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# Performance of a Server Cluster with Parallel Processing and Randomized Load Balancing

T. Bonald and C. Comte

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## Abstract

We consider a cluster of servers where each incoming job is assigned  $d$  servers chosen uniformly at random, for some fixed  $d \geq 2$ . Jobs are served in parallel and the resource allocation is balanced fairness. We provide a recursive formula for computing the exact mean service rate of each job. The complexity is polynomial in the number of servers.

## 1 Model

We consider a cluster of  $S$  servers, each with service rate  $\mu$ . Jobs arrive according to a Poisson process with intensity  $\lambda$ . Each incoming job is assigned  $d$  servers chosen uniformly at random, for some fixed  $d \geq 2$ . Each job is processed in parallel by its assigned servers, the overall service capacity being allocated according to balanced fairness [1].

There are  $N = \binom{S}{d}$  classes of jobs, defined by the assigned servers. Let  $\mathcal{I} = \{1, \dots, N\}$  be the set of classes. We denote by  $\mathcal{S}_i \subset \{1, \dots, S\}$  the set of servers assigned to each job of class  $i$  and, for each set  $\mathcal{A} \subset \mathcal{I}$  of classes, we denote by  $\mathcal{S}(\mathcal{A})$  the set of servers assigned to jobs whose class belongs to  $\mathcal{A}$ . Let  $x = (x_i)_{i \in \mathcal{I}}$  be the network state, where  $x_i$  is the number of ongoing class- $i$  jobs. We denote by  $\phi_i(x)$  the total service rate of class- $i$  jobs in state  $x$ . The corresponding vector lies in the capacity set:

$$\mathcal{C} = \left\{ \phi \in \mathbb{R}_+^N : \forall \mathcal{A} \subset \mathcal{I}, \quad \sum_{i \in \mathcal{A}} \phi_i \leq \mu |\mathcal{S}(\mathcal{A})| \right\},$$

where  $|\mathcal{A}|$  is the cardinal of the set  $\mathcal{A}$ . The capacity set is a polymatroid, and it follows from [3] that balanced fairness is Pareto-efficient. Specifically,

$$\forall i \in \mathcal{I}, \quad \phi_i(x) = \begin{cases} \frac{\Phi(x - e_i)}{\Phi(x)} & \text{if } x_i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the function  $\Phi$  is defined by the recursion  $\Phi(0) = 1$  and, using the notation  $\mathcal{A}_x = \{i \in \mathcal{I} : x_i > 0\}$ ,

$$\forall x \in \mathbb{N}^N \setminus \{0\}, \quad \Phi(x) = \frac{\sum_{i \in \mathcal{A}_x} \Phi(x - e_i)}{\mu |\mathcal{S}(\mathcal{A}_x)|}. \quad (1)$$

Under the stability condition  $\lambda < S\mu$ , the stationary distribution of the system state is given by

$$\forall x \in \mathbb{N}^N, \quad \pi(x) = \pi(0) \Phi(x) \left( \frac{\lambda}{N} \right)^{|x|}, \quad (2)$$

where  $|x| = \sum_{i \in \mathcal{I}} x_i$  is the total number of jobs. This stationary distribution is insensitive to the job size distribution beyond the mean.

## 2 Performance

We are interested in the mean service rate, defined by

$$\gamma = \frac{\mathbb{E}(\sum_{i \in \mathcal{I}} \phi_i(X))}{\mathbb{E}(\sum_{i \in \mathcal{I}} X_i)},$$

where  $X$  is a random variable distributed according to the stationary distribution  $\pi$ . Observe that, by symmetry,  $\gamma$  is the mean service rate of any job (whatever its class), and that  $\gamma \leq d\mu$ . Moreover, by work conservation,

$$\gamma = \frac{\lambda}{\mathbb{E}(\sum_{i \in \mathcal{I}} X_i)}. \quad (3)$$

In particular, it follows from Little's law that  $1/\gamma$  is the mean job duration.

There is no explicit formula for computing  $\gamma$  with a low complexity. In particular, the recursive formula of de Veciana and Shah [3] does not apply because the capacity set is *not* a symmetric polymatroid. We use the recent results of Gardner et. al. [2] to derive an explicit recursive formula, whose complexity is linear in  $SN$  (thus polynomial in  $S$ ). We denote the system load by

$$\rho = \frac{\lambda}{S\mu}.$$

**Proposition 1** *We have  $\gamma = G/F$  with*

$$G = \sum_{n=0}^N \sum_{m=0}^S G_{n,m} \quad \text{and} \quad F = \sum_{n=0}^N \sum_{m=0}^S F_{n,m},$$

where  $G_{n,m}$  and  $F_{n,m}$  are given by the recursions  $G_{0,0} = 1, F_{0,0} = 0$ ,

$$G_{n,m} = \frac{\frac{\rho S}{N}}{m - n \frac{\rho S}{N}} \left( \left[ \binom{m}{d} - n + 1 \right] G_{n-1,m} + \sum_{r=1}^{\min(d,m)} \binom{S-m+r}{r} \binom{m-r}{d-r} G_{n-1,m-r} \right),$$

$$F_{n,m} = \frac{\frac{\rho S}{N}}{m - n \frac{\rho S}{N}} \left( \left[ \binom{m}{d} - n + 1 \right] F_{n-1,m} + \sum_{r=1}^{\min(d,m)} \binom{S-m+r}{r} \binom{m-r}{d-r} F_{n-1,m-r} \right) + \frac{1}{\lambda} \frac{m}{m - n \frac{\rho S}{N}} G_{n,m}$$

if  $d \leq m \leq S$  and  $\lceil \frac{m}{d} \rceil \leq n \leq \binom{m}{d}$ ,  $G_{n,m} = F_{n,m} = 0$  otherwise.

*Proof.* By symmetry, we have for any  $i \in \mathcal{I}$ ,

$$\gamma = \frac{\lambda/N}{\mathbb{E}(X_i)}.$$

Let

$$G(\lambda_1, \dots, \lambda_N) = \sum_{x \in \mathbb{N}^N} \Phi(x) \prod_{i \in \mathcal{I}} \lambda_i^{x_i}$$

and

$$G = G\left(\frac{\lambda}{N}, \dots, \frac{\lambda}{N}\right).$$

In view of (2), we have  $\gamma = G/F$ , with

$$F = \frac{\partial G}{\partial \lambda_i} \left( \frac{\lambda}{N}, \dots, \frac{\lambda}{N} \right).$$

We first prove the recursion for computing  $G$ . For any  $\mathcal{A} \subset \mathcal{I}$ , let

$$G_{\mathcal{A}}(\lambda_1, \dots, \lambda_N) = \sum_{x \in \mathbb{N}^N: \mathcal{A}_x = \mathcal{A}} \Phi(x) \prod_{i \in \mathcal{I}} \lambda_i^{x_i}$$

and

$$G_{\mathcal{A}} = G_{\mathcal{A}} \left( \frac{\lambda}{N}, \dots, \frac{\lambda}{N} \right).$$

Observe that

$$G = \sum_{\mathcal{A} \subset \mathcal{I}} G_{\mathcal{A}}.$$

Now let  $\mathcal{S}(\mathcal{A})$  be the set of servers that can serve jobs of classes in  $\mathcal{A}$ . Let  $n = |\mathcal{A}|$  be the number of active classes and  $m = |\mathcal{S}(\mathcal{A})|$  be the number of busy servers. In view of (1), we have

$$G_{\mathcal{A}} = \frac{\sum_{i \in \mathcal{A}} \frac{\lambda}{N} G_{\mathcal{A} \setminus \{i\}}}{m\mu - n \frac{\lambda}{N}}. \quad (4)$$

For all  $n = 0, 1, \dots, N$  and  $m = 0, 1, \dots, S$ , let

$$G_{n,m} = \sum_{\substack{\mathcal{A} \subset \mathcal{I} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} G_{\mathcal{A}}.$$

Observe that

$$G = \sum_{n=0}^N \sum_{m=0}^S G_{n,m}$$

Moreover,  $G_{0,0} = 1$  and  $G_{n,m} = 0$  unless  $d \leq m \leq S$  and  $\lceil \frac{m}{d} \rceil \leq n \leq \binom{m}{d}$ . For such a pair  $n, m$ , we deduce from (4) that

$$G_{n,m} = \frac{\frac{\rho S}{N}}{m - n \frac{\rho S}{N}} \sum_{\substack{\mathcal{A} \subset \mathcal{I} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} \sum_{i \in \mathcal{A}} G_{\mathcal{A} \setminus \{i\}}. \quad (5)$$

We can rewrite the sum as follows:

$$\begin{aligned} \sum_{\substack{\mathcal{A} \subset \mathcal{I} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} \sum_{i \in \mathcal{A}} G_{\mathcal{A} \setminus \{i\}} &= \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} G_{\mathcal{A} \setminus \{i\}}, \\ &= \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{B} \subset \mathcal{I}, i \notin \mathcal{B} \\ |\mathcal{B}|=n-1, |\mathcal{S}(\mathcal{B} \cup \{i\})|=m}} G_{\mathcal{B}}, \\ &= \sum_{r=0}^{\min(d,m)} \sum_{\substack{\mathcal{B} \subset \mathcal{I}, \\ |\mathcal{B}|=n-1, |\mathcal{S}(\mathcal{B} \cup \{i\})|=m \\ |\mathcal{S}(\mathcal{B})|=m-r}} \sum_{i \in \mathcal{I} \setminus \mathcal{B}} G_{\mathcal{B}}, \\ &= \left[ \binom{m}{d} - n + 1 \right] G_{n-1,m} + \sum_{r=1}^{\min(d,m)} \binom{S-m+r}{r} \binom{m-r}{d-r} G_{n-1,m-r}, \end{aligned}$$

which is the announced result.

We now show the recursion for  $F$ . For any  $\mathcal{A} \subset \mathcal{I}$  and  $i \in \mathcal{I}$ , let

$$F_{\mathcal{A}} = \frac{\partial G_{\mathcal{A}}}{\partial \lambda_i} \left( \frac{\lambda}{N}, \dots, \frac{\lambda}{N} \right).$$

Observe that

$$F = \sum_{\mathcal{A} \subset \mathcal{I}} F_{\mathcal{A}}.$$

For all  $n = 0, 1, \dots, N$  and  $m = 0, 1, \dots, S$ , we can define by symmetry,

$$F_{n,m} = \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A}, \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} F_{\mathcal{A}} = \frac{1}{N} \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A}, \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} F_{\mathcal{A}}$$

and we have

$$F = \sum_{n=0}^N \sum_{m=0}^S F_{n,m}.$$

Similarly,  $F_{0,0} = 0$  and  $F_{n,m} = 0$  unless  $d \leq m \leq S$  and  $\lceil \frac{m}{d} \rceil \leq n \leq \binom{m}{d}$ .

For any  $i \in \mathcal{I}$  and  $\mathcal{A} \subset \mathcal{I}$  with  $|\mathcal{A}| = n$  and  $|\mathcal{S}(\mathcal{A})| = m$ , we have

$$F_{\mathcal{A}} = \frac{1}{\mu \left( m - n \frac{S\rho}{N} \right)} \left\{ G_{\mathcal{A}} + G_{\mathcal{A} \setminus \{i\}} + \frac{\lambda}{N} \sum_{j \in \mathcal{A}, j \neq i} F_{\mathcal{A} \setminus \{j\}} \right\}$$

so that

$$\begin{aligned} F_{n,m} = \frac{1}{N} \frac{1}{\mu \left( m - n \frac{\rho S}{N} \right)} & \left\{ \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} G_{\mathcal{A}} + \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} G_{\mathcal{A} \setminus \{i\}} \right. \\ & \left. + \frac{\lambda}{N} \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} \sum_{j \in \mathcal{A}, j \neq i} F_{\mathcal{A} \setminus \{j\}} \right\} \end{aligned} \quad (6)$$

The first term of the sum is

$$\sum_{\substack{\mathcal{A} \subset \mathcal{I} \\ |\mathcal{A}|=n \\ |\mathcal{S}(\mathcal{A})|=m}} \sum_{i \in \mathcal{A}} G_{\mathcal{A}} = n \sum_{\substack{\mathcal{A} \subset \mathcal{I} \\ |\mathcal{A}|=n \\ |\mathcal{S}(\mathcal{A})|=m}} G_{\mathcal{A}} = n G_{n,m}.$$

By equation (5), the second term of the sum is

$$\sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} G_{\mathcal{A} \setminus \{i\}} = \sum_{\substack{\mathcal{A} \subset \mathcal{I} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} \sum_{i \in \mathcal{A}} G_{\mathcal{A} \setminus \{i\}} = \frac{m - n \frac{S\rho}{N}}{\frac{\rho S}{N}} G_{n,m}.$$

Thus the first two terms of equation (6) are equal to

$$n G_{n,m} + \frac{m - n \frac{\rho S}{N}}{\frac{\rho S}{N}} G_{n,m} = \frac{m N}{\rho S} G_{n,m}.$$

Now the third term of (6) is

$$\begin{aligned}
& \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} \sum_{j \in \mathcal{A}, j \neq i} F_{\mathcal{A} \setminus \{j\}}, \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}, j \neq i} \sum_{\substack{\mathcal{A} \subset \mathcal{I}, i, j \in \mathcal{A} \\ |\mathcal{A}|=n, |\mathcal{S}(\mathcal{A})|=m}} F_{\mathcal{A} \setminus \{j\}}, \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}, j \neq i} \sum_{\substack{\mathcal{B} \subset \mathcal{I}, i \in \mathcal{B}, j \notin \mathcal{B} \\ |\mathcal{B}|=n-1, |\mathcal{S}(\mathcal{B} \cup \{j\})|=m}} F_{\mathcal{B}}, \\
&= \sum_{r=0}^{\min(d,m)} \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{B} \subset \mathcal{I}, i \in \mathcal{B} \\ |\mathcal{B}|=n-1, |\mathcal{S}(\mathcal{B})|=m-r}} \sum_{j \in \mathcal{I} \setminus \mathcal{B}, |\mathcal{S}(\mathcal{B} \cup \{j\})|=m} F_{\mathcal{B}}, \\
&= \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{B} \subset \mathcal{I}, i \in \mathcal{B} \\ |\mathcal{B}|=n-1, |\mathcal{S}(\mathcal{B})|=m}} \left[ \binom{m}{d} - n + 1 \right] F_{\mathcal{B}}, \\
&+ \sum_{r=1}^{\min(d,m)} \sum_{i \in \mathcal{I}} \sum_{\substack{\mathcal{B} \subset \mathcal{I}, i \in \mathcal{B} \\ |\mathcal{B}|=n-1, |\mathcal{S}(\mathcal{B})|=m-r}} \binom{m-r}{d-r} \binom{S-m+r}{r} F_{\mathcal{B}}, \\
&= N \left[ \binom{m}{d} - n + 1 \right] F_{n-1,m} + N \sum_{r=1}^{\min(d,m)} \binom{m-r}{d-r} \binom{S-m+r}{r} F_{n-1,m-r}.
\end{aligned}$$

Summing the two parts of equation (6) gives the announced result.  $\square$

## References

- [1] T. Bonald, L. Massoulié, A. Proutière, and J. Virtamo. A queueing analysis of max-min fairness, proportional fairness and balanced fairness. *Queueing Syst.*, 53(1-2):65–84, 2006.
- [2] K. Gardner, S. Zbarsky, M. Harchol-Balter, and A. Scheller-Wolf. Analyzing response time in the redundancy-d system. Technical report, Technical Report CMU-CS-15-141, 2015.
- [3] V. Shah and G. de Veciana. Performance evaluation and asymptotics for content delivery networks. In *Proceedings of IEEE Infocom*. IEEE, 2014.