \\ (11 \text{ ii}) \quad & |D_{(\theta, x)}^\rho \psi(t)| \leq \alpha(t) e^{\beta(\lambda)t} \quad (1 \leq |\rho| \leq k-1), \\ (11 \text{ iii}) \quad & |D_{(\theta, x)}^\rho \xi(t)| \leq \alpha(t) e^{(-\mu + \beta(\lambda))t} \quad (1 \leq |\rho| \leq k-1), \end{aligned}$$

where $\alpha(t)$ is a polynomial in t with positive coefficients, $\beta(\lambda) \geq 0$ is continuous in λ , $\beta(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$; these inequalities are uniform in θ , $v \in \tilde{\mathfrak{X}}_y^{k-1}$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, and (11 ii, iii) hold uniformly in $|x| < 1$ also.

Proof. The proof is accomplished in a finite number of steps wherein we find a succession of α 's and β 's. In (11 ii, iii) we take α and β to be the largest of the α 's and β 's constructed, respectively.

If $F = F(\theta, x)$ is any smooth vector-valued function of (θ, x) , then

$$F(\theta, x') - F(\theta, x'') = \int_0^1 F_x(\theta, sx' + (1-s)x'') ds \cdot \{x' - x''\}.$$

In particular if $F(\theta, 0) \equiv 0$, then

$$F(\theta, x) = \int_0^1 F_x(\theta, sx) ds \cdot x.$$

Thus from (2 iii), (9 iv), for $v \in \tilde{\mathfrak{X}}_v^1$, $w \in \tilde{\mathfrak{X}}_z^1$, $|x| < 1$, we have

$$\begin{aligned} X(\theta, x, v(\theta, x), w(\theta, x), \lambda) = & \int_0^1 \{X_x(\theta, sx, v(\theta, sx), w(\theta, sx), \lambda) \\ & + X_y(\theta, sx, \dots)v_x(\theta, sx) \\ & + X_z(\theta, sx, \dots)w_x(\theta, sx)\} ds \cdot x. \end{aligned}$$

Hence from (2 v),

$$|X(\theta, x, v(\theta, x), w(\theta, x), \lambda)| \leq \mu |x|, \quad (|x| < 1, |\lambda| < \delta_0) \quad (12)$$

for $\delta_0 > 0$ sufficiently small. From (4), (10)–(12) with $|x| < 1$, $|\lambda| < \delta_0$,

$$\begin{aligned} \dot{\xi} &= A\xi + X(\psi, \xi, v(\psi, \xi), w(\psi, \xi), \lambda), \\ (d/dt)|\xi|^2 &= 2\langle A\xi, \xi \rangle + 2\langle X(\psi, \xi, \dots), \xi \rangle, \\ (d/dt)|\xi|^2 &\leq -4\mu|\xi|^2 + 2|X||\xi| \leq -2\mu|\xi|^2, \end{aligned}$$

which implies inequality (11 i) and the existence of (ψ, ξ) as stated. From (10),

$$\dot{\psi}_{(\theta, x)} = [\Theta_\theta + \Theta_y v_\theta + \Theta_z w_\theta] \psi_{(\theta, x)} + [\Theta_x + \Theta_y v_x + \Theta_z w_x] \xi_{(\theta, x)}, \quad (13)$$

$$\dot{\xi}_{(\theta, x)} = [A + X_x + X_y v_x + X_z w_x] \xi_{(\theta, x)} + [X_\theta + X_y v_\theta + X_z w_\theta] \psi_{(\theta, x)},$$

where $\Theta_\theta = \Theta_\theta(\psi, \xi, v(\psi, \xi), w(\psi, \xi), \lambda)$, etc. Since the matrix [in (13)]

$$A' = A + X_x + X_y v_x + X_z w_x \rightarrow A$$

as $\lambda \rightarrow 0$, it follows from (4) for $|\lambda| < \delta_0$, δ_0 restricted further if necessary,

$$\langle A'p, p \rangle \leq -\mu|p|^2 \quad (14)$$

for all real vectors p , $\dim p = \dim x$. The procedure used to obtain (12) also yields

$$|X_\theta + X_y v_\theta + X_z w_\theta| \leq \beta_1(\lambda) |\xi| \leq \beta_1(\lambda) e^{-\mu t} \quad (15)$$

for $|x| < 1$ where $X_\theta = X_\theta(\psi, \xi, \dots)$, etc.; $\beta_1(\lambda) \geq 0$, $\beta_1(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. From (2 v) and (13)–(15),

$$\begin{aligned} |\dot{\psi}_{(\theta, x)}| &\leq \beta_2(\lambda) |\psi_{(\theta, x)}| + \beta_2(\lambda) |\xi_{(\theta, x)}|, \\ (d/dt) |\xi_{(\theta, x)}|^2 &\leq -2\mu |\xi_{(\theta, x)}|^2 + 2\beta_2(\lambda)e^{-\mu t} |\psi_{(\theta, x)}| |\xi_{(\theta, x)}|, \\ |\psi_{(\theta, x)}(0)| &= |\xi_{(\theta, x)}(0)| = 1, |x| < 1, |\lambda| < \delta_0, \end{aligned} \quad (16)$$

where $\beta_2(\lambda) \geq 0$, $\beta_2(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. From Lemma 2 for $0 \leq t < \infty$,

$$|\psi_{(\theta, x)}(t)| \leq e^{\beta_2(\lambda)t} \left\{ 1 + \int_0^t e^{-\beta_2(\lambda)\tau} |\xi_{(\theta, x)}(\tau)| d\tau \right\}, \quad (17)$$

so that

$$\begin{aligned} (d/dt) |\xi_{(\theta, x)}|^2 &\leq -2\mu |\xi_{(\theta, x)}|^2 + 2\beta_2(\lambda)e^{-\mu t} e^{\beta_2(\lambda)t} \\ &\quad \times \left\{ 1 + \int_0^t e^{-\beta_2(\lambda)\tau} |\xi_{(\theta, x)}(\tau)| d\tau \right\} |\xi_{(\theta, x)}|. \end{aligned} \quad (18)$$

By restricting δ_0 further if necessary, we may assume $\beta_2(\lambda) < \mu$ for $|\lambda| \leq \delta_0$ so that near $t = 0$, $|\xi_{(\theta, x)}(t)|$ is a decreasing function of t ; $|\xi_{(\theta, x)}(t)| \leq 1$ for $0 \leq t < \epsilon$, $\epsilon > 0$ sufficiently small. Now compare (18) with the real scalar equation

$$\frac{d}{dt} f^2 = -2\mu f^2 + 2\beta_2(\lambda)e^{(-\mu + \beta_2(\lambda))t} \left\{ 1 + \int_0^t e^{-\beta_2(\lambda)\tau} 2d\tau \right\} f, \quad (19)$$

with the initial condition $f(0) = 1$. As long as $|f(t)| \leq 2$ holds, it follows that

$$|\dot{\xi}_{(\theta, x)}(t)| \leq f(t). \quad (20)$$

From (19)

$$\dot{f} = -\mu f + \beta_2(\lambda)e^{(-\mu + \beta_2(\lambda))t} \left\{ 1 + \int_0^t e^{-\beta_2(\lambda)\tau} 2d\tau \right\}.$$

Since $f(0) = 1$, the inequality $\dot{f} \geq -\mu f$ implies $f(t) > 0$ for $0 \leq t < \infty$. Thus

$$\begin{aligned} \dot{f} &\leq (-\mu + \beta_2(\lambda))f + \beta_2(\lambda)e^{(-\mu + \beta_2(\lambda))t} \{1 + 2t\}, \\ f(t) &\leq \{1 + \beta_2(\lambda)(t + t^2)\} e^{(-\mu + \beta_2(\lambda))t}, \end{aligned}$$

and by restricting δ_0 further if necessary we have $|f(t)| \leq 2$ for $0 \leq t < \infty$ so that (20) holds for all $0 \leq t < \infty$. Hence we have proved (11 iii) for $|\rho| = 1$. The crude inequality $|\dot{\xi}_{(\theta, x)}(t)| \leq 2$ in (17) yields

$$|\psi_{(\theta, x)}(t)| \leq e^{\beta_2(\lambda)t} \left\{ 1 + \int_0^t e^{-\beta_2(\lambda)\tau} 2d\tau \right\} \leq \{1 + 2t\} e^{\beta_2(\lambda)t},$$

so that (11 ii) holds for $|\rho| = 1$. We now proceed inductively. Let $D_{(\theta,x)}^{\rho}\psi = \psi_{\rho}$, $D_{(\theta,x)}^{\rho}\xi = \xi_{\rho}$, and consider the case $\rho = \rho_1 + \rho_2$, $|\rho_1| = |\rho_2| = 1$, $|\rho| = 2$. Assuming $k \geq 3$ (otherwise we are done), we have from (10)

$$\begin{aligned}\dot{\psi}_{\rho} &= [\Theta_{\theta} + \Theta_y v_{\theta} + \Theta_z w_{\theta}] \psi_{\rho} \\ &\quad + [\Theta_x + \Theta_y v_x + \Theta_z w_x] \xi_{\rho} + \Theta_{\theta\theta} \psi_{\rho_2} \psi_{\rho_1} + \dots, \\ \dot{\xi}_{\rho} &= [A + X_x + X_y v_x + X_z w_x] \xi_{\rho} \\ &\quad + [X_{\theta} + X_y v_{\theta} + X_z w_{\theta}] \psi_{\rho} + X_{x\theta} \psi_{\rho_2} \xi_{\rho_1} + \dots.\end{aligned}\tag{21}$$

The notation $\Theta_{\theta\theta} \psi_{\rho_2} \psi_{\rho_1}$, etc., is defined by writing out (21) in complete detail. Since (11 ii, iii) hold for $|\rho_1| = |\rho_2| = 1$, we obtain from (21)

$$\begin{aligned}|\dot{\psi}_{\rho}| &\leq \beta_3(\lambda) |\psi_{\rho}| + \beta_3(\lambda) |\xi_{\rho}| + \beta_3(\lambda) \alpha_1(t) e^{\beta_3(\lambda)t} \\ (d/dt) |\xi_{\rho}|^2 &\leq -2\mu |\xi_{\rho}|^2 + 2\beta_3(\lambda) e^{-\mu t} |\psi_{\rho}| |\xi_{\rho}| + 2\beta_3(\lambda) \alpha_1(t) e^{(-\mu+\beta_3(\lambda))t} |\xi_{\rho}|, \\ |\psi_{\rho}(0)| &= |\xi_{\rho}(0)| = 0,\end{aligned}\tag{22}$$

where $\alpha_1(t)$ is a polynomial with positive coefficients and $\beta_3(\lambda) \geq \beta_2(\lambda) \geq 0$, $\beta_3(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. From Lemma 2

$$|\psi_{\rho}(t)| \leq \beta_3(\lambda) e^{\beta_3(\lambda)t} \int_0^t e^{-\beta_3(\lambda)\tau} \{ |\xi_{\rho}(\tau)| + \alpha_1(\tau) e^{\beta_3(\lambda)\tau} \} d\tau.$$

Since $|\xi_{\rho}(0)| = 0$, it follows that near $t = 0$

$$|\xi_{\rho}(t)| \leq 1,\tag{23}$$

and as long as this inequality holds, it follows that

$$|\psi_{\rho}(t)| \leq \beta_3(\lambda) \alpha_2(t) e^{\beta_3(\lambda)t}\tag{24}$$

where

$$\alpha_2(t) = \int_0^t \{1 + \alpha_1(\tau)\} d\tau.$$

Using inequality (24) in (22) we have

$$(d/dt) |\xi_{\rho}|^2 \leq -2\mu |\xi_{\rho}|^2 + 2[\beta_3(\lambda)]^2 \alpha_2(t) e^{-\mu t} |\xi_{\rho}| + 2\beta_3(\lambda) \alpha_1(t) e^{(-\mu+\beta_3(\lambda))t} |\xi_{\rho}|.$$

Comparing this differential inequality with the scalar equation

$$\begin{aligned}(d/dt) f^2 &= -2\mu f^2 + 4\beta_4(\lambda) \alpha_3(t) e^{(-\mu+\beta_4(\lambda))t} f, & f(0) &= \frac{1}{2}, \\ f &= -\mu f + 2\beta_4(\lambda) \alpha_3(t) e^{(-\mu+\beta_4(\lambda))t}, & f(0) &= \frac{1}{2},\end{aligned}$$

where $\alpha_3(t)$ is a polynomial in t with positive coefficients, $\alpha_3(t) \geq \alpha_2(t)$, $\alpha_3(t)$ for $0 \leq t < \infty$; $\beta_4(\lambda) \geq \beta_3(\lambda)$, $[\beta_3(\lambda)]^2 \geq 0$; $\beta_4(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$; it follows that

$$|\xi_\rho(t)| \leq f(t) \quad (25)$$

for $t \geq 0$ as long as $|\xi_\rho(t)| \leq 1$. Since

$$f \geq -\mu f, \quad f(0) = \frac{1}{2},$$

we have $f(t) > 0$ for $0 \leq t < \infty$, so that

$$\begin{aligned} f &\leq (-\mu + \beta_4(\lambda))f + 2\beta_4(\lambda)\alpha_3(t)e^{(-\mu+\beta_4(\lambda))t}, \\ f(t) &\leq e^{(-\mu+\beta_4(\lambda))t} \left\{ \frac{1}{2} + \int_0^t 2\beta_4(\lambda)\alpha_3(\tau) d\tau \right\}. \end{aligned} \quad (26)$$

Thus by restricting δ_0 further, if necessary, it follows that $f(t) \leq 1$ for $0 \leq t < \infty$ and hence (23)–(25) are valid for all $0 \leq t < \infty$. Inequalities (24)–(26) show that (11 ii, iii) hold for $1 \leq |\rho| \leq 2$. By continuing in this manner (a finite number of steps), one proves (11 ii, iii) valid for all $1 \leq |\rho| \leq k - 1$. This completes the proof of Lemma 3.

For $v \in \tilde{\mathfrak{X}}_y^{k-1}$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, define

$$(T_1 v)(\theta, x) = \int_{+\infty}^0 e^{-B\sigma} Y(v, w) d\sigma, \quad (27a)$$

$$(T_2 w)(\theta, x) = \int_{+\infty}^0 e^{-C\sigma} Z(v, w) d\sigma, \quad (27b)$$

where

$$\begin{aligned} Y(v, w) &= Y(\psi^{(v, w)}, \xi^{(v, w)}, v^{(v, w)}, w^{(v, w)}, \lambda), \\ Z(v, w) &= Z(\psi^{(v, w)}, \xi^{(v, w)}, v^{(v, w)}, w^{(v, w)}, \lambda), \\ v^{(v, w)} &= v(\psi^{(v, w)}, \xi^{(v, w)}), \\ w^{(v, w)} &= w(\psi^{(v, w)}, \xi^{(v, w)}), \end{aligned}$$

and $(\psi^{(v, w)}, \xi^{(v, w)})$ is the solution of (10) which is given explicitly in (11).

For $(v, w) \in \tilde{\mathfrak{X}}_y^{k-1} \times \tilde{\mathfrak{X}}_z^{k-1}$ define

$$T(v, w) = (T_1 v, T_2 w).$$

LEMMA 4. For $2 \leq k < \infty$ and for $|\lambda| \leq \delta_1 \leq \delta_0$, $\delta_1 > 0$ sufficiently small, the transformation T maps $\tilde{\mathfrak{X}}_y^{k-1} \times \tilde{\mathfrak{X}}_z^{k-1}$ into itself and is a contraction in the C^{k-2} topology:

$$\begin{aligned} \|(v, w)\| &= \max(\|v\|, \|w\|), \\ \|\cdot\| &= \|\cdot\|_{k-2} = \max_{0 \leq |\rho| \leq k-2} \sup_{\theta} \sup_{|x| \leq 1} |D_{(\theta, x)}^\rho \cdot|. \end{aligned}$$

Proof. The fact that T maps $\tilde{\mathfrak{X}}_y^{k-1} \times \tilde{\mathfrak{X}}_z^{k-1}$ into itself for $|\lambda| \leq \delta_1$, δ_1 sufficiently small, is an immediate consequence of (2 iii, v), (5), and Lemma 3.

To show that T is a contraction in the C^{k-2} topology, it is sufficient to show that T_1 is a contraction on $\tilde{\mathfrak{X}}_y^{k-1}$ in the C^{k-2} topology uniformly in $w \in \tilde{\mathfrak{X}}_z^{k-1}$, and similarly that T_2 is a contraction on $\tilde{\mathfrak{X}}_z^{k-1}$ in the C^{k-2} topology uniformly in $v \in \tilde{\mathfrak{X}}_y^{k-1}$. We will give the argument for T_1 ; the argument for T_2 is completely analogous. To show T_1 a contraction, it is sufficient to show uniformly in θ , $|x| < 1$, $|\lambda| \leq \delta_1$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, $0 \leq |\rho| \leq k-2$, that the inequality

$$|D_{(\theta, w)}^\rho(T_1 v^1 - T_1 v^2)| \leq \frac{1}{2} \|v^1 - v^2\| \quad (28)$$

holds for $v^1, v^2 \in \tilde{\mathfrak{X}}_y^{k-1}$.

Let $(\psi^j, \xi^j) = (\psi^{(v^j, w)}, \xi^{(v^j, w)})$, $(j = 1, 2)$, and let $Y^j = Y^{(v^j, w)} = Y(\psi^j, \xi^j, \dots)$, $(j = 1, 2)$. To prove (28) we will show that uniformly in θ , $|x| < 1$, $|\lambda| \leq \delta_1$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, $0 \leq |\rho| \leq k-2$, the inequalities

$$\begin{aligned} |D_{(\theta, w)}^\rho(\psi^1 - \psi^2)| &\leq \alpha(t)e^{\beta(\lambda)t} \|v^1 - v^2\|, \\ |D_{(\theta, w)}^\rho(\xi^1 - \xi^2)| &\leq \alpha(t)e^{(-\mu + \beta(\lambda))t} \|v^1 - v^2\| \end{aligned} \quad (29)$$

hold where $\alpha(t)$ is a polynomial in t with positive coefficients, $\beta(\lambda) \geq 0$ is continuous in $|\lambda| \leq \delta_1$, $\beta(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. If we suppose (29) valid, then, by restricting δ_1 to be sufficiently small, inequality (28) is immediate. One computes from (27)

$$D_{(\theta, w)}^\rho(T_1 v^1 - T_1 v^2) = \int_{+\infty}^0 e^{-B\sigma} D_{(\theta, w)}^\rho(Y^1 - Y^2) d\sigma$$

and then uses (2 iii, v), (5), Lemma 3, and (29) to verify that the interchange of differentiation and integration is valid and that (28) holds. The mean-value theorem as presented at the beginning of the proof of Lemma 3 is used repeatedly.

Hence it remains to prove (29). From (10),

$$\begin{aligned} (d/dt)(\psi^1 - \psi^2) &= \Theta^1 - \Theta^2, \\ (d/dt)(\xi^1 - \xi^2) &= A(\xi^1 - \xi^2) + X^1 - X^2, \end{aligned} \quad (30)$$

where $\Theta^1 = \Theta(\psi^1, \xi^1, v^1(\psi^1, \xi^1), w(\psi^1, \xi^1), \lambda)$, etc. Thus

$$\begin{aligned} |(d/dt)(\psi^1 - \psi^2)| &\leq |\Theta^1 - \Theta^2|, \\ (d/dt)|\xi^1 - \xi^2|^2 &= 2\langle A(\xi^1 - \xi^2), \xi^1 - \xi^2 \rangle + 2\langle X^1 - X^2, \xi^1 - \xi^2 \rangle \end{aligned}$$

and, from inequality (4),

$$(d/dt) |\xi^1 - \xi^2|^2 \leq -4\mu |\xi^1 - \xi^2|^2 + 2|X^1 - X^2| \cdot |\xi^1 - \xi^2|.$$

Using the mean value theorem, we obtain

$$|\Theta^1 - \Theta^2| \leq \beta_1(\lambda) \{ |\psi^1 - \psi^2| + |\xi^1 - \xi^2| + |v^1(\psi^1, \xi^1) - v^2(\psi^2, \xi^2)| \\ + |w(\psi^1, \xi^1) - w(\psi^2, \xi^2)| \},$$

where $\beta_1(\lambda) \geq 0$, $\beta_1(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. Since

$$v^1(\psi^1, \xi^1) - v^2(\psi^2, \xi^2) = v^1(\psi^1, \xi^1) - v^1(\psi^2, \xi^2) + v^1(\psi^2, \xi^2) - v^2(\psi^2, \xi^2)$$

and $v^1 \in \tilde{\mathfrak{X}}_y^{k-1}$, $w \in \tilde{\mathfrak{X}}_z^{k-1}$, we have

$$|v^1(\psi^1, \xi^1) - v^2(\psi^2, \xi^2)| \leq |\psi^1 - \psi^2| + |\xi^1 - \xi^2| + \|v^1 - v^2\|, \\ |w(\psi^1, \xi^1) - w(\psi^2, \xi^2)| \leq |\psi^1 - \psi^2| + |\xi^1 - \xi^2|. \quad (31)$$

Thus

$$|\Theta^1 - \Theta^2| \leq \beta_2(\lambda) \{ |\psi^1 - \psi^2| + |\xi^1 - \xi^2| + \|v^1 - v^2\| \},$$

where $\beta_2(\lambda) = 3\beta_1(\lambda) \geq 0$, $\beta_2(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. By using properties (2 iii, v) and applying the mean-value theorem, we obtain

$$|X^1 - X^2| \leq \beta_3(\lambda) \{ |\xi^1| + |\xi^2| \} \cdot \{ |\psi^1 - \psi^2| \\ + |\xi^1 - \xi^2| + |v^1(\psi^1, \xi^1) - v^2(\psi^2, \xi^2)| \\ + |w(\psi^1, \xi^1) - w(\psi^2, \xi^2)| \}, \quad (32)$$

where $\beta_3(\lambda) \geq 0$, $\beta_3(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. From (11 i) for $|x| < 1$,

$$|\xi^1(t)| + |\xi^2(t)| \leq 2e^{-\mu t}, \quad (33)$$

so that (31) and (33) in (32) yields

$$|X^1 - X^2| \leq \beta_4(\lambda) e^{-\mu t} \{ |\psi^1 - \psi^2| + |\xi^1 - \xi^2| + \|v^1 - v^2\| \},$$

where $\beta_4(\lambda) \geq \beta_3(\lambda) + \beta_2(\lambda) \geq 0$, $\beta_4(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. By restricting δ_1 to be sufficiently small we may assume $\beta_4(\lambda) \leq \mu$ for $|\lambda| \leq \delta_1$; then from (30) we finally obtain

$$|(d/dt)(\psi^1 - \psi^2)| \leq \beta_4(\lambda) \{ |\psi^1 - \psi^2| + |\xi^1 - \xi^2| + \|v^1 - v^2\| \}, \quad (34a)$$

$$(d/dt) |\xi^1 - \xi^2|^2 \leq -2\mu |\xi^1 - \xi^2|^2 \\ + 2\beta_4(\lambda) e^{-\mu t} \{ |\psi^1 - \psi^2| + \|v^1 - v^2\| \} \cdot |\xi^1 - \xi^2|. \quad (34b)$$

Since $\psi^1(0) - \psi^2(0) = 0$, we have from (34a) and Lemma 2, for $t \geq 0$,

$$|\psi^1(t) - \psi^2(t)| \leq \beta_4(\lambda) e^{\beta_4(\lambda)t} \int_0^t e^{-\beta_4(\lambda)\tau} \{|\xi^1(\tau) - \xi^2(\tau)| + \|v^1 - v^2\|\} d\tau.$$

Since $\xi^1(0) - \xi^2(0) = 0$, the inequality

$$|\xi^1(t) - \xi^2(t)| \leq \|v^1 - v^2\| \quad (35)$$

is valid in some neighborhood of $t = 0$, and as long as (35) remains valid ($t \geq 0$),

$$|\psi^1(t) - \psi^2(t)| \leq 2\beta_4(\lambda) t e^{\beta_4(\lambda)t} \|v^1 - v^2\|. \quad (36)$$

Using (36) in (34b) we obtain

$$\begin{aligned} (d/dt) |\xi^1 - \xi^2|^2 &\leq -2\mu |\xi^1 - \xi^2|^2 \\ &\quad + 2\beta_4(\lambda) \alpha_1(t) e^{(-\mu + \beta_4(\lambda))t} \|v^1 - v^2\| \cdot |\xi^1 - \xi^2|, \end{aligned} \quad (37)$$

where $\alpha_1(t) = 1 + 2\beta_4(\lambda)t$. Comparing (37) with the scalar equation

$$(d/dt) f^2 = -2\mu f^2 + 2\beta_4(\lambda) \alpha_1(t) e^{(-\mu + \beta_4(\lambda))t} \|v^1 - v^2\| \cdot f \quad (38)$$

with $f(0) = \frac{1}{2} \|v^1 - v^2\| > 0$, we see that

$$|\xi^1(t) - \xi^2(t)| \leq f(t) \quad (39)$$

holds so long as (35) remains valid. From (38),

$$\dot{f} = -\mu f + \beta_4(\lambda) \alpha_1(t) e^{(-\mu + \beta_4(\lambda))t} \|v^1 - v^2\|$$

and since, for $0 \leq t < \infty$,

$$\dot{f} \geq -\mu f, \quad f(0) > 0$$

is valid, it follows that $f(t) > 0$ for $0 \leq t < \infty$ and therefore

$$\begin{aligned} f &\leq (-\mu + \beta_4(\lambda))f + \beta_4(\lambda) \alpha_1(t) e^{(-\mu + \beta_4(\lambda))t} \|v^1 - v^2\|, \\ f(t) &\leq e^{(-\mu + \beta_4(\lambda))t} \left\{ \frac{1}{2} \|v^1 - v^2\| + \int_0^t \beta_4(\lambda) \alpha_1(\tau) d\tau \|v^1 - v^2\| \right\}, \\ f(t) &\leq \left\{ \frac{1}{2} + \beta_4(\lambda) \alpha_2(t) \right\} e^{(-\mu + \beta_4(\lambda))t} \|v^1 - v^2\|, \end{aligned}$$

where

$$\alpha_2(t) = \int_0^t \alpha_1(\tau) d\tau.$$

Therefore, by restricting δ_1 further if necessary, we have for $|\lambda| \leq \delta_1$ that

$$f(t) \leq \|v^1 - v^2\|$$

holds for all $0 \leq t < \infty$. Thus (35)–(37) and (39) are valid for all $0 \leq t < \infty$ and (29) is proved for the case $|\rho| = 0$.

Assuming that $k \geq 3$ (otherwise we are finished), we now want to show that (29) is valid for $|\rho| = 1$. Let $D_{(\theta, x)}^1 \psi^1 = \psi_\rho^1$, etc. Then from (10),

$$\begin{aligned} (d/dt)(\psi_\rho^1 - \psi_\rho^2) &= \Theta_\theta^1 \psi_\rho^1 - \Theta_\theta^2 \psi_\rho^2 + \Theta_x^1 \xi_\rho^1 - \Theta_x^2 \xi_\rho^2 + \dots, \\ (d/dt)(\xi_\rho^1 - \xi_\rho^2) &= A(\xi_\rho^1 - \xi_\rho^2) + X_\theta^1 \psi_\rho^1 - X_\theta^2 \psi_\rho^2 + \dots, \end{aligned}$$

and, by means of (2 iii, v), Lemma 1, Lemma 3, inequality (29) for the case $|\rho| = 0$, and the mean-value theorem, one achieves

$$\begin{aligned} |(d/dt)(\psi_\rho^1 - \psi_\rho^2)| &\leq \beta_5(\lambda) \{ |\psi_\rho^1 - \psi_\rho^2| + |\xi_\rho^1 - \xi_\rho^2| \} \\ &\quad + \beta_5(\lambda) \alpha_3(t) e^{\beta_5(\lambda)t} \|v^1 - v^2\|, \\ |(d/dt)|\xi_\rho^1 - \xi_\rho^2|^2 &\leq -2\mu |\xi^1 - \xi^2|^2 + 2\beta_5(\lambda) \alpha_3(t) e^{(-\mu + \beta_5(\lambda))t} \cdot \\ &\quad \cdot \{ |\psi_\rho^1 - \psi_\rho^2| + \|v^1 - v^2\| \} \cdot |\xi^1 - \xi^2|, \end{aligned}$$

where $\alpha_3(t)$ is a polynomial in t with positive coefficients, $\beta_5(\lambda) \geq 0$, $\beta_5(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. By restricting δ_1 further if necessary, one now readily establishes that (29) is valid for all $|\rho| \leq 1$. In an analogous manner one proceeds inductively (a finite number of steps) to establish (29) for all $0 \leq |\rho| \leq k-2$. This completes the proof of Lemma 4.

The fixed point of the transformation T is designated $v^+ = v^+(\theta, x, \lambda)$, $w^+ = w^+(\theta, x, \lambda)$, and these functions define the stable manifold M_λ^+ . The unstable manifold M_λ^- is constructed in an analogous manner.

The function $w^{*+} = w^{*+}(\theta, x, y, \lambda)$ which defines the center-stable manifold M_λ^{*+} is taken to be the unique solution to the differential-integral system

$$\begin{aligned} \dot{\theta} &= a + \Theta(\theta, x, y, w^{*+}(\theta, x, y, \lambda), \lambda), \\ \dot{x} &= Ax + X(\theta, x, y, w^{*+}(\theta, x, y, \lambda), \lambda), \\ \dot{y} &= By + Y(\theta, x, y, w^{*+}(\theta, x, y, \lambda), \lambda), \end{aligned} \quad (40a)$$

$$\begin{aligned} w^{*+}(\theta, x, y, \lambda) \\ = \int_{+\infty}^0 e^{-C\sigma} Z(\psi^{*+}, \xi^{*+}, \eta^{*+}, w^{*+}(\psi^{*+}, \xi^{*+}, \eta^{*+}, \lambda), \lambda) d\sigma, \end{aligned} \quad (40b)$$

where $\psi^{*+} = \psi^{*+}(t) = \psi^{*+}(t, \theta, x, y, w^{*+}, \lambda)$, $\xi^{*+} = \dots$, $\eta^{*+} = \dots$, represents the unique solution of (40a) with initial condition (θ, x, y) at $t = 0$. This

system is also solved by iteration, but with the following modification. For l , m positive integers define

$$\mathfrak{X}_m^l = \{\chi = \chi(\theta, x, y) \text{ satisfying (41 i-v) below}\}.$$

(41 i) χ is a real vector-valued function defined and C^l for all (θ, x, y) .

(41 ii) $\dim \chi = m$.

(41 iii) χ has multiple period ω in θ .

(41 iv) $\chi, \chi(x, y) \equiv 0$ when $(x, y) = 0$.

(41 v) $\|\chi\| = \max_{0 \leq |\rho| \leq l} \sup_{(\theta, x, y)} |D_{(\theta, x, y)}^\rho \chi(\theta, x, y)| < \infty$.

With the norm in (41 v), \mathfrak{X}_m^l is a Banach space. Define $\tilde{\mathfrak{X}}_m^l$ to be the closed unit ball in \mathfrak{X}_m^l . For conciseness let $\tilde{\mathfrak{X}}_z^l = \tilde{\mathfrak{X}}_{\dim z}^l$. Now, for $w \in \tilde{\mathfrak{X}}_z^k$ consider the system

$$\begin{aligned} \dot{\theta} &= a + \Theta(\theta, x, y, w(\theta, x, y), \lambda), \\ \dot{x} &= Ax + X(\theta, x, y, w(\theta, x, y), \lambda), \\ \dot{y} &= By + Y(\theta, x, y, w(\theta, x, y), \lambda). \end{aligned} \tag{42}$$

Let $\psi^w = \psi(t) = \psi(t, \theta, x, w, \lambda)$, $\xi^w = \dots$, $\eta^w = \dots$, represent the unique solution of (42) with initial condition (θ, x, y) at $t = 0$. Now define the transformation T acting on $\tilde{\mathfrak{X}}_z^k$ as follows. For $w \in \tilde{\mathfrak{X}}_z^k$,

$$(Tw)(\theta, x, y) = \int_{+\infty}^0 e^{-C\sigma} Z(\psi^w, \xi^w, \eta^w, w(\psi^w, \xi^w, \eta^w), \lambda) d\sigma.$$

Corresponding to Lemma 4, we can now prove

LEMMA 5. For $1 \leq k < \infty$ and $|\lambda| \leq \delta_2$, $\delta_2 > 0$ sufficiently small, the transformation T maps $\tilde{\mathfrak{X}}_z^k$ into itself and is a contraction in the C^{k-1} topology:

$$\|\cdot\| = \|\cdot\|_{k-1} = \max_{0 \leq |\rho| \leq k-1} \sup_{(\theta, x, y)} |D_{(\theta, x, y)}^\rho \cdot|.$$

The proof of Lemma 5 is analogous to the proof of Lemma 4. In fact, the details are even easier to carry out. The function $w^{*+} = w^{*+}(\theta, x, y, \lambda)$ is the unique fixed point of T .

The proof of the existence and smoothness of the center-unstable manifold is similar to that for the center-stable manifold. Once we have both M_λ^{*+} and M_λ^{*-} , then

$$M_\lambda^* = M_\lambda^{*+} \cap M_\lambda^{*-}.$$

However, we can also construct the center manifold M_λ^* directly by constructing the functions $u^* = u^*(\theta, y, \lambda)$, $w^* = w^*(\theta, y, \lambda)$ as the unique solution of the following differential-integral system:

$$\dot{\theta} = a + \Theta(\theta, u^*(\theta, y, \lambda), y, w^*(\theta, y, \lambda), \lambda), \quad (43a)$$

$$u^*(\theta, y, \lambda) = \int_{-\infty}^0 e^{-A\sigma} X(\psi^*, u^*(\psi^*, \eta^*, \lambda), \eta^*, w^*(\psi^*, \eta^*, \lambda), \lambda) d\sigma,$$

$$\dot{y} = By + Y(\theta, u^*(\theta, y, \lambda), y, w^*(\theta, y, \lambda), \lambda), \quad (43b)$$

$$w^*(\theta, y, \lambda) = \int_{+\infty}^0 e^{-C\sigma} Z(\psi^*, u^*(\psi^*, \eta^*, \lambda), \eta^*, w^*(\psi^*, \eta^*, \lambda), \lambda) d\sigma,$$

where $\psi^* = \psi^*(t) = \psi^*(t, \theta, u^*, y, w^*, \lambda)$, $\eta^* = \dots$, represents the unique solution of (43a, b) with initial condition (θ, y) at $t = 0$. The procedure followed here is similar to the procedure used to solve (40).

If another invariant manifold

$$M'_\lambda = \{(\theta, x, y, z) | \theta \text{ arbitrary, } |x| < 1, y = v'(\theta, x, \lambda), z = w'(\theta, x, \lambda)\}$$

satisfies all the properties of M_λ^+ , then M'_λ is composed of solutions of (2) which we designate $\psi', \xi', \eta' = v'(\psi', \xi', \lambda)$, $\zeta' = w'(\psi', \xi', \lambda)$. Since $|\xi'|$ goes exponentially to zero as $t \rightarrow 0$, so do $|v'(\psi', \xi', \lambda)|$ and $|w'(\psi', \xi', \lambda)|$. Therefore v', w' must satisfy (3); but since the solution of (3) is unique, $(v', w') = (v^+, w^+)$ and $M'_\lambda = M_\lambda^+$. A similar argument shows that M_λ^- is also unique. An example of non-uniqueness for the center manifold is given in Section 4 below, and this same counter-example can be used to show non-uniqueness for the center-stable and center-unstable manifolds also. This completes the proof of Theorems 1 and 2.

Let us point out what should already be obvious. Namely, the reason that the center-stable manifold has one more derivative than the stable manifold is because the factor $e^{-C\sigma}$ occurring in the integrand in (40b) is an exponentially converging factor whereas the factor $e^{-B\sigma}$ occurring in the integrand in (27a) is not. (See (5).) However, all the manifolds have one more derivative. This will be discussed in Section 5 below.

4. NON-UNIQUENESS OF THE CENTER MANIFOLD

Consider the pair of real scalar equations

$$\dot{x} = -x, \quad \dot{y} = y^2. \quad (44)$$

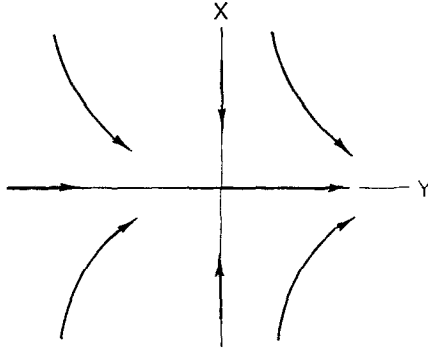
Dividing \dot{x} by \dot{y} we obtain

$$dx/dy = -(x/y^2)$$

which can be integrated to yield

$$x = ce^{y-1}$$

where c is the constant of integration. This give us the following phase portrait in the (y, x) plane for system (44).



Let

$$u(y, c) = \begin{cases} ce^{y-1} & \text{for } y < 0, \\ 0 & \text{for } y \geq 0. \end{cases}$$

Clearly, $u(0, c) = u_y(0, c) = 0$ so that

$$M(c) = \{(x, y) | x = u(y, c), y \text{ arbitrary}\}$$

is a center manifold for each real constant c . By adding the scalar equation $\dot{z} = z$ to (44) we see that the center-stable and center-unstable manifolds are also non-unique. If, however, the center-stable manifold is stable [the origin is (Lyapounov) stable with respect to the center-manifold], then it is not difficult to show that the center-stable manifold is unique. With respect to $-t$, the same is true of the center-unstable manifold. If both the center-stable and center-unstable manifolds are unique, then the center-manifold is also unique.

5. ADDITIONAL SMOOTHNESS

In our construction of M_λ^+ and M_λ^- , the procedure was to construct a mapping T on the closed unit ball $\tilde{\mathfrak{X}}^{k-1}$ of the appropriate Banach space with a C^{k-1} topology. Then we proved

$$T : \tilde{\mathfrak{X}}^{k-1} \rightarrow \tilde{\mathfrak{X}}^{k-1}$$

and T is a contraction in the C^{k-2} topology. Thus the fixed point (say, p) of T has $k - 2$ derivatives. But also p is the limit in the C^{k-2} topology of elements in $\tilde{\mathfrak{X}}^{k-1}$. Thus the $(k - 2)$ th derivatives of p are uniformly Lipschitzian. Using this fact and the proof method of Theorem 4.2. on page 333 of [I], one can show that, for λ sufficiently small, $p \in C^{k-1}$, and even more, $p \in \tilde{\mathfrak{X}}^{k-1}$. (The details of this program are quite laborious so we do not present them here.) Thus the manifolds $M_\lambda^+, M_\lambda^- \in C^{k-1}$ where system (2) is C^k , $2 \leq k < \infty$. An analogous argument shows that $M_\lambda^{*+}, M_\lambda^*, M_\lambda^{*-} \in C^k$ where system (2) is C^k , $1 \leq k < \infty$.

When the y -equation in (1) is absent, then $M^+ = M^{*+}$, $M^- = M^{*-}$. Therefore in this case M^+ , M^- are as differentiable as system (1). This fact will be used in Section 7 below.

Finally, we remark that multiple periodicity in θ for system (1) is not essential in the proof of Theorem 1. Rather, one needs only to be able to put the original system in a form similar to (2), but without multiple periodicity in θ . Of course, the invariant manifolds also will not exhibit multiple periodicity in θ .

6. PERTURBATION THEORY FOR M^{*+} , M^* , M^{*-}

For simplicity we will not discuss the perturbation theory of M^+ , M^- in the general case. But notice that when the center equation (y -equation) is absent from (1), then $M^{*+} = M^+$, $M^{*-} = M^-$ so that the results of this section apply to M^+ , M^- in that special case.

Consider the real C^k , $1 \leq k < \infty$, system of ordinary differential equations

$$\begin{aligned} \dot{\theta} &= a + \tilde{\Theta}(\theta, x, y, z, \epsilon), & \dot{x} &= Ax + \tilde{X}(\theta, x, y, z, \epsilon), \\ \dot{y} &= By + \tilde{Y}(\theta, x, y, z, \epsilon), & \dot{z} &= Cz + \tilde{Z}(\theta, x, y, z, \epsilon), \end{aligned} \quad (45)$$

where a, A, B, C are as in (1); θ, x , etc., are vectors; ϵ is a real (perturbation) scalar; $\tilde{\Theta}, \tilde{X}, \tilde{Y}, \tilde{Z}$ are defined and C^k in

$$N_\delta = \{(\theta, x, y, z, \epsilon) | \theta \text{ arbitrary, } |x| + |y| + |z| + |\epsilon| < \delta\}$$

for some δ positive and have multiple period ω in θ ; $\tilde{\Theta}, \tilde{X}, \tilde{Y}, \tilde{Z}$, $(\tilde{X}, \tilde{Y}, \tilde{Z})_{(x, y, z)} \equiv 0$ when $(x, y, z, \epsilon) = 0$. Thus when $\epsilon = 0$, (45) reduces to a system of the form (1).

THEOREM 3. *For system (45) there exists invariant manifolds*

$$\begin{aligned} M^{*+} &= \{(\theta, x, y, z) | \theta \text{ arbitrary, } |x| + |y| + |\epsilon| < \delta_1, \\ & \quad z = w^{*+}(\theta, x, y, \epsilon)\}, \end{aligned}$$

$$M^* = \{(\theta, x, y, z) | \theta \text{ arbitrary, } x = u^*(\theta, y, \epsilon), |y| + |\epsilon| < \delta_1, \\ z = w^*(\theta, y, \epsilon)\},$$

$$M^{*-} = \{(\theta, x, y, z) | \theta \text{ arbitrary, } x = u^{*-}(\theta, y, z, \epsilon), \\ |y| + |z| + |\epsilon| < \delta_1\},$$

where w^{*+}, u^*, w^*, u^{*-} are real vector-valued functions defined and C^k in some neighborhood N_{δ_1} for δ_1 sufficiently small; w^{*+}, u^*, w^*, u^{*-} have multiple period ω in θ ; $w^{*+}, u^*, w^*, u^{*-}, (w^{*+}, u^*, w^*, u^{*-})_{(x,y,z)} \equiv 0$ when $(x, y, z, \epsilon) = 0$.

The proof of this theorem is essentially a copy of the proof of Theorem 1. One merely introduces a scalar change of variables

$$(x, y, z, \epsilon) \rightarrow (\lambda x, \lambda y, \lambda z, \lambda^2 \epsilon)$$

and then changes the system outside a neighborhood of the (x, y, z, ϵ) origin similar as in the proof of Theorem 1. The essential property of the transformed system will be the analog of (2 v), namely,

$$D_{(\theta, x, y, z, \epsilon)}^0(\Theta, X, Y, Z) \rightarrow 0$$

uniformly in $(\theta, x, y, z, \epsilon)$ as $\lambda \rightarrow 0$ for $0 \leq |\rho| \leq k$, where analogous to the procedure in the proof of Theorem 1,

$$\Theta(\theta, x, y, z, \epsilon, \lambda) = \phi(|x|^2 + |y|^2 + |z|^2 + \epsilon^2 + K\lambda^2)\tilde{\Theta}(\theta, \lambda x, \lambda y, \lambda z, \lambda^2 \epsilon),$$

etc. With this property there is no difficulty in solving the appropriate differential-integral system for w^{*+} , etc., provided λ is sufficiently small.

If there is no center equation (y -equation) in (45), then $M^{*+} = M^+$, $M^{*-} = M^-$. Since we haven't defined M^+ , M^- in the general perturbation case, we can take this as a definition. Also, with no center equation in (45), the center manifold

$$M^* = \{(\theta, x, z) | \theta \text{ arbitrary, } x = u^*(\theta, \epsilon), z = w^*(\theta, \epsilon), |\epsilon| \leq \delta_1\}$$

is the same as what is known as the periodic surface. There is an extensive literature concerning this invariant manifold. See, for example, [2], [3], [5], and [8]. As a corollary to Theorem 3 we have that the periodic surface is as differentiable as the system of differential equations.

7. PERTURBATION THEORY WITHOUT CENTER

In this section we want to discuss how M^+ , M^- vary with respect to the perturbing function.

Consider the real C^k , $1 \leq k < \infty$, system of ordinary differential equations

$$\begin{aligned}\dot{\theta} &= a + \Theta(\theta, x, y) + \epsilon \hat{\Theta}(\theta, x, y), \\ \dot{x} &= Ax + X(\theta, x, y) + \epsilon \hat{X}(\theta, x, y), \\ \dot{y} &= By + Y(\theta, x, y) + \epsilon \hat{Y}(\theta, x, y),\end{aligned}\tag{46}$$

where A, B are constant matrices in real canonical form; A has eigenvalues with negative real parts; B has eigenvalues with positive real parts; θ, x , etc. are vectors; a is a constant vector; ϵ is a perturbation parameter; $\Theta, \hat{\Theta}, X, \hat{X}, Y, \hat{Y}$ are defined and C^k in

$$N_\delta = \{(\theta, x, y) | \theta \text{ arbitrary, } |x| + |y| < \delta\}$$

and have multiple period ω in θ ; $\Theta, X, Y, (X, Y)_{(x,y)} \equiv 0$ when $(x, y) = 0$.

Let P (for perturbation) represent the triple $(\hat{\Theta}, \hat{X}, \hat{Y})$. From Theorem 4 we know that locally (for (x, y, ϵ) sufficiently small) there exists stable and unstable manifolds $M^\pm = M^\pm(P, \epsilon)$. Since $M^\pm(P, 0)$ are independent of P , let $M^\pm(P, 0) = M_0^\pm$. Define $\mathfrak{X}^{k,l}$ to be the set of all triples $P = (\hat{\Theta}, \hat{X}, \hat{Y})$ which are defined and C^k in N_δ , have multiple period ω in θ , and satisfy

$$\max_{0 \leq |\rho| \leq l} \sup_{\theta^+} \sup_{|x|+|y| < \delta} |D_{(\theta,x,y)}^\rho P| \leq 1,$$

where $l \leq k$ is a positive integer. For $P \in \mathfrak{X}^{k,l}$ let

$$M^+(P, \epsilon) = \{(\theta, x, y) | \theta \text{ arbitrary, } y = v^+(\theta, x, P, \epsilon)\},$$

where P in the argument of v^+ denotes a functional dependence.

THEOREM 4. *Uniformly in $P \in \mathfrak{X}^{k,l}$, v^+ is defined and C^k on*

$$N^{\delta_0} = \{(\theta, x, \epsilon) | \theta \text{ arbitrary, } |x| < \delta_0, |\epsilon| < \delta_0\},$$

where δ_0 is a sufficiently small positive constant which is independent of P . Moreover,

$$\max_{0 \leq |\rho| \leq l} \sup_{\theta^+} \sup_{|x| < \delta_0} |D_{(\theta,x)}^\rho (v^+(\theta, x, P, \epsilon) - v^+(\theta, x))| \rightarrow 0$$

as $\epsilon \rightarrow 0$ uniformly in $P \in \mathfrak{X}^{k,l}$, where $v^+(\theta, x) \stackrel{\text{def}}{=} v^+(\theta, x, P, 0)$.

The proof of Theorem 4 is also essentially the same as the proof of Theorem 1. After introducing a scalar change of variables and changing the system outside a neighborhood of the origin, the assertions are readily proved. But the assertions are logically equivalent for both the original system (46) and the transformed system. A similar theorem holds for M^- .

Since A, B in (46) have eigenvalues with nonzero real parts, $M^* = M^+ \cap M^-$.

THEOREM 5. M^+, M^- have the following characterization for ϵ sufficiently small:

$$\begin{aligned} M^+ &= \{(\theta, x, y) | (\psi, \xi, \eta) \rightarrow M^* \text{ as } t \rightarrow +\infty\}, \\ M^- &= \{(\theta, x, y) | (\psi, \xi, \eta) \rightarrow M^* \text{ as } t \rightarrow -\infty\}, \end{aligned}$$

where $\psi = \psi(t, \theta, x, y, \epsilon)$, $\xi = \dots$, $\eta = \dots$, represents the unique solution of (46) with initial condition (θ, x, y) at $t = 0$.

Proof. It is sufficient to prove this theorem for a system which has been transformed from (46). Let

$$\begin{aligned} \dot{\theta} &= a + \Theta(\theta, x, y, \epsilon), \\ \dot{x} &= Ax + X(\theta, x, y, \epsilon), \\ \dot{y} &= By + Y(\theta, x, y, \epsilon) \end{aligned} \tag{47}$$

be a transformed system. If we introduce the change of variables

$$\begin{aligned} p &= x - u^-(\theta, y, \epsilon), \\ q &= y - v^+(\theta, x, \epsilon), \end{aligned} \tag{48}$$

then, in these new coordinates, (47) has the form

$$\begin{aligned} \dot{\theta} &= a + \tilde{\Theta}(\theta, p, q, \epsilon), \\ \dot{p} &= Ap + P(\theta, p, q, \epsilon), \\ \dot{q} &= Bq + Q(\theta, p, q, \epsilon), \end{aligned} \tag{49}$$

where

$$\begin{aligned} |P(\theta, p, q, \epsilon)| &\leq \mu |p|, \\ |Q(\theta, p, q, \epsilon)| &\leq \mu |q|, \end{aligned} \tag{50}$$

and μ is the positive constant given in (5). [B in (49) is C in (5).] Let us compute P , for example, to see that inequality (50) is true. From (48),

$$\begin{aligned} \dot{p} &= \dot{x} - \dot{v}^- = Ax + X - v_{\theta}^-\{a + \Theta\} - v_x^-\{Ax + X\} \\ &= Ax - v_{\theta}^-\{a + \Theta(\theta, v^-, y, \epsilon)\} - v_x^-\{Ax + X(\theta, v^-, y, \epsilon)\} \\ &\quad + X(\theta, p + v^-, y, \epsilon) + v_{\theta}^-\{\Theta(\theta, v^-, y, \epsilon) - \Theta(\theta, p + v^-, y, \epsilon)\} \\ &\quad + v_x^-\{X(\theta, v^-, y, \epsilon) - X(\theta, p + v^-, y, \epsilon)\} \\ &= Ax - Av^- - X(\theta, v^-, y, \epsilon) + X(\theta, p + v^-, y, \epsilon) \\ &\quad + v_{\theta}^-\{\Theta(\theta, v^-, y, \epsilon) - \Theta(\theta, p + v^-, y, \epsilon)\} \\ &\quad + v_x^-\{X(\theta, v^-, y, \epsilon) - X(\theta, p + v^-, y, \epsilon)\} \\ &= Ap + P(\theta, p, q, \epsilon). \end{aligned}$$

Thus we see that $P \in C^{k-1}$ and that P satisfies (50) provided θ, X, Y have sufficiently small first-order derivatives. From (5), (49), (50),

$$\begin{aligned} (d/dt)|p|^2 &= 2\langle Ap, p \rangle + 2\langle P, p \rangle \leq -4\mu|p|^2 + 2\mu|p|^2 \leq -2\mu|p|^2, \\ (d/dt)|q|^2 &= 2\langle Bq, q \rangle + 2\langle Q, q \rangle \geq 4\mu|q|^2 - 2\mu|q|^2 \geq 2\mu|q|^2. \end{aligned}$$

Our theorem follows immediately from these differential inequalities.

Finally we state a theorem which is closely related to Theorem 4. We want to show that $M^+(P, \epsilon) \rightarrow M^+(P_0, \epsilon)$ in the C^{l-1} topology as $P \rightarrow P_0$ in the C^l topology where ϵ is small but fixed. A similar statement will hold relative to M^- .

THEOREM 6. For $P, P_0 \in \mathfrak{X}^{k,l}$ if $P \rightarrow P_0$ in the C^l topology,

$$\max_{0 \leq |\rho| \leq l} \sup_{\theta} \sup_{|x|+|y| < \delta} |D_{(\theta, x, y)}^\rho(P - P_0)| \rightarrow 0,$$

then $v^+(\theta, x, P, \epsilon) \rightarrow v^+(\theta, x, P_0, \epsilon)$ in the C^{l-1} topology,

$$\max_{0 \leq |\rho| \leq l-1} \sup_{\theta} \sup_{|x| < \delta_0} |D_{(\theta, x)}^\rho\{v^+(\theta, x, P, \epsilon) - v^+(\theta, x, P_0, \epsilon)\}| \rightarrow 0.$$

The proof of Theorem 6 is obtained by introducing the change of variables (48) relative to P_0 ,

$$\begin{aligned} p &= x - u^-(\theta, y, P_0, \epsilon), \\ q &= y - v^+(\theta, x, P_0, \epsilon). \end{aligned}$$

System (49) will now only be C^{k-1} , but otherwise Theorem 6 reduces to Theorem 4 with only minor modifications.

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