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# STARK-WANNIER LADDERS AND CUBIC EXPONENTIAL SUMS

ALEXANDER FEDOTOV AND FRÉDÉRIC KLOPP

On  $L^2(\mathbb{R})$ , we consider the Schrödinger operator

$$(1.1) \quad H_\epsilon = -\frac{\partial^2}{\partial x^2} + v(x) - \epsilon x,$$

where  $v$  is a real analytic 1-periodic function and  $\epsilon$  is a positive constant. This operator is a model to study a Bloch electron in a constant electric field ([1]). The parameter  $\epsilon$  is proportional to the electric field. The operator (1.1) was studied both by physicists (see, e.g., the review [6]) and by mathematicians (see, e.g., [9]). Its spectrum is absolutely continuous and fills the real axis. One of main features of  $H_\epsilon$  is the existence of Stark-Wannier ladders. These are  $\epsilon$ -periodic sequences of resonances, which are poles of the analytic continuation of the resolvent kernel in the lower half plane through the spectrum (see, e.g., [2]). Most of the mathematical work studied the case of small  $\epsilon$  (see, e.g., [9, 3] and references therein). When  $\epsilon$  is small, there are ladders exponentially close to the real axis. Actually, only the case of finite gap potentials  $v$  was relatively well understood. For these potentials, there is only a finite number of ladders exponentially close to the real axis. It was further noticed that the ladders non-trivially “interact” as  $\epsilon$  changes, and conjectured that the behavior of the resonances strongly depends on number theoretical properties of  $\epsilon$  (see, e.g., [1]).

In the present note, we only consider the periodic potential  $v(x) = 2 \cos(2\pi x)$  and study the reflection coefficient  $r(E)$  of the Stark-Wannier operator (1.1) in the lower half of the complex plane of the spectral parameter  $E$ . The resonances are the poles of the reflection coefficient. We show that, as  $\text{Im } E \rightarrow -\infty$ , the function  $E \mapsto \frac{1}{r(E)}$  can be asymptotically described in terms of a regularized cubic exponential sum that is a close relative of the cubic exponential sums often encountered in analytic number theory. This explains the dependence of the reflection coefficient on the arithmetic

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nature of  $\epsilon$ . For  $\frac{\pi^2}{3\epsilon} \in \mathbb{Q}$ , we describe the asymptotics of the Stark-Wannier ladders situated far from the real axis.

Let us recall the definition of the reflection coefficient for (1.1) following [2]. Consider the equation

$$(1.2) \quad -\psi''(x) + (v(x) - \epsilon x)\psi(x) = E\psi(x), \quad x \in \mathbb{C},$$

For the sake of simplicity, assume that the potential  $v$  is entire. Assume also  $\int_0^1 v(x) dx = 0$ . For any  $E \in \mathbb{C}$ , there are unique solutions  $\psi_{\pm}$  to (1.2) that admit the asymptotic representations

$$(1.3) \quad \begin{aligned} \psi_-(x, E) &= \frac{1}{\sqrt[4]{-\epsilon x - E}} e^{-\int_x^{-E/\epsilon} \sqrt{-\epsilon t - E} dt + o(1)}, \quad x \rightarrow -\infty, \\ \psi_+(x, E) &= \frac{1}{\sqrt[4]{\epsilon x + E}} e^{i \int_{-E/\epsilon}^x \sqrt{\epsilon t + E} dt + o(1)}, \quad x \rightarrow +\infty, \end{aligned}$$

where the determinations of  $\sqrt{\cdot}$  and  $\sqrt[4]{\cdot}$  are analytic in  $\mathbb{C} \setminus \mathbb{R}_-$  and positive along  $\mathbb{R}_+$ . Consider also the solution  $\psi_+^*(x, E) = \overline{\psi_+(\bar{x}, \bar{E})}$ . The solutions  $\psi_+$  and  $\psi_+^*$  being linearly independent, one has

$$(1.4) \quad \psi_-(x, E) = w(E)\psi_+^*(x, E) + w^*(E)\psi_+(x, E), \quad x \in \mathbb{R},$$

where the coefficient  $w(E)$  is independent of  $x$  and the function  $E \mapsto w(E)$  is entire. The ratio  $r(E) = w^*(E)/w(E)$  is the reflection coefficient. It is an  $\epsilon$ -periodic meromorphic function of  $E$ . The reflection coefficient is analytic in  $\mathbb{C}_+$ , and, for  $E \in \mathbb{R}$ , one has  $|r(E)| = 1$ . The poles of  $r$  are the resonances of  $H_\epsilon$ .

Let us now state the first of our results. Represent  $1/r$  by its Fourier series  $1/r(E) = \sum_{m \in \mathbb{Z}} e^{2\pi n i E/\epsilon} p(m)$  for  $\text{Im } E \leq 0$ . Let  $a(\epsilon) = \sqrt{\frac{2}{\epsilon}} \pi e^{i\pi/4}$ . One has

**Theorem 1.** *Let  $v(x) = 2 \cos(2\pi x)$ . Then, as  $m \rightarrow \infty$ ,*

$$(1.5) \quad p(m) = a(\epsilon) \sqrt{m} e^{-2\pi i \omega m^3 - 2m \log(2\pi m/\epsilon) + \delta(m)}, \quad \omega = \left\{ \frac{\pi^2}{3\epsilon} \right\},$$

where, for  $x$  real,  $\{x\}$  denotes the fractional part of  $x$ , and  $\delta(m) = O(\log^2 m/m)$ . This estimate is locally uniform in  $\epsilon > 0$ .

Clearly, the asymptotic behavior of  $1/r(E)$  as  $\text{Im } E \rightarrow -\infty$  is determined by the Fourier series terms with large positive  $m$ , and so, roughly,

$$(1.6) \quad \frac{1}{r(E)} \approx a(\epsilon) \mathcal{P}(E/\epsilon), \quad \mathcal{P}(s) = \sum_{m \geq 1} \sqrt{m} e^{-2\pi i \omega m^3 - 2m \log(2\pi m/\epsilon) + 2\pi i m s}.$$

It is worth to compare the function  $\mathcal{P}$  with the cubic exponential sums  $\sum_{n=1}^N e^{-2\pi i \omega n^3}$ . Such sums were extensively studied in analytic number theory, see, e.g., [4]. They

were proved to depend strongly on the arithmetic nature of  $\omega$ . This appears to be true in our case too. We have

**Theorem 2.** *Let  $v(x) = 2 \cos(2\pi x)$ . Assume that  $\omega \in \mathbb{Q}$  and represent it in the form  $\omega = \frac{p}{q}$ , where  $0 \leq p < q$  are co-prime integers. If  $p = 0$ , we take  $q = 1$ . For  $\xi \in \mathbb{R}$ , we set  $I_q(\xi) := \{m \in \mathbb{Z} : |\xi - \frac{m}{q}| \leq 1/2\}$ . As  $\text{Im } E \rightarrow -\infty$ , one has*

$$(1.7) \quad r^{-1}(E) = \frac{b(\epsilon)\rho}{q} \sum_{m \in I_q(\xi)} S_q(p, m) e^{\rho e^{i\pi(\xi - m/q)} + i\pi(\xi - m/q) + O(\log^2 \rho/\rho)} + e^{O(\frac{\rho}{\text{Im } E})},$$

where  $b(\epsilon) = \pi^{\frac{3}{2}} e^{i\pi/4} / \sqrt{2\epsilon}$ ,  $\xi = \text{Re } E/\epsilon$ ,  $\rho = e^{-\pi \text{Im } E/\epsilon}$ , and

$$S_q(p, m) = \sum_{l=0}^{q-1} e^{-2\pi i \frac{pl^3 - ml}{q}}.$$

The error estimates are locally uniform in  $\epsilon > 0$ .

Let us discuss this result. First, assume that  $\omega = 0$ . By Theorem 2,

$$(1.8) \quad (b(\epsilon)r(E))^{-1} = \sqrt{z} e^{\sqrt{z} + O(\frac{\ln^2 z}{\sqrt{z}})} + e^{O(\frac{\sqrt{z}}{\ln z})}, \quad z = e^{2i\pi E/\epsilon},$$

where the determination of  $\sqrt{\cdot}$  is analytic in  $\mathbb{C} \setminus \mathbb{R}_-$  and positive along  $\mathbb{R}_+$ . Recall that  $1/r$  is  $\epsilon$ -periodic. Let  $B_\epsilon = \{E \in \mathbb{C} : \text{Im } E \leq 0, 0 \leq \text{Re } E \leq \epsilon\}$ . Representation (1.8) implies

**Corollary 1.** *Assume  $\omega = 0$ . The resonances located in  $B_\epsilon$  have the following properties :*

- for sufficiently large  $y > 0$ , the resonances with  $\text{Im } E < -\epsilon y$  are located in the domain  $|\text{Re } E - \epsilon/2| \leq C\epsilon^2/|\text{Im } E|$ , where  $C > 0$  is a constant;
- let  $n(y)$  be the number of resonances in the rectangle  $[0, \epsilon] - i[0, \epsilon y]$ ; then, one has

$$n(y) = \frac{1}{\pi} e^{\pi y + o(1)} \quad \text{as } y \rightarrow \infty.$$

The first statement immediately follows from Theorem 2; to prove the second one has to use Jensen formula and Levin lower bounds for the absolute values of entire functions, see, e.g., [8].

When  $\omega = 0$ , it is difficult to obtain the asymptotics of the resonances as, in a neighborhood of the line  $\text{Re } E/\epsilon = 1/2 \pmod{1}$ , they are determined by the first Fourier coefficients of  $1/r$ , i.e., by  $p(m)$  with  $m = 1, 2, 3, \dots$ . Hence, the problem is not asymptotic in nature.

If  $\omega \neq 0$ , then the description of the resonances is determined by the values of  $S_q(p, m)$  for  $m = 1, 2, \dots, q-1$  (the map  $m \rightarrow S_q(p, m)$  is  $q$ -periodic). The  $S_q(p, m)$  are *cubic complete rational exponential sums*, see, e.g., [7]. One easily checks

$$\text{Lemma 1. For any } q \in \mathbb{N}, \quad \sum_{m=0}^{q-1} |S_q(p, m)|^2 = q^2.$$

This implies that, for  $q \geq 1$ , there is at least one integer  $0 \leq m_0 < q - 1$  such that  $S_q(p, m_0) \neq 0$ .

If  $S_q(p, m)$  is non zero for only one  $0 \leq m_0 < q$  (this happens, for example, for  $q = 2, 3, 6$ ), then one can characterize the resonances as when  $\omega = 0$ . Now, they live near the lines  $\{\operatorname{Re} E/\epsilon = m_0/q + 1/2 + n\}$ ,  $n \in \mathbb{Z}$ .

For large  $q$ , there are actually many non-zero values  $S_q(p, m)$ :

**Lemma 2.** *There exists a constant  $C > 0$  such that, for any co-prime  $q > p > 0$ , one has  $\#\{0 \leq m < q : S_q(p, m) \neq 0\} \geq Cq^{\frac{2}{3}}$ .*

This statement follows from Lemma 1 and the well-known upper bound for general complete rational exponential sums of Hua ([7]).

In general, the behavior of  $m \mapsto S_q(p, m)$  is nontrivial; it is known to depend strongly on the prime factorization of  $q$ . Computer calculations lead to the following conjecture: *if  $q$  is prime,  $0 < p < q$ , and  $0 < m < q$ , then  $S_q(p, m) \neq 0$ .*

If  $S_q(p, m)$  is non zero for at least two values of  $m$  such that  $0 \leq m < q$ , then, using (1.5), one can describe asymptotically all the resonances with sufficiently negative imaginary part. One has

**Corollary 2.** *Assume that, for some integers  $m_1 < m_2$  such that  $m_2 - m_1 < q$ , one has  $S_q(p, m_1) \neq 0$ ,  $S_q(p, m_2) \neq 0$ , and  $S_q(p, m) = 0$  for all  $m_1 < m < m_2$ . Then, for sufficiently large  $y > 0$ , in the vertical half-strip*

$$\left\{ E \in \mathbb{C} : -\operatorname{Im} E \geq \epsilon y, \frac{m_1}{q} \leq \frac{\operatorname{Re} E}{\epsilon} \leq \frac{m_2}{q} \right\},$$

*there are resonances, and they are described by the asymptotic formulas:*

$$(1.9) \quad \frac{E}{\epsilon} = -i \left( \frac{\ln(\pi k)}{\pi} - \ln \sin \frac{\pi(m_2 - m_1)}{q} \right) + \frac{m_2 + m_1}{q} + o(1), \quad k \in \mathbb{N},$$

*where  $o(1) \xrightarrow[k \rightarrow +\infty]{} 0$ .*

This statement easily follows from Theorem 2.

Finally, let us describe very briefly the ideas leading to Theorems 1 and 2. Buslaev's solutions  $\psi_{\pm}$  used to define the reflection coefficient (see (1.3)) are entire functions of  $x$  and  $E$ ; they satisfy the relations  $\psi_{\pm}(x+1, E) = \psi_{\pm}(x, E + \epsilon)$ . It appears that the analytic properties of such solutions can naturally be described in terms of a system of two first order difference equations on the complex plane (see, for example, [5]). To get the asymptotics of the Fourier coefficients of the reflection coefficient, we study the solutions of this system far from the origin. The idea leading from Theorem 1 to Theorem 2 is analogous to one used to study the behavior of

the exponential sums  $\sum_{n=1}^N e^{-2\pi i \omega n^3}$  with  $\omega \in \mathbb{Q}$  for large  $N$ , see [4]. However, to use

it successfully, one has to carry out a non trivial analysis of properties of the error term in (1.5).

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