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Minimal time of controllability of two parabolic equations with disjoint control and coupling domains

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Abstract

We consider two parabolic equations coupled by a matrix $A(x) = q(x)A_0$, where A_0 is a Jordan block of order 1, and controlled by a single localized function, or by a single boundary control. The support of the coupling coefficient, q , and the control domain may be disjoint. We exhibit an explicit minimal time of null-controllability, $T_0(q) \in [0, +\infty]$.

Résumé

On considère deux équations paraboliques couplées par une matrice $A(x) = q(x)A_0$, où A_0 est un bloc de Jordan d'ordre 1, et contrôlées par un seul contrôle localisé en espace ou frontière. Le support du coefficient de couplage, q , et du contrôle peuvent être disjoints. Nous mettons en évidence un temps minimal de contrôlabilité à 0, $T_0(q) \in [0, +\infty]$.

Version française abrégée

L'objet de cette note est d'étudier la contrôlabilité à zéro du système parabolique (1). Il est connu (voir par exemple [15], [4] et [12]) que si $\text{Supp } q \cap \omega \neq \emptyset$ le système (1) avec $B \neq 0$ et $C = 0$ est contrôlable à zéro en tout temps $T > 0$. Lorsque $\text{Supp } q \cap \omega = \emptyset$, seuls quelques résultats ont été obtenus (voir [13], [8] et [2, 3], [14], [1], [9]). Dans [14], [1] et [9], les auteurs établissent la contrôlabilité à zéro en tout temps $T > 0$ dans le cas où le couplage q est positif. Dans cette note, on établit, en notant $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$ et $I_k(q) = \int_0^\pi q(x) \varphi_k^2(x) dx$, que, aussi bien dans le cas de la contrôlabilité interne que dans celui de la contrôlabilité par le bord, il peut exister un temps minimal de contrôle $T_0(q) > 0$.

On montre plus précisément le résultat suivant.

Théorème 0.1. *Supposons que $I_k(q) \neq 0$ pour tout $k \geq 1$ et soit*

$$T_0(q) = T_0 := \limsup_{k \rightarrow +\infty} \frac{\log \frac{1}{|I_k|}}{k^2} \in [0, \infty].$$

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1. **Contrôlabilité interne** ($B \neq 0$ et $C = 0$). Soit $\omega = (a, b)$ avec $0 < a < b < \pi$. Pour tout $T > T_0$, le système (1) est contrôlable à zéro au temps T . Sous l'hypothèse $\text{Supp } q \subset (0, a)$ ou $\text{Supp } q \subset (b, \pi)$, pour tout $T < T_0$ le système (1) n'est pas contrôlable à zéro au temps T .
2. **Contrôlabilité par le bord** ($B = 0$ et $C \neq 0$). Si $T > T_0$, le système (1) est contrôlable à zéro au temps T . Pour tout $T < T_0$, le système (1) n'est pas contrôlable à zéro au temps T .

Remarque 0.1. La condition $I_k(q) \neq 0$ pour tout $k \geq 1$ est nécessaire et suffisante pour la contrôlabilité approchée frontière ($B = 0$) du système (1) (voir [5]). Elle est aussi nécessaire et suffisante pour la contrôlabilité approchée interne ($C = 0$) du même système sous l'hypothèse géométrique (A1) (voir [8]).

On peut alors se demander s'il peut arriver que $T_0(q) > 0$. En fait, on a:

Théorème 0.2. Pour tout $\tau \in [0, +\infty]$, il existe $q \in L^\infty(0, \pi)$ tel que $T_0(q) = \tau$.

On notera que si $\int_0^\pi q(x) dx \neq 0$ alors $T_0(q) = 0$. C'est en particulier le cas dans [14]. Noter que pour tout $\tau \in [0, \infty]$, il existe $q \in L^\infty(0, \pi)$ tel que $\text{Supp } q = [0, \pi]$ et $T_0(q) = \tau$. Pour une telle fonction q le résultat de contrôlabilité frontière à zéro n'a pas lieu pour $T < \tau$.

1. Main results and comments

Let $T > 0$ and $\omega = (a, b) \subset (0, \pi)$ be fixed and let us consider the following control problem:

$$\begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T := (0, \pi) \times (0, T), \\ y(0, \cdot) = Cv, \quad y(\pi, \cdot) = 0 \text{ on } (0, T), \quad y(\cdot, 0) = y^0 \text{ in } (0, \pi), \end{cases} \quad (1)$$

where $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$, $B = \begin{pmatrix} 0 \\ b \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ c \end{pmatrix}$ are vectors of \mathbb{R}^2 , $q \in L^\infty(0, \pi)$, y^0 is the initial datum and $u \in L^2(Q_T)$ and $v \in L^2(0, T)$ are the control functions. We will consider two different issues: distributed control (i.e. $C = 0$, $B \neq 0$) and boundary control (i.e., $C \neq 0$, $B = 0$). In each case we ask if for every $y^0 \in L^2(0, \pi; \mathbb{R}^2)$ (resp. $y^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$) there exists u (resp. v) such that the solution y of (1) satisfies $y(T) = 0$ in $(0, \pi)$. In the sequel, we set $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$ for $x \in (0, \pi)$ and $k \geq 1$. To the function $q \in L^\infty(0, \pi)$ we associate the sequence $\{I_k(q)\}_{k \geq 1}$ and the number $T_0(q)$ defined by

$$I_k(q) = \int_0^\pi q(x)\varphi_k^2(x) dx, \quad T_0(q) = T_0 := \limsup_{k \rightarrow +\infty} \frac{\log \frac{1}{|I_k(q)|}}{k^2} \in [0, \infty]. \quad (2)$$

It is well known (see for instance [15], [4] and [12]) that when $\text{Supp } q \cap \omega \neq \emptyset$ the internal null-controllability result for System (1) ($B \neq 0$, $C = 0$) is valid for any time $T > 0$. When $\text{Supp } q \cap \omega = \emptyset$, only a few results are known (see [13], [8] and [2, 3], [14], [1], [9]). In [14], [1] and [9], the authors prove the internal null-controllability of System (1) for all time $T > 0$ in the case where the coupling coefficient $q \neq 0$ is non-negative.

Throughout this paper and in some situations, we are going to consider the following geometrical assumption

Assumption (A1): The function q satisfies $\text{Supp } q \subset (0, a)$ or $\text{Supp } q \subset (b, \pi)$.

Remark 1.1. Concerning the boundary controllability problem for System (1) ($B \neq 0$, $C = 0$), the first results were proved in [2, 3] for particular coupling matrices. In [5], it is proved that System (1) is boundary approximate controllable at any time $T > 0$ if and only if $I_k(q) \neq 0$ for all $k \geq 1$. When $\int_0^\pi q(x) dx \neq 0$, in [5] it is also proved that condition $I_k(q) \neq 0$ for any $k \geq 1$ characterizes boundary null-controllability property for System (1) at any time $T > 0$.

On the other hand, under Assumption (A1), System (1) is distributed approximately controllable at any time $T > 0$ if and only if $I_k(q) \neq 0$ for all $k \geq 1$ (see [8]).

The objective of this Note is to give a complete answer about the null-controllability properties of System (1) in the boundary case and under the geometrical assumption (A1) in the distributed case. One has:

Theorem 1.1. *Assume $I_k(q) \neq 0$ for all $k \geq 1$ and consider T_0 given in (2). Then,*

1. **Internal controllability** ($B \neq 0$ et $C = 0$). *If $T > T_0$, System (1) is null-controllable at time T . Under the geometrical assumption (A1), for any $T < T_0$, System (1) is not null-controllable at time T .*
2. **Boundary controllability** ($B = 0$ et $C \neq 0$). *If $T > T_0$, System (1) is null-controllable at time T . For any $T < T_0$, System (1) is not null-controllable at time T .*

Theorem 1.1 asserts that there is a minimal control time for both boundary and internal controllability. It remains to check that there exist functions $q \in L^\infty(0, \pi)$ for which $T_0(q) > 0$. Indeed, the following result shows that T_0 can be any non negative real number or even $+\infty$.

Theorem 1.2. *For any $\tau \in [0, +\infty]$, there exists $q \in L^\infty(0, \pi)$ such that $T_0(q) = \tau$.*

At this level, some consequences of Theorems 1.1 and 1.2 must be stressed. The null-controllability property of System (1) depends on the coupling function q . This dependence is described by the asymptotic behavior of $I_k(q)$. Observe that, when $I_k(q) \neq 0$ for any $k \geq 1$, System (1) is approximately controllable at any positive time. Nevertheless, the corresponding null-controllability property could fail at a given $T > 0$ or even at any positive time. We have already pointed out this fact for the boundary controllability of this kind of systems (see [6]). But, to our knowledge, this fact is new for internal controllability by L^2 -controls supported in space. In [5] it is shown that if $\int_0^\pi q(x) dx \neq 0$, then $T_0(q) = 0$. This is the case in [14]. Observe that for any $\tau \in [0, \infty]$, there exists $q \in L^\infty(0, \pi)$ such that $\text{Supp } q = [0, \pi]$ and $T_0(q) = \tau$. For this function q the boundary null-controllability result fails when $T < \tau$. This Note is part of the results on null-controllability for System (1) which will be developed in [7], a forthcoming work of the authors.

2. Tools for the proofs. Reduction to a problem of moments

Let us consider the operator $L := -\frac{d^2}{dx^2} Id + q(x)A_0 : D(L) \subset L^2(0, \pi; \mathbb{R}^2) \longrightarrow L^2(0, \pi; \mathbb{R}^2)$ with domain $D(L) = H^2(0, \pi; \mathbb{R}^2) \cap H_0^1(0, \pi; \mathbb{R}^2)$. We will assume in the sequel that $I_k(q) \neq 0$ for all $k \geq 1$. In this case, direct computations provide that the family $\mathcal{B} = \left\{ \Phi_{k,1} := \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \Phi_{k,2} := \begin{pmatrix} \psi_k \\ \frac{1}{I_k} \varphi_k \end{pmatrix} : k \geq 1 \right\}$ is a basis of root vectors (generalized eigenfunctions) of the operator $(L, D(L))$ in $L^2(0, \pi; \mathbb{R}^2)$. The family $\mathcal{B}^* = \left\{ \Phi_{k,1}^* := \begin{pmatrix} \varphi_k \\ I_k \psi_k \end{pmatrix}, \Phi_{k,2}^* := \begin{pmatrix} 0 \\ I_k \varphi_k \end{pmatrix} : k \geq 1 \right\}$ is biorthogonal to \mathcal{B} and satisfies $(L^* - k^2 Id) \Phi_{k,1}^* = \Phi_{k,2}^*$ and $(L^* - k^2 Id) \Phi_{k,2}^* = 0$, for $k \geq 1$. With the notation $h_k(x) = 1 - q(x)/I_k$, the function ψ_k is given by:

$$\psi_k(x) = \alpha_k \varphi_k(x) - \frac{1}{k} \int_0^x \sin(k(x-\xi)) h_k(\xi) \varphi_k(\xi) d\xi, \quad \alpha_k = \frac{1}{k} \int_0^\pi \int_0^x \sin(k(x-\xi)) h_k(\xi) \varphi_k(\xi) \varphi_k(x) d\xi dx. \quad (3)$$

With the previous notation, one has:

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$|I_k \alpha_k| \leq \frac{C}{k}, \quad \|I_k \psi_k\|_{L^\infty(0, \pi)} \leq \frac{C}{k}, \quad \|I_k \psi_k'\|_{L^\infty(0, \pi)} \leq C, \quad \forall k \geq 1. \quad (4)$$

Introduce the backward adjoint problem associated with System (1):

$$\begin{cases} -\theta_t - \theta_{xx} + q(x)A_0^* \theta = 0 & \text{in } Q_T, \\ \theta(0, \cdot) = \theta(\pi, \cdot) = 0 \text{ on } (0, T), \quad \theta(\cdot, T) = \theta^0 \text{ in } (0, \pi), \end{cases} \quad (5)$$

where $\theta^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$. If y is the solution of System (1) associated with $y^0 \in L^2(0, \pi; \mathbb{R}^2)$ ($y^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$ for the boundary problem) $u \in L^2(Q_T)$ and $v \in L^2(0, T)$, then it can be easily checked that $y(T) = 0$ in Ω if and only if

$$\iint_{Q_T} u 1_\omega B^* \theta \, dx \, dt + \int_0^T v(t) C^* \theta_x(0, t) \, dt = -\langle y^0, \theta(\cdot, 0) \rangle_{H^{-1}, H_0^1}, \quad \forall \theta^0 \in H_0^1(0, \pi; \mathbb{R}^2).$$

For all $k \geq 1$, if $\theta^0 = \Phi_{k,1}^*$, then $\theta_{k,1}(\cdot, t) = e^{-k^2(T-t)} \Phi_{k,1}^* - (T-t)e^{-k^2(T-t)} \Phi_{k,2}^*$ is the associated solution of (5) and if $\theta^0 = \Phi_{k,2}^*$, the associated solution of (5) is $\theta_{k,2}(\cdot, t) = e^{-k^2(T-t)} \Phi_{k,2}^*$. Thus:

- For $C = 0$ (internal controllability), we seek a control in the form $u(x, t) = f(x)\gamma(t)$. Let $f_{k,1} := \int_\omega f(x)\varphi_k(x) \, dx$ and $f_{k,2} := \int_\omega f(x)\psi_k(x) \, dx$ for all $k \geq 1$. Assuming that a function f can be found such that $f_{k,1} \neq 0$ for all $k \geq 1$ and proceeding as in [6] and [11] we reduce the null-controllability issue to the following problem of moments:

$$\begin{cases} \int_0^T e^{-k^2 t} \gamma(T-t) \, dt = -\frac{e^{-k^2 T}}{b I_k f_{k,1}} \int_0^\pi y^0 \cdot \Phi_{k,2}^* \, dx := M_{k,1}(y^0), \\ \int_0^T t e^{-k^2 t} \gamma(T-t) \, dt = \frac{e^{-k^2 T}}{b I_k f_{k,1}} \int_0^\pi y^0 \cdot \left(\Phi_{k,1}^* - \left(T + \frac{f_{k,2}}{f_{k,1}} \right) \Phi_{k,2}^* \right) \, dx := M_{k,2}(y^0), \quad \forall k \geq 1. \end{cases} \quad (6)$$

- For $B = 0$ (boundary controllability), we get in the same way the problem of moments:

$$\begin{cases} \int_0^T e^{-k^2 t} v(T-t) \, dt = -\frac{e^{-k^2 T}}{c I_k \varphi'_k(0)} \langle y^0, \Phi_{k,2}^* \rangle_{H^{-1}, H_0^1} := \widetilde{M}_{k,1}(y^0), \\ \int_0^T t e^{-k^2 t} v(T-t) \, dt = \frac{e^{-k^2 T}}{c I_k \varphi'_k(0)} \langle y^0, \Phi_{k,1}^* - \left(T + \frac{\psi'_k(0)}{\varphi'_k(0)} \right) \Phi_{k,2}^* \rangle_{H^{-1}, H_0^1} := \widetilde{M}_{k,2}(y^0), \quad \forall k \geq 1. \end{cases} \quad (7)$$

3. Internal null-controllability

Let us take $T > T_0$. In view of the relations (6), we first build a function $f \in L^2(0, \pi)$ such that $f_{k,1} \neq 0$ for all $k \geq 1$, where $f_{k,1} = \int_\omega f(x)\varphi_k(x) \, dx$.

Lemma 3.1. *There exists $f \in L^2(0, \pi)$ such that $\text{Supp } f \subset \omega$ and for all $\varepsilon > 0$ one has $\inf_{k \geq 1} f_{k,1} e^{\varepsilon k^2} > 0$.*

Sketch of the proof. Let $f = 1_{(a_0, b_0)}$ with $(a_0, b_0) \subset \omega$ and $r_1 := \frac{b_0 - a_0}{2\pi}$, $r_2 := \frac{b_0 + a_0}{2\pi} \notin \mathbb{Q}$. Then,

$$f_{k,1} = \int_{a_0}^{b_0} f(x)\varphi_k(x) \, dx = \frac{2\sqrt{2}}{k\sqrt{\pi}} \sin(\pi k r_1) \sin(\pi k r_2) \neq 0, \quad \forall k \geq 1.$$

If r_1 and r_2 are algebraic numbers of order $d \geq 2$, using diophantine approximations it can be proved that $|f_{k,1}|^2 \underset{k \rightarrow \infty}{\sim} \frac{1}{2\pi} \frac{c}{k^{4d-2}}$. ■

Now from the results in [11], the family $\left\{ e_{k,1} = e^{-k^2 t}, e_{k,2} = t e^{-k^2 t} \right\}_{k \geq 1}$ admits a biorthogonal family $\{q_{k,1}, q_{k,2}\}_{k \geq 1}$ in $L^2(0, T)$, i.e.,

$$\int_0^T e_{k,r} q_{j,s}(t) \, dt = \delta_{kj} \delta_{rs}, \quad \forall k, j \geq 1, \quad 1 \leq r, s \leq 2, \quad (8)$$

which moreover satisfies that for every $\varepsilon > 0$ there exists $C_{\varepsilon, T} > 0$ such that $\|(q_{k,1}, q_{k,2})\|_{L^2(0, T)} \leq C_{\varepsilon, T} e^{\varepsilon k^2}$ for any $k \geq 1$.

Looking for $\gamma \in L^2(0, T)$ in the form $\gamma(T-t) = \sum_{k \geq 1} (\gamma_k^1 q_{k,1}(t) + \gamma_k^2 q_{k,2}(t))$ and using (8), we see that γ satisfies (6) if and only if

$$\gamma_k^1 = M_{k,1}(y^0) \quad \text{and} \quad \gamma_k^2 = M_{k,2}(y^0), \quad \forall k \geq 1.$$

Taking into account Lemma 3.1, inequalities (4) and the definition of $T_0 = T_0(q)$ in (2), we get that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (independent of k) such that

$$|\gamma_k^1| + |\gamma_k^2| \leq C_\varepsilon e^{-k^2(T-T_0-2\varepsilon)} |y^0|, \quad \forall k \geq 1.$$

Taking for instance $\varepsilon = (T-T_0)/4$, the previous inequality ensures that the series which defines γ converges in $L^2(0, T)$. This gives the proof of the internal null-controllability of System (1) if $T > T_0(q)$.

Assume now that $T \in (0, T_0(q))$ and, in particular $I_k(q) \rightarrow 0$. We will prove that (1) is not null-controllable at time T by contradiction. Indeed, System (1) is null-controllable at time T if and only if there exists $C > 0$ such that any solution θ of the adjoint problem (5) satisfies the observability inequality:

$$\|\theta(0)\|_{L^2(0, \pi; \mathbb{R}^2)}^2 \leq C \int_0^T \int_\omega |\theta_2|^2 dx dt, \quad \forall \theta^0 \in L^2(0, \pi; \mathbb{R}^2). \quad (9)$$

Let us fix $k \geq 1$. For $\theta^0 = a_k \Phi_{k,1}^* + b_k \Phi_{k,2}^*$ with $(a_k, b_k)_{k \geq 1} \subset \mathbb{R}^2$, the previous inequality reads as $A_{k,1} \leq CA_{k,2}$, with

$$\begin{aligned} A_{k,1} &:= e^{-2k^2 T} \{ |a_k|^2 (1 + I_k^2 |\psi_k|^2 + T^2 I_k^2) + |b_k|^2 I_k^2 - 2a_k b_k T I_k^2 \}, \\ A_{k,2} &:= I_k^2 \int_0^T \int_\omega e^{-2k^2 t} |a_k \psi_k(x) + (b_k - ta_k) \varphi_k(x)|^2 dx. \end{aligned}$$

Now, we will use the following expression of $\psi_k(x)$ deduced from (3):

$$\begin{cases} \psi_k(x) = \tau_k(x) \varphi_k(x) + g_k(x), & \tau_k(x) = \alpha_k + \frac{1}{2kI_k} \int_0^x \sin(2k\xi) q(\xi) d\xi; \\ g_k(x) = -\frac{1}{k} \int_0^x \sin(k(x-\xi)) \varphi_k(\xi) d\xi - \frac{\sqrt{\pi/2}}{kI_k} \int_0^x q(\xi) \varphi_k^2(\xi) d\xi \cos(kx). \end{cases}$$

If we assume that $\text{Supp } q \cap \omega = \emptyset$, then the function τ_k is constant on $\omega = (a, b)$ and, thanks to Lemma 2.1, $\tau_k I_k \rightarrow 0$ uniformly on $(0, \pi)$. Moreover, if $\text{Supp } q \subset (0, a)$ or $\text{Supp } q \subset (b, \pi)$, then $\|g_k\|_{L^\infty(\omega)} \leq C/k$ for any $k \geq 1$. Indeed, for $x \in \omega$ one has:

$$g_k(x) = \begin{cases} -\frac{1}{k} \int_0^x \sin(k(x-\xi)) \varphi_k(\xi) d\xi - \frac{\sqrt{\pi/2}}{k} \cos(kx) & \text{if } \text{Supp } q \subset (0, a), \\ -\frac{1}{k} \int_0^x \sin(k(x-\xi)) \varphi_k(\xi) d\xi & \text{if } \text{Supp } q \subset (b, \pi). \end{cases}$$

Thus, in this case, we can choose $a_k = 1$ and $b_k = -\tau_k$, to get:

$$A_{k,2} = I_k^2 \int_0^T \int_\omega e^{-2k^2 t} |g_k(x) - t \varphi_k(x)|^2 dx dt \leq CI_k^2.$$

On the other hand, using again that $\tau_k I_k \rightarrow 0$, we also deduce the existence of $k_0 \geq 1$ such that

$$A_{k,1} = e^{-2k^2 T} \{ 1 + I_k^2 |\psi_k|^2 + T^2 I_k^2 + \tau_k^2 I_k^2 + 2T \tau_k I_k^2 \} \geq \frac{1}{2} e^{-2k^2 T}, \quad \forall k \geq k_0.$$

Inequality (9) leads to $1 \leq Ce^{2k^2 T} I_k^2$ for all $k \geq k_0$. From the definition of T_0 in (2), there exists a subsequence of $\{I_k\}_{k \geq k_0}$ (still denoted by $\{I_k\}_{k \geq k_0}$) satisfying: for any $\varepsilon > 0$ there is $k_1(\varepsilon) \geq 1$ such that $I_k^2 \leq e^{-2k^2(T_0-\varepsilon)}$ for all $k \geq k_1(\varepsilon)$. In particular, $1 \leq Ce^{2k^2(T-T_0+\varepsilon)}$ for any $k \geq k_1(\varepsilon)$. Taking $\varepsilon = (T_0-T)/2 > 0$, the previous inequality provides a contradiction and completes the proof of Theorem 1.1 for internal controllability.

4. Boundary null-controllability

We assume here that $B = 0$ and we have to solve the problem of moments (7). Using the previous arguments, it is not difficult to see that $v(T - t) = \sum_{k \geq 1} \left(\widetilde{M}_{k,1}(y^0)q_{k,1}(t) + \widetilde{M}_{k,2}(y^0)q_{k,2}(t) \right)$ is a formal solution of (7). Using the estimates (3), (4) and the definition of $T_0(q)$, it can be also checked that $v \in L^2(0, T)$ when $T > T_0(q)$. This finalizes the positive part of point 2 in Theorem 1.1.

If $T < T_0(q)$, we again reason by contradiction. In this case, the observability inequality for a solution θ to (5) is:

$$\|\theta(0)\|_{H_0^1(0, \pi; \mathbb{R}^2)}^2 \leq C \int_0^T \left| \frac{\partial \theta_2}{\partial x}(0, t) \right|^2 dt, \quad \forall \theta^0 \in H_0^1(0, \pi; \mathbb{R}^2).$$

For $\theta^0 = \Phi_{k,1}^* - (\psi'(0)/k) \Phi_{k,2}^*$ this inequality gives:

$$e^{-2k^2 T} \left\{ k^2 + I_k^2 \|\psi_k\|_{H_0^1(0, \pi)}^2 + [T^2 k^2 + \psi_k'(0)^2] I_k^2 + 2T \psi_k'(0) I_k^2 k \right\} \leq k^2 I_k^2 \int_0^T e^{-2k^2 t} t^2 dt \leq C k^2 I_k^2.$$

Then, as in previous computations and using once more (4), we infer the existence of $k_2 \geq 1$ such that $1 \leq C e^{2k^2 T} I_k^2$ for any $k \geq k_2$. As previously, this gives a contradiction with the definition of $T_0(q)$ and ends the proof of Theorem 1.1.

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