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► **To cite this version:**

Mohammed El Azhari. On uniform topological algebras. *Analele Stiintifice Ale Universitatii "Al. I. Cuza" Din Iasi Matematica*, 2017, 63 (1), pp.177-180. hal-01306458

HAL Id: hal-01306458

<https://hal.science/hal-01306458>

Submitted on 23 Apr 2016

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ON UNIFORM TOPOLOGICAL ALGEBRAS

M. EL AZHARI

Abstract. The uniform norm on a uniform normed Q-algebra is the only uniform Q-algebra norm on it. The uniform norm on a regular uniform normed Q-algebra with unit is the only uniform norm on it. Let A be a uniform topological algebra whose spectrum $M(A)$ is equicontinuous, then A is a uniform normed algebra. Let A be a regular semisimple commutative Banach algebra, then every algebra norm on A is a Q-algebra norm on A .

Mathematics Subject Classification 2010: 46H05, 46J05.

Keywords: Uniform topological algebra, regular algebra, uniform seminorm.

1. Preliminaries

A topological algebra is an algebra (over the complex field) which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra is a topological algebra whose topology is locally convex. A uniform seminorm on an algebra A is a seminorm p satisfying (i) : $p(x^2) = p(x)^2$ for all $x \in A$. It is shown in [3], that a seminorm satisfying (i) is submultiplicative. A uniform topological algebra (uT-algebra) is a topological algebra whose topology is determined by a family of uniform seminorms. A uniform normed algebra is a normed algebra $(A, \|\cdot\|)$ such that $\|x^2\| = \|x\|^2$ for all $x \in A$. A topological algebra is a Q-algebra if the set of quasi-invertible elements is open. A norm $\|\cdot\|$ on an algebra A is said to be an algebra norm (resp. Q-algebra norm) if $(A, \|\cdot\|)$ is a normed algebra (resp. normed Q-algebra). Let A be an algebra, if $x \in A$ we denote by $sp_A(x)$ the spectrum of x and by $r_A(x)$ the spectral radius of x . An algebra A is spectrally bounded if for each $x \in A$, $sp_A(x)$ is bounded. For a topological algebra A ; $M(A)$ denotes the set of all nonzero continuous multiplicative linear functionals on A , $M(A)$ is endowed with the weak topology induced by the topological dual of A . For an arbitrary topological algebra, $M(A)$ may be empty. A topological algebra is regular if given a closed subset $F \subset M(A)$ and $f_1 \in M(A)$, $f_1 \notin F$, there exists an $x \in A$ such that $\hat{x}|_F = 0$ and $\hat{x}(f_1) \neq 0$; $\hat{x} : M(A) \rightarrow \mathbb{C}$, $\hat{x}(g) = g(x)$, is the Gelfand transform of x . Let $(B, \|\cdot\|)$ be a uniform Banach algebra, $r_B(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|$ for all $x \in B$; B is commutative by [4, Corollary 1], hence $\|x\| = \sup\{|f(x)|, f \in M(B)\}$ for all $x \in B$. Let A be a uT-algebra; the completion of A is a uT-algebra and then an inverse limit of uniform Banach algebras, thus A is commutative.

2. Results

Theorem 2.1. Let $(A, \|\cdot\|)$ be a uniform normed Q-algebra. If $|\cdot|$ is a uniform norm on A , then $|x| \leq \|x\|$ for all $x \in A$.

Proof. By [7, Proposition 7.5], $M(A, \|\cdot\|)$ (resp. $M(A, |\cdot|)$) is topologically isomorphic to $M(\tilde{A}, \|\cdot\|)$ (resp. $M(\tilde{A}, |\cdot|)$), $(\tilde{A}, \|\cdot\|)$ and $(\tilde{A}, |\cdot|)$ are respectively the completions of $(A, \|\cdot\|)$ and $(A, |\cdot|)$. $(\tilde{A}, \|\cdot\|)$ and $(\tilde{A}, |\cdot|)$ are uniform Banach algebras, then for $x \in A$, we have $\|x\| = \sup\{|f(x)|, f \in M(\tilde{A}, \|\cdot\|)\} = \sup\{|f(x)|, f \in M(A, \|\cdot\|)\}$ and $|x| = \sup\{|f(x)|, f \in M(\tilde{A}, |\cdot|)\} = \sup\{|f(x)|, f \in M(A, |\cdot|)\}$, hence $|x| \leq \|x\|$ since $M(A, |\cdot|) \subset M(A, \|\cdot\|)$; the last inclusion is due to the fact that $(A, \|\cdot\|)$ is a Q-algebra.

Corollary 2.2. Let $(A, \|\cdot\|)$ be a uniform normed Q-algebra, then $\|\cdot\|$ is the unique uniform Q-algebra norm on A .

Theorem 2.3. Let $(A, \|\cdot\|)$ be a regular uniform normed Q-algebra with unit. If $|\cdot|$ is a uniform norm on A , then $\|\cdot\| = |\cdot|$.

Proof. Let $x \in A$, $|x| = \sup\{|f(x)|, f \in M(\tilde{A}, |\cdot|)\} = \sup\{|f(x)|, f \in M(A, |\cdot|)\}$. $M(A, |\cdot|)$ is topologically isomorphic to $M(\tilde{A}, |\cdot|)$, then $M(A, |\cdot|)$ is compact. Put $T = M(A, |\cdot|)$, T is a closed subset of $M(A, \|\cdot\|)$. Suppose that $T \neq M(A, \|\cdot\|)$, let $f_1 \in M(A, \|\cdot\|)$ such that $f_1 \notin T$. Since $(A, \|\cdot\|)$ is regular, there exists $x \in A$ such that $\hat{x}|_T = 0$ and $\hat{x}(f_1) \neq 0$, hence $|x| = 0$ with $x \neq 0$, a contradiction. For $x \in A$, $\|x\| = \sup\{|f(x)|, f \in M(\tilde{A}, \|\cdot\|)\} = \sup\{|f(x)|, f \in M(A, \|\cdot\|)\} = \sup\{|f(x)|, f \in M(A, |\cdot|)\} = |x|$.

Theorem 2.4. Let A be a uT-algebra whose spectrum $M(A)$ is equicontinuous. Then A is a uniform normed algebra.

Proof. The topology of A is determined by a family $\{p_u, u \in U\}$ of uniform seminorms. For each $u \in U$, let $N_u = \{x \in A, p_u(x) = 0\}$ and let A_u be the Banach algebra obtained by completing A/N_u in the norm $\|x_u\|_u = p_u(x)$, $x_u = x + N_u$, it is clear that A_u is a uniform Banach algebra. For each $u \in U$, let $M_u(A) = \{f \in M(A), |f(x)| \leq p_u(x) \text{ for all } x \in A\}$. Let $u \in U$ and $x \in A$, $p_u(x) = \|x_u\|_u = \sup\{|g(x_u)|, g \in M(A_u)\} = \sup\{|f(x)|, f \in M_u(A)\}$ by [7, Proposition 7.5]. Let $q(x) = \sup\{|f(x)|, f \in M(A)\}$, $x \in A$. Since $M(A)$ is equicontinuous, it follows that q is a continuous seminorm on A . Let $x \in A$ and suppose that $q(x) = 0$, since A is Hausdorff and $p_u(x) \leq q(x) = 0$ for all $u \in U$, it follows that $x = 0$. q is a continuous uniform norm on A , also $p_u(x) \leq q(x)$ for all $u \in U$ and $x \in A$, then the topology of A can be defined by the uniform norm q .

Corollary 2.5. Let A be a uT-algebra that is a Q-algebra. Then A is a uniform normed algebra.

Proof. By [6, Proposition II.7.1], every topological Q-algebra has an equicontinuous spectrum.

Corollary 2.6. Let A be a spectrally bounded, barrelled, uT-algebra. Then A is a uniform normed algebra.

Proof. $\sup\{|f(x)|, f \in M(A)\} \leq r_A(x)$ for all $x \in A$. $M(A)$ is bounded for the weak topology, then $M(A)$ is equicontinuous.

Remark. These results show that completeness is not necessary for some results of S. J. Bhatt on uniform topological algebras (see [1, Theorem 1(ii)], [1, the second affirmation in the corollary], [1, Theorem 2] and [2, Corollary 2.5]).

Theorem 2.7. Let A be a topological algebra. If p is a uniform seminorm on A , then $p(x) \leq r_A(x)$ for all $x \in A$. If A is a Q-algebra, then every uniform seminorm on A is continuous.

Proof. Let $N_p = \{x \in A, p(x) = 0\}$ and A_p be the Banach algebra obtained by completing A/N_p in the norm $\|x_p\|_p = p(x)$, $x_p = x + N_p$. $\|\cdot\|_p$ is a uniform norm on A_p since p is a uniform seminorm on A . Consider $G : A \rightarrow A_p$, $G(x) = x_p$, the quotient map. Let $x \in A$, $p(x) = \|x_p\|_p = r_{A_p}(x_p) \leq r_A(x)$. If A is a Q-algebra, r_A is continuous at 0 by [6, Lemma II.4.2], then p is continuous.

Remark. Theorem 2.7 shows that the hypothesis " A is a unital locally convex algebra and the inversion in A is continuous" is not necessary in [2, Theorem 2.3].

Now we generalize the first affirmation of [1, Corollary].

Theorem 2.8. Let $(A, \|\cdot\|)$ be a regular semisimple commutative Banach algebra. Then every algebra norm on A is a Q-algebra norm on A .

Proof. (1) A is unital: Let $|\cdot|$ be an algebra norm on A . Put $K_1 = M(A, |\cdot|)$ and $K = M(A, \|\cdot\|)$, $K_1 \subset K$, K_1 is homeomorphic to $M(\tilde{A}, |\cdot|)$ ($(\tilde{A}, |\cdot|)$ is the completion of $(A, |\cdot|)$), then K_1 is compact, hence K_1 is closed in K . Suppose that $K_1 \neq K$. $K \setminus K_1$ is a nonempty open set, there exists an open set G in K such that $\bar{G} \subset K \setminus K_1$ since $(A, \|\cdot\|)$ is regular. Also by regularity of $(A, \|\cdot\|)$ [5, Corollary 7.3.4], there exist $x \in A, y \in A, y \neq 0$, such that $\hat{x}|_{K_1} = 1, \hat{x}|_{\bar{G}} = 0$ and $\hat{y}|_{K \setminus G} = 0$. x is invertible in $(\tilde{A}, |\cdot|)$ since $\hat{x}|_{K_1} = 1$. $\widehat{xy} = \hat{x}\hat{y} = 0$ on K , then $xy \in Rad(A) = \{0\}$, hence $y = x^{-1}xy = 0$, a contradiction. Finally, $K_1 = K$. Let $x \in A$, $r_A(x) = \sup\{|f(x)|, f \in K\} = \sup\{|f(x)|, f \in K_1\} \leq |x|$, then $(A, |\cdot|)$ is a Q-algebra by [8, Lemma 2.1].

(2) A is not unital: Let A_1 be the algebra obtained from A by adjunction of an identity e . The elements of A_1 have the form $x + re$, where $x \in A$ and $r \in C$. A_1 is also regular and semisimple. Let p be an algebra norm on A . For $x \in A$

and $r \in C$, put $p_1(x + re) = p(x) + |r|$, it is easy to show that p_1 is an algebra norm on A_1 . By (1) and [8, Lemma 2.1], $r_{A_1}(x + re) \leq p_1(x + re)$ for all $x \in A$ and $r \in C$, consequently $r_A(x) = r_{A_1}(x) \leq p_1(x) = p(x)$ for all $x \in A$. Then p is a Q-algebra norm on A by [8, Lemma 2.1].

The results of this paper were communicated at the international conference on topological algebras and applications (ICTAA 2000).

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