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# THE HISTORY OF THE CONCEPT OF A FINITE-DIMENSIONAL VECTOR SPACE

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The purpose of this note is to draw attention to an interesting problem in the history of mathematics which the pressure of work will prevent me from taking up, and to make a suggestion about a possibly neglected aspect of the problem that should be explored. The problem is: to describe the origins of the concept of a vector space and its recognition as a central topic in mathematics. Moves towards the explicit treatment of topics, and their axiomatization, are always of interest; in addition, it seems that vector spaces have had a curious and convoluted development.

My specific suggestion, drawn from my research into the history of the monodromy group and differential equations in the complex domain, is that the theory of ordinary linear differential equations and geometric complex function theory provided a rich source of finite-dimensional vector spaces throughout the late 18th and 19th centuries. The existence of a finite-dimensional basis for the solutions of an  $n$ th-order linear ordinary differential equation with constant coefficients is described, for example, in Euler [1743], and the theory is developed by many subsequent writers, culminating in Fuchs [1865]. Fourier, in a study of heat diffusion among  $n$  bodies in a circle, derived  $n$  simultaneous linear ordinary differential equations, which he solved to obtain finite "Fourier Series" [Grattan-Guinness 1972]. He was well aware, as were his contemporaries, of the linear character of his problems and of the orthogonality properties of his basis of solutions, which are expressed by the familiar properties of integrals of the form

$$\int_0^{2\pi} \sin mx \sin nx.$$

Fourier's work is also important for the study of infinite matrices and function spaces which began at the end of the 19th century.

The matrices which appear explicitly in the study of the monodromy (and the monodromy group) of an algebraic function and in the related study of the singular points of a differential

equation operate implicitly on vectors (the  $n$ -tuples of possible values or independent solutions); see, for example, Hermite [1851], Riemann [1857], Jordan [1872], or most writers on complex function theory throughout the century. The sheer size of the literature on complex functions is daunting, and I wish to draw attention to an important source of ideas about vectors which might, perhaps, otherwise be ignored.

It does not seem that much has been written on the history of vector spaces, with the notable exceptions of Bernkopf [1966], on the origin of function spaces; Bourbaki [1960], which examines number-theoretic roots; Bourbaki [1975], which considers rather more topics; and Hawkins' considerable reconstruction of the theory of matrices [1977a, 1977b]. In a methodological way, Hawkins' papers are also germane to the topic raised in this note; they suggest a sequence of stages for the theory of matrices, from their origins in the study of systems of differential equations by Lagrange and Laplace, through an explicit but un-theoretical period, to a generic and finally rigorous theoretical treatment by Weierstrass and Frobenius. It is likely that the study of vector spaces passed through similar reformulations. Hawkins suggests that the modern pedagogic approach, which puts vector spaces first and matrices or, better, linear maps second, dates only from the 1930's [Hawkins 1977a, 109].

Two problem areas are important if one looks for early explicit formulations of the concept of a vector space. In the course of a study of systems of  $n$  linear first-order ordinary differential equations, Peano [1888] gave an explicit axiomatic definition of an  $n$ -dimensional vector space over the reals. He called his  $n$ -tuples "number complexes of order  $n$ ," which suggests affinity with the study of hyper-complex numbers. Significantly, he had just read Grassmann [1844], which contains a long-neglected treatment of vector spaces and exterior algebras. The study of differential equations with irregular singular points led directly to the study of infinite vector spaces, or rather, as it was then expressed, the study of linear equations in infinitely many unknowns and infinite determinants [Hill 1877; Appell 1884; Poincaré 1884, 1886]. These topics are treated in Bernkopf [1966], for they merged into the study of integral equations and thence function spaces, so decisively established by Hilbert from 1904 to 1910. However, as Bernkopf points out, Hilbert was more interested in integral equations and their kernels (generalizations, accordingly, of matrices) than in the spaces of functions. The study of the topological aspects of these spaces is due to Fréchet [1906], and the geometric aspects to Schmidt [1906]. It may well be that the unexpected richness and complexity of *infinite*-dimensional vector spaces (which need not be isomorphic to their duals, for example) first inclined mathematicians to study the finite-dimensional spaces explicitly. This interesting possibility merits further study.

The concept of a finite-dimensional vector space also appears explicitly in the development of Galois theory. Kronecker's constructive idea of a "Rationalitätsbereich" [1882] consists typically of finite linear combinations of algebraic numbers taken over the rationals. The totality of these linear combinations is an algebraic extension of the rationals and therefore a vector space over the rationals. Unhappily, Kronecker's ideas are notoriously hard to date (see Kiernan [1971]). The same idea of a field extension is quite explicit in Dedekind and Weber [1881], where it is called a "Modul." Here the module is over a field, usually the field of rational functions. Dedekind had long been interested in fields because of his interest in number theory, and in the paper he wrote with Weber he sought a common language to describe the similar domains of algebraic number theory and algebraic function theory. The first abstract treatment of fields is given in Weber [1893], but here vector spaces do not appear explicitly: algebraic extensions of a given field are called "congruence fields" by Weber and treated as fields. The concept of a vector space was, of course, of considerable importance for Artin's modern reformulation of Galois theory [Artin 1938, 1959] (and see also van der Waerden [1971]).

Hilbert [1894] took up the concepts of fields and vector spaces in number theory, as did Hensel and Landsberg [1902] in function theory. It is apparently from number theory that Hilbert extracted the theory of finite-dimensional vector spaces, which he said he was generalizing in his work in integral equations. (It is interesting to note that Dedekind and Peano, the two mathematicians of the period who are most prominently identified with the question of what a number is, should be among the first to identify vector spaces as objects worthy of interest.)

Finite-dimensional vector spaces also appeared in the work of S. Lie, F. Engel, W. Killing, and E. Cartan in the theory of continuous transformation groups. The infinitesimal action on a "Lie" group by its Lie algebra was the first object of study, and the successful classification of the complex semisimple algebras is described in Bourbaki [1975] and Helgason [1977].

Geometry, with its  $n$ -tuples of points in an  $n$ -dimensional space, is another obvious source of the concept. Although interest is not usually centred on the ambient space but upon configurations within it, linear combinations of figures do occur. For example, if  $S \equiv 0$  and  $S' \equiv 0$  define two conics, so does  $\lambda S + \lambda' S' \equiv 0$  for scalars  $\lambda, \lambda'$ . The investigations of matrix representations by Frobenius and others, the related study of hyper-complex numbers, and researches into finite-dimensional algebras would also have to be considered in any study of the origins of finite-dimensional vector spaces (see [Hawkins 1972 and Novy 1973, Chap. 4]). Finally, there is the physicist's vector algebra (see Crowe [1967]) and its generalization to tensors (see Weyl [1920]).

It is perhaps worth raising the related questions: when did the concept of a vector space with additional structure, such as an inner product, first emerge, and how and when did it gain general acceptance? The concept may first have emerged with the Killing form of a Lie algebra [Killing 1888, 1889], its final acceptance may have come with von Neumann's work on unitary operators in the late 1920's (see, e.g., [von Neumann 1927]).

It is possible that the concept of a vector space is one of many which mathematicians have found they have been using for many years without knowing it, or, perhaps one should say, needing to know it. Other examples taken at random are semigroups, long used in the theory of integral equations [Hille 1965], and even groups (see Wussing [1969]). If this is so, then subtle attention paid to the ideas in their historical context may prove to be instructive about the nature of mathematical activity and the necessity, or otherwise, of abstraction. For the problem is not merely to assess the neglect of Grassmann by mathematicians, but to respond to the fact that for 50 years mathematicians worked confidently without feeling the need to elaborate an abstract structure, when it is now so often assumed that making such abstractions is essential to mathematical work.

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