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SOME REMARKS ON REPRESENTATIONS OF JET GROUPS AND GAUGE GROUPS

S. Albeverio*, B.Torresani**

Abstract : We study irreducible unitary representations of the three first jet (or Leibniz) extensions G_m^k of a compact group G, focusing on the neighborhood of the trivial representation.
We also describe some associated irreducible unitary (continuous tensor product) representations of the gauge (or current) groups of mappings from a Riemannian manifold into G_m^k.

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* : Fakultät für Mathematik, Ruhr - Universität,
D-4630 Bochum 1 (Germany) ;
SFB - 237 - Essen - Bochum - Düsseldorf ;
Bibos - Bielefeld ;
CERFIM - Locarno (Switzerland).

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I - Introduction

Since the early seventies, a great attention has been paid to the representation theory of a special class of infinite-dimensional Lie groups, called gauge groups, i.e. groups of the form Map (M, G), M being a Riemannian manifold and G a locally compact Lie group. Here, "Map" stands for any functional setting of interest, but we will restrict ourselves throughout this paper to the groups $G^M = C_0^k(M, G)$ of smooth compactly supported mappings, for the sake of simplicity.

One of the main motivations for the study of these groups is that they play an important role in the context of gauge theories, and the understanding of their representation theory would make easier the construction of quantum theories. Another main motivation was the generalisation of distribution theory to a non-commutative setting. Indeed, I.M. Gelfand and his collaborators remarked that irreducible unitary representations of G^M give natural non-commutative generalisations of the unitary characters of $\mathbf{C}^M$, that are nothing but complex exponentials of distributions on M. Finally, the discovery of affine Kac-Moody algebras and groups (loop groups), special cases of gauge algebras and groups for which $M = T^1$, has generated an important activity in the field. There is now a quite well understood highest weight representation theory for affine Kac-Moody algebras, that integrates to a unitary representation theory for the corresponding Kac-Moody groups [1] [2].

Unfortunately, such algebraic techniques fail to generalize to higher-dimensional M manifolds [3], [4], except in some particular cases [2], [5], [6].

In this paper, we will be concerned with the non-commutative distributions point of view (see e.g. [7], [8], [9], [10], [11]; see also [12], [13] for topological background). In [14], the authors constructed an irreducible unitary representation of $SL(2, \mathbf{R})^M$, realised in six different ways. It was in particular realised as a continuous tensor product [15], [16],[17], and generalised later to any locally compact group with a non-trivial representation-valued first cohomology group (notice that such an assumption excludes all the simple groups, except $SO_0(n, 1)$ and $SU(n, 1)$). The construction was modified later, to apply to compact groups, yielding the so-called energy representations [18] [19] [20]. Such representations have been widely discussed in the literature (see [7] and references therein). They are essentially based on an extension of the compact group by a nilpotent group, yielding the first order Leibniz group of G, to which the previous construction can be applied. The use of Leibniz groups has also been proposed in [17]. We study here the three first

Leibniz extensions of a compact group G, to investigate the possibility of constructing associated continuous tensor product representations of G^M. Using Mackey's induced representations theory, we classify their irreducible unitary representations, and study the neighborhood of the trivial representation (which is closely connected to the representation-valued cohomology groups of order 1).

In particular, we show that the Leibniz extensions of order 2, G^2_m, do not yield non trivial continuous tensor product representations of G^M. In the case of the Leibniz extensions of order 3, G^3_m, we show that the trivial representation is not isolated in $\widehat{G^3_m}$ (where $\widehat{G}$ stands for the unitary dual of G). Unfortunately, we do not have for the moment any example of associated representation of G^M. Finally, returning to the case of the first order Leibniz extension, we describe some associated energy representations.

Dedication :

This paper was written after the untimely departure of our dearest friend Raphael Høegh-Krohn, early 88. It is based on discussions we had with him in Oslo, Bochum and Marseille, in the two last years of his life as part of a general project on the study of non-commutative distributions, in which Raphael had been a most inspiring force.

In fact, in the last years of his life he was deeply concerned with noncommutative distributions which he considered as basic structures of universal relevance, natural infinite-dimensional extensions of Lie groups. The present study, to which he also provided the basic ideas, was for him just a component in a greater design. It was extremely sad to carry through this work without him and we missed very much his deep insight.

We dedicate this paper to his beloved memory.

II - Non-commutative distributions and the Vershik-Gelfand-Graev construction :

Let us start with a short description of the basic concepts of representations of gauge groups. Let G be a locally compact Lie group, and let M be a Riemannian manifold. Consider the group (gauge group)

$$G^M = \mathcal{D}(M, G) = C_0^\infty(M, G)$$

with pointwise group multiplication :

$$(\varphi \cdot \psi)(x) = \varphi(x)\ \psi(x) \qquad \varphi, \psi \in G^M\ ,\ x \in M$$

Such an infinite-dimensional group can be provided with a Schwartz topology (see [7]). We will be interested in (strongly) continuous unitary representations

$$\mathcal{U}\ :\ G^M \to GL(\mathcal{H})$$

of G^M in a Hilbert space $\mathcal{H}$, which provide non-commutative distributions on M, as explained in the introduction.

The simplest example is a non-commutative version of the delta distribution : pick $x_0 \in M$, and let π be a unitary representation of G. Then

$$\mathcal{U} : f \in C_0^\infty(M, G) \to \mathcal{U}(f) = \pi[f(x_0)]$$

is a non-commutative distribution on M, located at x_0. The first non-located G-valued distribution was constructed in [14], [15] as follows. Let π be an orthogonal representation of G in the Hilbert space $\mathcal{K}$, and let $\beta \in Z^1(G, \mathcal{K})$ be a continuous $\mathcal{K}$-valued 1-cocycle (i.e. $\beta\ :\ G \to \mathcal{K}$ is such that $\beta(gg') = \beta(g) + \pi(g) \cdot \beta(g')$) for all $g, g' \in G$). Let μ be a positive non-atomic Radon measure on M, and introduce the Hilbert integral :

$$\widetilde{\mathcal{K}} = \int_M^\oplus d\mu(x)\ \mathcal{K}_x \tag{$II.1$}$$

where $\mathcal{K}_x \cong \mathcal{K}\ ,\ x \in M$.

One then has an action of $C_0^\infty(M, G)$ on $\widetilde{\mathcal{K}}$:

$$\widetilde{\pi}(f) = \int_M^\oplus d\mu(x) \pi[f(x)] \in GL(\mathcal{K}) \tag{$II.2$}$$

and an associated one-cocycle :

$$\widetilde{\beta}(f) = \int_M d\mu(x) \beta[f(x)] \in \widetilde{\mathcal{K}} \tag{$II.3$}$$

Let now $\widetilde{\mathcal{H}} = S\overline{\mathcal{K}}_{\mathbf{C}}$ be the symmetric Hilbert space of $\overline{\mathcal{K}}_{\mathbf{C}}$, the complexification of the closure $\overline{\mathcal{K}}$ of $\widetilde{\mathcal{K}}$ w.r.t. the norm $\| \cdot \| =< \cdot, \cdot >^{\frac{1}{2}}$

$$\widetilde{\mathcal{H}} = \sum^\oplus S_n \overline{\mathcal{K}}_{\mathbf{C}} = \sum^\oplus \overline{\mathcal{K}}_{\mathbf{C}}^{\oplus n} \quad , \tag{$II.4$}$$

spanned by the coherent vectors

$$Exp[k] = 1 \oplus k \oplus \frac{k \otimes k}{\sqrt{2!}} \oplus ..... \quad k \in \overline{\mathcal{K}}_{\mathbf{C}} \quad . \tag{II.5}$$

From $\widetilde{\pi}$ and $\widetilde{\beta}$ one can build a unitary (projective) representation of $C_0^\infty(M,G)$ on $\widetilde{\mathcal{H}}$, as follows :

$$\mathcal{U}[f] \cdot Exp[h] = e^{-\frac{1}{2}\|\widetilde{\beta}(f)\|^2} \; e^{-<\widetilde{\pi}(f)\cdot h, \widetilde{\beta}(f)>} \; Exp[\widetilde{\pi}(f) \cdot h + \widetilde{\beta}(f)] \quad , \tag{II.6}$$

the projective factor (or 2-cocycle) being given by

$$\omega(f,g) = Im < \widetilde{\beta}(f), \widetilde{\pi}(f) \cdot \widetilde{\beta}(g) > \tag{II.7}$$

Moreover, this projective representation can be turned into a true representation if one can find some function $\gamma \; : \; G^M \to \mathbf{R}$ such that :

$$\omega(f,g) = \gamma(f \cdot g) - \gamma(f) - \gamma(g) \quad . \tag{II.8}$$

Then $\mathcal{U}_\gamma = e^{-i\gamma} \cdot \mathcal{U}$ is a true representation of G.

Remark : If one chooses to work with real Hilbert spaces only, then the 2-cocycle vanishes identically, and $\mathcal{U} = \mathcal{U}_0$ is a true representation.

The starting point of the above construction was a unitary representation π of G, and a continuous 1-cocycle β. It has been shown in [14] and [21] that a necessary condition for $\mathcal{U}$ to be irreducible is that $\beta \notin B^1(G,\mathcal{K})$. However, the groups $H^1(G,\mathcal{K})$ are closely connected to the neighborhood of the trivial representation in the unitary dual $\widehat{G}$ of G.

G is said to satisfy the Γ property if the trivial representation is not isolated in $\widehat{G}$ [22], (we recall that the canonical topology in $\widehat{G}$ is not Hausdorff separated). In [23] and [21], it was shown that if G has the Γ property, then $H^1(G,\mathcal{H}) \neq \{0\}$ for some irreducible unitary representation $\pi \; : \; G \to \mathcal{U}(\mathcal{H})$.

Moreover, if the irreducible unitary representation $\pi \; : \; G \to \mathcal{U}(\mathcal{H})$ is such that $H^1(G,\mathcal{H} \neq \{0\}$, then π is non separated from the trivial representation $\mathbb{1}_G$ in $\widehat{G}$, [24] : (π is called in [24] an infinitely small representation of G). The neighborhood of $\mathbb{1}_G$ in $\widehat{G}$ is then an interesting object to study. Notice that in [15], the continuous tensor product representation is also realised as an inductive limit of finite tensor products fo "smaller

and smaller" representations of G (in that case in the supplementary series of $SL(2,\mathbf{R})$), converging to the trivial one.

III - The jet groups G_m^k and their representations :

In the case of compact groups, all unitary irreducible representations are square-integrable, and form a discrete series. The trivial representation is then isolated, and the previous construction fails. However, such a disease can be cured by the following trick : it is sufficient to find an embedding :

$$j \ : \ G \hookrightarrow G' \qquad (III.1)$$

into a bigger group G', such that $H'(G^1, \mathcal{H})$ is non trivial for some irreducible unitary representation in $\mathcal{H}$. Then one can build continuous tensor product representations of G'^M, and use the j embedding to get a representation of G^M. The irreducibility of such representations must be studied separately. In [18], and in [20], [25], the authors considered the first jet groups G_m^1 of G to start with, mentioning the possibility of using higher order extensions. We consider here jet groups of order 1, 2 and 3.

III - 1 Structure of jet groups :

Let G be a locally compact Lie group, and M a Riemannian manifold of dimension m. With G and M are canonically associated the corresponding jet extensions G_m^k, defined as follows. Consider the current group $C_0^k(M,G)$, and the subgroup

$$G_{x_0,k}^M = \{\varphi \in C_0^k(M,G) \ , \ \varphi(x_0) = e, (D^\alpha \varphi)(x_0) = 0 \quad \forall \ \alpha \in \mathbf{N}^m \, , 1 \leq \mid \alpha \mid \leq k\} \quad (III.2)$$

for some $x_0 \in M$. It follows from Leibniz's derivation rule that $G_{x_0,k}^M$ is a normal subgroup of the current group $C_0^k(M,G)$, so that the coset space

$$G_m^k = C_0^k(M,G)/G_{x_0,k}^M \qquad (III.3)$$

inherits a group structure[(1)]. Let $[\varphi] \in G_m^k$ denote the equivalence class of $\varphi \in C_0^k(M,G)$. Then if $\varphi, \psi \in C_0^k(M,G)$:

(1) It is easy to see that G_m^k does not depend on x_0, which justifies our notation.

$$[\varphi][\psi] = [\varphi \cdot \psi] \qquad (III.4)$$

Moreover, if $\varphi \in C_0^k(M,G)$, and modulo a right-trivialization, $[\varphi]$ is in a bijective correspondence with

$$[\varphi] \cong (\varphi(x_0), (d\varphi \cdot \varphi^{-1})(x_0), \ldots [d^{k-1}(d\varphi \cdot \varphi^{-1})](x_0)) \qquad (III.5)$$

where $(d\varphi \cdot \varphi^{-1})(x_0)$ denotes as usual the value obtained by right-translation by $\varphi^{-1}(x_0)$ of $d\varphi^{-1}(x_0) \in T_{\varphi^{-1}(x_0)}M$. To simplify the notations, we shall omit the x_0 - argument from now on.

To specify a bit more the structure of G_m^k, it is necessary to introduce some more ingredients.

Denote by $\mathcal{G} = \text{Lie }(G)$ the Lie algebra of G, and let $A(n)$ denote the following abelian group :

$$A(n) = \mathcal{G}^n = \mathcal{G} \times \mathcal{G} \times \ldots \times \mathcal{G} \quad (n \text{ times}) \ , \qquad (III.6)$$

the direct product of n copies of $\mathcal{G}$, considered as an abelian group. On $A(n)$ one has an action of G, as the direct product of n adjoint actions of G on $\mathcal{G}$:

$$ad(g) \cdot (x_1, \ldots x_n) = ((ad(g) \cdot x_1, \ldots ad(g) \cdot x_n) \quad g \in \mathcal{G} \qquad (III.7)$$

In what follows, we will need two composition laws, denoted by $[\ ,\ ]_\otimes$ and $[\ ,\ ]^\otimes$ respectively, and defined by :

$$[\ ,\ ]_\otimes \ : \ x, y \in A(n) \to [x,y]_\otimes \in A(n^2) \qquad (III.8)$$

$$\{[x,y]_\otimes\}_{i,j} = [x_i, y_j] \qquad (III.9)$$

and

$$[\ ,\ ]^\otimes \ : \ (x,y) \in A(n) \times A(n) \to [x,y]^\otimes \in A(n^2) \qquad (III.10)$$

by duality, using the (positive-definite) killing form :

$$< [x,y]^\otimes, z > = < x, [y,z]_\otimes > \quad \forall\ z \in A(m^2) \qquad (III.11)$$

We will also make use of the following property, the proof of which follows from simple computations:

Lemma 2 :

Let $\varphi \in C_0^k(M,G)$, and $X \in A(m)$. Then

$$d[ad(\varphi)\cdot X] = [d\varphi\cdot\varphi^{-1},\ ad(\varphi)\cdot X]_{\otimes} + ad(\varphi)\cdot dX \qquad (III.12)$$

We are now in position to show :

Proposition 3 :

i)

$$G_m^k \cong G\ltimes N_m^k \quad , \qquad (III.13)$$

where N_m^k is topologically isomorphic to

$$N_m^k \cong A(m)\times A(m^2)\times\ldots\times A(m^k) \qquad (III.14)$$

and the action of G on N_m^k is the adjoint action.

ii) N_m^1 is the abelian group $A(m)$.

iii) N_m^2 is a central extension of $A(m)$ by $A(m^2)$.

$$(x_1,y_1)(x_2,y_2) = (x_1+x_2,\ y_1+y_2+[x_1,x_2]_{\otimes}) \quad (x_1,y_1)(x_2,y_2)\in N_m^2 \qquad (III.15)$$

iv) N_m^3 is the semi-direct product of $A(m)$ by the (non-abelian) group $A(m^2)\times A(m^3)$

$$N_m^3 \cong A(m)\ltimes[A(m^2)\times A(m^3)] \qquad (III.16)$$

$$\begin{aligned}(x_1,y_1,z_1)(x_2,y_2,z_2) =&(x_1+x_2,\ y_1+y_2+[x_1,x_2]_{\otimes},\ z_1+z_2+2[x_1,y_2]_{\otimes}\\ &+[y_1,x_2]_{\otimes}+[x_1,\ [x_1,x_2]_{\otimes}]_{\otimes}\\ &(x_1,y_1,z_1),(x_2,y_2,z_2)\in N_m^3 \qquad (III.17)\end{aligned}$$

Proof : The proof follows from explicit computations

i) is a direct consequence of (III.5).

ii) Using (III.4) and (III.5), we have :

$$\begin{aligned}(\varphi, d\varphi\cdot\varphi^{-1})(\psi, d\psi\cdot\psi^{-1}) &= \big(\varphi\psi, d(\varphi\psi)(\varphi\psi)^{-1})\big)\\ &= \big(\varphi\psi, d\varphi\cdot\varphi^{-1} + ad(\varphi)\cdot(d\psi\cdot\psi^{-1})\big) \qquad (III.18)\end{aligned}$$

iii) We restrict to the third argument of $(\varphi, (d\varphi\cdot\varphi^{-1}), d(d\varphi\cdot\varphi^{-1})) \in N_m^2$, and use Lemma 2. Then

$$\begin{aligned}d[d\varphi\cdot\varphi^{-1} + ad(\varphi)\cdot(d\psi\cdot\psi^{-1})] &= d(d\varphi\cdot\varphi^{-}1)\\ &+ \big[d\varphi\cdot\varphi^{-1}, ad(\varphi)\cdot(d\psi\cdot\psi^{-1})\big]_{\otimes} + ad(\varphi)\cdot[d(d\psi\cdot\psi^{-1})] \quad (III.19)\end{aligned}$$

which yields (III.15).

iv) We now focus on the fourth component of the product $(\varphi, d\varphi\cdot\varphi^{-1}, d(d\varphi\cdot\varphi^{-1}), d^2(d\varphi\cdot\varphi^{-1}))\cdot(\psi, d\psi\cdot\psi^{-1}, d(d\psi\cdot\psi^{-1}), d^2(d\psi\cdot\psi^{-1}))$ that reads

$$\begin{aligned}&d[d(d\varphi\cdot\varphi^{-1}) + [(d\varphi\cdot\varphi^{-1}), ad(\varphi)\cdot(d\psi\cdot\psi^{-1})]_{\otimes} + ad(\varphi)\cdot d(d\psi\cdot\psi^{-1})]\\ &\quad = d^2(d\varphi\cdot\varphi^{-1}) + [d(d\varphi\cdot\varphi^{-1}), ad(\varphi)\cdot(d\psi\cdot\psi^{-1})]_{\otimes}\\ &\qquad + 2[(d\varphi\cdot\varphi^{-1}), ad(\varphi)\cdot[d(d\psi\cdot\psi^{-1})]]_{\otimes} + ad(\varphi)\cdot d^2(d\psi\cdot\psi^{-1})\\ &\qquad + [(d\varphi\cdot\varphi^{-1}), [d\varphi\cdot\varphi^{-1}, ad(\varphi)\cdot(d\psi\cdot\psi^{-1})]_{\otimes}]_{\otimes} \qquad (III.20)\end{aligned}$$

and this proves (III.17) and completes the proof of the proposition.

In the next section, we will also use the following properties of N_m^3, that are direct consequences of proposition 3 :

Lemma 4 :

i) Let $(x, y, z) \in N_m^3$. Then

$$\begin{aligned}(x, y, z)^{-1} = (&-x, -y + [x, x]_{\otimes}, -z + 2[x, y]_{\otimes} + [y, x]_{\otimes}\\ &-[x, [x, x]_{\otimes}]_{\otimes}) \qquad (III.21)\end{aligned}$$

ii) $A(m^2)\times A(m^3)$ is an abelian normal subgroup of N_m^3 : if $(b, c) \in A(m^2)\times A(m^3)$:

$$ad(x,y,z)\cdot(0,b,c) = (0,b,c+2[x,b]_{\otimes} - [b,x]_{\otimes}) \qquad (III.22)$$

$\forall\ (x,y,z) \in N_m^3$.

III - 2 Representation theory

We will now make use of the assumption that G is compact simple. We discuss in this section the representation theory of the previously considered jet groups. More precisely, we will focus on the neighborhood of the trivial representation in their unitary dual $\widehat{G_m^k}$. In all cases, we will use Mackey's induction theory (see [26] for a review).

a) First order group :

The case of the first order Leibniz group is fairly simple, and the result is already well known (see e.g. [26]). We sketch the construction here. Let

$$\chi_T = e^{i<T,\cdot>} \qquad (III.23)$$

be a multiplicative character of $A(m)$, and let $\mathcal{O}_T$ be the G-orbit through χ_T (here $T \in \mathcal{G}^\star \oplus \ldots \oplus \mathcal{G}^\star$, where $\mathcal{G}^*$ is the dual of $\mathcal{G}$). Let $G_T \subset G$ be the stability group of χ_T :

$$G_T = \{g \in G_m^1,\ ad^\star(g)\cdot\chi_T = \chi_T\} \qquad (III.24)$$

As usual, $A(m)$ is a normal subgroup of G_T, and

$$G_T = H_T \ltimes A(m) \qquad (III.25)$$

($\ltimes$ standing for semidirect product) where

$$H_T = G_T/A(m)$$

Let

$$T = (t_1,\ldots t_m) \in A(m)^\star = \mathcal{G}^\star \oplus \ldots \oplus \mathcal{G}^\star$$

Since we will focus on the neighborhood of the trivial representation, we can restrict ourselves to the case of regular t_i's. Then H_T is a torus of G, of dimension say n. It is easy to see that χ_T can be extended to a character of G_T, denoted by χ_T^e :

$$\chi_T^e(h,x) = \chi_T(x) \qquad (h,x) \in G_T \quad . \tag{III.26}$$

Let $\chi_\alpha \ : \ H_T \to \mathbf{C}$

$$\chi_\alpha(\xi) = \xi^\alpha \qquad \xi \in H_T \tag{III.27}$$

be an arbitrary character of the compact abelian group H_T and let χ_α^e be its extension to G_T. Let finally

$$\chi = \chi_\alpha^e \otimes \chi_T^e \tag{III.28}$$

denote the Kronecker product of χ_α^e and χ_T^e. Then, since N_m^1 is of type I, the correspondence

$$\chi_\alpha \leftrightarrow \mathcal{U} = \mathcal{I}nd_{G_T}^G(\chi) \tag{III.29}$$

is a one-to-one correspondence between

- The characters of H_T

- The unitary irreducible representations of G_m^1 whose quasi-orbits are concentrated on $\mathcal{O}_T$.

Let us now study the limit $T \to 0$ (i.e. $t_i \to 0, \ i = 1, \ldots m$). $\mathcal{U}$ acts on the Hilbert space

$$\mathcal{H} \cong L^2\left({}_{G_T}\backslash^{G_m^1}\right) \cong L^2\left({}_{H_T}\backslash^{G}\right) \tag{III.30}$$

Let $u, v \in \mathcal{H}$. A simple calculation shows that

$$< u, \ \mathcal{U}(g,x) \cdot v > \underset{T \to 0}{\longrightarrow} u(e)v(e) \tag{III.31}$$

so that $\mathcal{U} \to \mathbb{1}$ weakly. We have then exhibited a sequence of non-trivial irreducible unitary representations converging to the trivial one. Thus the trivial representation is not isolated in $\widehat{G_m^1}$.

Remark : Set directly $T = 0$; then χ_T extends to the trivial representation of G_m^1, and then any representation of G_m^1 that is trivial on N_m^1 is an extension π^e of an irreducible unitary representation π of G on $\mathcal{H}$. Let β be a continuous 1-cocycle associated with π^e. It then follows from the cocycle property that

$$(e,x) = \pi(g)\cdot\beta(e,\ ad(g)^{-1}\cdot x) \qquad \forall\ (g,x)\in G^1_m \tag{III.32}$$

This implies that the linear operator $U\ :\ x\in g\to U\cdot x=\beta(e,x)\in\mathcal{H}$ intertwines π and the adjoint representation, if β is non-trivial. Then the non-triviality of β implies that π is unitarily equivalent to the adjoint representation. An example of a non-trivial 1-cocycle is well known : the so-called Maurer-Cartan cocycle $\beta(g,x)=x$.

b) Second order group :

As stressed in Proposition 3, N^2_m is a central extension of $A(m)$ by $A(m^2)$. We shall now study the representations of N^2_m. Let

$$\chi_T = e^{i<T,\cdot>} \tag{III.33}$$

be a multiplicative character of $A(m^2)$. Here $T\in A(m^2)^\star$.
Denote by $N_T=\{n\in N^2_m,\ Ad^\star(n)\cdot T=0\}$ the stability group T. N_T is a normal subgroup. Set

$$H_T = N_T/A(m^2) \tag{III.34}$$

We set $H_0=A(m)/H_T$. Let B_T be the skew 2-form on $A(m)$, defined by

$$B_T(x,x') =< T,[x,x']_\otimes > \qquad x,x'\in A(m) \tag{III.35}$$

B_T is clearly non-degenerate on H_0. It is then possible to find a maximal isotropic H_0-subgroup, denoted by H_2 :

$$H_0 = H_1\times H_2 \tag{III.36}$$

Let $\chi_\alpha = e^{i<\alpha,\cdot>}$ and $\chi_\beta = e^{i<\beta,\cdot>}$ be two characters of H_2 and H_T respectively and let

$$\chi_{\alpha\beta} = \chi^e_\alpha\otimes\chi^e_\beta \tag{III.37}$$

where χ^e_α and χ^e_β are the extensions of χ_α and χ_β to H_0. Finally, if $\chi^e_{\alpha\beta}$ and χ^e_T are the extensions of $\chi_{\alpha\beta}$ and χ_T to N_T, the correspondence

$$\chi_{\alpha\beta}\to\chi = \chi^e_{\alpha\beta}\otimes\chi^e_T \tag{III.38}$$

is a one-to-one correspondence between the characters of H_2 and the characters of N_T whose restriction to $A(m^2)$ is χ_T.

To proceed further, let us induce from N_T to N_m^2. Let

$$\chi(h_2+z,y) = e^{i<\alpha,h_2>}\ e^{i<\beta,z>}\ e^{i<T,y>}, \quad h_2\in H_2,\ z\in H_T,\ y\in A(m^2) \qquad (III.39)$$

Let $H = H_2\times H_T\times A(m^2)$

$$\mathcal{H} = \Big\{f\ :\ N_m^2\to\mathbf{C},\ \text{measurable},\ f\in L^2\left({}_H\backslash^{N_m^2},\mu\right),$$
$$f(hn)=\chi(h)f(n)\quad \forall\ h\in H,\ n\in N_m^2\Big\} \qquad (III.40)$$

for some quasi-invariant measure μ on ${}_H\backslash^{N_m^2}$, and let $\mathcal{U} = Ind_H^{N_m^2}(\chi)$. Then

$$[\mathcal{U}(h_1,h_2,z,y)\cdot f](x,0,0,0) = e^{i<\alpha,h_2>}\ e^{i<\beta,z>}$$
$$e^{i<T,y+z[x,h_2]_\otimes+[x,h_1]_\otimes+[h_1,h_2]_\otimes>}\ f(x+h_1,0,0,0) \qquad (III.41)$$

Let us now look for representations of G_m^2. Starting from the previous representation $\mathcal{U}$, one looks for its stability G_m^2-subgroup :

$$G_{\mathcal{U}} = \{(g,x,y)\in G_m^2,\quad \mathcal{U}^{(g,x,y)}\cong\mathcal{U}\} \qquad (III.42)$$

where

$$\mathcal{U}^{(g,x,y)}(n) = \mathcal{U}[ad(g,x,y)\cdot n] \qquad n\in N_m^2,\ (g,x,y)\in G_m^2 \qquad (III.43)$$

Then, with $\cong$ denoting isomorphism:

$$G_{\mathcal{U}}\cong\{g\in G,\ ad^\star(g)\cdot\alpha=\alpha,\ ad^\star(g)\cdot\beta=\beta,\ ad^\star(g)\cdot T=T\}\ltimes N_m^2 \qquad (III.44)$$

and if α,β,T are regular elements of $A(m^2)$, $G_{\mathcal{U}}$ is a torus of G, times N_m^2.

In general, $\mathcal{U}$ does not extend to a representation of $G_{\mathcal{U}}$. However, by general results [26], there exists a (unique up to equivalence) 2-cocycle σ of $G_{\mathcal{U}}/N_m^2$ and a

σ-representation $\mathcal{U}^e$ of $G_{\mathcal{U}}$ that extends $\mathcal{U}$ to $G_{\mathcal{U}}$. Let $\mathcal{V}$ be a $\frac{1}{\sigma}$-representation of $G_{\mathcal{U}}/N_m^2$, let $\mathcal{V}^e$ the representation of $G_{\mathcal{U}}$ defined by

$$\mathcal{V}^e(ng) = \mathcal{V}^e(g) \quad \forall\ n \in N_m^2,\ g \in G \tag{III.45}$$

and define

$$\Gamma = \mathcal{V}^e \otimes \mathcal{U}^e$$

Γ is a unitary representation of $G_{\mathcal{U}}$ on some Hilbert space $\mathcal{K}$, and all the representations of G_m^2 are obtained as

$$\Pi = \mathcal{I}nd_{G_{\mathcal{U}}}^{G_m^2}(\Gamma) \tag{III.46}$$

The representation space of Π is

$$\mathcal{H}_\Pi = \{f : G_m^2 \to \mathcal{K}, \text{measurable};$$
$$f \in L^2\left({}_{G_{\mathcal{U}}}\backslash^{G_m^2}\right); f(hg) = \Gamma(h) \cdot f(g)\ \forall\ h \in G_{\mathcal{U}}\} \tag{III.47}$$

and Π acts on $f \in \mathcal{H}_\Pi$ as

$$[\Pi(g,x,y) \cdot f](h,x',y') = f(h,x',y')(g,x,y) \tag{III.48}$$

Setting $\alpha = \beta = 0$, and letting $T \to 0$, one can see that such a sequence of irreducible unitary representation converges to the extension of $\mathcal{U}$ to G_m^2, which is nontrivial.

Thus the trivial representation is an isolated point in $\widehat{G_m^2}$, and

$$H^1(G_m^2, \mathcal{H}) = \{0\} \tag{III.49}$$

for any irreducible unitary G_m^2-module $\mathcal{H}$.

Remark : J. Pichaud [27] has proved that if H is a closed cocompact subgroup of a locally compact locally connected separable group G, then

$$H^1(G, \mathcal{H}') \cong H^1(H, \mathcal{H})$$

for any separable irreducible unitary H-module $\mathcal{H}$, and with $\mathcal{H}'$ the G-module induced from $\mathcal{H}$ in the sense of Mackey. In the case of G^2_m, it is easy to see from (III.41) - (III.46) that the trivial representation is isolated in $\widehat{G_{\mathcal{U}}}$, which in turn implies (III.49).

c) Third order group :

Since N^3_m is connected and nilpotent one knows that $H^1(N^3_m, \mathcal{H}) = \{0\}$ for any irreducible unitary N^3_m-module $\mathcal{H}$. However, it is not possible to use directly such a result for the representations of G^3_m.

We first look at the representations of N^3_m. It follows from section III.1 that

$$N^3_m = A(m) \times A \tag{III.50}$$

where $A = A(m^2) \times A(m^3)$ is an abelian group.

Let $T = (\tau_y, \tau_z) \in A(m^2)^\star \oplus A(m^3)^\star$, and let

$$\chi_T(y,z) = e^{i<\tau_y,y>}\ e^{i<\tau_z,z>},\ (y,z) \in A \tag{III.51}$$

be the corresponding character of A. Let $n = (x,y,z) \in N^3_m$, and set

$$\begin{aligned} \chi^n_T(u,v) &= \chi_T(ad(x,y,z)\cdot(u,v)) \\ &= \chi_T(u, v + 3[x,u]_\otimes) \end{aligned} \tag{III.52}$$

Then $\chi^n_T \cong \chi_T$ if and only if $[x,\tau_z]_\otimes = 0$. Let $N_T = \{n \in N^3_m, \chi^n_T \cong \chi_T\}$, and assume that all the components of τ_z are regular elements of $\mathcal{G}$. Then N_T/A is an intersection of Cartan subalgebras of $\mathcal{G}$, and is thus abelian. Extend χ_T to N_T by

$$\chi^e_T(x,y,z) = \chi_T(y,z) \quad (x,y,z) \in N_T \tag{III.53}$$

A simple calculation shows that χ^e_T is indeed a representation of N_T. Now let χ_α be a character of $N_T(A)$, χ^e_α its extension to N_T, and set :

$$\chi = \chi_\alpha \otimes \chi_T \tag{III.54}$$

Let $\mathcal{U} = \mathcal{I}nd^{N^3_m}_{N_T}(\chi)$. $\mathcal{U}$ acts on

$$\mathcal{H} = \{ f : N_m^3 \to \mathbf{C}, \text{ measurable}, \int_{N_T \backslash N_m^3} \mid f \mid^2 d\mu < \infty, \\ f(hn) = \chi(h) f(n) \quad \forall\, h \in N_T \} \qquad (III.55)$$

as :

$$[\mathcal{U}(n+\xi, y, z) \cdot f](u) = \chi_\alpha(\xi)\; e^{i<\tau_y, y+2[u,\xi]_\otimes + [u,n]_\otimes - [\xi,n]_\otimes>} \\ e^{i<\tau_z, z+3[u,y]_\otimes + [n,y]_\otimes + 2[u[u,n+\xi]_\otimes]_\otimes + [n,[u,n+\xi]_\otimes]_\otimes - [\xi,[\xi,u+n]_\otimes]_\otimes>} \\ f(u+n) \qquad (III.56)$$

Let $G_T = H_T \rtimes A$ be the stability group of $\mathcal{U}$. Then H_T is a torus in G, and $\mathcal{U}$ extends to a σ-representation $\mathcal{U}^e$ of G_T (for some $\sigma \in H^2(H_T, S^1)$). Let $\tau \in H^2(H_T, S^1)$, cohomologous to σ^{-1}. Then if $\mathcal{V}$ is any irreducible τ-representation of H_T and $\mathcal{V}^e$ its extension to G_T, the correspondence

$$\mathcal{V} \leftrightarrow \Pi = \mathcal{I}nd_{G_T}^G (\mathcal{V}^e \otimes \mathcal{U}^e) \qquad (III.57)$$

is a one-to-one correspondence [26].

The same arguments as in section III.2.a permit to exhibit generalized sequences of nontrivial unitary irreducible representations that converge weakly to the trivial one. Nevertheless, it is not possible to deduce the triviality or non-triviality of the $H^1(G_m^3, \mathcal{H})$ cohomology groups from such a result alone, and we have to leave open this problem for the time being.

d) Summary :

The results of the current section can be summarized as follows :

Theorem 5 :

Let G_m^k be the Leibniz extension of order m and degree k of a compact simple group G.

i) There exist irreducible unitary G_m^1-modules $\mathcal{H}$ such that $H^1(G_m^1, \mathcal{H})$ is non-trivial.

ii) $H^1(G_m^2, \mathcal{H})$ is trivial for any irreducible unitary G_m^2-module $\mathcal{H}$.

iii) The trivial representation is not isolated in $\widehat{G_m^3}$.

IV - Conclusion

We have investigated in this paper the irreducible unitary representations of the three first Leibniz extensions G_m^k of a compact simple Lie group G, focusing on the neighborhood of the trivial representation.

The motivation was the generalization of the energy representation of $\mathcal{D}(M,G)$ to such contexts. By known results, such a generalization is only possible when there exists an irreducible unitary representation $\pi\ :\ G_m^k \to \mathcal{U}(\mathcal{H})$ such that $H^1(G_m^k,\mathcal{H})$ is nontrivial. Moreover the continuous cohomology groups $H^1(G_m^k,\mathcal{H})$ are closely related to the structure of $\widehat{G_m^k}$, and more precisely to that of the neighborhood of the trivial representation.

We have shown that the trivial representation is isolated in $\widehat{G_m^2}$, while it is not isolated in $\widehat{G_m^1}$ and $\widehat{G_m^3}$. This implies that all the unitary representation valued cohomology groups of G_m^2 are trivial, and that there does not exist a second-order generalization of the energy representation. The order one case was already studied before, and leads to the standard energy representation and some generalizations [7]. In the case of third order Leibniz group, our result, that the trivial representation is not isolated in $\widehat{G_m^3}$, does not permit by itself to deduce the triviality or nontriviality of the H^1 groups. This shows the limits of the approach we have taken. The trival representation can be non-isolated in $\widehat{G_m^k}$, without being able to deduce relevant informations about triviality or non-triviality of the corresponding H^1 groups. Another limit of our approach is that the structure of the nilpotent groups is too complex to allow explicit computations at any order. N_m^1 is quite simple, N_m^2 a bit more complex, and N_m^3 much more difficult.

It would be much better to try to compute directly the H^1 groups, using spectral sequences, but at present this has not been done.

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