



HAL
open science

COORDINATES AT STABLE POINTS OF THE SPACE OF ARCS

Ana J. Reguera

► **To cite this version:**

Ana J. Reguera. COORDINATES AT STABLE POINTS OF THE SPACE OF ARCS. *Journal of Algebra*, 2018. hal-01305997

HAL Id: hal-01305997

<https://hal.science/hal-01305997>

Submitted on 22 Apr 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COORDINATES AT STABLE POINTS OF THE SPACE OF ARCS

ANA J. REGUERA

Universidad de Valladolid. Dpto. de Álgebra, Análisis Matemático, Geometría y Topología.
47011 Valladolid, Spain. E-mail : areguera@agt.uva.es. Tel. : 34 983423048. Fax : 34 983423788

Abstract. Let X be a variety over a field k and let X_∞ be its space of arcs. Let P_E be the stable point of X_∞ defined by a divisorial valuation ν_E on X . Assuming $\text{char } k = 0$, if X is smooth at the center of P_E , we make a study of the graded algebra associated to ν_E and define a finite set whose elements generate a localization of the graded algebra modulo étale covering. This provides an explicit description of a minimal system of generators of the local ring $\mathcal{O}_{X_\infty, P_E}$. If X is singular, we obtain generators of P_E / P_E^2 and conclude that $\text{embdim } \mathcal{O}_{(X_\infty)_{\text{red}}, P_E} = \text{embdim } \widehat{\mathcal{O}_{X_\infty, P_E}} \leq \widehat{k}_E + 1$ where \widehat{k}_E is the Mather discrepancy of X with respect to ν_E .¹

1. INTRODUCTION

The space of arcs X_∞ of a reduced separated scheme of finite type X over a perfect field k has finiteness properties when we localize at its stable points. The stable points of X_∞ were introduced in [Re1], its definition is based on the stability property of the family of truncated arcs $\{j_n(X_\infty)\}_n$ in Denef and Loeser's foundational article [DL]. Stable points are those fat points P of X_∞ for which the truncations of its zero set $Z(P)$ determine trivial fibrations of fiber \mathbb{A}_k^d over an open set, being X equidimensional of dimension d . Equivalently, stable points of X_∞ are defined on an open subset of X_∞ by the radical of a finitely generated ideal, i.e. they are the generic points of the irreducible cylinders (see [EM], sec. 5). It was proved in [Re1] that the ideal of definition of a stable point of X_∞ on an open subset of X_∞ , with the reduced structure, is finitely generated (see 2.4). This implies that the complete ring $\widehat{\mathcal{O}_{X_\infty, P}}$, P being stable, is a Noetherian ring, which is the basis of a useful Curve Selection Lemma centered at the stable points ([Re1], corol. 4.8).

To compute the dimension of $\mathcal{O}_{X_\infty, P}$, P being stable, is an important problem. One upper bound of $\dim \mathcal{O}_{X_\infty, P}$ is the codimension as a cylinder of $Z(P)$ (see prop. 2.3). But the inequality is in general strict ([IR], ex. 2.8). Another approach we have followed in some concrete examples ([Re2], 5.6, 5.16) is to describe the ring $\widehat{\mathcal{O}_{X_\infty, P}}$ by generators and relations. The main purpose of this work is to provide coordinates in $\widehat{\mathcal{O}_{X_\infty, P}}$.

Divisorial valuations are closely related to stable points : the valuation ν_P associated to a stable point P of X_∞ is divisorial ([Re2], (vii) in prop. 3.7). On the other hand, every divisorial valuation ν_E defines a stable point P_E on X_∞ and moreover, if we consider a multiple e of ν_E , we also have a stable point P_{eE} which contains P_E (see 2.7). A study of the graded algebra associated to the divisorial valuation

¹ **Keywords :** Space of arcs, divisorial valuation, graded algebra.
MSC : 13A02, 13A18, 14B05, 14B25, 14E15, 14J17, 32S05.

ν_E will be crucial in our study.

If X is smooth at the center P_0 of a stable point P of X_∞ , then we prove (prop. 2.6) that the local ring $\mathcal{O}_{X_\infty, P}$ is regular and essentially of finite type over some field and $\dim \mathcal{O}_{X_\infty, P}$ equals the dimension as a cylinder of $Z(P)$. Hence, applying the Change of Variables in the Motivic Integration ([DL], lemma 3.4), it follows that $\dim \mathcal{O}_{X_\infty, P_{eE}} = e(k_E + 1)$ where k_E is the discrepancy of X with respect to E . Assuming that $\text{char } k = 0$, in this paper we recover this equality, and moreover, we give an explicit description of a minimal set of generators of P_{eE} , i.e. we give a regular system of parameters of $\mathcal{O}_{X_\infty, P_{eE}}$.

If X is not smooth at P_0 then, if $X \rightarrow \mathbb{A}_k^d$ is a general projection, we have that a set of generators of the image of P_{eE} in $(\mathbb{A}^d)_\infty$ provides a set of generators of P_{eE} ([Re2], prop. 4.5). From this it follows that the embedding dimension of the ring $\widehat{\mathcal{O}_{X_\infty, P_{eE}}}$ is bounded from above by $e(\widehat{k}_E + 1)$, where \widehat{k}_E is the Mather discrepancy of X with respect to E . A further work will be done to determine whether this defines a minimal system of coordinates of (X_∞, P_{eE}) .

One of the main ideas in our proof is to define some ‘‘approximate roots’’ in $\text{gr}_{\nu_E} \mathcal{O}_{\mathbb{A}^d, P_0}$, being $X \rightarrow \mathbb{A}^d$ a general projection. In fact, if X is a curve then we have that $\text{embdim } \widehat{\mathcal{O}_{X_\infty, P_{eE}}} = \widehat{k}_E + 1$, which in this case is equal to $\text{mult } X + 1$ ([Re2], corol. 5.7). We follow the same line as in the proof for the case of curves, being the more subtle part to define these approximate roots $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ (definition 3.4). Although they do not generate $\text{gr}_{\nu_E} \mathcal{O}_{\mathbb{A}^d, P_0}$ in general ($\text{gr}_{\nu_E} \mathcal{O}_{\mathbb{A}^d, P_0}$ is not in general finitely generated for $d \geq 3$), they generate a localization of $\text{gr}_{\nu_E} \mathcal{O}_{\mathbb{A}^d, P_0}$ modulo étale covering (theorem 3.8). This is done in section 3. In section 4 we describe minimal coordinates of $(\mathbb{A}^d)_\infty$ at the image $P_{eE}^{\mathbb{A}^d}$ of P_{eE} in $(\mathbb{A}^d)_\infty$ from the $q_{j,r}$ ’s. From this we obtain a regular system of parameters of $\mathcal{O}_{X_\infty, P_{eE}}$ if X is smooth at P_0 (theorem 4.8), and a system of coordinates of (X_∞, P_{eE}) for general X (corollary 4.10).

2. PRELIMINARIES

2.1. Let k be a perfect field. For any scheme over k , let X_∞ denote the *space of arcs* of X . It is a (not of finite type) k -scheme whose K -rational points are the K -arcs on X (i.e. the k -morphisms $\text{Spec } K[[t]] \rightarrow X$), for any field extension $k \subseteq K$. More precisely, $X_\infty := \lim_{\leftarrow} X_n$ where, for $n \in \mathbb{N}$, X_n is the k -scheme of finite type whose K -rational points are the K -arcs of order n on X (i.e. the k -morphisms $\text{Spec } K[[t]]/(t)^{n+1} \rightarrow X$). In fact, the projective limit is a k -scheme because the natural morphisms $X_{n'} \rightarrow X_n$, for $n' \geq n$, are affine morphisms. We denote by $j_n : X_\infty \rightarrow X_n$, $n \geq 0$, the natural projections.

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by $h_P : \text{Spec } \kappa(P)[[t]] \rightarrow X$ the induced $\kappa(P)$ -arc on X . The image in X of the closed point of $\text{Spec } \kappa(P)[[t]]$, or equivalently, the image P_0 of P by $j_0 : X_\infty \rightarrow X = X_0$ is called the *center* of P . Then, h_P induces a morphism of k -algebras $h_P^\sharp : \mathcal{O}_{X, j_0(P)} \rightarrow \kappa(P)[[t]]$; we denote by ν_P the function $\text{ord}_t h_P^\sharp : \mathcal{O}_{X, j_0(P)} \rightarrow \mathbb{N} \cup \{\infty\}$.

The space of arcs of $\mathbb{A}_k^N = \text{Spec } k[x_1, \dots, x_N]$ is $(\mathbb{A}_k^N)_\infty = \text{Spec } k[\underline{X}_0, \dots, \underline{X}_n, \dots]$ where for $n \geq 0$, $\underline{X}_n = (X_{1;n}, \dots, X_{N;n})$ is an N -uple of variables. For any $f \in k[x_1, \dots, x_N]$, let $\sum_{n=0}^{\infty} F_n t^n$ be the Taylor expansion of $f(\sum_n \underline{X}_n t^n)$, hence

$F_n \in k[\underline{X}_0, \dots, \underline{X}_n]$. If $X \subseteq \mathbb{A}_k^N$ is affine, and $I_X \subset k[x_1, \dots, x_N]$ is the ideal defining X in \mathbb{A}_k^N , then we have $X_\infty = \text{Spec } k[\underline{X}_0, \dots, \underline{X}_n, \dots] / (\{F_n\}_{n \geq 0, f \in I_X})$.

2.2. For $r, m \in \mathbb{N}$, $0 \leq r \leq m$, let $A_k^{r,m} := k[[x_1, \dots, x_r]][x_{r+1}, \dots, x_m]$ and let $X \subseteq \text{Spec } A_k^{r,m}$ be an affine irreducible k -scheme. A point P of X_∞ is *stable* if there exist $G \in \mathcal{O}_{X_\infty} \setminus P$, such that, for $n \gg 0$, the map $X_{n+1} \rightarrow X_n$ induces a trivial fibration

$$\overline{j_{n+1}(Z(P))} \cap (X_{n+1})_G \rightarrow \overline{j_n(Z(P))} \cap (X_n)_G$$

with fiber \mathbb{A}_k^d , where $d = \dim X$, $Z(P)$ is the set of zeros of P in X_∞ , $\overline{j_n(Z(P))}$ is the closure of $j_n(Z(P))$ in X_n and $(X_n)_G$ is the open subset $X_n \setminus Z(G)$ of X_n . This definition is extended to any element X in \mathcal{X}_k , being \mathcal{X}_k the subcategory of the category of k -schemes defined by all separated k -schemes which are locally of finite type over some Noetherian complete local ring R_0 with residue field k ([Re2] def. 3.3). Note that \mathcal{X}_k contains the separated k -schemes of finite type and it also contains the k -schemes $\text{Spec } \widehat{R}$, being \widehat{R} the completion of a local ring R which is a k -algebra of finite type. In [Re1] and [Re2] a theory of stable points of X_∞ is developed. One important property of these points is the following :

Proposition 2.3. ([Re2], prop. 3.7 (iv)) *Let P be a stable point of X_∞ . For $n \geq 0$, let P_n be the prime ideal $P \cap \mathcal{O}_{\overline{j_n(X_\infty)}}$, where $\overline{j_n(X_\infty)}$ is the closure of $j_n(X_\infty)$ in X_n , with the reduced structure. Then we have that $\dim \mathcal{O}_{\overline{j_n(X_\infty)}, P_n}$ is constant for $n \gg 0$, and since*

$$\dim \mathcal{O}_{X_\infty, P} \leq \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)}, P_n}$$

it implies that $\dim \mathcal{O}_{X_\infty, P} < \infty$.

And the main result in the theory of stable points is :

2.4. Finiteness property of the stable points ([Re1] th. 4.1, [Re2] 3.10).

Let P be a stable point of X_∞ , then the formal completion $\widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, P}}$ of the local ring of $(X_\infty)_{\text{red}}$ at P is a Noetherian ring.

Moreover, if X is affine, then there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P(\mathcal{O}_{(X_\infty)_{\text{red}}})_G$ is a finitely generated ideal of $(\mathcal{O}_{(X_\infty)_{\text{red}}})_G$. In particular $P\mathcal{O}_{(X_\infty)_{\text{red}}, P}$ is finitely generated.

Besides we have $\widehat{\mathcal{O}_{X_\infty, P}} \cong \widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, P}}$ ([Re2] th. 3.13). Hence, from 2.4 it follows that the maximal ideal of $\widehat{\mathcal{O}_{X_\infty, P}}$ is $P\widehat{\mathcal{O}_{X_\infty, P}}$, and even more, $\widehat{P}^n = P^n \widehat{\mathcal{O}_{X_\infty, P}}$ for every $n > 0$ (see [Bo] chap. III, sec. 2, no. 12, corol. 2). Therefore, if P is a stable point of X_∞ then

$$\text{embdim } \widehat{\mathcal{O}_{X_\infty, P}} = \text{embdim } \mathcal{O}_{(X_\infty)_{\text{red}}, P}.$$

Though this article we will consider étale morphisms. The following holds :

Proposition 2.5. *Let $X, Z \in \mathcal{X}_k$ and let $\sigma : X \rightarrow Z$ be an étale k -morphism. Then we have*

$$X_\infty \cong Z_\infty \times_Z X$$

in particular, X_∞ is étale over Z_∞ . Therefore, the morphism $\sigma_\infty : X_\infty \rightarrow Z_\infty$ induces a map

$$\{\text{stable points of } X_\infty\} \rightarrow \{\text{stable points of } Z_\infty\}$$

and, if Q is a stable point of X_∞ and P its image by the previous map, then $\mathcal{O}_{Z_\infty, P} \rightarrow \mathcal{O}_{X_\infty, Q}$ is étale and

$$(1) \quad \widehat{\mathcal{O}_{X_\infty, Q}} \cong \widehat{\mathcal{O}_{Z_\infty, P}} \otimes_{\kappa(P)} \kappa(Q).$$

Proof : We may suppose that $Z = \text{Spec } A$, $X = \text{Spec } B$ where $B = (A[x]/(f))_g$, $f, g \in A[x]$ and the class of $f'(x)$ in B is a unit ([Ra], chap. V, th. 1). Then the stability property in [DL] (see also [Re2] (8) in 3.4) implies that

$$X_\infty = \text{Spec } (A_\infty[X_0] / (F_0))_{G_0}$$

where $A_\infty = \mathcal{O}_{Z_\infty}$. From this it follows that $X_\infty \cong Z_\infty \times_Z X$. Moreover, for $n \geq 0$, we have

$$X_n = \text{Spec } (A_n[X_0] / (F_0))_{G_0}$$

that is, $X_n \cong Z_n \times_Z X$. From this, the stability property [DL], lemma 4.1, and the definition of stable point, it follows that, if Q is a stable point of X_∞ then its image P by σ_∞ is a stable point of Z_∞ .

For the last assertion note that, if $\widehat{X} := \text{Spec } \widehat{\mathcal{O}_{X, Q_0}}$, being Q_0 the center of Q in X , then Q induces a stable point \widehat{Q} in \widehat{X}_∞ because $h_Q : \text{Spec } \kappa(Q)[[t]] \rightarrow X$ factorizes through \widehat{X} , and we have

$$(2) \quad \widehat{\mathcal{O}_{X_\infty, Q}} \cong \widehat{\mathcal{O}_{\widehat{X}_\infty, \widehat{Q}}}.$$

Analogously, $\widehat{\mathcal{O}_{Z_\infty, P}} \cong \widehat{\mathcal{O}_{\widehat{Z}_\infty, \widehat{P}}}$, where $\widehat{Z} := \text{Spec } \widehat{\mathcal{O}_{Z, P_0}}$ and \widehat{P} is the stable point of \widehat{Z}_∞ induced by P . Therefore, in order to prove (1) we may suppose that $Z = \text{Spec } A$, $X = \text{Spec } B$ where A and B are complete local rings and $X \rightarrow Z$ is local étale, hence $B \cong A \otimes_{\kappa(P_0)} \kappa(Q_0)$ ([Ra] VIII corol. to lemme 2 and [Ha] III exer. 10.4). Now, $X_\infty \cong Z_\infty \times_Z X$, therefore

$$B_\infty \cong A_\infty \otimes_{\kappa(P_0)} \kappa(Q_0) \quad \text{and} \quad (B_\infty)_Q \cong (A_\infty)_P \otimes_{\kappa(P_0)} \kappa(Q_0).$$

Thus $(A_\infty)_P \rightarrow (B_\infty)_Q$ is étale and hence $\widehat{(A_\infty)_P} \rightarrow \widehat{(B_\infty)_Q}$ is also étale, and from [Ra], VIII corol. to lemme 2, it follows that $\widehat{(B_\infty)_Q} \cong \widehat{(A_\infty)_P} \otimes_{\kappa(P)} \kappa(Q)$, therefore (1) holds.

The inequality in prop. 2.3 may be strict (see [IR] example 2.8). However, if X is nonsingular at P_0 , then we will next show that equality holds.

Proposition 2.6. *Let P be a stable point of X_∞ . If X is nonsingular at the center P_0 of P , then the ring $\mathcal{O}_{X_\infty, P}$ is regular and essentially of finite type over a field, and we have*

$$\dim \mathcal{O}_{X_\infty, P} = \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)}, (P)_n}.$$

Proof : The first statement is prop. 4.2 in [Re2]. The second one also follows from the proof of [Re2], prop. 4.2. In fact, by prop. 2.5 and since there exists an étale morphism from a neighborhood of P_0 to a subset of $A_k^{r, d-r}$, where $d = \dim X$, we may suppose that $X \subseteq A_k^{r, d-r}$. In this case we have

$$\mathcal{O}_{X_\infty} = \mathcal{O}_X[\underline{X}_1, \dots, \underline{X}_n, \dots] \quad \text{and} \quad \mathcal{O}_{X_n} = \mathcal{O}_X[\underline{X}_1, \dots, \underline{X}_n], \quad n \geq 0$$

where $\underline{X}_n = (X_{1;n}, \dots, X_{d;n})$, $n \geq 1$. By 2.4, there exist a finite number of polynomials $G_1, \dots, G_s, G \in \mathcal{O}_{X_\infty}$ such that $P = ((G_1, \dots, G_s) : G^\infty)$. If $n_0 \in \mathbb{N}$ is such that $\mathcal{O}_{\overline{j_{n_0}(X_\infty)}}$ contains G_1, \dots, G_s, G , then $k(\underline{X}_{n_0+1}, \dots, \underline{X}_n, \dots) \subset \mathcal{O}_{X_\infty, P}$. This implies that

$$\mathcal{O}_{X_\infty, P} \cong k(\underline{X}_{n_0+1}, \dots, \underline{X}_n, \dots) \otimes_k \mathcal{O}_{\overline{j_{n_0}(X_\infty)}, P_{n_0}}$$

hence we conclude the result.

2.7. Let X be a reduced separated k -scheme of finite type and let ν be a divisorial valuation on X , i.e. ν is a divisorial valuation on an irreducible component of X . Then there exists a proper and birational morphism $\pi : Y \rightarrow X$, with Y normal, such that the center of ν on Y is a divisor E of Y . We also denote by ν_E the valuation ν . Let $\pi_\infty : Y_\infty \rightarrow X_\infty$ be the morphism on the spaces of arcs induced by π . Let $Y_\infty^{E_{\text{reg}}}$ be the inverse image of $E \cap \text{Reg}(Y)$ by the natural projection $j_0^Y : Y_\infty \rightarrow Y$, which is an irreducible subset of Y_∞ , and let N_E be the closure of $\pi_\infty(Y_\infty^{E_{\text{reg}}})$. Then N_E is an irreducible subset of X_∞ , let P_E be the generic point of N_E . More generally, for every $e \geq 1$, let $Y_\infty^{eE_{\text{reg}}} := \{Q \in Y_\infty / \nu_Q(I_E) = e\}$, where I_E is the ideal defining E in an open affine subset of $\text{Reg}(Y)$ (the set $Y_\infty^{eE_{\text{reg}}}$ will be also denoted by Y_∞^{eE} if Y is nonsingular). Then $Y_\infty^{eE_{\text{reg}}}$ is an irreducible subset of Y_∞ , let N_{eE} be the closure of $\pi_\infty(Y_\infty^{eE_{\text{reg}}})$ and P_{eE} (also denoted by P_{eE}^X) be the generic point of N_{eE} . Note that P_{eE} only depends on e and on the divisorial valuation $\nu = \nu_E$, more precisely, if $\pi' : Y' \rightarrow X$ is another proper and birational morphism, with Y' normal, such that the center E' of ν on Y' is a divisor, then the point $P_{eE'}$ defined by e and E' coincides with P_{eE} . We have that P_{eE} is a stable point of X_∞ ([Re2], prop. 4.1, see also [Re1], prop. 3.8).

2.8. With the notation in 2.7, the image of the canonical homomorphism $d\pi : \pi^*(\wedge^d \Omega_X) \rightarrow \wedge^d \Omega_Y$ is an invertible sheaf at the generic point of E . That is, there exists a nonnegative integer \widehat{k}_E such that the fibre at E of the sheaf $d\pi(\pi^*(\wedge^d \Omega_X))$ is isomorphic to the fibre at E of $\mathcal{O}_Y(-\widehat{k}_E E)$. We call \widehat{k}_E the *Mather discrepancy* of X with respect to the prime divisor E . Note that $\widehat{k}_E \neq 0$ if and only if π is an isomorphism at the generic point of E , and that \widehat{k}_E only depends on the divisorial valuation $\nu = \nu_E$. We have :

$$(3) \quad \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)}, (P_{eE})_n} = e(\widehat{k}_E + 1)$$

([DL], lemma 3.4, [FEI], theorem 3.9). Hence by prop. 2.3 we have

$$\dim \mathcal{O}_{X_\infty, P_{eE}} \leq e(\widehat{k}_E + 1).$$

Moreover, let P be a stable point of X_∞ and let P_0 be its center. If P_0 is the generic point of X then ν_P is trivial. Otherwise, ν_P is a divisorial valuation ([Re2], (vii) in prop. 3.7 and prop. 3.8), i.e. there exists $\pi : Y \rightarrow X$ birational and proper such that the center of ν_P on Y is a divisor E and there exists $e \in \mathbb{N}$ such that $\nu_P = e\nu_E$. There exists a stable point $P^Y \in Y_\infty$ whose image by π_∞ is P ([Re2], prop. 4.1). Therefore $P^Y \supseteq P_{eE}^Y$ and $P \supseteq P_{eE}$. Now, assume that X is nonsingular at P_0 , and recall that in this nonsingular case we have $\widehat{k}_E = k_E$, where k_E is the *discrepancy* of X with respect to E , which is defined to be the coefficient of E in the divisor $K_{Y/X}$ with exceptional support which is linearly equivalent to $K_Y - \pi^*(K_X)$ ([EM], appendix). Applying prop. 2.6 and lemma 4.3 in [DL] we conclude

Corollary 2.9. *Let P be a stable point of X_∞ . Suppose that X is nonsingular at the center P_0 of P , and that P_0 is not the generic point of X and $\nu_P = e\nu_E$. Then $\mathcal{O}_{X_\infty, P}$ is a regular ring of dimension*

$$\dim \mathcal{O}_{X_\infty, P} = ek_E + \dim \mathcal{O}_{Y_\infty, P^Y}.$$

In particular

$$\dim \mathcal{O}_{X_\infty, P_{eE}} = e(k_E + 1).$$

The following question is open :

Question 2.10. Let P be a stable point of X_∞ and suppose that the local ring $\mathcal{O}_{X_\infty, P}$ is regular. Is X nonsingular at the center P_0 of P ?

3. ON THE GRADED ALGEBRA OF THE LOCAL RING OF A SMOOTH SCHEME ASSOCIATED TO A DIVISORIAL VALUATION

From now on, let k be a field of characteristic 0. Through this article, we will denote by $k \langle y_1, \dots, y_r \rangle$ the henselization of the local ring $k[y_1, \dots, y_r]_{(y_1, \dots, y_r)}$, being y_1, \dots, y_r indeterminacies (see [Ra] for more details on henselization).

Let $\eta : Y \rightarrow \mathbb{A}_k^d$ be a k -morphism dominant and generically finite, where Y is a nonsingular k -scheme, let E be a divisor on Y and let P_0 be the center on \mathbb{A}_k^d of the valuation defined by E . In this section we will define elements $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ in the fraction field of $\mathcal{O}_{\mathbb{A}^d, P_0}$ (prop. 3.3) whose initial forms generate a localization of the graded algebra $gr_{\nu_E} \mathcal{O}_{\mathbb{A}^d, P_0}$ modulo étale covering. In section 4 we will prove that they have the property of determining a basis of $P_{eE}^{\mathbb{A}^d} / (P_{eE}^{\mathbb{A}^d})^2$, being $P_{eE}^{\mathbb{A}^d}$ the image by η_∞ of the generic point of Y_∞^{eE} (see 2.7). From this and applying prop. 2.5, we will conclude analogous results for a smooth surface X and a divisorial valuation on X (theorems 3.8 and 4.8).

Let us apply the description of the morphism η appearing in [Re2], proof of prop. 4.5 (see (4) below). First, we may suppose that Y is an affine k -scheme. In fact, we may replace Y by an open affine subset which contains the generic point ξ_E of E . Let $u \in \mathcal{O}_U$, U being an open subset of Y that contains ξ_E , such that u defines a local equation of E . Since η is dominant and generically finite, there exist local coordinates x_1, \dots, x_d in an open subset of \mathbb{A}^d that contains $\eta(\xi_E)$ such that the image of x_1 in \mathcal{O}_{Y, ξ_E} is $g u^{m_1}$, where $m_1 > 0$ and g is a unit in \mathcal{O}_{Y, ξ_E} . By restricting U and adding a m_1 -th root of g , we can define an étale morphism $\varphi : \tilde{U} \rightarrow U$ such that the image of x_1 in $\mathcal{O}_{\tilde{U}}$ is $u_1^{m_1}$ where u_1 is a local equation of the strict transform \tilde{E} of E in \tilde{U} . Moreover, since $\text{char } k = 0$, and $\Omega_{\mathbb{A}^d} \otimes K(Y) \cong \Omega_Y \otimes K(Y)$, we may restrict \tilde{U} and U and define $\{u_1, \dots, u_d\} \subset \mathcal{O}_{\tilde{U}}$, $\{x_1, \dots, x_d\} \subset \mathcal{O}_V$, where V is an open subset of X , determining respective regular systems of parameters in a closed point $y_0 \in \tilde{E}$ and in $\eta \circ \varphi(y_0)$, and such that, if we identify x_1, \dots, x_d with their images by $\eta^\sharp : \mathcal{O}_{V, \eta(y_0)} \rightarrow \mathcal{O}_{\tilde{U}, y_0}$, then

$$(4) \quad \begin{aligned} x_1 &= u_1^{m_1} \\ x_2 &= \sum_{1 \leq i \leq m_2} \lambda_{2,i} u_1^i + u_1^{m_2} u_2 \\ x_3 &= \sum_{1 \leq i \leq m_3} \lambda_{3,i}(u_2) u_1^i + u_1^{m_3} u_3 \\ \dots &\dots\dots \\ x_\delta &= \sum_{1 \leq i \leq m_\delta} \lambda_{\delta,i}(u_2, \dots, u_{\delta-1}) u_1^i + u_1^{m_\delta} u_\delta \\ x_{\delta+1} &= u_{\delta+1} \\ \dots &\dots\dots \\ x_d &= u_d \end{aligned}$$

where $\delta = \text{codim}_{\mathbb{A}^d} \overline{\eta(\xi_E)}$,

$$\begin{aligned} m_1 &\leq \text{ord}_{u_1} x_j = \min\{i \mid \lambda_{j,i} \neq 0\} \quad \text{for } 2 \leq j \leq d, \\ 0 &< m_1 \leq m_2 \leq \dots \leq m_d, \end{aligned}$$

$\lambda_{j,i}(u_2, \dots, u_{j-1}) \in k[[u_2, \dots, u_{j-1}]]$, for $2 \leq j \leq \delta$, $0 \leq i \leq m_j$, and, given $j' < j$, if $i < m_{j'}$ then $\lambda_{j,i} \in k[[u_2, \dots, u_{j'-1}]]$. Moreover, since $x_j - u_1^{m_j} u_j$ belongs to

$k[[u_1, \dots, u_{j-1}]]$ and is integral over $k[[u_1, \dots, u_d]]_{(u_1, \dots, u_d)}$, it is also integral over $k[[u_1, \dots, u_{j-1}]]_{(u_1, \dots, u_{j-1})}$. Therefore, after a possible replacement of y_0 by another point in an open subset of $\tilde{U} \cap \tilde{E}$, we may suppose that, for $2 \leq j \leq \delta$ and $0 \leq i \leq m_j$, $\lambda_{j,i}(u_2, \dots, u_{j-1})$ belongs to the henselization $k \langle u_2, \dots, u_{j-1} \rangle$ of the local ring $k[[u_2, \dots, u_{j-1}]]_{(u_2, \dots, u_{j-1})}$, and, if $i < m_{j'}$, $j' < j$, then $\lambda_{j,i}$ belongs to $k \langle u_2, \dots, u_{j'-1} \rangle$.

Besides, from the expression (4) it follows that there exists an open neighborhood of y_0 in \tilde{E} whose closed points y'_0 satisfy the same property, i.e. there exists a regular system of parameters of y'_0 and of $\eta \circ \varphi(y'_0)$ for which (4) holds. In fact, replace u_i by $u'_i = u_i + c_i \pmod{u_1}$, for $2 \leq i \leq d$, where $(c_i)_i$ lies in an open subset of k^{d-1} . Hence, we may suppose with no loss of generality that

$$(5) \quad \begin{aligned} & \lambda_{j,i}(u_2, \dots, u_{j-1}) \in k \langle u_2, \dots, u_{j-1} \rangle \text{ for } 2 \leq j \leq \delta, 0 \leq i \leq m_j \\ & \text{if } i < m_{j'}, j' < j, \text{ then } \lambda_{j,i} \in k \langle u_2, \dots, u_{j'-1} \rangle \\ & \text{if } \lambda_{j,i}(u_2, \dots, u_{j-1}) \neq 0 \text{ then it is a unit in } k \langle u_2, \dots, u_{j-1} \rangle \\ & \lambda_{j,m_j}(u_2, \dots, u_{j-1}) \text{ is a unit, for } 2 \leq j \leq d. \end{aligned}$$

Note that \tilde{U} is nonsingular. Note also that $\wedge^d \Omega_V$ is an invertible sheaf, hence the image of $d\eta : \eta^*(\wedge^d \Omega_V) \rightarrow \wedge^d \Omega_U$ is an invertible sheaf. The order a_E in E of the corresponding divisor is equal to the order in \tilde{E} of the image of $d(\eta \circ \varphi) : (\eta \circ \varphi)^*(\wedge^d \Omega_V) \rightarrow \wedge^d \Omega_{\tilde{U}}$. So, from now on, after a possible replacement of Y by \tilde{U} and of $\eta : Y \rightarrow \mathbb{A}^d$ by $\eta \circ \varphi : \tilde{U} \rightarrow V$, we will suppose that (4) is a local expression of η . Besides, from (4) it follows that :

$$(6) \quad a_E = m_1 + \dots + m_\delta - 1.$$

Lemma 3.1. *Let A be a finitely generated k -algebra and let $\theta : Y \rightarrow \text{Spec } A[x, y]$ be a k -morphism, where x, y are indeterminacies. Let j , $2 \leq j \leq d+1$ and suppose that there exists a multiplicative system S_{j-1} of $A[x]$ and there exist elements*

$$l_{j'} \in S_{j-1}^{-1} A[x] \quad \text{for } 2 \leq j' \leq j-1$$

such that, if we set $v_{j'} := \theta^\#(l_{j'})$ for $2 \leq j' \leq j-1$, then $\{u_1, v_2, \dots, v_{j-1}, u_j, \dots, u_d\}$ is a regular system of parameters of \mathcal{O}_{Y, y_0} . Suppose that the images of x, y by $\theta^\#$ are given by $x \mapsto u_1^{m_1}$ and

$$(7) \quad y \mapsto \sum_{m_1 \leq i \leq m} \lambda_i(v_2, \dots, v_{j-1}) u_1^i + u_1^m \varrho \pmod{(u_1)^{m+1}}$$

where $m \geq m_1$, $\varrho \in \mathcal{O}_{Y, y_0}$ and $\lambda_i(v_2, \dots, v_{j-1}) \in k \langle v_2, \dots, v_{j-1} \rangle$. Set

$$(8) \quad \begin{aligned} \mathbf{e} & := \text{g.c.d.}(\{m_1\} \cup \{i / \lambda_i \neq 0\}), & \beta_0 & := m_1, & e_0 & := \beta_0 \\ \beta_{r+1} & := \min \{i / \lambda_i \neq 0 \text{ and } \text{g.c.d.}\{\beta_0, \dots, \beta_r, i\} < e_r\} \text{ and} \\ e_{r+1} & := \text{g.c.d.}\{\beta_0, \dots, \beta_{r+1}\} \text{ for } 1 \leq r \leq g-1, \text{ being } g \text{ such that } e_g = \mathbf{e} \\ \beta_{g+1} & := m. \end{aligned}$$

Let $n_0 = 1$ and $n_r := \frac{e_{r-1}}{e_r}$ for $1 \leq r \leq g$ and let $\bar{\beta}_0 = \beta_0$ and $\bar{\beta}_r$, $1 \leq r \leq g+1$ be defined by

$$(9) \quad \bar{\beta}_r - n_{r-1} \bar{\beta}_{r-1} = \beta_r - \beta_{r-1},$$

hence we have

$$(10) \quad \begin{aligned} & \bar{\beta}_r > n_{r-1} \bar{\beta}_{r-1} \text{ for } 1 \leq r \leq g, \quad \text{and } \bar{\beta}_{g+1} \geq n_g \bar{\beta}_g; \\ & n_r \bar{\beta}_r \text{ belongs to the semigroup generated by } \bar{\beta}_0, \dots, \bar{\beta}_{r-1}, 1 \leq r \leq g+1. \end{aligned}$$

Then, there exist an open subset U of Y containing ξ_E and a sequence of integers $\{i_s\}_{s=1}^N$ such that

- (i) $i_1 < i_2 < \dots < i_N = \bar{\beta}_{g+1}$ and $\{i_s\}_{s=1}^N \subset \{\bar{\beta}_0\} \cup \cup_{r=1}^{g+1} (n_{r-1}\bar{\beta}_{r-1}, \bar{\beta}_r]$,
- (ii) $\{\bar{\beta}_r\}_{r=1}^{g+1}$ is contained in $\{i_1, \dots, i_N\}$, that is, there exist $s_1 < s_2 < \dots < s_{g+1} := N$ such that $i_{s_r} = \bar{\beta}_r$ for $1 \leq r \leq g+1$,
- (iii) for each closed point y'_0 in $U \cap E$ there exist a regular system of parameters $\{u_1, v'_2, \dots, v'_{j-1}, u'_j, \dots, u'_d\}$ of \mathcal{O}_{Y, y'_0} , where $v'_i = v_i + c_i$, $u'_i = u_i + c_i$, $(c_i)_i \in k^{d-1}$, and there exist $\{h_1 = y, h_2, \dots, h_N\}$ satisfying : given s , let r , $1 \leq r \leq g+1$, be such that $n_{r-1}\bar{\beta}_{r-1} < i_s \leq \bar{\beta}_r$ (or $r = 1$ if $s = 1$ and $i_1 = \bar{\beta}_0$), then
 - (a) $h_s \in T_{r-1}^{-1} \dots T_0^{-1} S_{j-1}^{-1} A[x, y]$, where $T_{r'}$ is the multiplicative part generated by $q_{r'} := h_{s_{r'}}$ (resp. $q_0 := x_1$) for $1 \leq r' \leq r-1$ (resp. $r' = 0$),
 - (b) the image of h_s in $K(\mathcal{O}_{Y, y'_0})$ belongs to \mathcal{O}_{Y, y'_0} , and if we identify h_s with its image in \mathcal{O}_{Y, y'_0} then

$$(11) \quad h_s = \sum_{i_s \leq i \leq m^{(r)}} \lambda_{s,i} (v'_2, \dots, v'_{j-1}) u_1^i + \gamma_{s,m^{(r)}} (v'_2, \dots, v'_{j-1}) u_1^{m^{(r)}} \varrho \pmod{(u_1)^{m^{(r)}+1}}$$

where $\lambda_{s,i}, \gamma_{s,m^{(r)}} \in k < v'_2, \dots, v'_{j-1} >$, $\lambda_{s,i_s} \neq 0$ for $1 \leq s < N$, $\gamma_{s,m^{(r)}}$ is a unit and $m^{(r)} := m + (n_1 - 1)\bar{\beta}_1 + \dots + (n_{r-1} - 1)\bar{\beta}_{r-1}$. Moreover, for $r \leq r' \leq g$, let $\beta_{r'}^{(r)} := \beta_{r'} + (n_1 - 1)\bar{\beta}_1 + \dots + (n_{r-1} - 1)\bar{\beta}_{r-1}$ then we have

$$(12) \quad \min \left\{ i / \lambda_{s,i} \neq 0 \text{ and } g.c.d. \{e_{r-1}, \beta_r^{(r)}, \dots, \beta_{r'-1}^{(r)}, i\} < e_{r'-1} \right\} = \beta_{r'}^{(r)}$$

and $\lambda_{s, \beta_{r'}^{(r)}}$ is a unit.

- (c) For $s \geq 2$, if $s = s_{r-1} + 1$ (resp. $s_{r-1} + 1 < s$), then

$$h_s := q_0^{b_0^s} \dots q_\rho^{b_\rho^s} P_s \left(\frac{\bar{\mu}_s h}{q_0^{b_0^s} \dots q_\rho^{b_\rho^s}}, l_2, \dots, l_{j-1} \right)$$

where $h = (q_{r-1})^{n_{r-1}}$ (resp. $h = h_{s-1}$), $\rho = r - 2$ (resp. $\rho = r - 1$), the integers $\{b_{r'}^s\}_{r'=0}^\rho$ are the unique nonnegative integers satisfying $b_{r'}^s < n_{r'}$, $1 \leq r' \leq \rho$, and $n_{r-1}\bar{\beta}_{r-1} = \sum_{0 \leq r' \leq r-2} b_{r'}^s \bar{\beta}_{r'}$ (resp. $i_{j,s-1} = \sum_{0 \leq r' \leq r-1} b_{j,r'}^s \bar{\beta}_{j,r'}$), $\bar{\mu}_s = (\lambda_{s_1, \bar{\beta}_1})^{b_1^s} \dots (\lambda_{s_\rho, \bar{\beta}_\rho})^{b_\rho^s}$ is a unit, and $P_s \in k[z, v'_2, \dots, v'_{j-1}]$ is such that

$$(13) \quad P_s(\lambda, v'_2, \dots, v'_{j-1}) = 0, \quad \frac{\partial P_s}{\partial z}(\lambda, v'_2, \dots, v'_{j-1}) \text{ is a unit in } k < v'_2, \dots, v'_{j-1} >, \quad \text{where } \lambda = (\lambda_{s-1, i_{s-1}})^{n_{r-1}} \text{ (resp. } \lambda = \lambda_{j, s-1, i_{j, s-1}}).$$

Proof : First note that (10) follows from (8) and (9) (see [Za] 2.2.1 in the Appendix). Note also that there exists an open neighborhood of y_0 in E such that if y'_0 is a closed point on it and $\{u_1, v'_2, \dots, v'_{j-1}, u'_j, \dots, u'_d\}$ is a regular system of parameters of \mathcal{O}_{Y, y'_0} , where $v'_i = v_i + c_i$, $u'_i = u_i + c_i$, $(c_i)_i \in k^{d-1}$, then the integers defined by (8) and (9) for the expression of the image of y in terms of $\{u_1, v'_2, \dots, v'_{j-1}, u'_j, \dots, u'_d\}$ are the same as the ones defined for the expression in (7). Thus, to prove the lemma, it suffices to show that, after a possible replacement of y_0 in an open subset $U \cap E$ of E , there exist $\{i_s\}_{s=1}^N$ and $\{h_s\}_{s=1}^N$ satisfying (i), (ii) and (a), (b) for the image of h_s in $K(\mathcal{O}_{Y, y_0})$ (hence $v'_2 = v_2, \dots, v'_{j-1} = v_{j-1}$ in (11)) and (c).

We will define $\{i_s\}_{s=1}^N$ and $\{h_s\}_{s=1}^N$ by induction on s . First, after a possible replacement of y_0 in an open subset of E , we may suppose that, for every i such

that $\lambda_i \neq 0$ in (7), λ_i is a unit in the ring

$$R_{j-1} := k < v_2, \dots, v_{j-1} > .$$

Then, for $s = 1$, let $i_1 := \min\{i / \lambda_i(v_2, \dots, v_{j-1}) \neq 0\}$ and $h_1 := y$. It is clear that $\bar{\beta}_0 \leq i_1 \leq \bar{\beta}_1$ and that (a) and (b) hold for $s = 1$. Now, let $s \geq 2$ and suppose that $i_1 < \dots < i_{s-1}$ and h_1, \dots, h_{s-1} are defined and satisfy the required conditions. If $i_{s-1} = \bar{\beta}_{g+1}$ then set $N := s - 1$. If not, then $i_{s-1} < \bar{\beta}_{g+1}$. Thus, there exist r , $1 \leq r \leq g + 1$ such that $i_{s-1} \in \{\bar{\beta}_{r-1}\} \cup (n_{r-1}\bar{\beta}_{r-1}, \bar{\beta}_r)$. Let $s_1 < s_2 < \dots < s_{r-1} \leq s - 1$ be such that $i_{s_{r'}} = \bar{\beta}_{r'}$ for $1 \leq r' \leq r - 1$ and let $q_0 := x$, $q_{r'} := h_{s_{r'}}$ for $1 \leq r' \leq r - 1$.

If $i_{s-1} = \bar{\beta}_{r-1}$, recall that $\lambda_{s-1, \bar{\beta}_{r-1}}(v_2, \dots, v_{j-1}) \in R_{j-1} \setminus \{0\}$, thus $(\lambda_{s-1, \bar{\beta}_{r-1}})^{n_r}$ belongs to $R_{j-1} \setminus \{0\}$ and hence there exists an irreducible monic polynomial $P_s \in k[z, v_2, \dots, v_{j-1}]$ such that

$$P_s((\lambda_{s-1, \bar{\beta}_{r-1}})^{n_r}, v_2, \dots, v_{j-1}) = 0 \quad \text{and} \quad \frac{\partial P_s}{\partial z}((\lambda_{s-1, \bar{\beta}_{r-1}})^{n_r}, v_2, \dots, v_{j-1}) \neq 0$$

Moreover, after a possible replacement of y_0 in an open subset of E , we may suppose that

$$(14) \quad \begin{aligned} &P_s((\lambda_{s-1, \bar{\beta}_{r-1}})^{n_r}, v_2, \dots, v_{j-1}) = 0 \quad \text{and} \\ &\frac{\partial P_s}{\partial z}((\lambda_{s-1, \bar{\beta}_{r-1}})^{n_r}, \dots, v_{j-1}) \text{ is a unit in } R_{j-1}. \end{aligned}$$

Analogously, if $i_{s-1} \in (n_{r-1}\bar{\beta}_{r-1}, \bar{\beta}_r)$, then after a possible replacement of y_0 in an open subset of E , we may suppose that there exists an irreducible monic polynomial $P_s \in k[z, v_2, \dots, v_{j-1}]$ such that

$$(15) \quad P_s(\lambda_{s-1, i_{s-1}}, v_2, \dots, v_{j-1}) = 0, \quad \frac{\partial P_s}{\partial z}(\lambda_{s-1, i_{s-1}}, \dots, v_{j-1}) \text{ is a unit in } R_{j-1}.$$

If $i_{s-1} = \bar{\beta}_{r-1}$, let $b_{r', s} = b_{r-1, r'}$, $0 \leq r' \leq r-2$, be the unique nonnegative integers satisfying $b_{r-1, r'} < n_{r'}$ for $1 \leq r' \leq r-2$, and $n_{r-1}\bar{\beta}_{r-1} = \sum_{0 \leq r' \leq r-2} b_{r-1, r'} \bar{\beta}_{r'}$, and let $\bar{\mu}_s := (\lambda_{s_1, \bar{\beta}_1})^{b_1^s} \dots (\lambda_{s_{r-2}, \bar{\beta}_{r-2}})^{b_{r-2}^s}$, which is a unit in R_{j-1} , such that the image of $q_0^{b_0^s} \dots q_{r-2}^{b_{r-2}^s}$ by θ^\sharp is equal to $\bar{\mu}_s u_1^{n_{r-1}\bar{\beta}_{r-1}} \pmod{(u_1)^{n_{r-1}\bar{\beta}_{r-1}+1}}$. Set

$$(16) \quad h_s := q_0^{b_{r-10}^s} \dots q_{r-2}^{b_{r-2}^s} P_s \left(\frac{\bar{\mu}_s (q_{r-1})^{n_{r-1}}}{q_0^{b_{r-10}^s} \dots q_{r-2}^{b_{r-2}^s}}, l_2, \dots, l_{j-1} \right)$$

and $i_s := (n_{r-1} - 1)\bar{\beta}_{r-1} + \min\{i / i > \bar{\beta}_{r-1}, \lambda_{s-1, i} \neq 0\}$, unless we have $\lambda_{s-1, i} = 0$ for all $i > \bar{\beta}_{r-1}$, which implies $r - 1 = g$, then set $i_s := \bar{\beta}_{g+1}$. From (14), (16) and Taylor's development for P_s it follows that, if $s < N$ (resp. $s = N$) then the ν_E -value of the image $\theta^\sharp(h_s)$ of h_s in \mathcal{O}_{Y, y_0} is $i_s > n_{r-1}\bar{\beta}_{r-1}$ (resp. $i_s \geq i_N = \bar{\beta}_{g+1}$), and the exponents of u_1 in $\theta^\sharp(h_s)$ with nonzero coefficient (see the left hand side of (11)) are determined by the ones in $\theta^\sharp(h_{s-1})$ by adding $(n_{r-1} - 1)\bar{\beta}_{r-1}$, therefore $n_{r-1}\bar{\beta}_{r-1} < i_s \leq \bar{\beta}_r$ and (11) and (13) hold for s . Moreover, for $r \leq r' \leq g$, the coefficient $\lambda_{s, \beta_{r'}}^{(r)}$ in $u_1^{\beta_{r'}}^{(r)}$ of $\theta^\sharp(h_s)$ is equal, modulo product by a unit, to $(\lambda_{s-1, \bar{\beta}_{r-1}})^{n_{r-1}-1} \lambda_{s-1, \beta_{r'}}^{(r-1)}$, therefore it is a unit, and (b) is satisfied. Besides, $h_s \in T_{r-2}^{-1} \dots T_0^{-1} S_{j-1}^{-1} A[x, y]$, hence (a) also holds.

If $n_{r-1}\bar{\beta}_{r-1} < i_{s-1} < \bar{\beta}_r$ then e_{r-1} divides i_{s-1} (by (b) applied to $s - 1$) and there exist unique nonnegative integers $\{b_{r'}^s\}_{r'=0}^{r-1}$ satisfying $b_{r'}^s < n_{r'}$ for $1 \leq$

$r' \leq r - 1$ and $i_{s-1} = \sum_{0 \leq r' \leq r-1} b_{r'}^s \bar{\beta}_{r'}$ (because $n_{r-1} \bar{\beta}_{r-1} \leq i_{s-1}$). Then, let $\bar{\mu}_s := (\lambda_{s_1, \bar{\beta}_1})^{b_1^s} \cdots (\lambda_{s_{r-1}, \bar{\beta}_{r-1}})^{b_{r-1}^s}$, which is a unit in R_{j-1} , such that the image of $q_0^{b_0^s} \cdots q_{r-1}^{b_{r-1}^s}$ by θ^\sharp is equal to $\bar{\mu}_s u_1^{i_{s-1}} \bmod (u_1)^{i_{s-1}+1}$, and set

$$(17) \quad h_s := q_0^{b_0^s} \cdots q_{r-1}^{b_{r-1}^s} P_s \left(\frac{\bar{\mu}_s h_{s-1}}{q_0^{b_0^s} \cdots q_{r-1}^{b_{r-1}^s}}, l_2, \dots, l_{j-1} \right)$$

and $i_s := \min \{i / i > i_{s-1}, \lambda_{s,i} \neq 0\}$, unless we have $\lambda_{s-1,i} = 0$ for all $i > \bar{\beta}_{r-1}$, which implies $r - 1 = g$ and then we set $i_s := \bar{\beta}_{g+1}$. It is clear that (a) holds and, from (15) and (17), it follows that, if $s < N$ (resp. $s = N$), then the ν_E -value of the image $\theta^\sharp(h_s)$ of h_s in \mathcal{O}_{Y, y_0} is $i_s > i_{s-1} > n_{r-1} \bar{\beta}_{r-1}$ (resp. $\geq i_N = \bar{\beta}_{g+1} > n_g \bar{\beta}_g$), and the exponents of u_1 in $\theta^\sharp(h_s)$ with nonzero coefficient are the same as the ones for by $\theta^\sharp(h_{s-1})$, hence $n_{r-1} \bar{\beta}_r < i_s \leq \bar{\beta}_r$ and (11) and (13) hold for s . Moreover, for $r \leq r' \leq g$, the coefficient $\lambda_{s, \beta_{r'}^{(r)}}$ in $u_1^{\beta_{r'}^{(r)}}$ of $\theta^\sharp(h_s)$ is the same, modulo product by a unit, as the coefficient $\lambda_{s-1, \beta_{r'}^{(r)}}$ of $\theta^\sharp(h_{s-1})$, therefore it is a unit, and (b) is satisfied. Besides note that $\beta_r^{(r)} = \bar{\beta}_r$ for $1 \leq r \leq g + 1$, hence from the previous construction it follows that $\{\bar{\beta}_r\}_{r=1}^{g+1} \subset \{i_s\}_{s=1}^N$, hence the result is proved.

Corollary 3.2. *Let j , $2 \leq j \leq \delta$. Set $A := k[x_2, \dots, x_{j-1}]$, $x = x_1$, $y = x_j$, and let $\theta : Y \rightarrow \text{Spec } A[x_1, x_j]$ be the composition of $\eta : Y \rightarrow \mathbb{A}^d$ with the projection $\mathbb{A}^d \rightarrow \text{Spec } A[x_1, x_j]$. Suppose that the hypothesis in lemma 3.1 holds and let the image by η^\sharp of x_j be given by*

$$(18) \quad x_j = \sum_{m_1 \leq i \leq m_j} \lambda'_{j,i}(v_2, \dots, v_{j-1}) u_1^i + u_1^{m_j} u_j \bmod (u_1)^{m_j+1}.$$

where $\lambda'_{j,i}(v_2, \dots, v_{j-1}) \in R_{j-1} = k \langle v_2, \dots, v_{j-1} \rangle$. Let $\{\beta_{j,r}\}_{r=0}^{g_j+1}$, $\{e_{j,r}\}_{r=0}^{g_j}$, $\{n_{j,r}\}_{r=0}^{g_j}$ and $\{\bar{\beta}_{j,r}\}_{r=0}^{g_j+1}$ be the integers defined by (8) and (9). Then there exist an open subset U of Y and, for each point y'_0 in $U \cap E$, a regular system of parameters $\{u_1, v'_2, \dots, v'_{j-1}, u'_j, \dots, u'_d\}$ of \mathcal{O}_{Y, y'_0} , where $v'_i = v + c_i$, $u'_i = u_i + c_i$, $(c_i)_i \in k^{d-1}$, and there exist elements $\{q_{j,0} = x_1, q_{j,1}, \dots, q_{j, g_j+1}\}$ where

$$q_{j,r} \in T_{r-1}^{-1} \cdots T_0^{-1} S_{j-1}^{-1} [x_1, x_2, \dots, x_{j-1}, x_j]$$

being $T_{r'}$ the multiplicative part generated by $q_{j,r'}$, such that the images of $\{q_{j,r}\}_{r=0}^{g_j+1}$ in \mathcal{O}_{Y, y'_0} are given by

$$(19) \quad \begin{aligned} q_{j,r} &= \mu_{j,r}(v'_2, \dots, v'_{j-1}) u_1^{\bar{\beta}_{j,r}} \bmod (u_1)^{\bar{\beta}_{j,r}+1} \quad \text{for } 0 \leq r \leq g_j \\ q_{j, g_j+1} &= \mu_{j, g_j+1}(v'_2, \dots, v'_{j-1}) u_1^{\bar{\beta}_{j, g_j+1}} u_j \bmod (u_1)^{\bar{\beta}_{j, g_j+1}+1} \end{aligned}$$

where $\mu_{j,r}(v'_2, \dots, v'_{j-1})$ is a unit in $k \langle v'_2, \dots, v'_{j-1} \rangle$ for $0 \leq r \leq g_j + 1$.

Proof : This is consequence of lemma 3.1. In fact, after a possible replacement of y_0 in an open subset of E , we may suppose that there exist $\{i_s\}_{s=1}^N$ and $\{h_s\}_{s=1}^N$ satisfying (i), (ii) and (a), (b) in lemma 3.1. Let $q_{j,0} := x_1$, $q_{j,1} := h_{s_1}, \dots, q_{j, g_j} := h_{s_{g_j}}$. If $\lambda_{s_{g_j+1}, \bar{\beta}_{g_j+1}} = 0$ in the expression (11) for $\eta^\sharp(h_{s_{g_j+1}})$ then let $q_{j, g_j+1} := h_{s_{g_j+1}}$. Otherwise, after a possible replacement of y_0 in an open subset of E , we may suppose that there exists an irreducible monic polynomial $P \in k[z, v_2, \dots, v_{j-1}]$ such that $P(\lambda_{s_{g_j+1}, \bar{\beta}_{j, g_j+1}}, v_2, \dots, v_{j-1}) = 0$ and $\frac{\partial P}{\partial z}(\lambda_{s_{g_j+1}, \bar{\beta}_{j, g_j+1}})$ is a unit in R_{j-1} .

Then we proceed as in (17), that is we set

$$q_{j,g_j+1} := q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}} P \left(\frac{\bar{\mu} h_{s_{g_j+1}}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}}, l_2, \dots, l_{j-1} \right)$$

where $b_{j,0}, \dots, b_{j,g_j}$ are nonnegative integers satisfying $b_{j,r} < n_{j,r}$, $1 \leq r \leq g_j$, and $\bar{\beta}_{j,g_j+1} = \sum_{0 \leq r \leq g_j} b_{j,r} \bar{\beta}_{j,r}$, and $\bar{\mu} = (\lambda_{s_1, \bar{\beta}_{j,1}})^{b_{j,1}} \cdots (\lambda_{s_{g_j}, \bar{\beta}_{j,1}})^{b_{j,g_j}}$. Then $\{q_{j,r}\}_{r=0}^{g_j+1}$ satisfy the required condition.

Proposition 3.3. *There exist a point $y_0 \in E$, a regular system of parameters $\{u, v_2, \dots, v_d\}$ of \mathcal{O}_{Y,y_0} and a regular system of parameters $\{x_1, \dots, x_d\}$ of $\mathcal{O}_{\mathbb{A}^d, \eta(y_0)}$ such that the following holds :*

(i) *If we identify x_1, \dots, x_d with their images in \mathcal{O}_{Y,y_0} then*

$$\begin{aligned} x_1 &= u^{m_1} \\ x_j &= \sum_{m_1 \leq i \leq m_j} \lambda_{j,i}(v_2, \dots, v_{j-1}) u^i + u^{m_j} v_j \pmod{(u)^{m_j+1}}, \text{ for } 2 \leq j \leq \delta \\ x_r &= v_r \quad \text{for } \delta + 1 \leq r \leq d. \end{aligned}$$

where $0 < m_1 \leq m_2 \leq \dots \leq m_d$ and, for $2 \leq j \leq \delta$, if we set $R_{j-1} := k \langle v_2, \dots, v_{j-1} \rangle$, then $\lambda_{j,i}(v_2, \dots, v_{j-1}) \in R_{j-1}$, $\lambda_{j,i} \neq 0$ implies that it is a unit in R_{j-1} , and $\lambda_{j,m_j}(v_2, \dots, v_{j-1})$ is a unit in R_{j-1} and

$$(20) \quad \text{if } i < m_{j'}, j' < j, \text{ then } \lambda_{j,i} \in R_{j'-1}.$$

(ii) For $2 \leq j \leq \delta$, let $B_j := R_{j-1}[x_1, x_j]_{(x_1, x_j)}$, let ν_j be the restriction of ν_E to B_j , let $\bar{\beta}_{j,0} = m_1, \bar{\beta}_{j,1}, \dots, \bar{\beta}_{j,g_j}$ be a minimal system of generators of the semigroup $\nu_j(B_j \setminus \{0\})$ and $\bar{\beta}_{j,g_j+1} = \nu_j(I_j)$, being I_j the complete ideal defined by the restriction of ν_j to a general fibre of $\text{Spec } B_j \rightarrow \text{Spec } R_{j-1}$. Set

$$\mathcal{J}^* := \{(1,0)\} \cup \{(j,r) / 2 \leq j \leq \delta, 1 \leq r \leq g_j\}, \quad \mathcal{J} := \mathcal{J}^* \cup \{(j,g_j+1) / 2 \leq j \leq \delta\}$$

let us consider the lexicographic order in \mathcal{J} and, for $(j,r) \in \mathcal{J}$, let

$$\mathcal{J}_{j,r}^* := \{(j',r') \in \mathcal{J}^* / (j',r') < (j,r)\}, \quad \mathcal{J}_{j,r} := \{(j',r') \in \mathcal{J} / (j',r') < (j,r)\}.$$

Then, there exist elements $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ in $k(x_1, \dots, x_j)$, more precisely,

$$(21) \quad q_{j,r} \in \prod_{(j',r') \in \mathcal{J}_{j,r}^*} T_{j',r'}^{-1} k[x_1, \dots, x_j]$$

where, for $(j',r') \in \mathcal{J}_{j,r}^*$, $T_{j',r'}$ is the multiplicative system generated by $q_{j',r'}$, such that :

(a.2) $q_{1,0} := x_1$ and, for $2 \leq j \leq \delta$, $0 \leq r \leq g_j + 1$, the image of $q_{j,r}$ in the fraction field $K(\mathcal{O}_{Y,y_0})$ of \mathcal{O}_{Y,y_0} belongs to \mathcal{O}_{Y,y_0} and, if we identify $q_{j,r}$ with its image, then

$$(22) \quad \begin{aligned} q_{j,r} &= \mu_{j,r}(v_2, \dots, v_{j-1}) u^{\bar{\beta}_{j,r}} \pmod{(u)^{\bar{\beta}_{j,r}+1}} \text{ for } 1 \leq r \leq g_j \\ q_{j,g_j+1} &= \mu_{j,g_j+1}(v_2, \dots, v_{j-1}) u^{\bar{\beta}_{j,g_j+1}} v_j \pmod{(u)^{\bar{\beta}_{j,g_j+1}+1}} \end{aligned}$$

where $\mu_{j,r}(v_2, \dots, v_{j-1})$ is a unit in R_{j-1} for $1 \leq r \leq g_j + 1$.

(b.2) For $2 \leq j \leq \delta$, set $q_{j,0} := q_{1,0} = x_1$, $e_{j,r} := \text{g.c.d.}\{\beta_{j,0}, \dots, \beta_{j,r}\}$, $n_{j,r} := \frac{e_{j,r}-1}{e_{j,r}}$ for $1 \leq r \leq g_j$, and let $b_{j,0}, \dots, b_{j,g_j}$ be the unique nonnegative integers satisfying

$$(23) \quad b_{j,r} < n_{j,r} \text{ for } 1 \leq r \leq g_j \quad \text{and} \quad \bar{\beta}_{j,g_j+1} = \sum_{0 \leq i \leq g_j} b_{j,r} \bar{\beta}_{j,r}$$

then, identifying $q_{j,r}$ with its image in \mathcal{O}_{Y,y_0} , we have

$$\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}} = v_j \in \mathcal{O}_{Y,y_0}.$$

(iii) Even more, for $2 \leq j \leq \delta$, there exist nonnegative integers N_j and $s_{j,1} < s_{j,2} < \cdots < s_{j,g_j+1} = N_j$, and elements $\{h_{j,s}\}_{s=1}^{N_j}$, such that $q_{j,r} = h_{j,s_{j,r}}$ for $1 \leq r \leq g_j + 1$, and besides the following holds : given s , let $r, 1 \leq r \leq g_j + 1$ be such that $s_{j,r-1} < s \leq s_{j,r}$ (resp. $r = 1$ if $s \leq s_{j,1}$), then we have :

(a.3) $h_{j,s} \in \prod_{(j',r') \in \mathcal{J}_{j,r}^*} T_{j',r'}^{-1} k[x_1, \dots, x_j]$

(b.3) the image of $h_{j,s}$ in $K(\mathcal{O}_{Y,y_0})$ belongs to \mathcal{O}_{Y,y_0} and, if we identify $h_{j,s}$ with its image in \mathcal{O}_{Y,y_0} then

$$h_{j,s} = \sum_{i_{j,s} \leq i \leq m_j^{(r)}} \lambda_{j,s,i}(v_2, \dots, v_{j-1}) u^i + \gamma_{j,s,m_j^{(r)}}(v_2, \dots, v_{j-1}) u^{m_j^{(r)}} \text{ mod } (u)^{m_j^{(r)}+1}$$

where $n_{j,r-1} \bar{\beta}_{j,r-1} < i_{j,s} \leq \bar{\beta}_{j,r}$, $i_{j,s-1} < i_{j,s}$, $i_{j,s} = \bar{\beta}_{j,r}$ iff $s = s_r$,

$\lambda_{j,s,i}, \gamma_{j,s,m_j^{(r)}} \in R_{j-1}$, $\lambda_{j,s,i_{j,s}}, \gamma_{j,s,m_j^{(r)}}$ is a unit, and $m_j^{(r)} := m_j +$

$(n_{j,1} - 1) \bar{\beta}_{j,1} + \cdots + (n_{j,r-1} - 1) \bar{\beta}_{j,r-1}$.

(c.3) If $s = s_{j,r-1} + 1$ (resp. $s_{r-1} + 1 < s$), then $h_{j,s}$ is equal to

$$q_{j,0}^{b_{j,0}^s} \cdots q_{j,\rho}^{b_{j,\rho}^s} P_{j,s} \left(\frac{\bar{\mu}_{j,s} h}{q_{j,0}^{b_{j,0}^s} \cdots q_{j,\rho}^{b_{j,\rho}^s}}, \frac{q_{2,g_2+1}}{q_{2,0}^{b_{2,0}} \cdots q_{2,g_2}^{b_{2,g_2}}}, \dots, \frac{q_{j-1,g_{j-1}+1}}{q_{j-1,0}^{b_{j-1,0}} \cdots q_{j-1,g_{j-1}}^{b_{j-1,g_{j-1}}}} \right)$$

where $h = q_{j,r-1}^{n_{j,r-1}}$ (resp. $h = h_{j,s-1}$), $\rho = r - 2$ (resp. $\rho = r - 1$), the

integers $\{b_{j,r'}^s\}_{r'=0}^\rho$ satisfy $b_{j,r'}^s < n_{j,r'}$, $1 \leq r' \leq \rho$, and $n_{j,r-1} \bar{\beta}_{j,r-1} =$

$\sum_{0 \leq r' \leq r-2} b_{j,r'}^s \bar{\beta}_{j,r'}$ (resp. $i_{j,s-1} = \sum_{r' \leq r-1} b_{j,r'}^s \bar{\beta}_{j,r'}$), $\bar{\mu}_{j,s} = \mu_{j,0}^{b_{j,0}^s} \cdots \mu_{j,\rho}^{b_{j,\rho}^s}$ is a unit, and $P_{j,s} \in k[z, v_2, \dots, v_{j-1}]$ is irreducible and satisfies

$$(24) \quad P_{j,s}(\lambda, v_2, \dots, v_{j-1}) = 0, \quad \frac{\partial P_{j,s}}{\partial z}(\lambda, v_2, \dots, v_{j-1}) \text{ is a unit in } R_{j-1};$$

being $\lambda = (\lambda_{j,s-1, i_{j,s-1}})^{n_{j,r-1}}$ (resp. $\lambda = \lambda_{j,s-1, i_{j,s-1}}$).

Proof : The result is a consequence of lemma 3.1 and its corollary 3.2. First note that, given j , $2 \leq j \leq \delta$, if there exist $\{q_{j,r}\}_{r=1}^{g_j+1}$ in $k(x_1, \dots, x_j)$ satisfying (22) and we define

$$(25) \quad l_j := \frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}} \in k(x_1, \dots, x_j)$$

where $q_{j,0} = x_1$ and $\{b_{j,r}\}_{r=0}^{g_j}$ satisfy (23), and v_j to be the image of l_j , then v_j belongs to \mathcal{O}_{Y,y_0} and besides

$$(26) \quad v_j = \gamma_j u_j \text{ mod } (u) \quad \text{where } \gamma_j \text{ is a unit in } R_{j-1}.$$

In fact, with the notation in (22) we may take $\gamma_j = \frac{\mu_{j,g_1+1}}{\mu_{j,1}^{b_{j,1}} \cdots \mu_{j,g_j}^{b_{j,g_j}}}$.

Note also that, fixed j , $2 \leq j \leq \delta$, if (26) holds for every $j' \leq j - 1$, then the image of x_j in \mathcal{O}_{Y,y_0} is given by

$$x_j = \sum_{m_1 \leq i \leq m_j} \lambda_{j,i}(v_2, \dots, v_{j-1}) u_1^i + u_1^{m_j} u_j \text{ mod } (u_1)^{m_j+1}$$

where $\lambda_{j,i} \in R_{j-1}$, m_j is the integer in (4), $\lambda_{j,i} \neq 0$ implies that it is a unit in R_{j-1} , λ_{j,m_j} is a unit in R_{j-1} and (20) holds (recall the conditions in (5)). Moreover, the integers $\{\bar{\beta}_{j,r}\}_{r=0}^{g_j}$ (resp. $\bar{\beta}_{j,g_j+1}$) defined in (8) and (9) for the image of x_j are a

minimal system of generators of the semigroup $\nu_j(B_j \setminus 0)$ (resp. equal to $\nu_j(I_j)$). From this, and setting $u := u_1$ and v_r to be the image of x_r for $\delta + 1 \leq r \leq d$ (i.e. $v_r = u_r$, $\delta + 1 \leq r \leq d$, with the notation in (4)), (i) would follow.

Hence, in order to prove (i) and (ii), it suffices to show that, for $2 \leq j \leq \delta$, there exist $\{q_{j,r}\}_{r=1}^{g_j+1}$ satisfying (21) and (22), where R_{j-1} is defined taking $v_{j'}$ to be the image of $l_{j'}$ for $2 \leq j' \leq j-1$ (see (25)). We argue by induction on j . For $j = 2$ the hypothesis in corol. 3.2 is clearly satisfied (we may take $S_1 = \{1\}$). Thus, by corol. 3.2, there exist $\{q_{2,r}\}_{r=1}^{g_2+1}$ satisfying (21) and (22). Now, let j , $2 \leq j \leq \delta$ and suppose that, for $2 \leq j' \leq j-1$, there exist $\{q_{j',r}\}_{r=0}^{g_{j'}+1}$ satisfying (21) and (22). Since $v_{j'}$ is defined to be the image of $l_{j'}$, $2 \leq j' \leq j-1$, the hypothesis of corol. 3.2 is satisfied. In fact, there exists a multiplicative part S_{j-1} of $k[x_1, \dots, x_{j-1}]$ such that $\prod_{(j',r') \in \mathcal{J}_{j-1}^*} T_{j',r'}^{-1} k[x_1, \dots, x_{j-1}] \cong S_{j-1}^{-1} k[x_1, \dots, x_{j-1}]$, hence $l_{j'} \in S_{j-1}^{-1} k[x_1, \dots, x_{j-1}]$ for $2 \leq j' \leq j-1$. Thus, corol. 3.2 assures the existence of $\{q_{j,r}\}_{r=1}^{g_j+1}$ satisfying (21) and (22). From this, we conclude (i) and (ii). Besides, from the proof of corol. 3.2 (see the proof of lemma 3.1), (iii) follows.

Definition 3.4. The local expression in prop.3.3 (i) (or in (4) at the beginning of this section) will be called a *general transverse expression of $\eta : Y \rightarrow \mathbb{A}_k^d$ with respect to E* . The elements $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ obtained in prop. 3.3 (ii) will be called a *system of transverse generators for $\eta : Y \rightarrow \mathbb{A}_k^d$ with respect to E* .

Remark 3.5. For $j = 2$, $B_2 = k[x_1, x_2]_{(x_1, x_2)}$ is a two-dimensional regular local ring. Then $q_{2,0}, q_{2,1}, \dots, q_{2,g_2}, q_{2,g_2+1} \in B_2$ is a minimal generating sequence for ν_2 ([Sp], theorem 8.6). In fact, since $R_1 = k$, if we apply lemma 3.1 to $y = x_2$ then all the $\lambda_{s,i}$'s in (11) belong to k , hence we can take $P_s(z) = z - (\lambda_{s-1, i_{s-1}})^{n_r}$ (resp. $P_s(z) = z - \lambda_{s-1, i_{s-1}}$) in (13). Hence $q_{2,r} \in k[x_1, x_2]$ for $0 \leq r \leq g_2 + 1$, moreover we have $q_{2,0} = x_1$, $q_{2,1} = x_2 - \sum_{i < \bar{\beta}_{2,1}} \lambda_{2,i} q_{2,0}^{\bar{\beta}_{2,0}^i}$ and, for $1 \leq r \leq g_2$,

$$q_{2,r+1} = q_{2,r}^{n_{2,r}} - c_{2,r} q_{2,0}^{b_{2,r,0}} \cdots q_{2,r-1}^{b_{2,r,r-1}} - \sum_{\gamma=(\gamma_0, \dots, \gamma_r)} c_{2,\gamma} q_{2,0}^{\gamma_0} \cdots q_{2,r}^{\gamma_r}$$

$b_{2,r,i}$'s are the unique nonnegative integers satisfying $b_{2,r,i} < n_{2,i}$ for $1 \leq i \leq r-1$, and $n_{2,r} \bar{\beta}_{2,r} = \sum_{0 \leq i < r} b_{2,r,i} \bar{\beta}_{2,i}$, the γ 's are nonnegative integers satisfying $\gamma_i < n_{2,i}$ for $1 \leq i \leq r$ and $n_{2,r} \bar{\beta}_{2,r} < \sum_i \gamma_i \bar{\beta}_{2,i}$, and $c_{2,r}, c_{2,\gamma} \in k$, $c_{2,r} \neq 0$ and $c_{2,\gamma} \neq 0$ only for a finite number of γ 's.

Remark 3.6. Let j , $2 \leq j \leq \delta$. Set $A := k[v_2, \dots, v_{j-1}]$, $x = x_1$, $y = x_j$ and let $\theta : Y \rightarrow \text{Spec } A[x_1, x_j]$ be defined by the morphism of k -algebras given by $v_{j'} \mapsto v_{j'}, 2 \leq j' \leq j-1$, $x_i \mapsto \eta^\#(x_i)$, $i = 1, j$ (see (18)). Setting $l_{j'} = v_{j'}$, $2 \leq j' \leq j-1$, and $S_{j-1} = \{1\}$, the hypothesis in lemma 3.1 is satisfied. Let us apply lemma 3.1, then the integers defined in (8) and (9) are $\{\beta_{j,r}\}_{r=0}^{g_j+1}$, $\{e_{j,r}\}_{r=0}^{g_j}$, $\{n_{j,r}\}_{r=0}^{g_j}$ and $\{\bar{\beta}_{j,r}\}_{r=0}^{g_j+1}$ (see prop. 3.3 or corol. 3.2). We denote by $\{q'_{j,r}\}_{r=0}^{g_j+1}$ the elements $\{q_r = h_{s_r}\}_{r=0}^{g_j+1}$ in 3.1 (iii).(a), hence satisfying

$$q'_{j,r} \in T'_{j,r-1}^{-1} \cdots T'_{j,0}^{-1} k[v_2, \dots, v_{j-1}, x_1, x_j]$$

being $T'_{j,r}$ the multiplicative part generated by $q'_{j,r'}$, and such that the images by $\theta^\#$ of $\{q'_{j,r}\}_{r=0}^{g_j+1}$ are $\{\eta^\#(q_{j,r})\}_{r=0}^{g_j+1}$, thus given in (19). In fact, note that $q_{j,r}$ is obtained from $q'_{j,r}$ by replacing v_i by $\frac{q_{i,g_i+1}}{q_{i,0} \cdots q_{i,g_i}}$, for $1 \leq i \leq j-1$.

On the other hand, for $2 \leq j \leq \delta$, there exists a domain B_{j-1} which is an étale extension of $k[v_2, \dots, v_{j-1}]$ and contains $\lambda_{j,i}(v_2, \dots, v_{j-1})$, $m_1 \leq i \leq m_j$ (see (i) in prop. 3.3). Let $\tilde{\nu}_j$ be the valuation on $B_{j-1}[x_1, x_j]$ extending ν_j and such that $\tilde{\nu}_j(\ell) = 0$ for all $\ell \in B_{j-1}$ (see (ii) in prop. 3.3). Let $\tilde{q}_{j,1}, \dots, \tilde{q}_{j,g_j+1} \in B_{j-1}[x_1, x_j]$ be a minimal generating sequence for $\tilde{\nu}_j$ defined as in remark 3.4, i.e. $\tilde{q}_{j,0} = x_1$, $\tilde{q}_{j,1} = x_j - \sum_{i < \bar{\beta}_{j,1}} \lambda'_{j,i} (\tilde{q}_{j,0})^{\bar{\beta}_{j,0}^i}$ and, for $1 \leq r \leq g_j$,

$$(27) \quad \tilde{q}_{j,r+1} = \tilde{q}_{j,r}^{n_{j,r}} - \tilde{c}_{j,r} \tilde{q}_{j,0}^{b_{j,r,0}} \dots \tilde{q}_{j,r-1}^{b_{j,r,r-1}} - \sum_{\gamma=(\gamma_0, \dots, \gamma_r)} \tilde{c}_{j,\gamma} \tilde{q}_{j,0}^{\gamma_0} \dots \tilde{q}_{j,r}^{\gamma_r}, \quad 1 \leq r \leq g_j$$

where $b_{j,r,i} = b_{j,i}^{s_{j,r}+1}$, $1 \leq i \leq r-1$, i.e. $b_{j,r,i} < n_{j,i}$ and $n_{j,r} \bar{\beta}_{j,r} = \sum_{0 \leq i < r} b_{j,r,i} \bar{\beta}_{j,i}$, we have $\tilde{\nu}_{j-1}(\tilde{q}_{j,0}^{\gamma_0} \dots \tilde{q}_{j,r}^{\gamma_r}) > n_{j,r} \bar{\beta}_{j,r}$ for each sequence γ of nonnegative integers in the right hand side, and $\tilde{c}_{j,r}, \tilde{c}_{j,\gamma} \in B_{j-1}$, $\tilde{c}_{j,r} \neq 0$ and $\tilde{c}_{j,\gamma} \neq 0$ only for a finite number of γ 's.

Note that, for $1 \leq r \leq g_j + 1$, in the ring $\prod_{r'=0}^{r-1} T_{j,r'}^{-1} B_{j-1}[x_1, x_j]$ we have

$$(28) \quad q'_{j,r} = \tilde{q}_{j,r} \cdot \tilde{\ell} + \tilde{h}$$

where $\tilde{\ell}, \tilde{h} \in \prod_{s=0}^{r-1} T_{j,s}^{-1} B_{j-1}[x_1, x_j]$, $\tilde{\ell}$ is a unit and $\tilde{\nu}(\tilde{h}) > \bar{\beta}_{j,r}$.

3.7. Now, let X be a smooth k -scheme and let ν be a divisorial valuation on an irreducible component X_0 of X . Let P_0 be the center of ν on X and let $R := \mathcal{O}_{X, P_0}$. We consider the graded algebra associated with ν , that is, $gr_{\nu} R := \bigotimes_{n \in \Phi^+} \wp_n / \wp_n^+$ where $\Phi^+ := \nu(R \setminus \{0\})$ is the semigroup of the valuation and, for $n \in \Phi^+$,

$$\wp_n = \{h \in R / \nu(h) \geq n\}, \quad \wp_n^+ = \{h \in R / \nu(h) > n\}.$$

Let $\pi : Y \rightarrow X_0$ be a proper and birational morphism such that the center of ν on Y is a divisor E , and let $\eta : Y \rightarrow \mathbb{A}_k^d$ be the composition of π with an étale morphism $X_0 \rightarrow \mathbb{A}_k^d$, where $d = \dim X_0$. Let us consider the notation introduced in this section for the morphism η , in particular, let $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ be a system of transverse generators for $\eta : Y \rightarrow \mathbb{A}_k^d$ with respect to E , (prop. 3.3 (ii)). Recall that the center of ν on \mathbb{A}_k^d is (x_1, \dots, x_{δ}) and let $S := k[x_1, \dots, x_d]_{(x_1, \dots, x_{\delta})}$.

There exists a proper and birational morphism $Z \rightarrow \mathbb{A}_k^d$ with Z smooth such that the center of ν on Z is a divisor F . Since $\mathcal{O}_{Z,F}$ is the valuation ring of the restriction of ν to $K(S)$, we have that $\mathcal{O}_{Z,F} \prec \mathcal{O}_{Y,E}$, i.e. $\mathcal{O}_{Y,E}$ dominates $\mathcal{O}_{Z,F}$, hence, after restricting to some open subset of Y , we may suppose that Y dominates Z , let $\sigma : Y \rightarrow Z$ denote the corresponding morphism. Note that we have

$$\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \dots q_{j,g_j}^{b_{j,g_j}}} \in \mathcal{O}_{Z,F} \quad \text{for } 2 \leq j \leq \delta.$$

because these elements belong to $K(S)$ and have ν -value equal to 0; we also denote by v_j the element $\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \dots q_{j,g_j}^{b_{j,g_j}}}$ of $\mathcal{O}_{Z,F}$ (see prop. 3.3 (ii)). Besides, the ramification

index \mathfrak{e} of $\mathcal{O}_{Y,E}$ over $\mathcal{O}_{Z,F}$ is equal to $g.c.d.(\{\bar{\beta}_{j',r'}\}_{(j',r') \in \mathcal{J}^*})$. Thus there exist $\{\mathfrak{a}_{j,r}\}_{(j,r) \in \mathcal{J}^*}$, $\mathfrak{a}_{j,r} \in \mathbb{Z}$, such that

$$(29) \quad z := \prod_{(j',r') \in \mathcal{J}^*} q_{j',r'}^{\mathfrak{a}_{j',r'}} \in \mathcal{O}_{Z,F} \quad \text{and} \quad \nu(z) = \sum_{(j',r') \in \mathcal{J}^*} \mathfrak{a}_{j',r'} \bar{\beta}_{j',r'} = \mathfrak{e}.$$

Then,

$$\nu(\sigma^*(dz \wedge dv_2 \wedge \dots \wedge dv_d)) = \mathfrak{e} - 1$$

and hence, if $k_F(\mathbb{A}^d)$ denotes the discrepancy of \mathbb{A}^d with respect to F , we have

$$(30) \quad a_E = \mathfrak{e}k_F(\mathbb{A}^d) + \mathfrak{e} - 1$$

Since $S \prec R$, the initial forms of the elements of $k[x_1, \dots, x_d]$ are well defined elements in $gr_\nu R$, and since $q_{1,0} = x_1$, applying (21) in prop. 3.3, by recurrence on (j, r) we can define the initial form $\mathbf{q}_{j,r}$ of $q_{j,r}$ for every $(j, r) \in \mathcal{J}$. We have

$$\mathbf{q}_{j,r} \in \prod_{(j',r') \in \mathcal{J}_{j,r}^*} \mathbf{T}_{j',r'}^{-1}(gr_\nu R)$$

where, for $(j', r') \in \mathcal{J}_{j,r}^*$, $\mathbf{T}_{j',r'}$ is the multiplicative system generated by $\mathbf{q}_{j,r}$. Let $k[\{\mathbf{q}_{j,r}\}_{(j,r) \in \mathcal{J}}]$ be the k -subalgebra of the fraction field $K(gr_\nu R)$ of $gr_\nu R$ generated by the $\mathbf{q}_{j,r}$'s and, for $\delta + 1 \leq j \leq d$, let \mathbf{x}_j be the initial form of x_j . With this notation, the following holds :

Theorem 3.8. *The system of transverse generators $\{\mathbf{q}_{j,r}\}_{(j,r) \in \mathcal{J}}$ satisfy the following properties :*

(i) *We have an isomorphism of graded rings*

$$G := \prod_{(j,r) \in \mathcal{J}^*} \mathbf{T}_{j,r}^{-1} k[\{\mathbf{q}_{j,r}\}_{(j,r) \in \mathcal{J}}, \mathbf{x}_{\delta+1}, \dots, \mathbf{x}_d] \stackrel{\Phi}{\cong} A[\mathbf{u}^\mathfrak{e}, \mathbf{u}^{-\mathfrak{e}}]$$

where $\deg(\mathbf{u}) = 1$, and A is a k -algebra which is étale over the polynomial ring in $d - 1$ variables $k[\mathbf{v}_2, \dots, \mathbf{v}_d]$, being $\deg(\mathbf{v}_j) = 0$, $2 \leq j \leq d$.

(ii) *We have an isomorphism*

$$\prod_{(j,r) \in \mathcal{J}^*} \mathbf{T}_{j,r}^{-1} gr_\nu R \cong B[\mathbf{u}^\mathfrak{e}, \mathbf{u}^{-\mathfrak{e}}]$$

whose restriction to G is Φ , where $A \otimes_k \kappa(P_0) \subseteq B$ and the extension is étale. Besides, the fraction field $K(B)$ of B is $\kappa(E)$.

(iii) *For $2 \leq j \leq \delta$, the isomorphism Φ in (i) restricts to*

$$G_j := \prod_{(j,r) \in \mathcal{J}_{j,g_j+1}^*} \mathbf{T}_{j,r}^{-1} k[\{\mathbf{q}_{j',r'}\}_{(j',r') \in \mathcal{J}_{j,g_j+1} \cup \{(j,g_j+1)\}}] \stackrel{\Phi}{\cong} A_{j-1}[\mathbf{v}_j][\mathbf{u}^{\mathfrak{e}_j}, \mathbf{u}^{-\mathfrak{e}_j}]$$

where $\mathfrak{e}_j := \text{g.c.d.}\{\bar{\beta}_{j',r'} / (j', r') \in \mathcal{J}_{j,g_j+1}^*\}$, $A_1 = k$ and A_{j-1} is étale over $k[\mathbf{v}_2, \dots, \mathbf{v}_{j-1}]$ for $2 < j \leq \delta$.

(iv) *For $2 \leq j \leq \delta$, there exists a domain B_{j-1} étale over A_{j-1} such that*

$$B_{j-1} \left[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{j,r}\}_{r=1}^{g_j+1} \right] \cong B_{j-1} [y_{1,0}, y_{j,2}, \dots, y_{j,g_j+1}] / J_j$$

where the $y_{j,r}$'s are indeterminacies and J_j is a prime ideal which is generated by $\{y_{j,r}^{n_{j,r}} - \tilde{c}_{j,r} y_{1,0}^{b_{j,r,0}} \cdot y_{j,1}^{b_{j,r,1}} \cdot \dots \cdot y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$, being $\tilde{c}_{j,r} \in B_{j-1}$. In particular, the previous ring is a domain which is a complete intersection over B_{j-1} .

Moreover, for any domain C , any ideal in $C[y_{1,0}, y_{j,2}, \dots, y_{j,g_j+1}]$ generated by $\{y_{j,r}^{n_{j,r}} - c_{j,r} y_{1,0}^{b_{j,r,0}} \cdot y_{j,1}^{b_{j,r,1}} \cdot \dots \cdot y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$, $c_{j,r} \in C$, is a prime ideal.

Proof : First, we have that $R = \mathcal{O}_{X,P_0} \supseteq k[x_1, \dots, x_d]_{(x_1, \dots, x_\delta)} =: S$ is étale, hence $\widehat{R} \cong \widehat{S} \otimes_k \kappa(P_0)$ where we denote by \widehat{R} (resp. \widehat{S}) the completion with respect to the maximal ideal. Since the valuation ν on R (resp. on S) can be extended to a valuation $\widehat{\nu}$ on \widehat{R} (resp. on \widehat{S}) and we have $gr_\nu R = gr_{\widehat{\nu}} \widehat{R}$ (resp. $gr_\nu S = gr_{\widehat{\nu}} \widehat{S}$) we conclude that $gr_\nu R \cong gr_\nu S \otimes_k \kappa(P_0)$. Therefore, in (ii) we may suppose that

$X = \mathbb{A}_k^d$, i.e. $R = S$.

Keep the notation in prop. 3.3. The morphism $S \hookrightarrow \mathcal{O}_{Z,F}$ induces an inclusion

$$\Phi: gr_\nu S \hookrightarrow gr_\nu \mathcal{O}_{Z,F} \cong \kappa(F) [\mathbf{u}^\epsilon]$$

where $\kappa(F)$ is the residue field of F on Y , which contains $k(\mathbf{v}_2, \dots, \mathbf{v}_d)$, and $\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_d$ are indeterminacies, \mathbf{v}_j , $2 \leq j \leq d$ (resp. \mathbf{u}) is the initial form of v_j (resp. u), hence $\deg(\mathbf{v}_j) = 0$, $\deg(\mathbf{u}) = 1$. We have

$$\prod_{r'=0}^{g_2} \mathbf{T}_{2,r'}^{-1} k[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{2,r'}\}_{r'=1}^{g_2}] \stackrel{\Phi}{\cong} k[\mathbf{u}^{e_2, g_2}, \mathbf{u}^{-e_2, g_2}]$$

and hence $G_2 \stackrel{\Phi}{\cong} k[\mathbf{v}_2][\mathbf{u}^{e_2, g_2}, \mathbf{u}^{-e_2, g_2}]$. More precisely, the image of the ring in the left hand side in the fraction field $K(gr_\nu \mathcal{O}_{Z,F})$ of $gr_\nu \mathcal{O}_{Z,F}$ is in fact in $gr_\nu \mathcal{O}_{Z,F}$ and is equal to the ring in the hand side. Arguing by recurrence on j , $2 \leq j \leq \delta$, it follows that

$$\prod_{(j', r') \in \mathcal{J}_{j, g_{j+1}}^*} \mathbf{T}_{j', r'}^{-1} k[\{\mathbf{q}_{j', r'}\}_{(j', r') \in \mathcal{J}_{j, g_{j+1}}^*}] \stackrel{\Phi}{\cong} A_{j-1}[\mathbf{u}^{\epsilon_j}, \mathbf{u}^{-\epsilon_j}]$$

where $\epsilon_j := g.c.d.\{e_{2, g_2}, \dots, e_{j, g_j}\} = g.c.d.\{\bar{\beta}_{j', r'} / (j', r') \in \mathcal{J}_{j, g_{j+1}}^*\}$ and A_{j-1} is the minimal subring of $\kappa(F)$ containing $k[\mathbf{v}_2, \dots, \mathbf{v}_{j-1}]$ and $\mu_{j', r'}(\mathbf{v}_2, \dots, \mathbf{v}_{j'-1})$, $\mu_{j', r'}(\mathbf{v}_2, \dots, \mathbf{v}_{j'-1})^{-1}$ for $(j', r') \in \mathcal{J}_{j, g_{j+1}}^*$, hence A_{j-1} is étale over $k[\mathbf{v}_2, \dots, \mathbf{v}_{j-1}]$. Therefore

$$G_j \stackrel{\Phi}{\cong} A_{j-1}[\mathbf{v}_j][\mathbf{u}^{\epsilon_j}, \mathbf{u}^{-\epsilon_j}] \quad \text{and} \quad G = G_\delta \otimes_k k[x_{\delta+1}, \dots, x_d] \stackrel{\Phi}{\cong} A[\mathbf{u}^{\epsilon_\delta}, \mathbf{u}^{-\epsilon_\delta}]$$

where $A = A_{\delta-1}[\mathbf{v}_\delta, \dots, \mathbf{v}_d]$, hence (i) and (iii) hold.

In order to prove (ii), let B be the minimal subring of $\kappa(F)$ containing $k[\mathbf{v}_2, \dots, \mathbf{v}_d]$ and $\{\lambda_{j,i}(\mathbf{v}_2, \dots, \mathbf{v}_{j-1})\}_{2 \leq j \leq d, m_1 \leq i \leq m_j}$. From the construction of the $h_{j,s}$'s in prop. 3.3 (iii) (see the proof of (iii) in lemma 3.1) it follows that, for every (j, i) , $2 \leq j \leq d$, $m_1 \leq i \leq m_j$, there exists $h \in \prod_{(j,r) \in \mathcal{J}^*} T_{j,r}^{-1} S$ such that the initial form of h is $\lambda_{j,i}(\mathbf{v}_2, \dots, \mathbf{v}_d) \mathbf{u}^\epsilon$. Now, let $h \in S = k[x_1, \dots, x_\delta]_{(x_1, \dots, x_\delta)}$ and let $a := \nu(h)$. Then ϵ_δ divides a and the image of h in \mathcal{O}_{Y, y_0} is equal to $\lambda(v_2, \dots, v_\delta) u^a$ modulo u^{a+1} , where $\lambda(\mathbf{v}_2, \dots, \mathbf{v}_\delta) \in B$. Hence the initial form of h belongs to $B[\mathbf{u}^{\epsilon_\delta}]$. Besides, it follows that the set of elements of $K(S)$ of degree 0 is precisely $K(B)$, that is, $\kappa(F) = K(B)$. From this (ii) follows.

For (iv), recall that, given $n \in \mathbb{N}$, a field F containing a primitive n -th root of unity ξ and an element $b \in F^* = F \setminus \{0\}$, if the class of b in F^*/F^{*n} has order m , then there exists $d \in F$ such that $X^m - d$ is an irreducible polynomial in $F[X]$ and moreover $X^n - b = \prod_{i=0}^{n/m} (X^m - \xi^i d)$ is the decomposition in $F[x]$ of $X^n - b$ in irreducible factors (see for instance prop. 9.6 in [Mo]). In particular, if A is a domain containing a primitive n th root of unity and $b \in A$ is such that

$$(31) \quad b^{\frac{1}{n'}} \notin A \text{ for every } n' > 1, n' | n, \text{ then } X^n - b \text{ is irreducible in } A[x].$$

For $j = 2$, with the notation in remark 3.5, let J_2 is the ideal of $k[y_{1,0}, y_{2,1}, \dots, y_{2, g_2}]$ generated by $\{y_{2,r}^{n_{2,r}} - c_{2,r} y_{1,0}^{b_{2,r,0}} y_{2,1}^{b_{2,r,1}} \dots y_{2, r-1}^{b_{2,r, r-1}}\}_{r=1}^{g_2}$, where the $y_{2,r}$'s are indeterminacies. Let $B_1 = A_1 = k$ and let us consider the morphism of k -algebras

$$k[y_{1,0}, y_{2,1}, \dots, y_{2, g_2+1}] / J_2 \rightarrow k[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{2,r}\}_{r=1}^{g_2+1}]$$

sending $y_{2,r}$, $1 \leq r \leq g_2 + 1$ (resp. $y_{1,0}$) to $\mathbf{q}_{2,r}$ (resp. $\mathbf{q}_{1,0}$). Since $k[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{2,r}\}_{r=1}^{g_2+1}]$ is a 2-dimensional domain, to prove the isomorphism it suffices to show that for $1 \leq r \leq g_2$ the element $y_{2,r}^{n_{2,r}} - c_{2,r} y_{1,0}^{b_{2,r,0}} y_{2,1}^{b_{2,r,1}} \cdots y_{2,r-1}^{b_{2,r,r-1}}$ is irreducible in

$$\left(k[y_{1,0}, \dots, y_{2,r-1}] / \left(\{y_{2,r'}^{n_{2,r'}} - c_{2,r'} y_{1,0}^{b_{2,r',0}} \cdots y_{2,r'-1}^{b_{2,r',r'-1}}\}_{r'=1}^{r-1} \right) \right) [y_{2,r}]$$

i.e. $y_{1,0}^{b_{2,r,0}} \cdots y_{2,r-1}^{b_{2,r,r-1}}$ does not have a n' -root for any $n' > 1$ dividing $n_{2,r}$. In fact, suppose that

$$(32) \quad y_{1,0}^{b_{2,r,0}} \cdots y_{2,r-1}^{b_{2,r,r-1}} = \left(\sum_{\underline{a} \in \mathbb{Z}_{\geq 0}^r} \lambda_{\underline{a}} y_{1,0}^{a_0} \cdots y_{2,r-1}^{a_{r-1}} \right)^{n'} \pmod{\left(\{y_{2,r'}^{n_{2,r'}} - c_{2,r'} y_{1,0}^{b_{2,r',0}} \cdots y_{2,r'-1}^{b_{2,r',r'-1}}\}_{r'=1}^{r-1} \right)}$$

where $n'|n_{2,r}$, $\lambda_{\underline{a}} \in k$, the sum in the right hand side term is finite, then we may suppose that (32) is homogeneous with respect to the degree, that is, for each \underline{a} in (32), we have $n' \left(\sum_{i=0}^{r-1} a_i \bar{\beta}_{2,i} \right) = n_{2,r} \bar{\beta}_{2,r}$. Since there exists at least one \underline{a} in (32) and we have $n_{2,r} = \frac{e_{2,r-1}}{e_{2,r}}$ where $e_{2,l} = g.c.d.(\bar{\beta}_{1,0}, \dots, \bar{\beta}_{2,i})$, $i = r-1, r$, and $n'|n_{2,r}$, we conclude that $n'e_{2,r}$ divides $\bar{\beta}_{2,r}$ and also $e_{2,r-1}$, hence $n'e_{2,r}$ divides $e_{2,r}$, that is $n' = 1$.

Now, let j , $2 < j \leq \delta$. Let us consider the notation in remark 3.6. We have $B_{j-1}[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{j,r}\}_{r=1}^{g_j+1}] \cong B_{j-1}[\{\mathbf{q}'_{1,0}\} \cup \{\mathbf{q}'_{j,r}\}_{r=1}^{g_j+1}]$. Besides, from (28) it follows that, for $1 \leq r \leq g_j+1$, the initial form $\mathbf{q}'_{j,r}$ of $q'_{j,r}$ belongs to $gr_{\tilde{\nu}_j}(B_{j-1}[x_1, x_j])$, although $q'_{j,r} \in \prod_{r'=0}^{r-1} T_{j,r'}^{-1} B_{j-1}[x_1, x_j]$. It also follows that

$$B_{j-1}[\{\mathbf{q}'_{1,0}\} \cup \{\mathbf{q}'_{j,r}\}_{r=1}^{g_j+1}] \cong B_{j-1}[y_{1,0}, y_{j,1}, \dots, y_{j,g_j+1}] / J_j$$

where J_j is the ideal generated by $\{y_{j,r}^{n_{j,r}} - \tilde{c}_{j,r} y_{1,0}^{b_{j,r,0}} \cdots y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$. In fact, from the same argument as in before it follows that, for $1 \leq r \leq g_j$ and for any n' dividing $n_{j,r}$, $\tilde{c}_{j,r} y_{1,0}^{b_{j,r,0}} \cdots y_{j,r-2}^{b_{j,r,r-2}}$ does not have a n' -root in the ring

$$B_{j-1}[y_{1,0}, \dots, y_{j,r-2}] / \left(\{y_{j,r'}^{n_{j,r'}} - \tilde{c}_{j,r'} y_{1,0}^{b_{j,r',0}} \cdots y_{j,r'-1}^{b_{j,r',r'-1}}\}_{r'=1}^{r-1} \right)$$

More precisely, $(b_{j,r,0}, \dots, b_{j,r,r-2}) \neq (0, \dots, 0)$ and $y_{1,0}^{b_{j,r,0}} \cdots y_{j,r-2}^{b_{j,r,r-2}}$ does not have a n' -root in any ring of the form

$$C[y_{1,0}, \dots, y_{j,r-2}] / \left(\{y_{j,r'}^{n_{j,r'}} - c_{j,r'} y_{1,0}^{b_{j,r',0}} \cdots y_{j,r'-1}^{b_{j,r',r'-1}}\}_{r'=1}^{r-1} \right)$$

where C is a domain and the $c_{j,r'}$'s are in C . Hence J_j is a prime ideal and (iv) holds. This concludes the proof.

Remark 3.9. Similar ideas to the ones in (ii) in theorem 3.8 appear in [Pi], proof of th. 1.3.8.

Restricting to dimension 3, but considering any valuation ν of rational rank 1 and dimension 3, i.e. ν centered in a regular 3-dimensional ring R , in [Ka] an (infinite) generating sequence $\{q_n\}_{n \in \mathbb{N}}$ of ν in R is constructed. Our construction in prop. 3.3 is different to the one in [Ka] and we do not reach a generating sequence. Generating sequences in higher dimensional complete local rings are considered in [LMSS].

4. DEFINING COORDINATES AT STABLE OF THE SPACE OF ARCS

Let $\eta : Y \rightarrow \mathbb{A}_k^d$ be a k -morphism dominant and generically finite, where Y is a nonsingular k -scheme, let E be a divisor on Y and $e \geq 1$, and keep the notation in section 3.

Let P_{eE}^Y be the generic point of Y_∞^{eE} (see 2.7), and let $P_{eE}^{\mathbb{A}^d}$ be the image by η_∞ of P_{eE}^Y , which is a stable point of $(\mathbb{A}^d)_\infty$ ([Re2] prop. 4.5). We will first prove (prop. 4.5) that a system of transverse generators for η with respect to E induces a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$. Then we will conclude theorem 4.8 and corollary 4.10.

Given a finitely generated k -algebra A , let us denote by A_∞ the ring of $(\text{Spec } A)_\infty$. Given $l \in A$, we denote by $\sum_{n=0}^\infty L_n t^n$ the image of l by the morphism of k -algebras $A \rightarrow A_\infty[[t]]$.

Lemma 4.1. (*[Re2] proof of prop. 4.1 (iii)*) *Let $A \subseteq B$ be finitely generated k -algebras and let $\theta : \text{Spec } B \rightarrow \text{Spec } A$ be the induced dominant morphism. Let P' be a stable point of $\text{Spec } B_\infty$ and let P be its image by θ_∞ in $\text{Spec } A_\infty$. Let $h \in B$ belonging to the fraction field $K(A)$ of A , $h = l/q$ where $l, q \in A$. Then, there exist $\{\overline{H}_n\}_{n \geq 0}$ in $(A_\infty)_P$ such that*

$$(33) \quad H_n \equiv \overline{H}_n \pmod{P'}$$

(recall that $H_n \in B_\infty$ for $n \geq 0$). Even more, there exists $c \in \mathbb{N}$ such that $Q_0, \dots, Q_{c-1} \in P$, $Q_c \notin P$ and there exist polynomials S_n on $2(n+1)$ indeterminacies with coefficients in k , for $n \geq 0$, such that,

$$\overline{H}_n := \frac{S_n(L_c, \dots, L_{n+c}, Q_c, \dots, Q_{n+c})}{(Q_c)^{n+1}} \in (A_\infty)_P$$

satisfies (33).

Proof : First note that P is a stable point of $\text{Spec } A_\infty$ ([Re2] prop. 4.5), hence the existence of c such that $Q_0, \dots, Q_{c-1} \in P$, $Q_c \notin P$ ([Re2], th. 3.7 (i)). Then, the result follows from the following observation : given $h = l/q$, $l, q \in A$, if $Q_0, \dots, Q_{c-1} \in P$, $Q_c \notin P$, then we have

$$Q_c H_n + \dots + Q_{n+c} H_0 \equiv L_{n+c} \pmod{P'} \quad \text{for } n \geq 0.$$

([Re2] proof of prop. 4.1).

Lemma 4.2. *Suppose that the assumptions in lemma 3.1 hold and suppose besides that $\theta : Y \rightarrow \text{Spec } A[x, y]$ is dominant. Let $P = P_{eE}^{A[x, y]}$ be the image of P_{eE}^Y by θ_∞ , which is a stable point of $\text{Spec } A[x, y]_\infty$. Let y_0 , the regular system of parameters $\{u, v_2, \dots, v_d\}$ of \mathcal{O}_{Y, y_0} and $\{h_1 = y, h_2, \dots, h_N\}$ satisfy (a) to (c) in 3.1. For $2 \leq j' \leq j-1$, let $\{\overline{L}_{j', n}\}_{n \geq 0}$ in $(A[x, y]_\infty)_P$ be such that $L_{j', n} \equiv \overline{L}_{j', n} \pmod{P_{eE}^Y}$ (see lemma 4.1). Then, there exists a multiplicative system \overline{S}_{j-1} of $A[x]_\infty$ such that $\overline{L}_{j', n} \in \overline{S}_{j-1}^{-1} A[x]_\infty$ for $2 \leq j' \leq j-1, n \geq 0$ and there exist elements $\{\overline{H}_{s; n}\}_{1 \leq s \leq N, n \geq 0}$ in $(A[x, y]_\infty)_P$, $n \geq 0$, satisfying :*

(i) $H_{s; n} \equiv \overline{H}_{s; n} \pmod{P_{eE}^Y}$, therefore

$$\overline{H}_{s; n} \in P(A[x, y]_\infty)_P \quad \text{for } 0 \leq n \leq ei_s - 1 \quad \text{and} \quad \overline{H}_{s; ei_s} \notin P(A[x, y]_\infty)_P.$$

(ii) Let r , $1 \leq r \leq g+1$ be such that $n_{r-1} \overline{\beta}_{r-1} < i_s \leq \overline{\beta}_r$ (resp. $r=1$ if $s=1$ and $i_1 = \overline{\beta}_0$). Set $\overline{Q}_{0; n} := X_n$ for $n \geq 0$, $\overline{Q}_{r'; n} := \overline{H}_{s; r'; n}$, for $1 \leq r' < r, n \geq 0$

and let $\bar{T}_{r'}$ is the multiplicative part generated by $\bar{Q}_{r';e\bar{\beta}_{r'}}$, $0 \leq r' < r$. Then, for $n \geq e(\bar{\beta}_r - \beta_r)$, we have :

$$\bar{H}_{s;n} \in \bar{T}_{r-1}^{-1} \cdots \bar{T}_0^{-1} \bar{S}_{j-1}^{-1} A_\infty [X_0, X_1, \dots, X_n, \dots, Y_0, Y_1, \dots, Y_{n-e(\bar{\beta}_r - \beta_r)}].$$

(iii) If $s = 1$ then $\bar{H}_{1;n} = Y_n$ for $n \geq 0$. If $s > 1$ then

$$\bar{H}_{s;n} \in (\{\bar{Q}_{r';n}\}_{\substack{r' \leq r-1 \\ n < e\bar{\beta}_{r'}}} \cup \{\bar{H}_{s-1;n}\}_{n < ei_{s-1}}) \quad \text{for } 0 \leq n < \max \{en_{r-1}\bar{\beta}_{r-1}, ei_{s-1}\}$$

$$\bar{H}_{s;n} = u_{s,n} Y_{n-(\bar{\beta}_r - \beta_r)} + \rho_{s,n} \quad \text{for } n > \max \{en_{r-1}\bar{\beta}_{r-1}, ei_{s-1}\}$$

where $u_{s,n}, \rho_{s,n} \in B_\infty^r [Y_{e\bar{\beta}_{r-1}+1}, \dots, Y_{n-e(\bar{\beta}_r - \beta_r)-1}]$ and $u_{s,n}$ is a unit.

(iv) Suppose that $s > 1$. If $i_{s-1} = \bar{\beta}_{r-1}$ (resp. $i_{s-1} \in (n_{r-1}\bar{\beta}_{r-1}, \bar{\beta}_r)$) then $\bar{H}_{s;en_{r-1}\bar{\beta}_{r-1}}$ (resp. $\bar{H}_{s;ei_{s-1}}$) is equal to

$$\bar{Q}_{0;e\bar{\beta}_0}^{b_0^s} \cdots \bar{Q}_{\rho;e\bar{\beta}_\rho}^{b_\rho^s} \cdot P_s \left(\frac{c_s \bar{H}}{\bar{Q}_{0;e\bar{\beta}_0}^{b_0^s} \cdots \bar{Q}_{\rho;e\bar{\beta}_\rho}^{b_\rho^s}}, \bar{L}_{2;0}, \dots, \bar{L}_{j-1;0} \right)$$

where $\bar{H} = (\bar{Q}_{r-1;e\bar{\beta}_{r-1}})^{nr-1}$ (resp. $\bar{H} = \bar{H}_{s-1;ei_{s-1}}$), $c_s \in k \setminus \{0\}$ and $\rho, \{b_{r'}^s\}_{r'=0}^\rho$ and P_s are as in (c) in 3.1.

(v) Fixed r , $1 \leq r \leq g+1$, the following ideals in $\bar{T}_{r-1}^{-1} \cdots \bar{T}_0^{-1} \bar{S}_{j-1}^{-1} A[x, y]_\infty$ are equal :

$$\left(\{\bar{Q}_{r';n}\}_{\substack{0 \leq r' \leq r \\ 0 \leq n \leq e\bar{\beta}_{r'-1}}} \right) = \left(\{\bar{Q}_{r';n}\}_{\substack{0 \leq r' \leq 1 \\ 0 \leq n \leq e\bar{\beta}_{r'-1}}} \cup \{\bar{Q}_{r';n}\}_{\substack{2 \leq r' \leq r \\ en_{r'-1}\bar{\beta}_{r'-1} \leq n \leq e\bar{\beta}_{r'-1}}} \right)$$

and also the ideal generated by

$$\{\bar{Q}_{0;n}\}_{n=0}^{em_1-1} \cup \{\bar{H}_{1;n}\}_{n=0}^{ei_1-1} \cup \left(\bigcup_{s=2}^{s_1} \{\bar{H}_{s;n}\}_{n=ei_{s-1}}^{ei_s-1} \right) \cup \bigcup_{r'=2}^r \left(\{\bar{H}_{s_{r'-1}+1;n}\}_{n=en_{r'-1}\bar{\beta}_{r'-1}}^{ei_{s_{r'-1}+1}-1} \cup \left(\bigcup_{s=s_{r'-1}+2}^{s'_r} \{\bar{H}_{s;n}\}_{n=ei_{s-1}}^{ei_s-1} \right) \right).$$

Proof : The existence of \bar{S}_{j-1} follows from lemma 4.1 ; in fact, it suffices to ask \bar{S}_{j-1} to contain the elements Q_c where $q \in S_{j-1}$ and c is such that $Q_0, \dots, Q_{c-1} \in P$ and $Q_c \notin P$. Now, let us define the elements $\{\bar{H}_{s;n}\}_{n \geq 0}$, $1 \leq s \leq N$, by induction on s . For $s = 1$, $h_1 = y \in A[x, y]$, so $H_{1;n} \in A[x, y]_\infty$ for $n \geq 0$. We set $\bar{H}_{1;n} := H_{1;n} = Y_n \in A[x, y]_\infty$ for $n \geq 0$. It is clear that (i) to (iii) are satisfied. Now, let s , $2 \leq s \leq N$, and suppose that $\bar{H}_{s';n} \in (A[x, y]_\infty)_P$ are defined, for $1 \leq s' < s$, $n \geq 0$, and satisfy the conditions. Let r , $1 \leq r \leq g_j + 1$ be such that $i_{s-1} \in \{\bar{\beta}_{r-1}\} \cup (n_{r-1}\bar{\beta}_{r-1}, \bar{\beta}_r)$. Therefore $\{\bar{Q}_{r';n}\}_{0 \leq r' < r, n \geq 0}$ in $(A[x, y]_\infty)_P$ are defined, and satisfy :

$$\bar{Q}_{r';n} \in P(A[x, y]_\infty)_P \quad \text{for } 0 \leq n \leq e\bar{\beta}_{r'} - 1 \quad \text{and} \quad \bar{Q}_{r';e\bar{\beta}_{r'}} \notin P(A[x, y]_\infty)_P.$$

Hence, for every l in the k -algebra $k[\{q_{r'}\}_{0 \leq r' < r} \cup \{h_{s-1}\}]$ generated by $q_{r'}$, $0 \leq r' < r$, and h_{s-1} , and for every $n \geq 0$, there exists a polynomial function \bar{L}_n on $\{\bar{Q}_{r';n}\}_{r' < r, n \geq e\bar{\beta}_{j',r'}} \cup \{\bar{H}_{s-1;n}\}_{n \geq ei_{s-1}}$ such that $L_n \equiv \bar{L}_n \pmod{P_{eE}^Y}$. Moreover, given

$$(34) \quad h = \frac{l}{q} \in \mathcal{O}_{Y, y_0} \quad \text{where } l \in k[\{q_{r'}\}_{0 \leq r' < r} \cup \{h_{s-1}\}], \quad q = \prod_{0 \leq r' < r} q_{r'}^{a_{r'}}$$

being $a_{r'} \in \mathbb{N} \cup \{0\}$, let $c = \sum_{0 \leq r' < r} a_{r'} e\bar{\beta}_{r'}$, so that $\bar{Q}_0, \dots, \bar{Q}_{c-1} \in P$, $\bar{Q}_c \notin P$ and set

$$\bar{H}_n := \frac{S_n(\bar{L}_c, \bar{L}_{c+1}, \dots, \bar{L}_{n+c}, \bar{Q}_c, \bar{Q}_{c+1}, \dots, \bar{Q}_{n+c})}{(\bar{Q}_c)^{n+1}} \in (A[x, y]_\infty)_P$$

where S_n is the polynomial in lemma 4.1; then $H_n \equiv \overline{H}_n \pmod{P_{eE}^Y}$. From this and (c) in lemma 3.1, which expresses h_s as a polynomial in elements of the form (34), the definition of $\{\overline{H}_{s;n}\}_{n \geq 0} \subset (A[x, y]_\infty)_P$ follows. They satisfy (i) and, from the expression in 3.1 (c) and the induction hypothesis, it follows that (ii) holds and that the first statement in (iii) and also (iv) are satisfied. In (iv), c_s is the class of $\overline{\mu}_s \in R_{j-1}$, hence $c_s \neq 0$. The second statement in (iii) is obtained from the expression in 3.1 (c) and the induction hypothesis, applying also (13) in lemma 3.1. Finally, (v) can also be proved by induction, applying the same argument as before.

Let $A_\infty^{1,0} := k$ and, for $2 \leq j \leq \delta$, let

$$A_\infty^{j,1} := k[\underline{X}_0^{j-1}, \dots, \underline{X}_n^{j-1}, \dots], \quad A_\infty^{j,r} := A_\infty^{j,1}[X_{j;0}, \dots, X_{j,e\beta_{j,r-1}}], \quad 2 \leq r \leq g_j + 1,$$

where $\underline{X}_n^{j-1} := (X_{1;n}, \dots, X_{j-1;n})$. Let $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ be a system of transverse generators for $\eta: Y \rightarrow \mathbb{A}_k^d$ with respect to E , as in 3.3 (ii). Even more, for $2 \leq j \leq \delta$, let us consider the elements $\{h_{j,s}\}_{s=1}^{N_j}$ in 3.3 (iii) and set $h_{1,0} := q_{1,0} = x_1 \in A$. Let

$$\mathcal{I} := \{(1, 0)\} \cup \{(j, s) / 2 \leq j \leq \delta, 1 \leq s \leq N_j\}.$$

Then we have :

Lemma 4.3. *There exist elements $\{\overline{H}_{j,s;n}\}_{(j,s) \in \mathcal{I}, n \geq 0}$ in $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$, $n \geq 0$, satisfying :*

- (i) $H_{j,s;n} \equiv \overline{H}_{j,s;n} \pmod{P_{eE}^Y}$, therefore $\overline{H}_{j,s;n} \in P_{eE}^{\mathbb{A}^d}$ for $0 \leq n \leq ei_{j,s} - 1$ and $\overline{H}_{j,s;ei_{j,s}} \notin P_{eE}^{\mathbb{A}^d}$
- (ii) We have $\overline{H}_{1,0;n} = X_{1;n}$ for $n \geq 0$. For $2 \leq j \leq \delta$, let $r, 1 \leq r \leq g_j + 1$ be such that $n_{j,r-1}\overline{\beta}_{j,r-1} < i_{j,s} \leq \overline{\beta}_{j,r}$ (resp. $r = 1$ if $s = 1$ and $i_{j,1} = \overline{\beta}_{j,0}$). For $(j', r') \in \mathcal{J}_{j,r}$, set $\overline{Q}_{j',r';n} := \overline{H}_{j',s_{r'};n}$, $n \geq 0$ and, for $(j', r') \in \mathcal{J}_{j,r}^*$, let $\overline{T}_{j',r'}$ be the multiplicative system generated by $\overline{Q}_{j',r';e\overline{\beta}_{j',r'}}$. Then, for $n \geq e(\overline{\beta}_{j,r} - \beta_{j,r})$ we have :

$$\overline{H}_{j,s;n} \in \prod_{(j',r') \in \mathcal{J}_{j,r}^*} \overline{T}_{j',r'}^{-1} A_\infty^{j',r}[X_{j;e\beta_{j,r-1}+1}, \dots, X_{j;n-e(\overline{\beta}_{j,r}-\beta_{j,r})}]$$

(if $r = 1$, replace $X_{j;e\beta_{j,r-1}+1}$ by $X_{j;0}$ in the previous equality).

- (iii) For $2 \leq j \leq \delta$, if $s = 1$ then $\overline{H}_{j,s;n} = X_{j;n}$ for $n \geq 0$. If $s > 1$ then :

$$\overline{H}_{j,s;n} \in \left(\left\{ \overline{Q}_{j,r';n} \right\}_{\substack{r' \leq r-1 \\ n < e\overline{\beta}_{r'}}} \cup \left\{ \overline{H}_{j,s-1;n} \right\}_{n < ei_{s-1}} \right)$$

for $0 \leq n < \max \{en_{j,r-1}\overline{\beta}_{j,r-1}, ei_{j,s-1}\}$ and

$$\overline{H}_{j,s;n} = u_{j,s,n} X_{j;n-e(\overline{\beta}_{j,r} - \beta_{j,r})} + \rho_{j,s,n} \quad \text{for } n > \max \{en_{j,r-1}\overline{\beta}_{j,r-1}, ei_{j,s-1}\}$$

where $u_{j,s,n}, \rho_{j,s,n} \in \prod_{(j',r') \in \mathcal{J}_{j,r}^*} \overline{T}_{j',r'}^{-1} A_\infty^{j',r}[X_{j;e\beta_{j,r-1}+1}, \dots, X_{j;n-e(\overline{\beta}_{j,r}-\beta_{j,r})-1}]$ and $u_{j,s,n}$ is a unit.

- (iv) Suppose that $j, s \geq 2$. If $i_{j,s-1} = \overline{\beta}_{j,r-1}$ (resp. $i_{j,s-1} \in (n_{j,r-1}\overline{\beta}_{j,r-1}, \overline{\beta}_{j,r})$) then $\overline{H}_{j,s;en_{j,r-1}\overline{\beta}_{j,r-1}}$ (resp. $\overline{H}_{j,s;ei_{j,s-1}}$) is equal to

$$\overline{Q}_{1,0;e\overline{\beta}_{j,0}}^{1_{j,0}} \cdot \overline{Q}_{j,1;e\overline{\beta}_{j,1}}^{b_{j,1}^s} \cdots \overline{Q}_{j,\rho;e\overline{\beta}_{j,\rho}}^{b_{j,\rho}^s} \cdot P_{j,s} \left(\frac{c_{j,s}\overline{H}}{\overline{Q}_{1,0;e\overline{\beta}_{j,0}}^{b_{j,0}^s} \cdots \overline{Q}_{j,\rho;e\overline{\beta}_{j,\rho}}^{b_{j,\rho}^s}}, \dots, \frac{\overline{Q}_{j-1,g_{j-1}+1;e\overline{\beta}_{j-1,g_{j-1}+1}}}{\overline{Q}_{1,0;e\overline{\beta}_{j-1,0}}^{b_{j-1,0}^s} \cdots \overline{Q}_{j-1,g_{j-1};e\overline{\beta}_{j-1,g_{j-1}}}^{b_{j-1,g_{j-1}}^s}} \right)$$

where $\overline{H} = (\overline{Q}_{j,r-1;e\overline{\beta}_{j,r-1}})^{n_{j,r-1}}$ (resp. $\overline{H} = \overline{H}_{j,s-1;ei_{j,s-1}}$), $c_{j,s} \in k \setminus \{0\}$ and $\rho, \{b_{j,r'}^s\}_{r'=0}^\rho$ and P_s are as in (c.3) in 3.3.

(v) Set $\mathcal{G}_{1,0} := \{\overline{H}_{1,0;n} / 0 \leq n \leq e m_1 - 1\}$ and, for $2 \leq j \leq \delta$,

$$\begin{aligned} \mathcal{G}_{j,1} &:= \{\overline{H}_{j,1;n} / 0 \leq n \leq e i_{j,1} - 1\} \cup \cup_{s=2}^{s_1} \{\overline{H}_{j,s;n} / e i_{j,s-1} \leq n \leq e i_{j,s} - 1\} \\ \mathcal{G}_{j,r} &:= \{\overline{H}_{j,s_{r-1}+1;n} / e n_{j,r-1} \overline{\beta}_{j,r-1} \leq n \leq e i_{j,s_{r-1}+1} - 1\} \cup \\ &\quad \cup_{s=s_{r-1}+2}^{s_r} \{\overline{H}_{j,s;n} / e i_{j,s-1} \leq n \leq e i_{j,s} - 1\} \quad \text{for } 2 \leq r \leq g_j + 1. \end{aligned}$$

then, for $2 \leq j \leq \delta$ and $1 \leq r \leq g_1 + 1$, we have

$$\begin{aligned} &\left(\left\{ \overline{Q}_{j,r';n} \right\}_{\substack{0 \leq r' \leq 1 \\ 0 \leq n \leq e \overline{\beta}_{j,r'} - 1}} \cup \left\{ \overline{Q}_{j,r';n} \right\}_{\substack{e n_{j,r'} - 1 \overline{\beta}_{j,r'} - 1 \leq n \leq e \overline{\beta}_{j,r'} - 1}} \right) \prod_{(j',r') \in \mathcal{J}_{j,r}^*} \overline{T}_{j',r'}^{-1} A_\infty^{j+1,1} = \\ &= (\mathcal{G}_{1,0} \cup \mathcal{G}_{j,1} \cup \dots \mathcal{G}_{j,r}) \prod_{(j',r') \in \mathcal{J}_{j,r}^*} \overline{T}_{j',r'}^{-1} A_\infty^{j+1,1}. \end{aligned}$$

Proof : Let us prove, by induction on j , $1 \leq j \leq \delta$, the existence of $\{\overline{H}_{j,s;n}\}_{\substack{(j,s) \in \mathcal{I} \\ n \geq 0}}$ satisfying the required conditions. For $j = 1$, $(j,s) = (1,0)$, $h_{1,0} := q_{1,0} = x_1 \in \mathcal{O}_{\mathbb{A}^d, \eta(y_0)}$, so, if we set $\overline{H}_{1,0;n} := H_{1,0;n} = X_{1;n} \in \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ for $n \geq 0$ then it is clear that (i) to (iii) are satisfied. Now, let j , $2 \leq j \leq \delta$, and suppose that $\overline{H}_{j',s';n} \in \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ are defined, for $j' < j$, $(j',s') \in \mathcal{I}$, $n \geq 0$, and satisfy the conditions. Then the result follows applying lemma 4.2 to $Y \rightarrow \text{Spec } A[x_1, x_j]$, where $A = k[x_2, \dots, x_{j-1}]$, and the following remark : since

$$l_{j'} = \frac{q_{j',g_{j'}+1}}{q_{1,0}^{b_{j',0}} q_{1,1}^{b_{j',1}} \dots q_{j',g_{j'}}^{b_{j',g_{j'}}}} \quad \text{for } 2 \leq j' \leq j-1$$

we may take $\overline{S}_{j-1} = \{\overline{Q}_{j',r';e\overline{\beta}_{j',r'}}\}_{(j',r') \in \mathcal{J}_{j-1,g_{j-1}+1}^*}$ and

$$\overline{L}_{j';0} = \frac{\overline{Q}_{j',g_{j'}+1;e\overline{\beta}_{j',g_{j'}+1}}}{\overline{Q}_{1,0;e\overline{\beta}_{j',0}}^{b_{j',0}} \cdot \overline{Q}_{j',1;e\overline{\beta}_{j',1}}^{b_{j',1}} \dots \overline{Q}_{j',g_{j'};e\overline{\beta}_{j',g_{j'}}}^{b_{j',g_{j'}}}}.$$

From this, (i) to (iv) follow for j . This concludes the proof.

Remark 4.4. Let j , $2 \leq j \leq \delta$. Let $\{\tilde{q}_{j,r}\}_{r=0}^{g_j+1}$ in $B_{j-1}[x_1, x_j]$ be as in remark 3.6, and $\tilde{Q}_{j,r;n} \in B_{j-1}[x_1, x_j]_\infty$, $n \geq 0$, as in the beginning of this section. Arguing by recurrence and applying (27), we obtain that, for $1 \leq r \leq g_j + 1$,

$$(35) \quad \tilde{Q}_{j,r;n} \in \left(\left\{ \tilde{Q}_{j',r';n} \right\}_{\substack{0 \leq r' \leq r-1 \\ 0 \leq n \leq \overline{\beta}_{j',r'} - 1}} \right)^2 B_{j-1}[x_1, x_j]_\infty$$

for $0 \leq n < e((n_{r-1} - 1)\overline{\beta}_{r-1} + \dots + (n_1 - 1)\overline{\beta}_1) = e(\overline{\beta}_{j,r} - \beta_{j,r})$. Set $\epsilon(\tilde{q}_{j,r}) := e(\overline{\beta}_{j,r} - \beta_{j,r})$.

Analogously, for $\{q'_{j,r}\}_{r=0}^{g_j+1}$, $q'_{j,r} \in T'_{j,r-1} \dots T'_{j,0} k[v_2, \dots, v_{j-1}, x_1, x_j]$ (see remark 3.6), let $\{\overline{Q}'_{j,r;n}\}_{n \geq 0}$ in $\prod_{0 \leq s \leq r-1} T'_{j,s} k[v_2, \dots, v_{j-1}, x_1, x_j]_\infty$ be obtained applying lemma 4.2. Given r , $0 \leq r \leq g_j + 1$, let $\{a_s\}_{0 \leq s \leq r-1}$ be nonnegative integers such that

$$z'_{j,r} := q'_{j,r} \cdot \prod_{0 \leq s \leq r-1} q'_{j,s}^{a_s} \in k[v_2, \dots, v_{j-1}, x_1, x_j].$$

and let $Z'_{j,r;n} \in k[v_2, \dots, v_{j-1}, x_1, x_j]_\infty[[t]]$, $n \geq 0$, as before. Arguing by recurrence and applying (c) in lemma 3.1 it follows that

$$(36) \quad Z'_{j,r;n} \in \left(\left\{ \overline{Q}'_{j,s;n} \right\}_{\substack{0 \leq s \leq r-1 \\ 0 \leq n \leq \overline{\beta}_{j,s}-1}} \right)^2 \prod_{0 \leq s \leq r-1} \overline{T}'_{j,s}^{-1} k[v_2, \dots, v_{j-1}, x_1, x_j]_\infty$$

for $0 \leq n < \epsilon(z'_{j,r}) := e(\nu(z'_{j,r}) - \beta_{j,r})$.

Now, with the assumptions and notation in lemma 4.3, given $(j, r) \in \mathcal{J}$, let $\{a_{j',r'}(j, r)\}_{(j',r') \in \mathcal{J}_{j,r}^*}$ be any sequence of nonnegative integers such that

$$z_{j,r} := q_{j,r} \cdot \prod_{(j',r') \in \mathcal{J}_{j,r}^*} q_{j',r'}^{a_{j',r'}(j,r)} \in k[x_1, \dots, x_j]$$

and let $\overline{\alpha}_{j,r} := \nu(z_{j,r})$ and let $Z_{j,r;n} \in k[x_1, \dots, x_j]_\infty$, $n \geq 0$, as before. Then we have

$$\begin{aligned} & \left(\left\{ Z_{j',r';n} \right\}_{\substack{(j',r') \in \mathcal{J}_{j,r} \\ 0 \leq n \leq e\overline{\alpha}_{j',r'}-1}} \right) \prod_{(j',r') \in \mathcal{J}_{j,r}^*} S_{j',r'}^{-1} k[x_1, \dots, x_j]_\infty = \\ & = \left(\left\{ \overline{Q}_{j',r';n} \right\}_{\substack{(j',r') \in \mathcal{J}_{j,r} \\ 0 \leq n \leq e\overline{\beta}_{j',r'}-1}} \right) \prod_{(j',r') \in \mathcal{J}_{j,r}^*} T_{j',r'}^{-1} k[x_1, \dots, x_j]_\infty \end{aligned}$$

where $S_{j',r'}$ is the multiplicative part generated by $Z_{j',r';e\overline{\alpha}_{j',r'}}$. Moreover, arguing by recurrence and applying (c.2) in prop. 3.3 and also the condition (20), it follows that

$$(37) \quad Z_{j,r;n} \in \left(\left\{ Z_{j',r';n} \right\}_{\substack{(j',r') \in \mathcal{J}_{j,r} \\ 0 \leq n \leq e\overline{\alpha}_{j',r'}-1}} \right)^2 \prod_{(j',r') \in \mathcal{J}_{j,r}^*} S_{j',r'}^{-1} k[x_1, \dots, x_j]_\infty$$

for $0 \leq n < \epsilon(z_{j,r}) := e(\nu(z_{j,r}) - \beta_{j,r})$. In fact, the proof is based on the one for (36), taking into account condition (20).

Let $\mathcal{G} := \cup_{(j,r) \in \mathcal{J}} \mathcal{G}_{j,r}$ where the $\mathcal{G}_{j,r}$'s are defined in lemma 4.3 (v). Note that the cardinal of $\mathcal{G}_{1,1}$ is em_1 and, for $2 \leq j \leq \delta$,

$$\begin{aligned} \# \left(\bigcup_{r=1}^{g_j+1} \mathcal{G}_{j,r} \right) &= e \overline{\beta}_{j,1} + (e \overline{\beta}_{j,2} - e n_{j,1} \overline{\beta}_{j,1}) + \dots + (e \overline{\beta}_{j,g_j+1} - e n_{j,g_j} \overline{\beta}_{j,g_j}) \\ &= e (\beta_{j,1} + (\beta_{j,2} - \beta_{j,1}) + \dots + (\beta_{j,g_j+1} - \beta_{j,g_j})) = e \beta_{j,g_j+1} = e m_j. \end{aligned}$$

Hence, applying (6) and (30) we obtain

$$\#\mathcal{G} = e (a_E + 1) = e \mathfrak{e} (k_F(\mathbb{A}^d) + 1).$$

Proposition 4.5. *We have*

$$P_{eE}^{\mathbb{A}^d} \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}} = (\mathcal{G}) \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$$

moreover, there exists $L \in \mathcal{O}_{(\mathbb{A}^d)_\infty} \setminus P_{eE}^{\mathbb{A}^d}$ such that $P_{eE}^{\mathbb{A}^d} (\mathcal{O}_{(\mathbb{A}^d)_\infty})_L = (\mathcal{G}) (\mathcal{O}_{(\mathbb{A}^d)_\infty})_L$. Besides, the images of the elements of \mathcal{G} in $P_{eE}^{\mathbb{A}^d} / (P_{eE}^{\mathbb{A}^d})^2 \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ are independent, hence define a basis as $\kappa(P_{eE}^{\mathbb{A}^d})$ -vector space. In particular, we obtain $\dim \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}} = \#\mathcal{G} = e (a_E + 1)$.

Proof : First note that, by (i) in lemma 4.3, we have $\mathcal{G} \subset P_{eE}^{\mathbb{A}^d}$. Let us prove that $(\mathcal{G}) \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a prime ideal. By (ii) in lemma 4.3, for $(j, r) \in \mathcal{J}$, we have

$$\mathcal{G}_{j,r} \subset \prod_{(j',r') \in \mathcal{J}_{j,r}^*} \overline{T}_{j',r'}^{-1} A_\infty^{j,r} [X_{j;e\beta_{j,r-1}+1}, \dots, X_{j;e\beta_{j,r-1}}]$$

(if $r = 0$ or 1 , replace $X_{j;e\beta_{j,r-1}+1}$ by $X_{j;0}$ and set $\beta_{1,0} := m_1$). Then, for each j , $2 \leq j \leq \delta$, there exists $M_j \in \mathbb{N}$ such that

$$\overline{Q}_{j',r'} \in \prod_{(j'',r'') \in \mathcal{J}_{j',r'}^*} \overline{T}_{j'',r''}^{-1} k[\underline{X}_0^j, \dots, \underline{X}_{M_j}^j] \quad \text{for every } (j', r') \in \mathcal{J}_{j,g_j+1}^*$$

and, if we set

$$B_\infty^j := \prod_{(j',r') \in \mathcal{J}_{j,g_j+1}^*} \overline{T}_{j',r'}^{-1} k[\underline{X}_0^j, \dots, \underline{X}_{M_j}^j]$$

then

$$\mathcal{G}_j := \bigcup_{(j',r') \in \mathcal{J}_{j,g_j+1} \cup \{(j,g_j+1)\}} \mathcal{G}_{j',r'} \subset B_\infty^j$$

(in fact, M_j can be taken to be equal to em_j). Set $P_j := B_\infty^j \cap P_{eE}^{\mathbb{A}^d}$. We will prove, by induction on j , $2 \leq j \leq \delta$, that there exists $L_j \in B_\infty^j \setminus P_j$ such that the ring $(B_\infty^j)_{L_j} / (\mathcal{G}_j)$ is a domain. For $j = 2$, we have $h_{1,0} = x_1$, thus $\mathcal{G}_{1,0} = \{X_{1;0}, \dots, X_{1;em_1-1}\}$ and, applying remark 3.5 and (iii) in lemma 4.3 to $\overline{Q}_{2,r;n}$, $en_{j,r-1}\overline{\beta}_{2,r-1} < n < e\overline{\beta}_{2,r}$ and (iv) in lemma 4.3 to $\overline{Q}_{2,r;e\overline{\beta}_{2,r}}$, we obtain that $B_\infty^2 / (\mathcal{G}_2)$ is isomorphic to

$$(S_2^{-1}k[y_{2,0}, y_{2,2}, \dots, y_{2,g_2+1}] / J_2) [\{X_{1;n}\}_{em_1 < n \leq M_2} \cup \{X_{2;n}\}_{e\beta_{2,g_2+1} < n \leq M_2}]$$

where the image of $y_{2,r}$, $1 \leq r \leq g_2 + 1$ (resp. $y_{2,0}$) is $Q_{2,r;e\overline{\beta}_{2,r}}$ (resp. X_{1,em_1}), J_2 is the ideal in th. 3.8 (iv) and S_2 is the multiplicative part generated by $\{y_{2,r}\}_{r=0}^{g_2}$, therefore $B_\infty^2 / (\mathcal{G}_2)$ is a domain by th. 3.8.

Let j , $3 \leq j \leq \delta$, and suppose that the result holds for $j - 1$. Applying (iii) in lemma 4.3 to $\overline{H}_{j,s;n}$, for $ei_{j,s-1} < n \leq ei_{j,s} - 1$ (resp. $en_{j,r-1}\overline{\beta}_{j,r-1} < n \leq ei_{j,sr-1} + 1 - 1$) if $s_{r-1} + 2 \leq s \leq s_r$ (resp. $s = s_{r-1} + 1$) and applying (iv) in 4.3 to $\overline{H}_{j,s;ei_{j,s-1}}$ (resp. $\overline{H}_{j,s;en_{j,r-1}\overline{\beta}_{j,r-1}}$), we obtain that there exists an étale extension \tilde{B}_∞^j of B_∞^j containing the image of P_{eE}^Y , i.e. the contraction of P_{eE}^Y to \tilde{B}_∞^j is a prime ideal $\tilde{P}_j \neq \tilde{B}_\infty^j$, and such that $\tilde{B}_\infty^j / (\mathcal{G}_j)\tilde{B}_\infty^j$ is isomorphic to a localization of

$$(S_j^{-1}\tilde{D}_{j-1}[y_{j,1}, \dots, y_{j,g_j+1}] / J_j) [\{X_{j;n}\}_{e\beta_{j,g_j+1} < n \leq M_j}]$$

where \tilde{D}_{j-1} is a domain which is an étale extension of $B_\infty^{j-1} / (\mathcal{G}_{j-1})$, S_j is the multiplicative part generated by $\{y_{j,r}\}_{r=1}^{g_j}$ and J_j is an ideal generated by $\{y_{j,r}^{n_{j,r}} - \tilde{c}_{j,r} y_{1,0}^{b_{j,r,0}} \cdot y_{j,1}^{b_{j,r,1}} \cdots y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$, being $\tilde{c}_{j,r} \in \tilde{D}_{j-1}$ and $y_{1,0} = X_{1,em_1} \in \tilde{D}_{j-1}$. Here $y_{j,r}$ is identified with $\overline{Q}_{j,r;e\overline{\beta}_{j,r}}$. Applying th. 3.8 (iv) we conclude that $\tilde{B}_\infty^j / (\mathcal{G}_j)$ is a domain. Since the morphism $(B_\infty^j)_{P_j} / (\mathcal{G}_j) \rightarrow (\tilde{B}_\infty^j)_{\tilde{P}_j} / (\mathcal{G}_j)\tilde{B}_\infty^j$ is local étale, hence an inclusion of local rings, we conclude that $(B_\infty^j)_{P_j} / (\mathcal{G}_j)$ is a domain. Therefore, there exists $L_j \in B_\infty^j \setminus P_j$ such that $(B_\infty^j)_{L_j} / (\mathcal{G}_j)$ is a domain (recall that B_∞^j is the localization of a finitely generated k -algebra).

In particular, it follows that there exists $L_\delta \in B_\infty^\delta \setminus P_\delta \subset \mathcal{O}_{(\mathbb{A}^d)_\infty} \setminus P_{eE}^{\mathbb{A}^d}$ such that the ideal generated by \mathcal{G} in $(\prod_{(j,r) \in \mathcal{J}^*} \overline{T}_{j,r}^{-1} \mathcal{O}_{(\mathbb{A}^d)_\infty})_{L_\delta}$ is a prime ideal. From

this it follows that there exists $L \in \mathcal{O}_{(\mathbb{A}^d)_\infty} \setminus P_{eE}^{\mathbb{A}^d}$ such that $(\mathcal{G}) (\mathcal{O}_{(\mathbb{A}^d)_\infty})_L$ is a prime ideal, in fact, we may take $L = L_\delta \cdot \prod_{(j,r) \in \mathcal{J}^*} \overline{Q}_{j,r}^{a_{j,r}} \overline{\beta}_{j,r}$ for some positive integers $\{a_{j,r}\}_{(j,r) \in \mathcal{J}^*}$. Hence $(\mathcal{G}) \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a prime ideal.

Let us denote by P' the prime ideal of $\mathcal{O}_{\mathbb{A}^d}$ such that $(\mathcal{G}) \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}} = P' \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$. We will next prove that $P' = P_{eE}^{\mathbb{A}^d}$. In fact, with the notation in 2.7 and 3.7, let $P_{e\epsilon F}^Z$ be the generic point of $Z_\infty^{e\epsilon F}$ and let $P_{e\epsilon F}^{\mathbb{A}^d}$ be the image of $P_{e\epsilon F}^Z$ by the morphism $Z_\infty \rightarrow (\mathbb{A}^d)_\infty$. Since ϵ is the ramification index of $\mathcal{O}_{Y,E}$ over $\mathcal{O}_{Z,F}$, $P_{e\epsilon F}^Z$ is the image of P_{eE}^Y by $\sigma_\infty : Y_\infty \rightarrow Z_\infty$ and hence $P_{e\epsilon F}^{\mathbb{A}^d} = P_{eE}^{\mathbb{A}^d}$. Now, by the definition of \mathcal{G} , and since $P' \subseteq P_{eE}^{\mathbb{A}^d}$, we have

$$e\overline{\beta}_{j,r} \leq \nu_{P'}(q_{j,r}) \leq \nu(q_{j,r}) = e\overline{\beta}_{j,r} \quad \text{for } (j,r) \in \mathcal{J}.$$

Therefore $\nu_{P'}(q_{j,r}) = e\overline{\beta}_{j,r}$ for every $(j,r) \in \mathcal{J}$ and hence

$$\nu_{P'} \left(\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}} \right) = 0 \quad \text{for } 2 \leq j \leq \delta \quad \text{and} \quad \nu_{P'}(z) = \sum_{(j',r') \in \mathcal{J}^*} a_{j',r'} e \overline{\beta}_{j',r'} = e \epsilon$$

(recall (29) in 3.7). From this it follows that the morphism of k -algebras $h_{P'}^\# : \mathcal{O}_{X,P_0} \rightarrow \kappa(P')[[t]]$ induced by the arc $h_{P'}$ extends to $\mathcal{O}_{Z,F}$. That is, $h_{P'} : \text{Spec } \kappa(P')[[t]] \rightarrow X$ lifts to (Z,F) , more precisely, since $\nu_{P'}(z) = e\epsilon$, this lifting defines a point in $Z_\infty^{e\epsilon F}$. Therefore $P' \in \overline{\{P_{e\epsilon F}^{\mathbb{A}^d}\}}$, hence we conclude that $P' = P_{e\epsilon F}^{\mathbb{A}^d} = P_{eE}^{\mathbb{A}^d}$.

Finally, since $\# \mathcal{G} = e \epsilon (k_F(\mathbb{A}^d) + 1)$, the end of the proof follows from prop. 2.6 and equality (3), which is in fact lemma 3.4 in [DL].

Remark 4.6. Alternatively, in the proof of prop. 4.5 it can be proved by induction on j , $2 \leq j \leq \delta$, and applying (iii) and (iv) in lemma 4.3, not only that (\mathcal{G}_j) is a prime ideal of B_∞^j , but also that the elements in \mathcal{G}_j are independent in $(\mathcal{G}_j) / (\mathcal{G}_j)^2$. Then, lemma 3.4 in [DL] can be recovered (at least for X smooth) from propositions 4.5 and 2.6. Therefore, prop. 4.5 can be seen as a new version of lemma 3.4 in [DL], which is in fact the change of variables theorem in the motivic integration.

Definition 4.7. Let $\eta : Y \rightarrow \mathbb{A}_k^d$ be a k -morphism dominant and generically finite, where Y is a nonsingular k -scheme, let E be a divisor on Y and let $e \geq 1$. Let $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ be a system of transverse generators for η with respect to E (def. 3.4), and let $\{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, n \geq 0}$ defined as in lemma 4.3. We call

$$\mathcal{Q} := \{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, e n_{j,r-1} \overline{\beta}_{j,r-1} \leq n \leq e \overline{\beta}_{j,r} - 1}$$

a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ associated to $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$.

In fact, note that by prop. 4.5 (see also lemma 4.3 (v)), $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a regular local ring of dimension the cardinal of \mathcal{Q} whose maximal ideal $P_{eE}^{\mathbb{A}^d} \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is generated by \mathcal{Q} .

Theorem 4.8. Assume that $\text{char } k = 0$. Let X be a nonsingular k -scheme, let ν be a divisorial valuation on an irreducible component X_0 of X , and let $e \in \mathbb{N}$. Let $\pi : Y \rightarrow X_0$ be a proper and birational morphism such that the center of ν on Y is a divisor E , and let $\eta : Y \rightarrow \mathbb{A}_k^d$ be the composition of π with an étale morphism

$X_0 \rightarrow \mathbb{A}_k^d$, where $d = \dim X_0$. Let $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, 0 \leq n \leq e\overline{\beta}_{j,r}-1}$ be a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ associated to a system of transverse generators for η with respect to E . Then \mathcal{Q} is also a regular system of parameters of $\mathcal{O}_{X_\infty, P_{eE}^X}$, that is

$$P_{eE}^X \mathcal{O}_{X_\infty, P_{eE}^X} = \left(\{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, e n_{j,r-1} \overline{\beta}_{j,r-1} \leq n \leq e\overline{\beta}_{j,r}-1} \right) \mathcal{O}_{X_\infty, P_{eE}^X}.$$

and $\mathcal{O}_{X_\infty, P_{eE}^X}$ is a regular local ring of dimension

$$\dim \mathcal{O}_{X_\infty, P_{eE}^X} = \#\mathcal{Q} = e(k_E + 1).$$

where k_E is the discrepancy of X with respect to E .

Moreover, there exist elements $z_{j,r} \in \mathcal{O}_{X, P_0}$, $(j, r) \in \mathcal{J}$, and $L \in \mathcal{O}_{X_\infty} \setminus P_{eE}^X$ such that

$$(38) \quad P_{eE}^X(\mathcal{O}_{X_\infty})_L = \left(\{Z_{j,r;n}\}_{(j,r) \in \mathcal{J}, 0 \leq n < e\overline{\alpha}_{j,r}} \right) (\mathcal{O}_{X_\infty})_L$$

where $\overline{\alpha}_{j,r} = \nu(z_{j,r})$ for $(j, r) \in \mathcal{J}$.

Proof : Recall that P_{eE}^Y is the generic point of Y_∞^{eE} (see 2.7) and that P_{eE}^X (resp. $P_{eE}^{\mathbb{A}^d}$) is the image of P_{eE}^Y by π_∞ (resp. η_∞). By prop. 2.5 (see also corol. 2.9) it suffices to prove the result for the point $P_{eE}^{\mathbb{A}^d}$ in $(\mathbb{A}^d)_\infty$. Then it follows from prop. 4.5. In fact, for the first assertion note that in this case $k_E(\mathbb{A}_k^d)$ is equal to the discrepancy k_E of X with respect to E . For the second assertion, let $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ be a system of transverse generators for η with respect to E . For each $(j, r) \in \mathcal{J}$ there exists a sequence of nonnegative integers $\{a_{j',r'}(j, r)\}_{(j',r') \in \mathcal{J}_{j,r}^*}$, such that

$$z_{j,r} := q_{j,r} \cdot \prod_{(j',r') \in \mathcal{J}_{j,r}^*} q_{j',r'}^{a_{j',r'}(j,r)} \in \mathcal{O}_{\mathbb{A}^d, P_0}.$$

(see prop. 3.3). Then, from prop. 4.5, (38) follows. This concludes the proof.

Remark 4.9. Let P be any stable point of X_∞ and suppose that X is non-singular at the center P_0 of P and that P_0 is not the generic point of X . There exists a birational and proper morphism $\pi : Y \rightarrow X$ such that the center of ν_P on Y is a divisor E , and $e \in \mathbb{N}$ such that $\nu_P = e\nu_E$ ([Re2], (vii) in prop. 3.7). Let $P^Y \in Y_\infty$ whose image by π_∞ is P , then we have $\dim \mathcal{O}_{X_\infty, P} = ek_E + \dim \mathcal{O}_{Y_\infty, P^Y}$ (corol. 2.9). Since $P^Y \supseteq P_{eE}^Y$ and $P \supseteq P_{eE}^X$, with the notation in theorem 4.8 and prop. 3.3, $\{U_0, \dots, U_{e-1}\}$ is part of a regular system of parameters of $\mathcal{O}_{Y_\infty, P^Y}$ and $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, 0 \leq n \leq e\overline{\beta}_{j,r}-1}$ is part of a regular system of parameters of $\mathcal{O}_{X_\infty, P}$. Moreover, suppose that $\{U_0, \dots, U_{e-1}, G_1, \dots, G_s\}$ is a regular system of parameters of $\mathcal{O}_{Y_\infty, P^Y}$. To describe a regular system of parameters of $\mathcal{O}_{X_\infty, P}$ we add to \mathcal{Q} the following elements : By lemma 4.1 and since π is birational, for each $y \in \mathcal{O}_Y$ and for each n , there exists $\overline{Y}_n \in \mathcal{O}_{X_\infty, P}$ such that

$$Y_n \equiv \overline{Y}_n \pmod{P}.$$

Then, let $\overline{G}_i \in \mathcal{O}_{X_\infty, P}$, $1 \leq i \leq s$ be obtained from G_i by replacing U_n and $V_{j;n}$ by \overline{U}_n and $\overline{V}_{j;n}$, for $n \geq 0$, $2 \leq j \leq d$. We have

$$G_i \equiv \overline{G}_i \pmod{P}.$$

and $\mathcal{Q} \cup \{\overline{G}_1, \dots, \overline{G}_s\}$ is a regular system of parameters of $\mathcal{O}_{X_\infty, P}$.

Now let us consider a reduced separated k -scheme of finite type X and a divisorial valuation ν on X centered on $\text{Sing } X$. There exists a resolution of singularities $\pi : Y \rightarrow X$ (i.e. π is a proper, birational k -morphism, with Y smooth, such that the induced morphism $Y \setminus \pi^{-1}(\text{Sing } X) \rightarrow X \setminus \text{Sing } X$ is an isomorphism) such that the center of ν on Y is a divisor E .

Corollary 4.10. *Assume that $\text{char } k = 0$. Let X be a reduced separated k -scheme of finite type, let ν be a divisorial valuation on an irreducible component X_0 of X centered on $\text{Sing } X$ and let $e \in \mathbb{N}$. Let $\pi : Y \rightarrow X$ be a resolution of singularities such that the center of ν on Y is a divisor E , and let $\eta : Y \rightarrow \mathbb{A}_k^d$ be the composition of π with a general projection $\mu : X_0 \rightarrow \mathbb{A}^d$, where $d = \dim X_0$. Let $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, 0 \leq n \leq e\overline{\beta}_{j,r}-1}$ be a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ associated to a system of transverse generators for η with respect to E . Then \mathcal{Q} is a system of coordinates of (X_∞, P_{eE}^X) , that is,*

$$P_{eE}^X \mathcal{O}_{X_\infty, P_{eE}^X} = \left(\{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, e n_{j,r-1} \overline{\beta}_{j,r-1} \leq n \leq e\overline{\beta}_{j,r}-1} \right) \mathcal{O}_{X_\infty, P_{eE}^X}.$$

Therefore

$$\text{embdim } \widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, P_{eE}}} = \text{embdim } \mathcal{O}_{(X_\infty)_{\text{red}}, P_{eE}} \leq \#\mathcal{Q} = e(\widehat{k}_E + 1).$$

where \widehat{k}_E is the Mather discrepancy of X with respect to E .

Moreover, there exist elements $z_1, \dots, z_s \in \mathcal{O}_{X, P_0}$ and $L \in \mathcal{O}_{X_\infty} \setminus P_{eE}^X$ such that

$$P_{eE}^X(\mathcal{O}_{X_\infty})_L = (Z_{1;0}, \dots, Z_{1;e\overline{\alpha}_1-1}, \dots, Z_{s;0}, \dots, Z_{s;e\overline{\alpha}_s-1})(\mathcal{O}_{X_\infty})_L$$

where $\overline{\alpha}_i = \nu(z_i)$ for $1 \leq i \leq s$.

Proof : We may suppose that $\pi : Y \rightarrow X$ dominates the Nash blowing up of X . We may suppose that X is affine, let $X \subseteq \mathbb{A}_k^N = \text{Spec } k[y_1, \dots, y_N]$. Then, a general projection $\rho : X \subseteq \mathbb{A}_k^N \rightarrow \mathbb{A}_k^d$, $\underline{y} \rightarrow (x_1, \dots, x_d)$ satisfies

$$\text{ord}_E \pi^*(dx_1 \wedge \dots \wedge dx_d) = \widehat{k}_E.$$

Let $P_{eE}^{\mathbb{A}^d}$ be the image of P_{eE}^Y by η_∞ . Then the result follows from prop. 4.5 applied to $P_{eE}^{\mathbb{A}^d}$ and prop. 4.5 (iii) in [Re2] applied to $\rho : X \rightarrow \mathbb{A}_k^d$.

Acknowledgements : I thank O. Piltant for his suggestions and comments.

References.

- [Bo] N. Bourbaki, *Commutative Algebra*, Chapter III, Springer (1989).
- [DL] J. Denef, F. Loeser, *Germes of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135, 201-232 (1999).
- [EM] L. Ein and M. Mustata, *Jet schemes and singularities*, Proc. Symp. Pure Math. 80.2, 505-546 (2009).
- [FEI] T. de Fernex, L. Ein and S. Ishii, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. 44 no. 2, 425-448 (2008).
- [Ha] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag 1977.
- [IR] S. Ishii, A. Reguera, *Singularities with the highest Mather minimal log discrepancy*, Math. Zeitschrift 275 no. 3, 1255-1274 (2013).
- [Ka] O. Kashcheyeva, *A construction of generating sequences for valuations centered in dimension 3 regular local rings*, Preprint.
- [LMSS] F. Lucas, J. Madden, D. Schaub, M. Spivakovsky, *M. Approximate roots of a valuation and the Pierce-Birkhoff conjecture*, Ann. Fac. Sci. Toulouse Math. (6) 21, no. 2, 259-342 (2012).
- [Mo] P. Morandi, *Field and Galois Theory*, Springer-Verlag GTM 167, 1996.
- [Pi] O. Piltant, *Gradués associés aux valuations Frobenius pour les sommes de Selberg*, Thesis Ecole Polytechnique (1994).
- [Ra] M. Raynaud, *Anneaux Locaux Henséliens*, Springer-Verlag LNM 169, 1956.
- [Re1] A.J. Reguera, *A curve selection lemma in spaces of arcs and the image of the Nash map*, Compositio Math. 142, 119-130, (2006).
- [Re2] A.J. Reguera, *Towards the singular locus of the space of arcs*, Amer. J. Maths. 131, n. 2, 313-350 (2009).
- [Sp] M. Spivakovsky, *Valuations in function fields of surfaces*, Amer. J. Math. 112, 107-156, (1990).
- [Za] O. Zariski with an appendix by B. Teissier, *The moduli problem for plane branches*, AMS 39, (2006).