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# INTERSECTION NORMS ON SURFACES AND BIRKHOFF SECTIONS FOR GEODESIC FLOWS

PIERRE DEHORNOY

ABSTRACT. For every finite collection of curves on a surface, we define an associated (semi-)norm on the first homology group of the surface. The unit ball of the dual norm is the convex hull of finitely many integer points. We give an interpretation of these points in terms of certain coorientations of the original collection of curves. Our main result is a classification statement: when the surface has constant curvature and the curves are geodesics, integer points in the interior of the unit ball of the dual norm classify isotopy classes of Birkhoff sections for the geodesic flow (on the unit tangent bundle to the surface) whose boundary is the symmetric lift of the collection of geodesics. These Birkhoff sections also yield numerous open-book decompositions of the unit tangent bundle.

This article has two goals. We first introduce an elementary family of norms, called *intersection norms*, on the first homology group of a real surface. These norms can be seen as (parametrized) 2-dimensional analogs of the Thurston norm on the second homology group of a 3-manifold. Secondly we use these intersection norms to classify up to isotopy certain 2-dimensional objects in some 3-manifolds, namely Birkhoff sections with prescribed boundary for the geodesic flow in the unit tangent bundle to a negatively curved surface (or to the torus with a flat metric).

**Intersection norms.** Let  $\Sigma$  be a real compact surface with or without boundary. A *closed multi-curve* (or *multi-curve* for short) on  $\Sigma$  is a finite collection of immersed closed curves in general position, meaning that the immersion is smooth and has only double points. (If  $\Sigma$  has some boundary, we also allow the curves to be arcs connecting two boundary components.) The geometric intersection number of two multi-curves is usually defined as the minimum of the number of intersection points of two representants of the free homotopy classes of the two multi-curves that have disjoint double points. We propose here a variation of this notion where we fix one multi-curve, but minimize over the homology class of the second.

So let  $\gamma$  denote a fixed multi-curve on  $\Sigma$ . For  $\alpha$  another multi-curve on  $\Sigma$ , the geometric intersection  $i_\gamma(\alpha)$  is the minimal number of intersection points of a multi-curve homotopic to  $\alpha$  and in general position with respect to  $\gamma$  with  $\gamma$ . Beware that this definition is not symmetric since the curve  $\gamma$  is fixed and not allowed to change in its homotopy class. Given a homology class  $a$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , we then minimise the intersection number over all closed multi-curves in  $a$ . This defines a function  $x_\gamma : H_1(\Sigma, \partial\Sigma; \mathbb{Z}) \rightarrow \mathbb{N}$  by

$$x_\gamma(a) := \min_{[\alpha]=a} i_\gamma(\alpha) = \min_{\substack{[\alpha]=a \\ \alpha \cap \gamma}} |\{\alpha \cap \gamma\}|.$$

**Theorem A.** *Let  $\Sigma$  be a compact oriented surface and  $\gamma$  a multi-curve on  $\Sigma$ . The function  $x_\gamma$  extends canonically into a continuous function  $x_\gamma : H_1(\Sigma, \partial\Sigma; \mathbb{R}) \rightarrow \mathbb{R}_+$  which is convex and*

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linear on rays through the origin. If, furthermore, the multi-curve  $\gamma$  fills  $\Sigma$  in the sense that the complement  $\Sigma \setminus \gamma$  is the union of topological discs, then  $x_\gamma$  is a norm.

Theorem A is an exact transposition of W. Thurston's result defining a norm on the second homology of a 3-manifold [Thu86, Thm 1]. In particular the (semi-)norm  $x_\gamma$  has the property of taking integer values on integer classes. Thurston showed [Thu86, Thm2] that this implies that the unit ball, denoted here by  $B_{x_\gamma}$ , is very peculiar: it is a polyhedron with finitely many sides, which are all given by linear equations with integer coefficients. Equivalently, this means that the closed unit ball of the dual norm on  $H^1(\Sigma; \mathbb{R})$ , denoted here by  $B_{x_\gamma}^*$ , is the convex hull of finitely many integer points.<sup>1</sup>

In the case of the Thurston norm, some of these extremal points of the dual ball could be interpreted using Euler classes of fibrations on the circle [Thu86, Thm 3]. An interpretation of all extremal points was then given in terms of Euler class of taut foliations by D. Gabai (unpublished, see [Yaz16, Thm 3.3]), and in terms of flows by D. Calegari [Cal06].

The analog statement for intersection norms is simpler. Considering the multi-curve  $\gamma$  as a graph whose vertices are the double-points and whose edges are the simple arcs of  $\gamma$ , a *coorientation* of  $\gamma$  is the a choice of a coorientation for every edge of  $\gamma$ . A given multi-curve has only finitely many coorientations. A coorientation is *Eulerian* if around every double point, there are two positively and two negatively cooriented edges. A coorientation  $\nu$  can be paired with an oriented curve  $\alpha$  using signed intersection. If  $\nu$  is Eulerian, it turns out that the pairing  $\nu(\alpha)$  depends only on the homology class of  $\alpha$ , so that a Eulerian coorientation  $\nu$  induces an integral cohomology class  $[\nu] \in H^1(\Sigma; \mathbb{Z})$ . One can wonder which classes are represented by such Eulerian coorientations. A first remark is that, representing a class  $a$  by a curve  $\alpha$  which minimises the geometric intersection with  $\gamma$ , one sees that  $|\nu(\alpha)|$  is not larger than  $x_\gamma(a)$ . A second remark is that the parity of  $\nu(\alpha)$  is fixed by  $\gamma$ : indeed, since all intersection points are counted with a coefficient  $\pm 1$ , the parity of  $\nu(\alpha)$  is determined by the parity of  $i_\gamma(\alpha)$  and does not depend on  $\nu$ ; since  $\gamma$  is a graph of even degree, the parity of  $i_\gamma(\alpha)$  does not change if we replace  $\alpha$  by a homologous curve. Our second result states that these restrictions are the only ones: the classes of the Eulerian coorientations are exactly the integer points in  $B_{x_\gamma}^*$  that are *congruent to  $[\gamma]_2 \bmod 2$* . More interestingly, the extremal points of  $B_{x_\gamma}^*$  correspond to some Eulerian coorientations.

**Theorem B.** *Let  $\Sigma$  be a compact oriented surface and  $\gamma$  a multi-curve on  $\Sigma$ . The dual unit ball  $B_{x_\gamma}^*$  in  $H^1(\Sigma; \mathbb{R})$  is the convex hull of the points in  $H^1(\Sigma; \mathbb{Z})$  given by all Eulerian coorientations of  $\gamma$ . Equivalently, for every  $a$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , we have*

$$x_\gamma(a) = \min_{[\alpha]=a} i_\gamma(\alpha) = \max_{\substack{\nu \text{ Eulerian} \\ \text{coor. of } \gamma}} \nu(a).$$

*Moreover every point in  $B_{x_\gamma}^* \cap H^1(\Sigma; \mathbb{Z})$  that is congruent to  $[\gamma]_2 \bmod 2$  is the class of some Eulerian coorientation (see Figure 1).*

Not only does this result provide an interpretation of the integer points inside the unit ball of the dual norm, it also gives an effective way of computing the norm  $x_\gamma$ , since it reduces the minimisation over an infinite number of curves into a maximisation over a finite number of coorientations.

<sup>1</sup>Thurston's proof of this fact is not as natural as one might expect. M. De La Salle recently gave a more direct proof [Sal16].

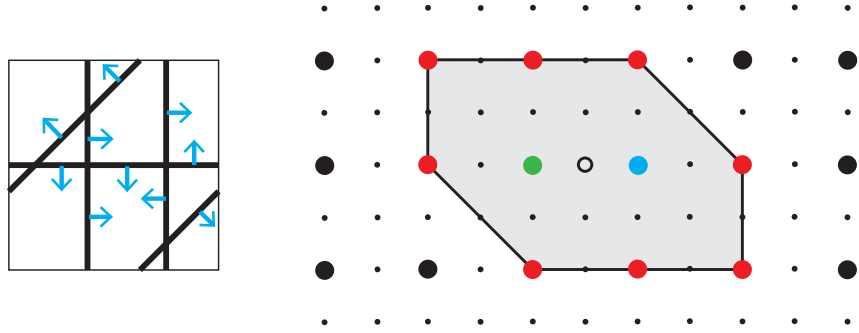


FIGURE 1. Illustration of Theorems B and D. On the left a collection  $\gamma$  of four geodesics on the torus  $T^2$ , and a Eulerian coorientation (blue arrows). On the right the dual unit ball  $B_{x_\gamma}^* \subset H^1(T^2, \mathbb{R})$  of the associated intersection norm. The empty circle denotes the origin. The big dots denote those classes in  $H^1(T^2, \mathbb{Z})$  congruent to  $[\gamma]_2 \bmod 2$ . Among these classes, 10 (in blue, green and red) are in the dual unit ball  $B_{x_\gamma}^*$  and correspond to all cohomology classes of Eulerian coorientations of  $\gamma$  (Theorem B). For example the class corresponding the blue coorientation is the blue point. The blue and green points lie in the interior of  $B_{x_\gamma}^*$ , hence describe the two isotopy classes of Birkhoff sections for  $\varphi_{\text{geod}}$  bounded by  $-\overleftrightarrow{\gamma}$ , while the 8 red points are on the boundary of  $B_{x_\gamma}^*$  and describe isotopy classes of surfaces transverse to  $\varphi_{\text{geod}}$ , but not intersecting every orbit, and bounded by  $-\overleftrightarrow{\gamma}$  (Theorem D).

**Classification of Birkhoff sections for geodesic flows.** Let  $M$  be a real compact, orientable  $n$ -manifold without boundary, and  $X$  be a non-singular vector field on  $M$ . In order to understand the dynamics of  $X$  it is desirable to find a *global section* for  $(M, X)$ , namely a compact, orientable hypersurface  $S$  without boundary such that

- $S$  is embedded in  $M$ ,
- $S$  is transverse to  $X$ ,
- every orbit of  $X$  intersects  $S$  after a bounded time.

When such a section exists, there is a well-defined first-return map on  $S$  and the first-return time is bounded from above by definition and from below by compactness. In this case the manifold  $M$  fibers over the circle with fiber  $S$ . The pair  $(M, X)$  is homeomorphic to  $(S \times [0, 1] / (p, 1) \sim (f(p), 0), \tau_p \frac{d}{dz})$ , where  $\tau_p$  is the first-return time on  $S$  and  $\frac{d}{dz}$  denotes the vector field tangent to the  $[0, 1]$ -coordinate. The dynamics of  $X$  is then, up to the time-reparametrisation function  $\tau$ , the dynamics of the first-return map  $f$  on  $S$ .

A standard argument shows that two global sections are isotopic if and only if they are homologous. Indeed the flow then realizes the isotopy between such homologous sections (see for example the discussion at the beginning of [Thu86, Section 3]). Therefore questions of existence and classification of global sections are of algebraic nature. Indeed, a necessary and sufficient condition for a given homology class  $\sigma$  in  $H_2(M; \mathbb{Z})$  to contain a global section has been described by S. Schwartzmann and F. Fuller: the set of *Schwarzmann asymptotic cycles* [Sch57] in  $H_1(M; \mathbb{R})$  has to lie in the half-space  $\{ \langle \sigma, \cdot \rangle > 0 \}$ , where  $\langle \cdot, \cdot \rangle$  denotes the algebraic intersection pairing

$H_2(M; \mathbb{R}) \times H_1(M; \mathbb{R}) \rightarrow \mathbb{R}$ . This implies for example that vector fields on  $\mathbb{S}^3$  never admit global sections. Further results of W. Thurston [Thu86] and D. Fried [Fri82] imply that in the case of a pseudo-Anosov flow, the set of homology classes of global sections is an open cone with finitely many extremal rays.

For  $\Sigma$  a Riemannian surface, the unit tangent bundle  $T^1\Sigma$  is the subset of  $T\Sigma$  of norm 1-vectors. It is a 3-manifold whose points are of the form  $(p, v)$  for  $p$  a point of  $\Sigma$  and  $v$  a tangent vector at  $p$  of norm 1. The geodesic flow  $\varphi_{\text{geod}}$  on  $T^1\Sigma$  is the vector field whose orbits are lifts of geodesics: for  $g$  an arbitrary geodesic of  $\Sigma$  travelled at speed 1, the orbit of  $\varphi_{\text{geod}}$  going through  $(g(0), \dot{g}(0))$  is given by  $\varphi_{\text{geod}}^t(g(0), \dot{g}(0)) = (g(t), \dot{g}(t))$ . The geodesic flow on a negatively curved surface has been studied since Hadamard who remarked its sensibility to initial condition [Had1898]. It even became the paradigm of 3-dimensional chaotic systems when Anosov showed its hyperbolic character [Ano67]. In general the geodesic flow depends heavily on the metric given on the surface. However Gromov remarked [Gro76] that the geodesic flows corresponding to any two negatively curved metrics on a surface are actually *topologically conjugated*, meaning that there is a homeomorphism of the tangent bundle sending the orbits the first on the orbit of the second. This is a consequence of the structural stability of Anosov flows. Therefore, as long as we are only interested in the topological properties of the orbits, one can speak of *the* geodesic flow on a negatively curved surface.

Since the antipodal map  $(p, v) \mapsto (p, -v)$  preserves the geodesic flow, its set of asymptotic cycles is symmetric with respect to the origin in  $H_1(T^1\Sigma; \mathbb{R})$ , so that geodesic flows on Riemannian surfaces do not admit global sections.

In order to make it useful, a relaxation of the notion of global section is desirable. For  $M$  a real compact, oriented 3-manifold and  $X$  a non-singular vector field on  $M$ , a *Birkhoff section* for  $(M, X)$  is compact orientable surface  $S$  with boundary such that

- $S$  is embedded in  $M$ ,
- the interior  $\text{int}(S)$  is transverse to  $X$ ,
- the boundary  $\partial S$  is tangent to  $X$ ,
- every orbit of  $X$  intersects  $S$  after a bounded time.

The third condition implies that the boundary of  $S$  is the union of finitely many periodic orbits of  $X$ . The second and third condition may look hard to realize at the same time, but actually it is not the case: in a flow box oriented so that the vector field is vertical, the general picture of a Birkhoff section near its boundary is that of a helicoidal staircase (see Figure 2). Since the interior of a Birkhoff section  $S$  is transverse to  $X$ , it is cooriented by  $X$ . Since  $M$  is oriented, this induces an orientation on  $S$ , and in turn an orientation of  $\partial S$ . On the other hand,  $\partial S$  is a collection of periodic orbits of  $X$ , so it is oriented by  $X$ . For every component of  $\partial S$ , these two orientations may coincide or be opposed (also on Figure 2). We say that  $S$  is a *positive* Birkhoff section if they coincide for every boundary component, *negative* if they are opposed (on the left), and *mixed* if they sometime agree and sometime disagree. If the fourth condition is not satisfied, namely if some orbits do not intersect the surface, we simply speak of a *transverse* surface.

It turns out that Birkhoff sections exist much more often than global sections. In particular H. Poincaré noticed that the geodesic flow on a sphere often admits an annulus as Birkhoff section. This remark was generalized by G. Birkhoff who gave a family of Birkhoff sections for the

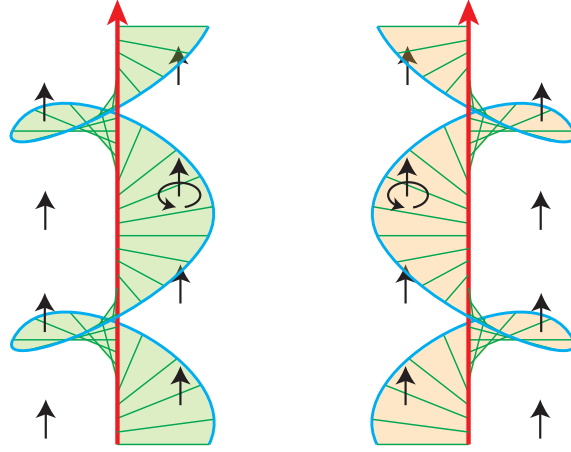


FIGURE 2. Two pieces of Birkhoff sections for the vertical flow around boundary components. The section on the left is negative while the section on the right is positive. It turns out that the geodesic flow admits no positive Birkhoff section.

geodesic flow [Bir17] (popularized in [Fri83]). Birkhoff's example was then given another presentation by M. Brunella [Bru94, Description 2]. Our first result is a generalization of Birkhoff's and Brunella's examples.

For  $\gamma$  an unoriented collection of geodesics on a surface  $\Sigma$ , we denote by  $\vec{\gamma}$  the *amphithetic lift* of  $\gamma$  in  $T^1\Sigma$ , that is, the set of unit tangent vectors based on  $\gamma$  and tangent to  $\gamma$ . The set  $\vec{\gamma}$  forms a link that is invariant by the involution  $(p, v) \mapsto (p, -v)$ . It is the union of  $2|\gamma|$  periodic orbits of  $\varphi_{\text{geod}}$ , each component being oriented by the flow.

**Theorem C.** *Let  $\Sigma$  be a compact oriented Riemannian surface and  $\gamma$  a finite collection of geodesics on  $\Sigma$ . There is canonical a map  $S^{BB}$  (for Birkhoff-Brunella) that associates to every Eulerian coorientation  $\nu$  of  $\gamma$  an oriented surface  $S^{BB}(\nu)$  in  $T^1\Sigma$  which is positively transverse to the geodesic flow and whose oriented boundary is  $-\vec{\gamma}$ . For every  $\nu$ , the Euler characteristic of  $S^{BB}(\nu)$  is minus twice the number of double points of  $\gamma$ .*

*If two Eulerian coorientations  $\nu_1, \nu_2$  of  $\gamma$  are cohomologous, then the associated surfaces  $S^{BB}(\nu_1)$  and  $S^{BB}(\nu_2)$  are isotopic (fixing their common boundary).*

The main interest of this new construction is that it actually gives a description of all isotopy classes of negative Birkhoff sections with boundary  $\vec{\gamma}$ , instead of one with the previously known constructions:

**Theorem D.** *Let  $\Sigma$  be a torus with a flat metric or a higher genus-surface with a negatively curved metric. Let  $\gamma$  be a finite collection of geodesics on  $\Sigma$ . Then the map  $[\nu] \mapsto \{S^{BB}(\nu)\}$  is a one-to-one correspondance between integer points in the closed unit ball  $B_{x_\gamma}^*$  congruent to  $[\gamma]_2 \pmod{2}$  and isotopy classes of surfaces in  $T^1\Sigma$  transverse to  $\varphi_{\text{geod}}$  with oriented boundary  $-\vec{\gamma}$ .*

*The restriction of this map to the integer points in the open unit ball  $\text{int}(B_{x_\gamma}^*)$  induces a one-to-one correspondance with isotopy classes of negative Birkhoff sections.*

Theorem D implies that the collection  $\overleftrightarrow{\gamma}$  bounds a negative Birkhoff section if and only if the polyhedron  $B_{x_\gamma}^*$  contains an integer point congruent to  $[\gamma]_2 \bmod 2$  in its interior. This is the case for most choices of  $\gamma$ , but not for all. For example, if there is a closed curve that intersects  $\gamma$  once or zero time, then  $\overleftrightarrow{\gamma}$  does not bound a Birkhoff section for  $\varphi_{\text{geod}}$ .

**Remark 1.** It may look strange to deal with negative Birkhoff sections and not with positive ones, *i.e.*, with surfaces such that the orientation of the boundary inherited from the orientation of the surface (itself inherited from the coorientation of the interior surface by the flow) is opposed to the direction of the flow. The reason is that there is actually no positive Birkhoff section for the geodesic flow. One could then look at *mixed* sections, namely transverse surfaces some of whose boundary components are positively tangent to  $\varphi_{\text{geod}}$  and some others are negatively transverse. It is likely that there are more mixed sections than negative. We do not have analogs of Theorems C and D in this more general case, namely we do not have any elementary way to construct them all.

**Remark 2.** A Birkhoff surface for the geodesic flow in  $T^1\Sigma$  bounded by  $\overleftrightarrow{\gamma}$  is a global section for the restriction of the flow to  $T^1\Sigma \setminus \overleftrightarrow{\gamma}$ . The assumption that the boundary is  $-\overleftrightarrow{\gamma}$  can be seen as a restriction on the homology class of the section: it has to lie in a certain affine subspace of  $H_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}; \mathbb{Z})$  (see Section 3.c). On the other hand, as explained before, the geodesic flow on  $T^1\Sigma$  for  $\Sigma$  a hyperbolic surface is of Anosov-type. Its restriction to  $T^1\Sigma \setminus \overleftrightarrow{\gamma}$  is then of pseudo-Anosov type, with singularities along the removed orbits. Thurston fibered faces Theory [Thu86, Section 3] then says that the homology classes of global sections to such a flow (and therefore of isotopy classes) is a cone in  $H_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}; \mathbb{R})$  whose extremal rays are directed by integral vectors. D. Fried [Fri82] gives an algorithm to explicitly compute these vectors, starting from a Markov partition of the flow. So one deduces directly that the set of negative Birkhoff sections is given by the intersection of a cone with an affine plane: it is a polyhedron. However, the determination of this polyhedron using Fried's approach requires an explicit Markov partition for the geodesic flow on  $T^1\Sigma \setminus \overleftrightarrow{\gamma}$ , which does not exist yet. So the interest of our paper lies in the elementary and explicit characters of all constructions.

**Remark 3.** Another possible interest of the paper is that it suggests that there may exist an object that would describe all Birkhoff sections for a given flow simultaneously (this role is played here by the intersection norm  $x_\gamma$ ), in the same spirit as Ghys proved [Ghy09] that Gauss linking forms describe all linking numbers between periodic orbits (and even invariant measures) for a vector field in a homology sphere.

**Plan of the article.** Section 1 presents the intersection norm and its elementary properties (Theorem A). Section 2 then presents Eulerian coorientations and the connection with the integer points in the dual unit ball (Theorem B). Section 3 then goes to dimension 3: we associate to every Eulerian coorientation a *BB-surface* in the unit tangent bundle, and then prove Theorems C and D.

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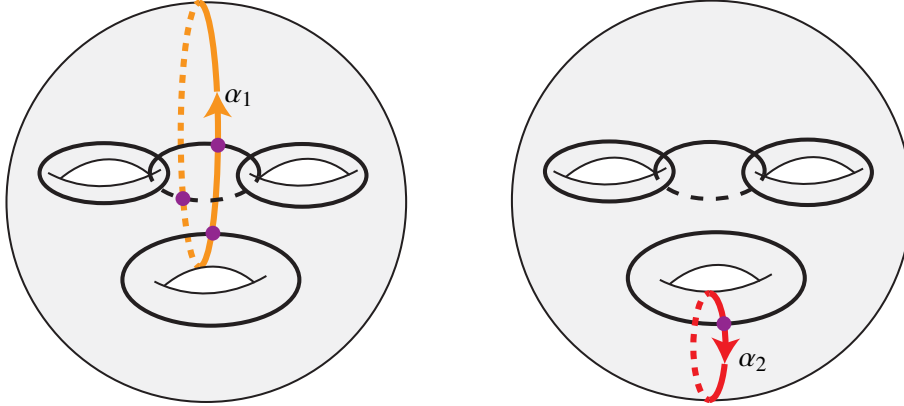


FIGURE 3. A genus 3 surface with a collection  $\gamma$  made of four closed curves (black). On the left the curve  $\alpha_1$  (orange and bold) is transverse to  $\gamma$  and intersects it three times. On the right  $\alpha_2$  (red) is homologous to  $\alpha_1$  since their difference bounds a subsurface, namely the right hemisphere. The curve  $\alpha_2$  intersects  $\gamma$  only once. This number cannot be reduced to 0 in the same homology class, hence  $x_\gamma([\alpha_1]) = x_\gamma([\alpha_2]) = i_\gamma(\alpha_2) = 1$ .

## 1. INTERSECTION NORMS

In this section we define intersection norms and prove Theorem A. All statements are transcriptions of results of Thurston [Thu86] in the 2-dimensional context of a surface with a multi-curve on it. Although the original 3-dimensional proofs are rather easy, their transcriptions are even more elementary.

For the whole section we fix a compact surface  $\Sigma$  of genus  $g$  with or without boundary. A *curve* on  $\Sigma$  is either an immersion of the circle in  $\Sigma$  or an immersion of the segment such that the extremities belong to  $\partial\Sigma$ . A *multi-curve* is a finite collection of closed curves that are mutually transverse. For the whole section we also fix a closed multi-curve  $\gamma$  on  $\Sigma$ .

Given another closed multi-curve  $\alpha$  transverse to  $\gamma$  and such that the multiple points of  $\alpha$  and  $\gamma$  are disjoint, there is a finite number of intersection points between  $\alpha$  and  $\gamma$ . What we do here is to minimize it over the homology class of  $\alpha$ :

**Definition 4.** (see Figure 3) The multi-curve  $\gamma$  being fixed on  $\Sigma$ , the function  $x_\gamma : H_1(\Sigma, \partial\Sigma; \mathbb{Z}) \rightarrow \mathbb{N}$  is defined by

$$x_\gamma(a) := \min_{[\alpha]=a} i_\gamma(\alpha) = \min_{\substack{[\alpha]=a \\ \alpha \pitchfork \gamma}} |\alpha \cap \gamma|.$$

Since the number of intersection points is an integer, the lower bound is always realized and  $x_\gamma$  takes integral values. A multi-curve that realizes the minimum is declared  *$x_\gamma$ -minimizing*.

The function  $x_\gamma$  has two properties that will turn it into a semi-norm, namely it is linear on rays and convex. For proving the first point we need an elementary remark. Let us recall that a multi-curve is *simple* if it has no double point, that is, if it is an embedding.

**Lemma 5** (simplification). *For every multi-curve  $\gamma$  in  $\Sigma$  and for every class  $a$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , there exists a  $x_\gamma$ -minimizing multi-curve in  $a$  that is simple.*



*Proof.* Starting from an arbitrary  $\alpha_0$  in  $a$  that is minimizing, we can smooth the double points of  $\alpha_0$  away from  $\gamma$



thus turning  $\alpha_0$  into a new multi-curve  $\alpha$  which is simple. The two multi-curves are not homotopic but they are homologous, hence the result.  $\square$

**Lemma 6** (linearity on rays). *For every  $a$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$  and for all  $n \in \mathbb{Z}$  one has*

$$x_\gamma(n \cdot a) = |n| x_\gamma(a).$$

*Proof.* Since one does not change the number of intersection points by reversing the orientation of a curve, one has  $x_\gamma(-a) = x_\gamma(a)$ .

We then assume  $n \geq 0$ . Given  $a \in H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , consider a minimizing multi-curve  $\alpha$  in  $a$ . Since  $n$  parallel copies of  $\alpha$  intersect  $\gamma$  in  $n x_\gamma(a)$  points, we have  $x_\gamma(n \cdot a) \leq n x_\gamma(a)$ .

For the other inequality, consider a multi-curve  $\alpha^{(n)}$  that minimizes  $x_\gamma(n \cdot a)$ . By the simplification Lemma, we can suppose  $\alpha^{(n)}$  simple. Since  $[\alpha^{(n)}]$  is a class that is a multiple of  $n$ , the class  $[\alpha^{(n)}]$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/n\mathbb{Z})$  is zero. This implies that the regions of the complement  $\Sigma \setminus \alpha^{(n)}$  can be colored by the numbers  $0, 1, \dots, n-1$  in such a way that the label increases by 1 mod  $n$  every time one crosses an arc of  $\alpha^{(n)}$  positively. Therefore  $\alpha^{(n)}$  is the union of  $n$  simple multi-curves: the one separating the regions labelled 0 from the regions labelled 1, the ones separating the 1 from the 2, etc. Since they pairwise bound a subsurface of  $\Sigma$ , all of these  $n$  multi-curves are homologous, hence they all belong to the class  $a$ . If one of them had an intersection smaller than  $x_\gamma(a)$  with  $\gamma$ , then we could replace all the others by a parallel copy of it, thus contradicting the minimality of  $\alpha^{(n)}$ . Hence each of these  $n$  multi-curves cuts  $\gamma$  in  $x_\gamma(a)$  points, and therefore we have  $x_\gamma(n \cdot a) = n x_\gamma(a)$ .  $\square$

**Lemma 7** (convexity). *For every  $a, b$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$  one has*

$$x_\gamma(a + b) \leq x_\gamma(a) + x_\gamma(b).$$

*Proof.* The union of two multi-curves that realize  $x_\gamma(a)$  and  $x_\gamma(b)$  crosses  $\gamma$  in  $x_\gamma(a) + x_\gamma(b)$  points, giving  $x_\gamma(a + b) \leq x_\gamma(a) + x_\gamma(b)$ .  $\square$

*Proof of Theorem A.* Every class in  $H_1(\Sigma, \partial\Sigma; \mathbb{Q})$  is of the form  $\frac{1}{q} a$  with  $a \in H_1(\Sigma, \partial\Sigma; \mathbb{Z})$  and  $q \in \mathbb{N}^*$ . We then define  $x_\gamma(\frac{1}{q} a)$  as  $\frac{1}{q} x_\gamma(a)$ , and the linearity on rays (Lemma 6) ensures that this definition does not depend on the choice of  $q$  and  $a$  and that it yields a well-defined function (also denoted by  $x_\gamma$ ) from  $H_1(\Sigma, \partial\Sigma; \mathbb{Q})$  to  $\mathbb{Q}_+$  that is linear on rays. Now convexity (Lemma 7) implies that this function is continuous, so it can be extended into a continuous function (always denoted by  $x_\gamma$ ) from  $H_1(\Sigma, \partial\Sigma; \mathbb{R})$  to  $\mathbb{R}_+$ . Since it is linear on rays and continuous, the function  $x_\gamma$  is a semi-norm on  $H_1(\Sigma, \partial\Sigma; \mathbb{R})$ .

If the collection  $\gamma$  decomposes  $\Sigma$  into simply-connected regions, then  $\gamma$  intersects every curve that is not null-homotopic at least once. This implies that  $x_\gamma$  is at least 1 on non-zero integral homology classes, hence  $x_\gamma$  is positive on  $H_1(\Sigma, \partial\Sigma; \mathbb{R}) \setminus \{0\}$ . Therefore  $x_\gamma$  is a norm.  $\square$

**Remark 8.** When  $\Sigma$  has some boundary, we can also define a similar norm  $x'_\gamma$  on  $H_1(\Sigma; \mathbb{R})$ , thus forgetting the classes represented by arcs that connect the boundary components but considering

the curves parallel to the boundary. Since all our examples are actually in the case of an empty boundary, we have no opinion on which norm is more interesting in the non-empty case.

**Remark 9.** One can wonder how the intersection norms compare with other known norms on the first homology of a surface. For example the *stable norm*  $x_g$  is defined in terms of a metric  $g$  by  $x_g(a) = \liminf_{n \rightarrow \infty} \min_{\alpha^{(n)} \in a^n} \frac{g(\alpha^{(n)})}{n}$ . When  $g$  is negatively curved, the stabilisation is not necessary, so that  $x_g(a) = \min_{\alpha \in a} g(\alpha)$ . One can check that if  $(\gamma_k)_{k \in \mathbb{N}}$  is a sequence of filling geodesics for  $g$ , meaning that the sequence of invariant measures on  $T^1\Sigma$  that are concentrated on the lift  $\tilde{\gamma}_k$  tend in the weak sense to the Liouville measure defined by  $g$  on  $T^1\Sigma$ , then the rescaled norms  $\frac{1}{g(\gamma_k)} x_{\gamma_k}$  tend to the stable norm of  $g$ . Equivalently, the rescaled unit balls  $g(\gamma_k) B_{x_{\gamma_k}}$  tend to the unit ball of the stable norm.

## 2. UNIT BALLS AND COORIENTATIONS

For the whole section we fix a surface  $\Sigma$  of genus at least 1 and a multi-curve  $\gamma$  on it. The norm  $x_\gamma$  defined in the previous section has a very peculiar property: it takes integral values on integral classes. This property is shared for example by the  $\ell_1$ - and  $\ell_\infty$ -norms on  $\mathbb{R}^d$  for which one remarks that the unit ball is polyhedral. Moreover all faces of these unit balls are of the form  $\{(x_1, \dots, x_d) \mid \sum x_i y_i = 1\}$  for some  $(y_1, \dots, y_d) \in \mathbb{Z}^d$ . This is not a coincidence as was remarked by Thurston.

**Theorem 10** (Thm 2 of [Thu86]). *If  $N$  is a seminorm on  $\mathbb{R}^d$  taking integral values on  $\mathbb{Z}^d$ , then there is a finite subset  $F$  of  $\mathbb{Z}^d$  such that  $N(x) = \max_{y \in F} \langle x, y \rangle$  for all  $x$  in  $\mathbb{R}^d$ .*

Let us recall that a norm  $N$  on a vector space induces a dual norm  $N^*$  on the dual by  $N^*(y) = \max_{x \in B} \langle x, y \rangle$  where  $B$  denotes the unit ball of  $N$ . Thurston's result can be restated by saying that the unit ball of the dual norm is the convex hull of finitely many integral points.

In our context, denote by  $x_\gamma^*$  the norm on  $H_1(\Sigma, \partial\Sigma; \mathbb{R})^* \simeq H^1(\Sigma; \mathbb{R})$  dual to  $x_\gamma$ , by  $B_{x_\gamma}$  the unit ball of  $x_\gamma$ , and by  $B_{x_\gamma}^*$  the unit ball of  $x_\gamma^*$ . A direct consequence of Theorem 10 is

**Corollary 11** (see Figure 4). *For  $\Sigma$  a compact surface and  $\gamma$  a multi-curve on it, the unit ball  $B_{x_\gamma}^*$  is the convex hull in  $H^1(\Sigma; \mathbb{R})$  of finitely many points in  $H^1(\Sigma; \mathbb{Z})$ .*

A natural question is whether the vertices of  $B_{x_\gamma}^*$  (or equivalently the faces of  $B_{x_\gamma}$ ) have a nice interpretation. For example in the context of the Thurston norm, the vertices correspond to the Euler classes of certain taut foliations (Gabai), or of certain vector fields [Fri79, Mos92, Cal06]. Here we also have such an interpretation in terms of Eulerian coorientations (Theorem B), and it is the goal of this section to prove it.

**2.a. Coorientations and signed intersections.** Recall that the multi-curve  $\gamma$  is assumed to be immersed with only double points. We denote by  $V(\gamma)$  the set of double points, that we call *vertices* of  $\gamma$ . Consequently we denote by  $E(\gamma)$  the set of connected components of  $\gamma \setminus V(\gamma)$ , that we call *edges* of  $\gamma$ . This turns  $\gamma$  into a graph of degree 4 embedded in  $\Sigma$ .

**Definition 12.** For  $e$  an edge of  $\gamma$ , a *coorientation on  $e$*  is the choice of one of the two possible ways of crossing  $e$ : from left to right, or from right to left. A *partial coorientation on  $\gamma$*  is the choice of a coorientation on a subset of  $E(\gamma)$ . The *support* of a partial coorientation is the set of edges that

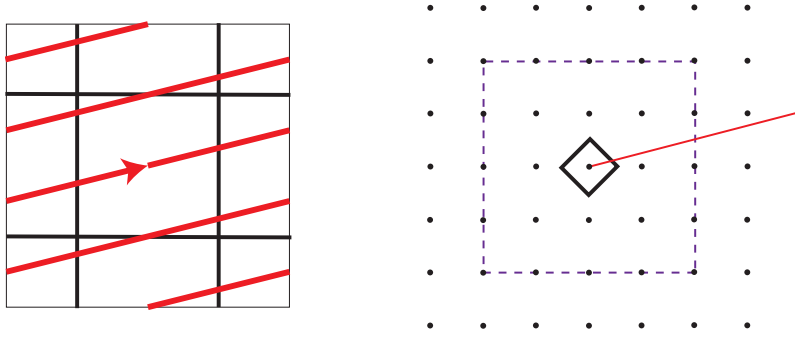


FIGURE 4. A torus with a collection  $\gamma$  (black) made of four curves, two vertical and two horizontal. The curve  $\alpha$  (red and bold) intersects  $\gamma$  in 10 points. It is the best for a curve whose homology class is  $(4, 1)$ . The norm  $x_\gamma$  is actually given by  $x_\gamma(p, q) = 2|p| + 2|q|$  in the canonical coordinates. The unit balls  $B_{x_\gamma}$  (bold) and  $B_{x_\gamma}^*$  (dotted) are shown on the right. The faces of  $B_{x_\gamma}$  are defined by integral equations while the vertices of  $B_{x_\gamma}^*$  belong to  $\mathbb{Z}^2$ , as predicted by Thurston's result.

are cooriented. A *global coorientation on  $\gamma$*  (or *coorientation* for short) is a coorientation whose support is all of  $E(\gamma)$ .

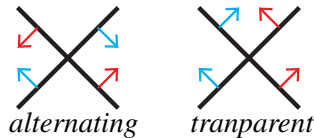
There are  $2^{|E(\gamma)|}$  global coorientations and  $3^{|E(\gamma)|}$  partial coorientations.

A partial coorientation  $\nu$  may be evaluated on an oriented immersed curve  $\alpha$  transverse to  $\gamma$ : one counts  $+1$  for every intersection point of  $\alpha$  with the support of  $\nu$  if the orientation of  $\alpha$  coincides with the coorientation of the edge, and  $-1$  if the orientations disagree. Denoting by  $\nu(\alpha)$  this intersection pairing, one sees that  $\nu(\alpha)$  is an integer satisfying  $|\nu(\alpha)| \leq |\{\alpha \cap \gamma\}|$ .

**2.b. Eulerian coorientations.** The question now is whether the above inequality may be turned into an equality for  $x_\gamma$ -minimizing curves on the one hand, and whether  $\nu(\alpha)$  may depend only on the homology class of  $\alpha$  so that one can compute  $\nu$  on a single representative. Both questions admit a positive answer if we restrict to some special coorientations, called Eulerian.

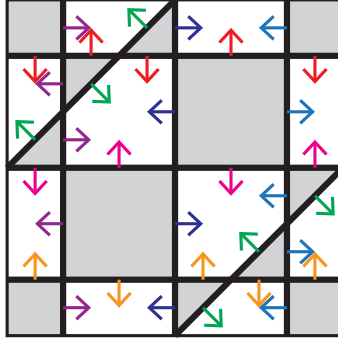
**Definition 13.** A partial coorientation on  $\gamma$  is *Eulerian* (or *closed*) if it vanishes on boundaries, that is, if for every region  $D \subset \Sigma$  whose boundary is transverse to  $\gamma$  one has  $\nu(\partial D) = 0$ . The set of all partial Eulerian coorientations is denoted by  $\mathcal{EulCo}(\gamma)$ .

The set  $\mathcal{EulCo}(\gamma)$  is an affine subspace of  $\{-, 0, +\}^{E(\gamma)}$ . Actually the closing condition is local: for  $\nu$  to be Eulerian it is enough that around every vertex of  $\gamma$  there are as many positively cooriented edges than negatively cooriented. Hence, for a global coorientation, up to rotation, there are only two types local Eulerian coorientations:



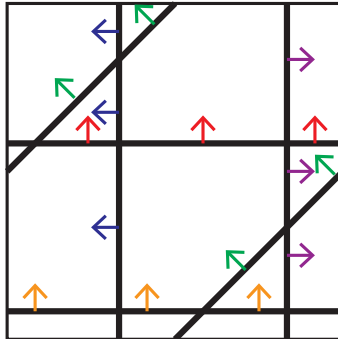
When one travels straight along  $\gamma$  and encounters a vertex of the first type the coorientation changes, hence the name. For the second type on the other hand, it is as if the coorientation does not see the vertex.

**Example 14.** If  $[\gamma]_2 \in H(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$  is zero (meaning that every closed curve intersects  $\gamma$  an even number of times), then the regions of  $\Sigma \setminus \gamma$  can be colored in black and white in such a way that adjacent regions have different colors. In this case we can coorient all edges toward the white regions. The obtained global coorientation is Eulerian:



We call it a *Birkhoff coorientation*, since the surface we will construct from it is isotopic to the historical construction of [Bir17].

**Example 15.** There always exist global Eulerian coorientations, even when  $[\gamma]_2 \in H^1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$  is non-zero. Indeed if the multi-curve  $\gamma$  is the immersion of  $c$  curves, it admits at least  $2^c$  Eulerian coorientations, obtained by choosing a coorientation for each component and having only transparent vertices:



We call such coorientations *Brunella coorientation*, since the surfaces we will construct from it correspond to those of [Bru94, Description 2]<sup>2</sup>.

**Lemma 16.** *If  $\nu$  is a Eulerian coorientation of  $\gamma$ , then for every multi-curve  $\alpha$ , the pairing  $\nu(\alpha)$  depends only of the homology class  $[\alpha] \in H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ .*

*Proof.* If two multi-curves  $\alpha, \alpha'$  are homologous, then their difference bounds a singular subsurface in  $\Sigma$ . Definition 13 implies that the pairing of the boundary of the image of a surface with a Eulerian coorientation is zero. Hence  $\nu(\alpha - \alpha') = 0$ , so  $\nu(\alpha) = \nu(\alpha')$ .  $\square$

Lemma 16 states that every Eulerian coorientation  $\nu$  induces a well-defined cohomology class  $[\nu]$  in  $H^1(\Sigma; \mathbb{Z})$ . We denote by  $[\mathcal{EulCo}(\gamma)]$  the subset of  $H^1(\Sigma; \mathbb{Z})$  formed by the classes of partial Eulerian coorientations on  $\gamma$ . Note the class of a Eulerian coorientation is easily computed since it is

<sup>2</sup>The starting point of this paper is the remark by E. Kudryavtseva that these surfaces are in general not isotopic to the ones of Birkhoff.

enough to evaluate its pairing with  $2g$  curves that generate the homology of  $\Sigma$ . Moreover Eulerian coorientations give lower bounds on  $x_\gamma$ :

**Lemma 17.** *For every partial Eulerian coorientation  $\nu$  of  $\gamma$  and for every  $a$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , we have  $x_\gamma(a) \geq \nu(a)$ .*

*Proof.* Let  $\alpha$  be a curve in  $a$  that realizes  $x_\gamma(a)$ . Then  $\nu(\alpha)$  counts every intersection point of  $\alpha$  and  $\gamma$  with a coefficient  $-1$ ,  $0$ , or  $+1$ , while  $x_\gamma(a)$  counts all these intersection points with a coefficient  $+1$ , hence the inequality.  $\square$

Lemma 17 is equivalent to the inclusion  $[\mathcal{EulCo}(\gamma)] \subset B_{x_\gamma}^* \cap H^1(\Sigma; \mathbb{Z})$ . The main content of Theorem B is the reverse inclusion. The proof goes in two steps: first we prove that the extremal points of  $B_{x_\gamma}^*$  are given by some Eulerian coorientations (Proposition 18), then we prove that every integer point inside  $B_{x_\gamma}^*$  (satisfying the parity condition) is also given by such a coorientation (Proposition 22).

### 2.c. Completing a coorientation.

**Proposition 18.** *For every  $a$  in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , there exists a global Eulerian coorientation  $\nu_a$  of  $\gamma$  such that  $x_\gamma(a) = \nu_a(a)$ .*

The scheme of the proof is as follows. Let  $\alpha$  be a minimising curve  $\alpha$  in  $a$ . Since we have  $\nu(\alpha) = \nu(a) \leq x_\gamma(a)$  for every curve and  $x_\gamma(a) = i_\gamma(\alpha)$  since  $\alpha$  is  $x_\gamma$ -minimizing, we see that one must have  $\nu(\alpha) = i_\gamma(\alpha)$ , namely all intersection points of  $\alpha$  with  $\gamma$  have to be counted positively. This defines a partial coorientation  $\nu_\alpha$  on all edges of  $\gamma$  that are intersected by  $\alpha$ . What we have to do is to prove that one can complete such a partial coorientation into a global Eulerian coorientation.

Note that if we rotate a coorientation of  $\gamma$  by an angle  $\pi/2$ , we obtain an orientation of  $\gamma$ , and this operation turns Eulerian coorientations into Eulerian orientations. Therefore our statement is about whether a certain partial orientation can be completed into a global Eulerian orientation. This is similar to classical problems in graph theory which can be solved using max-flow min-cut and Menger Theorems (see [Hof60] and [Sch03, Chapter 11]<sup>3</sup>). For completeness, we give a self-contained proof that does not use more technology than needed.

For  $\beta$  a multi-curve transverse to  $\gamma$  and  $\nu$  a partial coorientation, we denote by  $\text{indet}_\nu(\beta)$  the number of intersection points of  $\beta$  with  $\gamma$  belonging to edges of  $\gamma$  where  $\nu$  is not defined. We always have  $-|\{\beta \cap \gamma\}| \leq \nu(\beta) - \text{indet}_\nu(\beta) \leq \nu(\beta) + \text{indet}_\nu(\beta) \leq |\{\beta \cap \gamma\}|$  since the two central terms correspond to put  $\pm$ -signs on the intersection points of  $\beta$  and  $\gamma$ .

**Definition 19.** A partial coorientation  $\nu$  is *pre-Eulerian* if for every boundary  $\partial D$  of a subsurface one has  $-\text{indet}(\partial D) \leq \nu(\partial D) \leq \text{indet}(\partial D)$ .

Not being pre-Eulerian is an obvious obstruction to being extendable into a Eulerian coorientation. Indeed in this case there are not enough signs left to decide in order to cancel the already defined coorientation on curves that are boundaries.

**Lemma 20** (extension). *If  $\nu$  is a pre-Eulerian coorientation, then it can be extended into a global Eulerian coorientation.*

<sup>3</sup>I thank Arnaud de Mesmay for pointing these references to me.

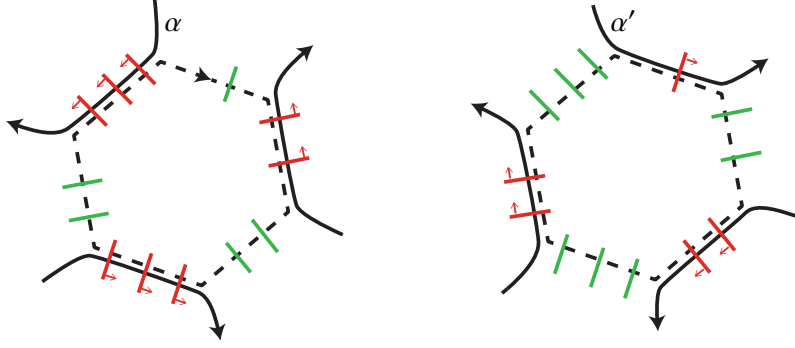


FIGURE 5. Proof of Lemma 21. On the left a curve  $\alpha$  that induces a partial coorientation  $\nu$  and a hexagon  $D$  such that  $|\nu_\alpha(\partial D)| > \text{indet}_{\nu_\alpha}(\partial D)$ . Replacing the arcs of  $\alpha \cap \partial D$  by  $\partial D \setminus \alpha$  we decrease the intersection of  $\alpha$  with  $\gamma$  (from 8 to 5 here).

For  $\alpha$  a minimizing multi-curve, we notice that  $\alpha$  may not intersect an edge of  $\gamma$  in both possible directions. If it were the case, then we could replace  $\alpha$  with a homologous multi-curve that would intersect  $\gamma$  in two points less. Therefore  $\alpha$  induces a partial coorientation of  $\gamma$  denoted by  $\nu_\alpha$  whose support in the set of edges intersected by  $\alpha$ .

**Lemma 21** (minimal implies pre-Eulerian). *If  $\alpha$  is a minimizing multi-curve, then the partial coorientation  $\nu_\alpha$  is pre-Eulerian.*

Before proving Lemmas 20 and 21, let us see how they imply Proposition 18.

*Proof of Proposition 18.* Let  $\alpha$  be a minimizing curve and  $\nu_\alpha$  the partial coorientation that it defines. By Lemma 21,  $\nu_\alpha$  is pre-Eulerian, so, by Lemma 20, it can be extended into a global Eulerian coorientation  $\bar{\nu}_\alpha$ . We then have  $\bar{\nu}_\alpha(\alpha) = \nu_\alpha(\alpha) = i_\gamma(\alpha) = x_\gamma(\alpha)$ , so  $\bar{\nu}_\alpha$  indeed gives the intersection norm of  $\alpha$ .  $\square$

*Proof of Lemma 21.* Suppose that  $\nu_\alpha$  is not pre-Eulerian. Then there exist a subsurface  $D$  such that  $|\nu_\alpha(\partial D)| > \text{indet}_{\nu_\alpha}(\partial D)$ . Since  $\nu_\alpha$  is only defined on edges of  $\gamma$  that intersect  $\alpha$ , this means that  $\partial D$  and  $\alpha$  intersect some common edges of  $\gamma$ . Consider the curve  $\alpha'$  obtained by suppressing the arcs of  $\alpha$  that cut the segments of  $\gamma$  also cut by  $\partial D$  and by adding the arcs of  $\partial D$  that cut some edges of  $\gamma$  not cut by  $\alpha$  (see Figure 5). Concerning homology classes we have  $[\alpha'] = [\alpha] - [\partial D] = [\alpha]$ . The assumption  $|\nu_\alpha(\partial D)| > \text{indet}_{\nu_\alpha}(\partial D)$  implies that  $\alpha'$  cuts  $\gamma$  in less points than  $\alpha$ , which contradicts the minimality of  $\alpha$ .  $\square$

*Proof of Lemma 20.* Let us show the result by induction on the number of edges of  $\gamma$  whose coorientation is undefined. If this number is zero, then all edges are cooriented, and being pre-Eulerian then means being Eulerian.

Now, if for every subsurface  $D$  we have  $\text{indet}_{\nu}(\partial D) > |\nu(\partial D)|$ , then we can choose an arbitrary edge on  $\partial D$  and fix arbitrarily its coorientation. The obtained coorientation is still pre-Eulerian.

The key-step is if there exists a subsurface  $D$  of  $\Sigma$  such that  $\text{indet}_{\nu}(\partial D) = |\nu(\partial D)|$ . At the expense of exchanging  $D$  and  $\Sigma \setminus D$ , that is to change the orientation of  $\partial D$ , we can suppose  $\nu(\partial D) > 0$ . Then there exists a unique extension  $\bar{\nu}$  of  $\nu$  to the segments that cut  $\partial D$  such that  $\bar{\nu}(\partial D) = 0$ . Let us show that  $\bar{\nu}$  is still pre-Eulerian. If it is not the case, then there exists  $D'$  such that  $\text{indet}_{\bar{\nu}}(\partial D') < |\bar{\nu}(\partial D')|$ . We will deduce a contradiction by finding a curve that testifies that  $\nu$  is actually not pre-Eulerian.

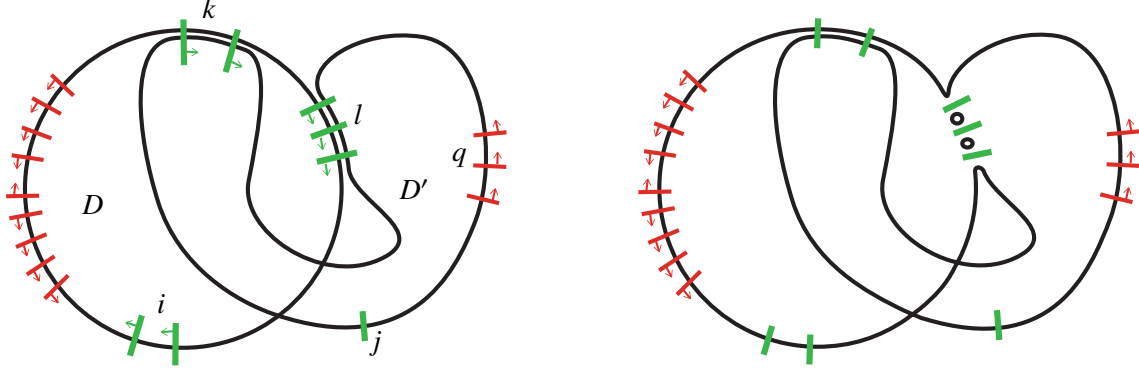


FIGURE 6. Proof of Lemma 20. On the left the curves  $\partial D$  and  $\partial D'$ . The numbers  $p$  and  $q$  correspond to the partial coorientation  $\nu$  (red), while  $i, j, k, l$  count the number of edges of  $\gamma$  not cooriented by  $\nu$  (green). The hypothesis  $p = i + j + k$  implies that there is a unique coorientation  $\bar{\nu}$  that colorient the  $i + k + l$  edges cut by  $\partial D$  so that  $\bar{\nu}(\partial D) = 0$ . If  $\partial D'$  is such that  $\text{indet}_{\bar{\nu}}(\partial D') < \bar{\nu}(\partial D')$ , we find a curve  $\partial D''$  (on the right) for which we have  $\text{indet}_{\nu}(\partial D'') < \nu(\partial D'')$ , a contradiction.

Since  $\nu$  is pre-Eulerian, we have  $\text{indet}_{\nu}(\partial D') \geq |\nu(\partial D')|$ . The comparison of the two previous inequalities shows that  $\partial D'$  cuts some edges whose coorientation has just been defined. For the time being, we have no orientation on  $\partial D'$ , but we will suppose we have one: this choice will be made later. Let us introduce some notations (see Figure 6): write  $p = \nu(\partial D) > 0$  and  $q = \nu(\partial D') \in \mathbb{Z}$ . Write  $i$  for the number of intersection points of  $\partial D$  with edges of  $\gamma$  that are not cut by  $\partial D'$ , and  $j$  for the number of intersection points of  $\partial D'$  with edges of  $\gamma$  that are not cut by  $\partial D$ . Finally write  $k$  for the number of intersection points of  $\partial D$  with edges of  $\gamma$  that are also cut by  $\partial D'$  and such that the orientations of  $\partial D$  and  $\partial D'$  coincide, and write  $l$  for the number of such points such that the orientations are opposite. Since changing the orientation to  $\partial D'$  amounts to exchange  $k$  and  $l$ , and to change  $q$  into  $-q$ , we can chose the orientation of  $\partial D'$  in such a way that the number  $q + l - k$  is nonnegative. The assumption  $\text{indet}_{\nu}(\partial D) = \nu(\partial D)$  translates into  $i + k + l = p$  and the assumption (which will lead to a contradiction)  $\text{indet}_{\bar{\nu}}(\partial D') < |\nu(\partial D')|$  into  $j < |q + l - k| = q + l - k$  by our choice of orientation of  $\partial D'$ .

Consider now the region  $D''$  obtained as the union  $D \cup D'$ , plus  $l$  connections around the  $l$  edges of  $\gamma$  cut by  $\partial D$  and  $\partial D'$  with opposite orientations. Its boundary  $\partial D''$  is a curve that travels along  $\partial D$  and  $\partial D'$  alternatively. In particular the intersection points of  $\partial D''$  with  $\gamma$  are those of  $\partial D \cap \gamma$  and those of  $\partial D' \cap \gamma$ , except the  $2l$  points that correspond to edges of  $\gamma$  where the orientations of  $\partial D$  and  $\partial D'$  are opposite. Thus we have

$$\text{indet}_{\nu}(\partial D'') = i + j + 2k = (p - k - l) + j + 2k = p + k - l + j < p + k - l + (q + l - k) = p + q = \nu(\partial D'').$$

This is a contradiction since  $\nu$  is pre-Eulerian.  $\square$

## 2.d. Averaging coorientations.

**Proposition 22.** *Every point in  $B_{x_\gamma}^* \cap H^1(\Sigma; \mathbb{Z})$  that is congruent to  $[\gamma]_2 \pmod{2}$  is the class of a global Eulerian coorientation.*

*Proof.* It is enough to prove that if  $\nu_1, \nu_2$  are two Eulerian coorientations such that  $n_{p/q} := \frac{p}{q}[\nu_1] + (1 - \frac{p}{q})[\nu_2]$  is a point in  $H^1(\Sigma; \mathbb{Z})$  congruent to  $[\gamma]_2 \pmod{2}$ , then there is a Eulerian coorientation whose class is  $n_{p/q}$ . We will define such a coorientation  $\nu$ . First consider the edges of  $\gamma$  where  $\nu_1$  and  $\nu_2$  agree and define  $\nu$  in the same way. Now consider  $\gamma_{p/q}$  the graph obtained by removing these edges from  $\gamma$ . Since  $[\nu_1] - n_{p/q}$  and  $[\nu_2] - n_{p/q}$  are integer classes multiple of 2, we deduce that  $[\nu_1] - [\nu_2]$  is actually an integer class that is a multiple of  $2q$ . Therefore the coorientation  $\nu_1$  on the graph  $\gamma_{p/q}$  induces a class multiple of  $q$ . This implies that  $\gamma_{p/q}$  can be decomposed as the union of  $q$  cooriented graphs, all cohomologous. We then just chose  $p$  of them and define  $\nu$  like  $\nu_1$  on them, and define  $\nu$  like  $\nu_2$  on the  $q - p$  others.  $\square$

Actually one can derive from the proof that every point in  $B_{x_\gamma}^* \cap H^1(\Sigma; \mathbb{Z})$  is the class of some *partial* Eulerian coorientation.

*Proof of Theorem B.* By Lemma 17 and Proposition 18, the convex ball  $B_{x_\gamma}^*$  is the convex hull of the set  $[\text{EulCo}(\gamma)]$ , and, by Proposition 22, all interior points of  $B_{x_\gamma}^*$  congruent to  $[\gamma]_2 \pmod{2}$  correspond to some Eulerian coorientation.  $\square$

### 3. BIRKHOFF SECTIONS WITH ANTITHETIC BOUNDARY FOR THE GEODESIC FLOW

In this part, we make an additional assumption: now  $\Sigma$  denotes a Riemannian surface that may be a torus with constant curvature or a higher-genus surface with strictly negative curvature. The collection  $\gamma$  now consists of finitely many periodic geodesics on  $\Sigma$ .

In this setting, the geodesic flow  $(\varphi_{\text{geod}}^t)_{t \in \mathbb{R}}$  on the unit tangent bundle  $T^1\Sigma$  is the flow whose orbits are lifts of geodesics. Namely for  $g$  a geodesic parametrized at speed one, the orbit of  $\varphi_{\text{geod}}$  going through the point  $(g(0), \dot{g}(0)) \in T^1\Sigma$  is  $\varphi_{\text{geod}}^t((g(0), \dot{g}(0))) = (g(t), \dot{g}(t))$ . For every oriented periodic geodesic  $g$  on  $\Sigma$ , there is one periodic orbit of  $\varphi_{\text{geod}}$  corresponding to the oriented lift of  $g$  and denoted by  $\vec{g}$ . Then if  $g$  now denotes an unoriented geodesic on  $\Sigma$ , there are two associated periodic orbits of  $\varphi_{\text{geod}}$ , one for each orientation. We denote by  $\overleftrightarrow{g}$  the union of these two periodic orbits, it is an oriented link in  $T^1\Sigma$  that is invariant under the involution  $(p, v) \mapsto (p, -v)$ . A link of the form  $\overleftrightarrow{g}_1 \cup \dots \cup \overleftrightarrow{g}_k$  is called an *antithetic link*.

Let us recall from the introduction that, given a complete flow  $(\phi^t)_{t \in \mathbb{R}}$ , a compact surface  $S$  with boundary is *transverse* to  $\phi^t$  if its interior is transverse to the orbits of the flow and its boundary is the union of finitely many periodic orbits<sup>4</sup>. A *Birkhoff section* for  $\phi^t$  is then a transverse surface  $S$  that intersects every orbit of  $\phi^t$ . A small analysis and a compactness argument show that around the boundary  $S$  necessarily looks like a helix (see Figure 2), so that the first-return time on  $\text{int}(S)$  is bounded.

In this section, we give a construction that associates to every Eulerian coorientation a surface transverse to the geodesic flow (3.a). Then we recall some facts on the existence of global sections for vector fields (3.b), before making some elementary algebraic topology for describing homology classes of surfaces with boundary (3.c). Finally we put pieces together to prove that the construction actually exhaust all possible surfaces, thus proving Theorems C and D (3.d).

<sup>4</sup>Often in the literature a transverse surface is only defined locally. The condition we add here on the boundary is not standard. However we keep the name for avoiding a heavier expression.



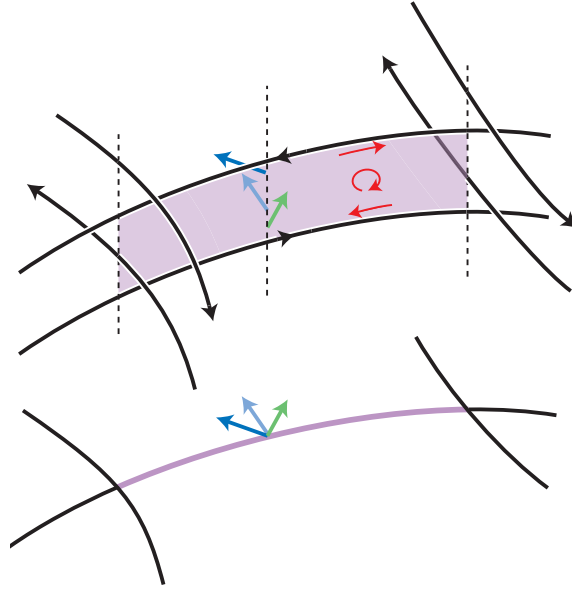


FIGURE 7. Bottom: an edge  $e$  of  $\gamma$  and a coorientation  $\nu$  on it. Top: the corresponding rectangle  $R^{e,\nu}$  in  $T^1\Sigma$ . The dotted lines represent the fibers of some points of  $\Sigma$ , that is, each point on these lines represent a unit tangent vector to  $\Sigma$ . Since the fibers are actually circles, the top and bottom extremities of the dotted lines should be glued.  $R^{e,\nu}$  is transverse to  $\varphi_{\text{geod}}$  and the induced coorientation is shown in red. The induced orientation of the horizontal boundary of  $R^{e,\nu}$  (red) is opposed to the orientation of the flow (black). Thus the surfaces we will construct are negative Birkhoff sections.

**3.a. Constructions of Birkhoff sections with antithetic boundary.** We now explain how to associate to every Eulerian coorientation of  $\gamma$  a surface bounded by  $\overleftrightarrow{\gamma}$  and transverse to  $\varphi_{\text{geod}}$ , thus proving the first part of Theorem C.

From now on we fix a global coorientation  $\nu$  (not yet Eulerian) of  $\gamma$ . For every edge  $e$  of  $\gamma$  (*i.e.* segment between two double points), we consider the set  $R^{e,\nu}$  of those tangent vectors based on  $e$  and pairing positively with  $\nu$ . This is an rectangle in  $T^1\Sigma$  of the form  $e \times [-\pi, \pi]$  (see Figure 7). It is bounded by the two lifts of  $e$  in  $T^1\Sigma$  (called the *horizontal part* of  $\partial R^{e,\nu}$ ) and two halves of the fibers of the extremities of  $e$  (called the *vertical part* of  $\partial R^{e,\nu}$ ). Note the interior of  $R^{e,\nu}$  is transverse to the geodesic flow  $\varphi_{\text{geod}}$  while the horizontal part of  $\partial R^{e,\nu}$  is tangent to it. We then orient  $R^{e,\nu}$  so that  $\varphi_{\text{geod}}$  intersects it positively. One checks that then the induced orientation on  $\partial R^{e,\nu}$  is opposite to the one given by  $\varphi_{\text{geod}}$ .

Consider now the 2-dimensional complex  $S^\times(\nu)$  that is the union of the rectangles  $R^{e,\nu}$  for all edges  $e$  of  $\gamma$ .

**Lemma 23.** *The 2-complex  $S^\times(\nu)$  described above has boundary  $-\overleftrightarrow{\gamma}$  if and only if the coorientation  $\nu$  is Eulerian.*

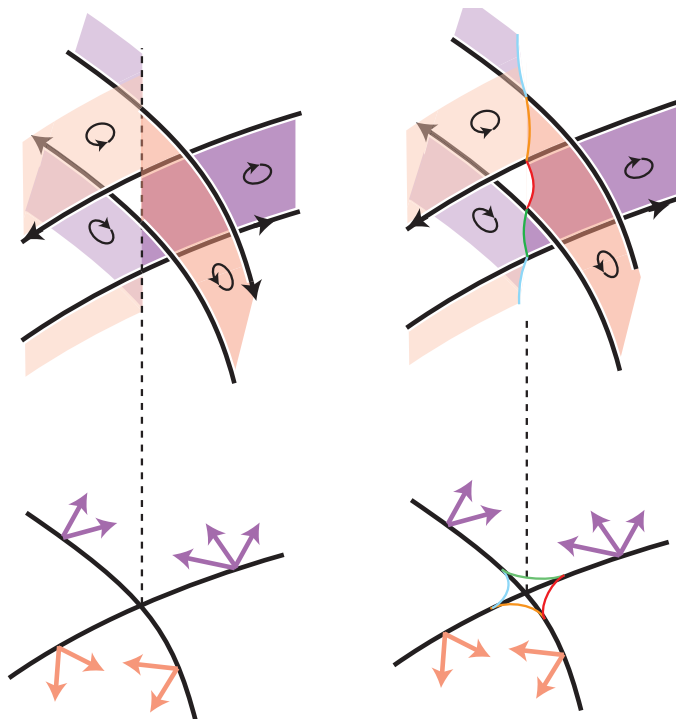


FIGURE 8. On the left, the complex  $S^{\times}(\nu)$  around the fiber of an alternating double point of  $\gamma$ . Every point of the fiber of  $\nu$  is adjacent to exactly two rectangles. On the right the surface  $S^{BB}(\nu)$  is obtained by smoothing  $S^{\times}(\nu)$ .

*Proof.* Since  $S^{\times}(\nu)$  is the union of one rectangle per edge of  $\gamma$ , the horizontal boundary of  $S^{\times}(\nu)$  is always  $\vec{\gamma}$ . Since the orientation is opposite to the geodesic flow, it is actually  $-\vec{\gamma}$ .

What we have to check is that the vertical boundary is empty if and only if  $\nu$  is Eulerian. At every double point  $\nu$  of  $\gamma$  there are four incident rectangles, corresponding to the four adjacent edges. Now the vertical boundary of a rectangle  $R^{e,\nu}$  is oriented upwards (that is, trigonometrically) at the right extremity of  $e$  (when cooriented by  $\nu$ ) and downwards at the left extremity. Then the vertical boundary in a vertex of  $\gamma$  is empty if only if two adjacent edges are cooriented in a direction, and two others in the opposite direction: this means that  $\nu$  is Eulerian around  $\nu$ . Conversely, if  $\nu$  is Eulerian, then up to rotation there are two local configurations around  $\nu$  (that we called alternating and transparent), and one checks that in both cases, the vertical boundary is empty (see the left parts of Figures 8 and 9).  $\square$

When  $\nu$  is Eulerian, the complex  $S^{\times}(\nu)$  is not a topological surface if  $\nu$  has some transparent points: as depicted on Figure 9, there are edges adjacent to four faces. But it is the only obstruction and we can desingularize such segments. Also if we want a smooth surface, we have to smooth  $S^{\times}(\nu)$  is a neighborhood of the fibers of the double points. In this way, we obtain a smooth surface, transverse to  $\varphi_{\text{geod}}$ .

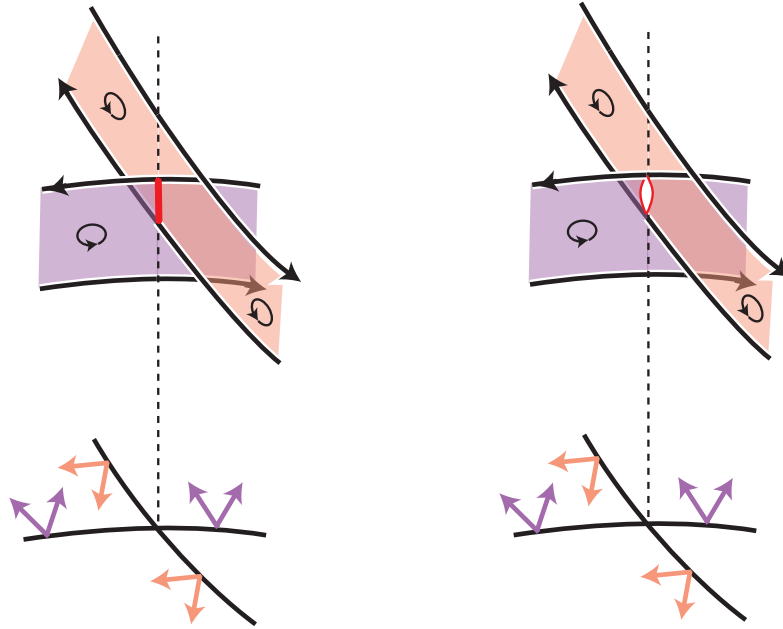


FIGURE 9. On the left, the complex  $S^\times(\nu)$  around the fiber of a transparent double point of  $\gamma$ . Every point of the fiber of  $\nu$  is adjacent to an even number of rectangles. On the right the surface  $S^{BB}(\nu)$  is obtained by desingularizing  $S^\times(\nu)$  on the portion of the fiber where four rectangles meet.

**Definition 24.** For  $\nu$  a Eulerian coorientation, the associated *BB-surface* is the surface  $S^{BB}(\nu)$  obtained from  $S^\times(\nu)$  by desingularizing and smoothing the fibers of the double points of  $\gamma$  (see the right parts of Figures 8 and 9).

**3.b. Asymptotic cycles and existence of sections.** The question whether a given vector field admit a global section (*i.e.*, with empty boundary) has been given a very satisfactory answer by Schwarzmann and Fuller [Sch57, Ful65], then expanded by Fried [Fri82].

A preliminary remark: if two surfaces  $S_1$  and  $S_2$  in a manifold  $M$  are global sections to a flow  $\phi$  and they are homologous, then they are isotopic, and the isotopy is realized by the flow. Indeed<sup>5</sup> one can consider the infinite cyclic covering of  $\hat{M} \rightarrow M$  associated to the morphism  $\pi_1(M) \rightarrow \mathbb{Z}$  given by the intersection with  $[S_1] = [S_2]$ . Then  $S_1$  and  $S_2$  lift into  $\mathbb{Z}$  disjoint copies  $t^n \hat{S}_1$  and  $t^n \hat{S}_2$  in  $\hat{M}$ , all transverse to the lift of the flow. Now following the flow starting from  $\hat{S}_1$ , one reaches  $\hat{S}_2$ , so we have a surjective map  $\hat{S}_1 \rightarrow \hat{S}_2$  of local degree 1, and since  $\hat{S}_1$  is transverse to the flow it is of local degree 1. Similarly we have a surjection  $\hat{S}_2 \rightarrow \hat{S}_1$  of local degree 1. By composing the two, we get of surjection  $\hat{S}_1 \rightarrow \hat{S}_1$  of local degree 1, hence a bijection. Therefore the maps  $\hat{S}_1 \rightarrow \hat{S}_2$  and  $\hat{S}_2 \rightarrow \hat{S}_1$  are actually bijections, and the flow hence induces an isotopy  $\hat{S}_1 \rightarrow \hat{S}_2$ . Projecting back in  $M$ , we obtained the desired isotopy  $S_1 \rightarrow S_2$ .

For  $X$  a vector field in a compact manifold  $M$ , we denote by  $k_X(p, t)$  a closed curve obtained by concatenating the piece of orbit  $\phi^{[0,t]}(p)$  starting at  $p$  of length  $t$  with an arc connecting  $\phi^t(p)$

<sup>5</sup>This mimics the folklore argument that the fiber of a fibration minimizes the genus, but it is not so easy to find a reference of this statement

to  $\phi^0(p)$  of bounded length. The class  $[k_X(p, t)]$  in  $H_1(M; \mathbb{Z})$  then depends on the choice of the closing segment, but only in a bounded way, so that the limit  $\lim_{t \rightarrow \infty} \frac{1}{t} [k_X(p, t)]$ , if it exists, does not depend on this choice. An *asymptotic cycle* of  $X$  is then the limit of a sequence of the form  $\{\frac{1}{t_n} [k_X(p_n, t_n)] \mid p_n \in M, t_n \rightarrow \infty\}$  in  $H_1(M; \mathbb{R})$ . The set of asymptotic cycles is denoted  $\mathcal{S}_X$ . Sullivan [Sul76] reinterpreted it by showing that every  $X$ -invariant measure  $\mu$  induces a *foliated cycle*  $c_\mu$  that is actually a positive barycenter of asymptotic cycles.

**Theorem 25.** [Sch57, Ful65] *A vector field  $X$  on a closed  $M$  admits a global section whose homology class is  $\sigma \in H_2(M, \partial M; \mathbb{Z})$  if and only if  $\sigma$  intersects positively every asymptotic cycle, namely for every  $c \in \mathcal{S}_X$  one has  $\langle \sigma, c \rangle > 0$ .*

This theorem is beautiful, but unfortunately, for many vector fields  $X$ , the point 0 belongs to  $\text{Conv}(\mathcal{S}_X)$ , so that  $X$  admits no global section at all. This is where Birkhoff sections come in.

**3.c. Classes of surfaces with given boundary.** Now we work in our restricted setting:  $\Sigma$  is a negatively curved surface,  $\gamma$  is a finite collection of periodic geodesics and  $\overleftrightarrow{\gamma}$  denotes the antithetic lift of  $\gamma$ . In order to apply Theorem 25 for finding Birkhoff section, we need to work in the complement  $T^1\Sigma \setminus \overleftrightarrow{\gamma}$  and in particular to determine the space  $H_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}; \mathbb{Z})$ . In this section we show that the homology classes of surfaces bounded by  $-\overleftrightarrow{\gamma}$  form an affine space and we give a canonical origin to this space.

**Lemma 26.** *The sequence  $0 \rightarrow H_2(T^1\Sigma; \mathbb{Z}) \xrightarrow{i} H_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}; \mathbb{Z}) \xrightarrow{\partial} H_1(\overleftrightarrow{\gamma}; \mathbb{Z})$ , where the first map is the inclusion map and the second is the boundary map, is exact.<sup>6</sup>*

*Proof.* The inclusion map is injective since a boundary in  $C_2(T^1\Sigma)$  is also a boundary in  $C_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma})$ . Then if a class in  $H_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}; \mathbb{Z})$  has image 0 in  $H_1(\overleftrightarrow{\gamma}; \mathbb{Z})$ , it means that it has no boundary, hence comes from a class in  $H_2(T^1\Sigma; \mathbb{Z})$ . This proves the lemma.  $\square$

The homology classes of those surfaces whose boundary is  $-\overleftrightarrow{\gamma}$  correspond to the preimages by  $\partial$  of the point  $\{-1, -1, \dots, -1\} \in H_1(\overleftrightarrow{\gamma}; \mathbb{Z}) \simeq \mathbb{Z}^{2|\gamma|}$ . Hence they form an affine space directed by  $H_2(T^1\Sigma; \mathbb{Z})$ . Indeed given two surfaces with the same boundary, their difference induces a class in  $H_2(T^1\Sigma; \mathbb{Z})$ . Now using the fact that  $T^1\Sigma$  is a circle bundle with non-zero Euler class, we get  $H_2(T^1\Sigma; \mathbb{Z}) \simeq H_1(\Sigma)$ : a non-trivial class in  $H_2(T^1\Sigma; \mathbb{Z})$  can be represented by the set of the fibers over a cycle in  $H_1(\Sigma)$ .

From the previous discussion we deduce that if we are given a explicit surface  $S_0$  bounded by  $-\overleftrightarrow{\gamma}$ , the classes of the other surfaces bounded by  $-\overleftrightarrow{\gamma}$  differ from  $[S_0]$  by a class in  $H_1(\Sigma)$ . In our context, there is a natural choice of such an origin, for which the computation of the intersection numbers with asymptotic cycles of the geodesic flow will be easy. We denote by  $S_\pm^\times$  the rational chain in  $H_2(T^1\Sigma \setminus \overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}; \mathbb{Q})$  that is half the sum of all rectangles of the form  $R^{e, \nu}$  (see Figure 10):

$$S_\pm^\times := \frac{1}{2} \sum_{e \in \gamma, \nu_e = \pm} R^{e, \nu_e}.$$

<sup>6</sup>An erroneous version of this statement is in [Fri82, Lemma 6], where it is claimed that the boundary map is surjective and admits a section. It is not true in general, unless the manifold is a homology sphere. This makes the statement of [Fri82, Theorem N] meaningless in general, to our regret. This does not alter the fact the rest of the paper is correct and very interesting.

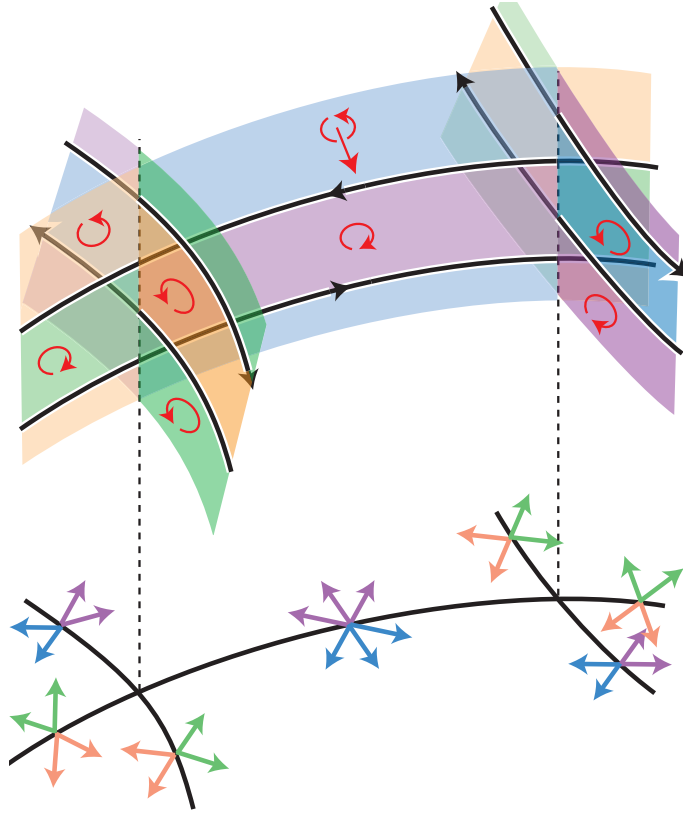


FIGURE 10. The 2-chain  $S_{\pm}^{\times}$  is half of the sum of all rectangles  $R^{e \cdot \nu_e}$ . It is cooriented by the geodesic flow, hence oriented (in red). Its boundary, taking orientations into account, is then  $-\overleftrightarrow{\gamma}$ .

In other words, we consider the set of all tangent vectors base at points of  $\gamma$ . Remember that every rectangle is cooriented by the geodesic flow, hence oriented. Therefore,  $S_{\pm}^{\times}$  is also oriented. Its boundary is then exactly  $-\overleftrightarrow{\gamma}$ . The chain  $S_{\pm}^{\times}$  is not a surface since the fibers of the double points of  $\gamma$  are singular. Its double is an integer class. As it is rational the class  $\sigma_{\pm} := [S_{\pm}^{\times}]$  might not be realized by a surface.<sup>7</sup>

**Lemma 27.** *For  $\alpha$  an oriented periodic geodesic on  $\Sigma$  that is not a component of  $\gamma$ , the algebraic intersection  $\langle \sigma_{\pm}, \overrightarrow{\alpha} \rangle$  is equal to  $+\frac{1}{2}|\{\alpha \cap \gamma\}|$ .*

This lemma appears in a different form in [DIT15] where it is used to prove that the linking number of two collections  $\overleftrightarrow{\gamma}, \overleftrightarrow{\gamma}'$  in  $T^1\Sigma$  is actually equal to  $|\{\gamma \cap \gamma'\}|$ .

<sup>7</sup>Actually,  $\sigma_{\pm}$  is realized by a surface if and only if  $[\gamma]_2$ , the class of  $\gamma$  with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients, is 0. In this case, the homology class of Birkhoff's coorientation  $\nu_B$  (Example 14) is 0, and  $S^{BB}(\nu_B)$  lies in the class  $\sigma_{\pm}$ . Also the class  $\sigma_{\pm}$  is equal to  $\frac{1}{2}[S^{BB}(\nu) + S^{BB}(-\nu)]$  for every Eulerian  $\nu$ . Hence it is always realized as the mean of two surfaces without any assumption on  $[\gamma]_2$ .

*Proof.* Since  $S_{\pm}^{\times}$  is positively transverse to the geodesic flow, all intersection points of  $\vec{\alpha}$  with  $S_{\pm}^{\times}$  counts positively. Since every rectangle has coefficients  $\frac{1}{2}$  in  $S_{\pm}^{\times}$ , every intersection point contributes for  $+\frac{1}{2}$  to the algebraic intersection. Finally  $\vec{\alpha}$  intersects  $S_{\pm}^{\times}$  exactly in the fiber of the intersection points of  $\underline{\alpha}$  and  $\gamma$ .  $\square$

3.d. **Main proof.** Denote by  $\mathcal{S}_{\vec{\gamma}} \subset H_1(T^1\Sigma \setminus \vec{\gamma}; \mathbb{R})$  the set of asymptotic cycles of the geodesic flow  $\varphi_{\text{geod}}$  restricted to  $T^1\Sigma \setminus \vec{\gamma}$ . Also denote by  $\pi$  the canonical projection from  $H_2(T^1\Sigma; \mathbb{R})$  to  $H_1(\Sigma; \mathbb{R})$ . The next statement is the key to our main result.

**Lemma 28.** *A class  $\sigma \in H_2(T^1\Sigma \setminus \vec{\gamma}, \vec{\gamma}; \mathbb{R})$  intersects positively every element of  $\mathcal{S}_{\vec{\gamma}}$  if and only if the class  $\pi(\sigma - \sigma_{\pm}) \in H_1(\Sigma; \mathbb{R})$  lies in the interior of  $\frac{1}{2}B_{x_{\gamma}}^*$ .*

*Proof.* By the shadowing property for pseudo-Anosov flows, the projectivization of  $\mathcal{S}_{\vec{\gamma}}$  is the convex hull of the cycles given of periodic orbits. Hence it is enough to estimate the intersection of  $\sigma$  with periodic orbits of  $\varphi_{\text{geod}}$ .

We use the bracket to denote the intersection, and the index reminds the space where the objects live. For every periodic orbit  $\vec{\alpha}$  of  $\varphi_{\text{geod}}$ , by Lemma 27, we have

$$\begin{aligned} \langle \sigma, \vec{\alpha} \rangle_{T^1\Sigma \setminus \vec{\gamma}} &= \langle \sigma - \sigma_{\pm}, \vec{\alpha} \rangle_{T^1\Sigma \setminus \vec{\gamma}} + \langle \sigma_{\pm}, \vec{\alpha} \rangle_{T^1\Sigma \setminus \vec{\gamma}} \\ &= \langle \sigma - \sigma_{\pm}, \vec{\alpha} \rangle_{T^1\Sigma \setminus \vec{\gamma}} + \frac{1}{2} |\{\underline{\alpha} \cap \gamma\}| \\ &= \langle \pi(\sigma - \sigma_{\pm}), \underline{\alpha} \rangle_{\Sigma} + \frac{1}{2} |\{\underline{\alpha} \cap \gamma\}|. \end{aligned}$$

Hence  $\langle \sigma, \vec{\alpha} \rangle_{T^1\Sigma \setminus \vec{\gamma}}$  is positive if and only if  $-\langle \pi(\sigma - [\underline{S}_{\pm}^{\times}]), \underline{\alpha} \rangle_{\Sigma}$  is smaller than  $\frac{1}{2} |\{\underline{\alpha} \cap \gamma\}|$ .

Now the term  $-\langle \pi(\sigma - \sigma_{\pm}), \underline{\alpha} \rangle_{\Sigma}$  depends only on the class  $[\underline{\alpha}] \in H_1(\Sigma, \partial\Sigma; \mathbb{Z})$ , while the term  $\frac{1}{2} |\{\underline{\alpha} \cap \gamma\}|$  is larger than  $\frac{1}{2} x_{\gamma}([\underline{\alpha}])$ , with equality if  $\underline{\alpha}$  is  $x_{\gamma}$ -minimizing. Hence the inequality  $-\langle \pi(\sigma - \sigma_{\pm}), \underline{\alpha} \rangle_{\Sigma} \leq \frac{1}{2} |\{\underline{\alpha} \cap \gamma\}|$  is equivalent to  $-\langle \pi(\sigma - \sigma_{\pm}), [\underline{\alpha}] \rangle_{\Sigma} \leq \frac{1}{2} x_{\gamma}([\underline{\alpha}])$ .

Therefore  $\sigma$  intersects positively every element of  $\mathcal{S}_{\vec{\gamma}}$  if and only if for every class  $a \in H_1(\Sigma; \mathbb{Z})$  we have  $-\langle \pi(\sigma - \sigma_{\pm}), a \rangle_{\Sigma} \leq \frac{1}{2} x_{\gamma}(a)$ , which means exactly that the point  $-\pi(\sigma - \sigma_{\pm})$  belongs to  $\frac{1}{2}B_{x_{\gamma}}^*$ . Since the latter is symmetric, this amounts to  $\pi(\sigma - \sigma_{\pm})$  belonging to  $\frac{1}{2}B_{x_{\gamma}}^*$ .  $\square$

As a byproduct of the proof, we obtain that a class  $\sigma \in H_2(T^1\Sigma \setminus \vec{\gamma}, \vec{\gamma}; \mathbb{R})$  intersects non-negatively every asymptotic cycle if and only if  $\pi(\sigma - \sigma_{\pm}) \in H_1(\Sigma; \mathbb{R})$  lies in the closed unit ball  $\frac{1}{2}B_{x_{\gamma}}^*$ .

*Proof of Theorem C.* For  $\nu$  a Eulerian coorientation, we consider the surface  $S^{BB}(\nu)$  (Definition 24). By construction it is transverse to the geodesic flow. One easily checks that every rectangle of the form  $R^{e,\nu}$  contributes to  $-1$  to the Euler characteristics, hence  $\chi(S^{BB}(\nu))$  is  $-|E(\gamma)|$ . Since  $\gamma$  is a graph of degree 4, one has  $|E(\gamma)| = 2|V(\gamma)|$ , so that  $\chi(S^{BB}(\nu)) = -2|V(\gamma)|$ .

Now if  $\nu_1$  and  $\nu_2$  are cohomologous, the class  $[S^{BB}(\nu_1) - S^{BB}(\nu_2)] \in H_2(T^1\Sigma; \mathbb{Z})$  projects by  $\pi$  onto  $[\nu_1 - \nu_2] = 0$ . Since  $\pi$  is actually an isomorphism,  $[S^{BB}(\nu_1) - S^{BB}(\nu_2)] = 0$ , which in turn implies  $[S^{BB}(\nu_1)] = [S^{BB}(\nu_2)]$  in  $H_2(T^1\Sigma \setminus \vec{\gamma}, \vec{\gamma}; \mathbb{Z})$ . Now since  $S^{BB}(\nu_1)$  and  $S^{BB}(\nu_2)$  are both transverse to  $\varphi_{\text{geod}}$  and homologous, the flow actually realizes an isotopy between them.  $\square$

*Proof of Theorem D.* Let us first treat the case when  $\Sigma$  has genus 2 or higher. Lemma 26 and the paragraph after implies that real homology classes of surfaces bounded by  $-\overleftrightarrow{\gamma}$  form an affine space directed by  $H_2(T^1\Sigma; \mathbb{R})$ . The class  $\sigma_{\pm}$  also defined in 3.c gives a canonical origin to this space. It is a half-integer class, and its double  $2\sigma_{\pm}$  is congruent to  $[\gamma]_2 \bmod 2$ . Therefore the double of all integer classes correspond to the sublattice of  $H_2(T^1\Sigma; \mathbb{Z})$  of those points congruent to  $[\gamma]_2 \bmod 2$ .

Now we have to determine which of these integer class yield Birkhoff sections. By Schwarzmann-Fuller Theorem 25, a class  $\sigma$  contains a Birkhoff section if and only if it intersects positively every asymptotic cycles. By Lemma 28 this means that the difference  $\sigma - \sigma_{\pm}$  lies inside  $\frac{1}{2}B_{x_{\gamma}}^*$ , or equivalently that  $2(\sigma - \sigma_{\pm})$  lies inside  $B_{x_{\gamma}}^*$ .

Now surfaces that are transverse to  $\varphi_{\text{geod}}$  correspond to homology classes that intersects non negatively every asymptotic cycle, allowing certain intersection to be zero. This means that the boundary of  $B_{x_{\gamma}}^*$  is now authorized. This end the proof for surface of genus at least 2.

For the case of the torus, the only difference is that the bundle  $T^1\Sigma$  is trivial, *i.e.*, of the form  $\Sigma \times \mathbb{S}^1$ . Therefore we no longer have  $H_2(T^1\Sigma; \mathbb{R}) \simeq H_1(\Sigma; \mathbb{R})$ , but instead  $H_2(T^1\Sigma; \mathbb{R}) \simeq H_1(\Sigma; \mathbb{R}) \times \mathbb{R}$ , since the fibers are no longer boundaries. However this extra-factor does not change the proof, since all asymptotic cycles of the geodesic flow on the flat torus are actually horizontal, meaning that the extra-coordinate is zero. Hence the positivity condition depends only on the coordinate in  $H_1(\Sigma; \mathbb{R})$ , and all the arguments can be translated.  $\square$

#### 4. QUESTIONS

**4.a. On intersection norms.** If  $\Sigma$  is a flat torus, then the minimal intersection is always realized by geodesics, which are unique in their homology class. Hence if  $\gamma$  is the union of  $k$  geodesics  $\gamma_1, \dots, \gamma_k$ , then  $i_{\gamma}(\alpha) = \sum_{i=1}^k i_{\gamma_i}(\alpha)$ . This implies that the dual ball  $B_{\gamma}^*$  coincides with the Minkowski sum  $B_{\gamma_1}^* + \dots + B_{\gamma_k}^*$ . Since the segment  $[-1, 1] \times \{0\} \subset \mathbb{R}^2$  is the dual unit ball  $B_{x_{\gamma}}^*$  for  $\gamma$  the vertical circle on the torus, every segment containing 0 in the middle is the dual unit ball of some closed circle on the torus. Therefore every convex polygon in  $\mathbb{R}^2$  whose vertices are integral and congruent mod 2 is of the form  $B_{x_{\gamma}}^*$  for some  $\gamma$ . In higher dimension the situation is probably more intricate.

**Question 29.** Which polyhedra of  $\mathbb{R}^{2g}$  with integer vertices can be realized as the dual unit ball  $B_{x_{\gamma}}^*$  for some  $\gamma$  in  $\Sigma_g$ ?

Also, if  $\Sigma$  is a torus and  $\gamma$  is a union of geodesics, then the above remarks imply that the number of self-intersection points of  $\gamma$  is exactly 1/4 of the area of  $B_{x_{\gamma}}^*$  (check on Figure 1). Is there an analog statement in higher genus?

**Question 30.** Which information concerning  $\gamma$  can be read on  $B_{x_{\gamma}}^*$ ? Is the number of self-intersection points of  $\gamma$  a certain function defined on  $B_{x_{\gamma}}^*$ ?

This information is interesting since the this number is exactly the opposite of the Euler characteristic of every Birkhoff section bounded by  $\overleftrightarrow{\gamma}$ . Note the the number of self-intersection points is homogenous of degree 2, so we should look for degree 2 functions on polyhedra in  $\mathbb{R}^{2g}$ : does it correspond to some symplectic capacity?

Motivated by our application we only defined the intersection norm for a collection of immersed curves, but one can directly extend it for an arbitrary embedded graph. One can wonder which properties extend to this case and which information on the embedded graphs are encoding in this

norm. For example when the graph is Eulerian (*i.e.*, all vertices have even degree) the connection with Eulerian coorientations remains.

**4.b. On Birkhoff sections.** Our constructions and our classification result deal only with Birkhoff sections bounded by an *amphithetic* collection of periodic orbits of the geodesic flow, that is, invariant under the involution  $(p, v) \mapsto (p, -v)$ . However the only restriction *a priori* for being the boundary of a Birkhoff section is to be a boundary, that is, to be null-homologous. Our results here say nothing about the classification, or even the existence, of Birkhoff sections with arbitrary null-homologous boundary. In this case, the theory of Schwarzmann-Fuller-Thurston-Fried and the remarks of Sections 3.b and 3.c still apply, so that these sections still correspond to the point inside a certain polytope in  $H^1(\Sigma; \mathbb{R})$ . However we have no analog for the coorientations and the explicit constructions derived from them.

**Question 31.** Is there a natural generalization of the polytope  $B_{x_\gamma}^*$  to non-amphithetic collections  $\vec{\gamma}$  of orbits of the geodesic flow  $\varphi_{\text{geod}}$ , so that integer points in this polytope classify surfaces bounded by  $\vec{\gamma}$  and transverse to  $\varphi_{\text{geod}}$ ?

In the case of the flat torus, this question was answered in [Deh15a, Thm 3.12] where a polygon  $P_{\vec{\gamma}}$  classifying transverse surfaces bounded by  $\vec{\gamma}$  was defined for *every* null-homologous collection  $\vec{\gamma}$ .

What would probably unlock the situation in the higher genus case would be to have, for every null-homologous collection  $\vec{\gamma}$ , *one* explicit surface bounded by  $\vec{\gamma}$  (not necessarily transverse), that is, a analog of  $\sigma_\pm$  when  $\vec{\gamma}$  is not antithetic. Such an explicit point allows to compute its intersection with every other periodic orbit  $\vec{\alpha}$  of  $\varphi_{\text{geod}}$ . These intersection numbers are all we need in order to describe explicitly the asymptotic directions of  $\varphi_{\text{geod}}$  in  $T^1\Sigma \setminus \vec{\gamma}$ . Generalising the constructions of [Deh15b] is a possibility here.

More generally, one can wonder whether there exists a generalization to all flows of the intersection norm  $x_\gamma$  in the following sense:

**Question 32.** For every 3-dimensional flow  $X$ , is there an object that describes all isotopy classes of Birkhoff sections?

A starting point would be to try with an Anosov flow that is not the geodesic flow, and see whether Gauss linking forms [Ghy09] could play this role.

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