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# An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two

Nils Berglund, Giacomo Di Gesù and Hendrik Weber

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## Abstract

We study spectral Galerkin approximations of an Allen–Cahn equation over the two-dimensional torus perturbed by weak space-time white noise of strength  $\sqrt{\varepsilon}$ . We introduce a Wick renormalisation of the equation in order to have a system that is well-defined as the regularisation is removed. We show sharp upper and lower bounds on the transition times from a neighbourhood of the stable configuration  $-1$  to the stable configuration  $1$  in the asymptotic regime  $\varepsilon \rightarrow 0$ . These estimates are uniform in the discretisation parameter  $N$ , suggesting an Eyring–Kramers formula for the limiting renormalised stochastic PDE. The effect of the “infinite renormalisation” is to modify the prefactor and to replace the ratio of determinants in the finite-dimensional Eyring–Kramers law by a renormalised Carleman–Fredholm determinant.

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## 1 Introduction

Metastability is a common physical phenomenon in which a system spends a long time in metastable states before reaching its equilibrium. One of the most classical mathematical models where this phenomenon has been studied rigorously is the overdamped motion of a particle in a potential  $V$ , given by the Itô stochastic differential equation

$$dx(t) = -\nabla V(x(t)) dt + \sqrt{2\varepsilon} dw(t). \quad (1.1)$$

For small noise strength  $\varepsilon$  solutions of (1.1) typically spend long stretches of time near local minima of the potential  $V$  with occasional, relatively quick transitions between these local minima. The mean transition time between minima is then governed by the Eyring–Kramers law [12, 22]: If  $\tau$  denotes the expected hitting time of a neighbourhood of a local minimum  $y$  of the solution of (1.1) started in another minimum  $x$ , and under suitable assumptions on the potential  $V$ , the Eyring–Kramers law gives the asymptotic expression

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0(z)|} \sqrt{\frac{|\det D^2V(z)|}{\det D^2V(x)}} e^{[V(z)-V(x)]/\varepsilon} [1 + \mathcal{O}_\varepsilon(1)], \quad (1.2)$$

where  $z$  is the *relevant saddle* between  $x$  and  $y$ , and  $\lambda_0(z)$  is the (by assumption) unique negative eigenvalue of the Hessian  $D^2V(z)$  (more precise bounds on the error term  $\mathcal{O}_\varepsilon(1)$  are available). The right exponential rate in this formula was established rigorously using

large deviation theory [15]. Rigorous proofs including the prefactor were given in [30, 8, 20], see [4] for a recent survey.

It is natural to study metastability in high- or infinite-dimensional systems and to seek an extension of the Eyring–Kramers law. In this direction the Allen–Cahn equation perturbed by a small noise term is a natural model to study. It is given by the stochastic PDE

$$\partial_t \phi(t, x) = \Delta \phi(t, x) - (\phi(t, x)^3 - \phi(t, x)) + \sqrt{2\varepsilon} \xi(t, x), \quad (1.3)$$

where  $\xi$  is a noise term to be described below. Just like in (1.1) the deterministic part of this equation (set  $\xi = 0$  in (1.3)) has gradient form and the relevant potential is given by

$$V(\phi) = \int \left( \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} \phi^2 + \frac{1}{4} \phi^4 \right) dx. \quad (1.4)$$

The natural choice of noise term  $\xi$  is (at least formally) given by space-time white noise because this choice is compatible with the scalar product used to define the deterministic gradient flow and it makes the dynamics given by (1.3) reversible (in the sense that the system satisfies detailed balance when in its statistical equilibrium state). The constant profiles  $\phi_{\pm}(x) = \pm 1$  are stable solutions of the deterministic system and it is natural to ask how long a small noise term typically needs to move the system from one of these stable profiles to the other one.

In the case where equation (1.3) is solved over a  $1 + 1$ -dimensional time-space domain  $(t, x) \in [0, \infty) \times [0, L]$  this question was first studied in [13] on the level of large deviations, yielding the correct exponent in (1.2). The problem of obtaining sharper asymptotics with the correct prefactor was first considered in [3], and infinite-dimensional versions of the Eyring–Kramers formula were established in [2, 6]. Let  $\tau$  denote the first-hitting time of a solution of (1.3) starting near the constant profile  $\phi_-$  of a neighbourhood of the constant profile  $\phi_+ = 1$ . In [6] it was shown, for example in the case where (1.3) is endowed with periodic boundary conditions on  $[0, L]$  and  $L < 2\pi$ , that

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0|} \prod_{k \in \mathbb{Z}} \sqrt{\frac{|\lambda_k|}{\nu_k}} e^{[V(\phi_0) - V(\phi_-)]/\varepsilon} [1 + \mathcal{O}_\varepsilon(1)], \quad (1.5)$$

where  $k$  plays the role of a wave number in a Fourier decomposition. The purpose of the condition  $L < 2\pi$  is to ensure that the constant profile  $\phi_0 = 0$  is the relevant saddle between the two stable minima  $\phi_{\pm}$ ; but situations for longer intervals and different choices of boundary conditions are also described in [6]. The sequences  $\lambda_k, \nu_k$  appearing in this expression are the eigenvalues of the Hessian of  $V$  around  $\phi_0$  and  $\phi_-$ , the operators  $-\partial_x^2 - 1$  and  $-\partial_x^2 + 2$ , both endowed with periodic boundary condition (the corresponding eigenfunctions being simply Fourier modes). All of these eigenvalues are strictly positive, except for  $\lambda_0 = -1$ . Leaving out the factor  $k = 0$ , the infinite product in (1.5) can be written as

$$\prod_{k \neq 0} \sqrt{\frac{\lambda_k}{\nu_k}} = \prod_{k \neq 0} \sqrt{\left( 1 + \frac{\nu_k - \lambda_k}{\lambda_k} \right)^{-1}} = \frac{1}{\sqrt{\det(\text{Id} + 3P_{\perp}(-\partial_x^2 - 1)^{-1})}}, \quad (1.6)$$

where  $P_{\perp}$  projects on the complement of the  $k = 0$  mode and the operator  $(-\partial_x^2 - 1)$  acts on zero-mean functions. This expression converges, because  $P_{\perp}(-\partial_x^2 - 1)^{-1}$  is a trace-class operator, so that the infinite-dimensional (Fredholm) determinant is well-defined (see for instance [14, 23]).

When trying to extend this result to higher spatial dimensions two problems immediately present themselves. First, for spatial dimension  $d \geq 2$  the Allen–Cahn equation as stated in (1.3) fails to be well-posed: in this situation already the linear stochastic heat equation (drop the non-linear term  $-(\phi^3 - \phi)$  in (1.3)) has distribution-valued solutions due to the irregularity of the white noise  $\xi$ . In this regularity class  $-(\phi^3 - \phi)$  does not have a canonical definition. As an illustration for the problems caused by this irregularity, it was shown in [18] that for fixed noise strength  $\varepsilon$  finite-dimensional spectral Galerkin approximations<sup>1</sup>

$$\partial_t \phi_N = \Delta \phi_N - (P_N \phi_N^3 - \phi_N) + \sqrt{2\varepsilon} \xi_N \quad (1.7)$$

defined over a two-dimensional torus converge to a trivial limit as the approximation parameter  $N$  goes to  $\infty$  (precise definitions of the finite-dimensional noise  $\xi_N$  and the projection operator  $P_N$  are given in Section 2 below). A second related problem is, that for  $d \geq 2$  the infinite product appearing in (1.5) converges to 0, corresponding to the fact that for  $d \geq 2$  the operator  $3P_{\perp}(-\Delta - 1)^{-1}$  fails to be trace-class so that the Fredholm determinant  $\det(\text{Id} + 3P_{\perp}(-\Delta - 1)^{-1})$  is not well-defined.

On the level of the  $N \rightarrow \infty$  limit for fixed  $\varepsilon$  the idea of renormalisation, inspired by ideas from Quantum Field Theory (see e.g. [16]), has been very successful over the last years. Indeed, in [9] it was shown that in the two-dimensional case, if the approximations in (1.7) are replaced by

$$\partial_t \phi_N = \Delta \phi_N - (P_N \phi_N^3 - 3\varepsilon C_N \phi_N - \phi_N) + \sqrt{2\varepsilon} \xi_N, \quad (1.8)$$

for a particular choice of logarithmically divergent constants  $C_N$  (see (2.6) below), the solutions do converge to a non-trivial limit which can be interpreted as *renormalised solutions* of (1.3). This result (for a different choice of  $C_N$ ) was spectacularly extended to three dimensions in Hairer’s pioneering work on regularity structures [17]. For spatial dimension  $d \geq 4$ , equation (1.3) fails to satisfy a *subcriticality condition* (see [17]) and non-trivial renormalised solutions are not expected to exist.

Note that formally the extra term  $3\varepsilon C_N \phi_N$  moves the stable solutions further apart to  $\pm\sqrt{3\varepsilon C_N + 1}$  (and ultimately to  $\pm\infty$  as  $N \rightarrow \infty$ ). Note furthermore that while the constants  $C_N$  diverge as  $N$  goes to  $\infty$ , for fixed  $N$  they are multiplied with a small factor  $\varepsilon$ . This suggests that in the small-noise regime the renormalised solutions may still behave as perturbations of the Allen–Cahn equation, despite the presence of the infinite renormalisation constant. In [19] this intuition was confirmed on the level of large deviations. There it was shown that both in two and three dimensions the renormalised stochastic PDE satisfies a large-deviation principle as  $\varepsilon \rightarrow 0$ , with respect to a suitable topology and with rate functional given by

$$\mathcal{I}(\phi) = \int_0^T \int (\partial_t \phi - (\Delta \phi - (\phi^3 - \phi)))^2 dx dt, \quad (1.9)$$

which is exactly the “correct” rate functional one would obtain by formally applying Freidlin–Wentzell theory to (1.3) without any regard to renormalisation. Results in a similar spirit had previously been obtained in [21, 1].

The purpose of this article is to show that the renormalised solutions have the right asymptotic small-noise behaviour even beyond large deviations, and to establish an Eyring–Kramers formula in this framework. As remarked also in [28] nothing seems to be known

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<sup>1</sup>In fact, in [18, 9] the nonlinearity  $\phi^3$  is not projected onto a finite dimensional space, but this does not affect the result.

at this level so far. The key observation is that the introduction of the infinite constant not only permits to define the dynamics, but that it also fixes the problem of vanishing prefactor in the Eyring–Kramers law (1.5). More precisely, we argue that in two dimensions the correct Eyring–Kramers formula for the renormalised SPDE is

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{|\lambda_k|}\right\}} e^{[V(\phi_0) - V(\phi_-)]/\varepsilon} [1 + \mathcal{O}_\varepsilon(1)], \quad (1.10)$$

where as above the  $\lambda_k$  and  $\nu_k$  are the eigenvalues of  $-\Delta - 1$  and  $-\Delta + 2$ , now indexed by a vectorial wave number  $k \in \mathbb{Z}^2$ . In functional-analytic terms this means that due to the presence of the infinite renormalisation constant the regular determinant from (1.5) is replaced by a renormalised or Carleman–Fredholm determinant of the operator  $\text{Id} + 3P_1(-\Delta - 1)^{-1}$ . Unlike the “usual” determinant, the Carleman–Fredholm determinant is defined for the class of Hilbert–Schmidt perturbations of the identity and not only for the smaller class of trace-class perturbations of the identity. Recall, that  $(-\Delta - 1)^{-1}$  is Hilbert–Schmidt both in two and three dimensions, but not for  $d \geq 4$ . It is striking to note that these are exactly the dimensions in which a renormalised solution to the Allen–Cahn (or  $\Phi^4$ ) equation can be constructed.

In order to illustrate our result in the easiest possible situation we only consider the case of the Allen–Cahn equation in a small domain  $\mathbb{T}^2 = [0, L]^2$  of size  $L < 2\pi$  with periodic boundary conditions. As in the one-dimensional case this assumption guarantees that the constant profile  $\phi_0$  is the relevant saddle. We make use of the  $\pm 1$  symmetry of the system to simplify some arguments. Throughout the article, we work in the framework of the finite-dimensional spectral Galerkin approximation (1.8) and derive asymptotic bounds for the expected transition time as  $\varepsilon \rightarrow 0$  which are uniform in the approximation parameter  $N \rightarrow \infty$ .

On the technical level, our analysis builds on the potential-theoretic approach developed in [8], which allows to express expected transition times in terms of committor functions and capacities, that can be estimated using a variational principle. As we work in finite dimensions throughout, we can avoid making any use of the analytic tools developed in recent years to deal with singular SPDEs. A crucial idea is to change point of view with respect to the usual finite dimensional setting as presented in [7, 8] and to regard capacities and partition functions as expectations of random variables under Gaussian measures, which are well-defined in infinite dimension. This idea already appeared in [11] for the analysis of metastability of the Allen–Cahn equation in space dimension  $d = 1$ . In the present setting ( $d = 2$ ) the new point of view is particularly powerful, since expectations under Gaussian measures can be estimated using Wick calculus, and in particular the classical Nelson argument [25] which permits to bound expectations of exponentials of Hermite polynomials. Another key argument is the observation from [5] that the field  $\phi$  can be decomposed into its average and fluctuating part and that the (non-convex) potential  $V$  is convex in the transverse directions (see Lemma 5.3). An additional key idea, following [11, Section 3.2], is to avoid using Hausdorff–Young inequalities in the discussion of Laplace asymptotics (see [3, 6]) and rather use Taylor expansions and global estimates, which lead to much better error estimates, both in  $\varepsilon$  and in  $N$ . The rest of this paper is structured as follows: in Sections 2 and 3 we give the precise assumptions, state our main theorem and give the necessary background from potential theory. Lower bounds on the expected transition time are proved in Section 4, upper bounds are proved in Section 5. Some well-known facts about Hermite polynomials and Wick powers are collected in Appendix A.

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## 2 Results

Let  $\mathbb{T}^2 = \mathbb{R}^2 / (L\mathbb{Z})^2$  denote the two-dimensional torus of size  $L \times L$ . We are interested in the renormalised Allen–Cahn equations

$$\partial_t \phi = \Delta \phi + [1 + 3\varepsilon C_N] \phi - \phi^3 + \sqrt{2\varepsilon} \xi_N \quad (2.1)$$

for  $\phi = \phi(t, x) : \mathbb{R}^+ \times \mathbb{T}^2 \rightarrow \mathbb{R}$ , where  $\xi_N$  approximates space-time white noise on the scale  $1/N$ . In fact, we will consider spectral Galerkin approximations of the above equation. Let  $e_k(x) = L^{-1} e^{i(2\pi/L)k \cdot x}$  denote  $L^2$ -normalised Fourier basis vectors of  $L^2(\mathbb{T}^2)$ , where  $k \in \mathbb{Z}^2$  is the wave vector. Denote by  $P_N$  the projection on Fourier modes with wave number  $k$  satisfying  $|k| = \max\{|k_1|, |k_2|\} \leq N$ , that is,

$$(P_N \phi)(x) = \sum_{k \in \mathbb{Z}^2: |k| \leq N} \langle \phi, e_k \rangle e_k(x). \quad (2.2)$$

Then we consider the sequence of equations

$$\partial_t \phi = \Delta \phi + [1 + 3\varepsilon C_N] \phi - P_N \phi^3 + \sqrt{2\varepsilon} \xi_N \quad (2.3)$$

where  $\xi_N = P_N \xi$  is the spectral Galerkin approximation of space-time white noise  $\xi$ .

We will assume periodic boundary conditions (b.c.), with a domain size satisfying  $0 < L < 2\pi$ . This assumption guarantees that the identically zero function plays the role of the transition state, which separates the basins of attraction of the two stable solutions  $\phi_{\pm} = \pm 1$  of the deterministic Allen-Cahn equation.

Note that the deterministic system (take  $\varepsilon = 0$  in (2.3)) is a gradient system, with potential

$$V_N[\phi] = \frac{1}{2} \int_{\mathbb{T}^2} [\|\nabla \phi(x)\|^2 - \phi^2(x)] dx + \frac{1}{4} \int_{\mathbb{T}^2} [\phi^4(x) - 6\varepsilon C_N \phi^2(x) + 3\varepsilon^2 C_N^2] dx. \quad (2.4)$$

The measure  $e^{-V_N/\varepsilon}$  is an invariant, reversible measure of the stochastic system (2.3), and we will denote by  $\mathcal{Z}_N(\varepsilon)$  its normalisation (the partition function of the system). The constant term  $3\varepsilon^2 C_N^2$  in the second integral is of course irrelevant for the dynamics, but it will simplify notations. This is related to the fact that

$$\phi^4(x) - 6\varepsilon C_N \phi^2(x) + 3\varepsilon^2 C_N^2 = H_4(\phi(x), C_N) \quad (2.5)$$

is the so-called Wick renormalisation of  $\phi^4(x)$  with respect to the centered Gaussian measure having covariance  $|P_N[-\Delta - 1]^{-1}|$ , often denoted  $:\phi^4(x):$ , where  $H_4$  is the fourth

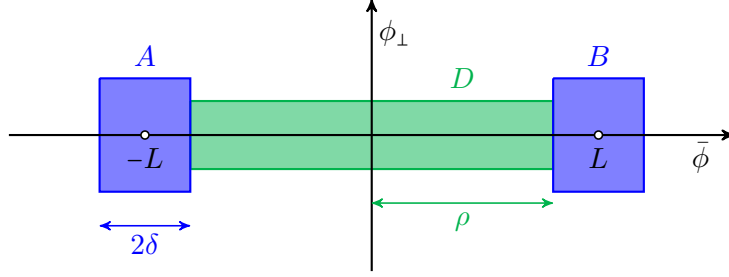


FIGURE 1. Geometry of the neighbourhoods  $A$  and  $B$  of the deterministic stable solutions  $\phi_{\pm}(x)$ . The set  $D$  will be needed later in the proof, cf. Section 5.3.

Hermite polynomial (see Section 4 and Appendix A). The renormalisation constant  $C_N$  is given by

$$C_N = \frac{1}{L^2} \text{Tr}(|P_N[-\Delta - 1]^{-1}|) := \frac{1}{L^2} \sum_{k \in \mathbb{Z}^2: |k| \leq N} \frac{1}{|\lambda_k|} \quad (2.6)$$

where  $\lambda_k = (2\pi/L)^2(k_1^2 + k_2^2) - 1$ . Therefore,  $C_N$  diverges logarithmically as

$$C_N \asymp \frac{2\pi}{L^2} \log(N). \quad (2.7)$$

The choice of  $C_N$  is somewhat arbitrary, as adding a constant independent of  $N$  to  $C_N$  will also yield a well-defined limit equation as  $N \rightarrow \infty$ . See Remark 2.5 below for the effect of such a shift on the results.

In the deterministic case  $\varepsilon = 0$ , the Allen–Cahn equation (2.3) has two stable stationary solutions given by  $\phi_-(x) = -1$  and  $\phi_+(x) = 1$ . We are interested in obtaining sharp asymptotics on the expectation of the first-hitting time  $\tau_B$  of a neighbourhood  $B$  of  $\phi_+$ , when starting near  $\phi_-$ . The neighbourhood  $B$  should have a minimal size. More precisely, we decompose any function  $\phi: \mathbb{T}^2 \rightarrow \mathbb{R}$  into its mean and oscillating part by setting

$$\phi(x) = \bar{\phi} + \phi_{\perp}(x) \quad (2.8)$$

where  $\bar{\phi} = L^{-2} \int_{\mathbb{T}^2} \phi(x) dx$  and the integral of  $\phi_{\perp}$  over  $\mathbb{T}^2$  is zero. Then we define two symmetric sets  $A$  and  $B$  as follows (see Figure 1).

**Definition 2.1.** Fix constants  $\delta \in (0, 1)$  and  $s < 0$ , and let  $D_{\perp}$  be a ball of radius  $r\sqrt{\varepsilon \log(\varepsilon^{-1})}$  in the Sobolev space  $H^s(\mathbb{T}^d)$ , where  $r$  is a sufficiently large numerical constant. Then

$$\begin{aligned} A &= \{ \phi \in H^s(\mathbb{T}^d): \bar{\phi} \in [-1 - \delta, -1 + \delta], \phi_{\perp} \in D_{\perp} \}, \\ B &= \{ \phi \in H^s(\mathbb{T}^d): \bar{\phi} \in [1 - \delta, 1 + \delta], \phi_{\perp} \in D_{\perp} \}. \end{aligned} \quad (2.9)$$

**Remark 2.2.** The definition of  $D_{\perp}$  ensures that  $A \cup B$  contains most of the mass of the invariant probability measure  $\mathcal{Z}_N^{-1} e^{-V_N/\varepsilon}$  of the equation. In fact, the result remains true if we replace  $D_{\perp}$  by any sufficiently regular set containing  $D_{\perp}$ , see Lemma 5.9.

Our main result for periodic b.c. is as follows (recall from the introduction that  $\nu_k = (2\pi/L)^2(k_1^2 + k_2^2) + 2 = \lambda_k + 3$ ).

**Theorem 2.3.** *Assume  $0 < L < 2\pi$ . There exists a sequence  $\{\mu_N\}_{N \geq 1}$  of probability measures concentrated on  $\partial A$  such that in the case of periodic b.c.,*

$$\begin{aligned} \limsup_{N \rightarrow \infty} \mathbb{E}^{\mu_N}[\tau_B] &\leq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{|\lambda_k|}\right\}} e^{[V(\phi_0) - V(\phi_-)]/\varepsilon} [1 + c_+ \sqrt{\varepsilon}], \\ \liminf_{N \rightarrow \infty} \mathbb{E}^{\mu_N}[\tau_B] &\geq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{|\lambda_k|}\right\}} e^{[V(\phi_0) - V(\phi_-)]/\varepsilon} [1 - c_- \varepsilon], \end{aligned} \quad (2.10)$$

where the constants  $c_{\pm}$  are uniform in  $\varepsilon$ .

Since  $\nu_k = \lambda_k + 3$  and  $V(\phi_0) - V(\phi_-) = L^2/4$ , the leading term in (2.10) can also be written as

$$2\pi \left( \frac{e^{3/|\lambda_0|}}{|\lambda_0|(\lambda_0 + 3)} \prod_{k \neq 0} \frac{e^{3/\lambda_k}}{1 + 3/\lambda_k} \right)^{1/2} e^{L^2/4\varepsilon}. \quad (2.11)$$

The infinite product indeed converges, since

$$\log\left(\frac{e^x}{1+x}\right) = x - \log(1+x) \leq \frac{1}{2}x^2, \quad (2.12)$$

and the sum over  $\mathbb{Z}^2$  of  $\lambda_k^{-2}$  converges, unlike the sum of  $\lambda_k^{-1}$  that would arise without the regularising term  $e^{3/\lambda_k}$ . On a more abstract level, as already mentioned in the introduction, this is due to the fact that we have replaced the usual Fredholm determinant  $\det(\text{Id} + T)$  (with  $T = 3|(-\Delta - 1)^{-1}|$ ) by the Fredholm–Carleman determinant

$$\det_2(\text{Id} + T) = \det(\text{Id} + T) e^{-\text{Tr} T}, \quad (2.13)$$

which is defined for every Hilbert–Schmidt perturbation  $T$  of the identity, without the requirement of  $T$  to be trace-class [29, Chapter 5].

**Remark 2.4.** An analogous result holds for zero-flux Neumann b.c.  $\partial_{x_i} \phi(t, x) = 0$  whenever  $x_i \in \{0, L\}$  for  $i = 1$  or  $2$ , provided  $0 < L < \pi$ . The only difference is that one has to replace  $2\pi/L$  by  $\pi/L$  in the definition of the  $\lambda_k$ , and that the sums are over  $k \in \mathbb{N}_0^2$  instead of  $k \in \mathbb{Z}^2$ .

**Remark 2.5.** The definition (2.6) of the renormalisation constant  $C_N$  is not unique, and one would still obtain a well-defined limit for (2.3) if  $C_N$  were replaced by  $C_N + \theta/L^2$  for some constant  $\theta \in \mathbb{R}$  (or even by  $C_N + \theta_N/L^2$ , where  $\theta_N$  converges to a limit  $\theta$  as  $N \rightarrow \infty$ ). One easily checks that the effect of such a shift in the renormalisation constant is to multiply the expected transition time by a factor  $e^{3\theta/2|\lambda_0|}$ .

**Remark 2.6.** The measures  $\mu_N$  appearing in Theorem 2.3 are the normalised equilibrium measures  $\mu_{A,B}$  of the finite dimensional dynamics of  $\phi_N$ . This measure appears in the result, because the method of proof is based on an asymptotic analysis as  $\varepsilon \rightarrow 0$  of the exact formula (3.5) for  $\mathbb{E}^{\mu_N}[\tau_B]$  which is available with this initial distribution. One would expect that the precise starting distribution does not affect the final result and that equation (2.10) remains true with  $\mu_N$  replaced by any distribution which is supported on a sufficiently small neighbourhood of  $\phi_-$ , in particular a deterministic initial condition at  $\phi_-$ . For the one-dimensional SPDE such a result was indeed obtained in [6], by post-processing a bound involving the equilibrium measure, building on a coupling technique developed in [24]. It would be interesting to see if a similar post-processing is also possible in the context of our renormalised SPDE.



**Remark 2.7.** In a similar spirit, it might seem more natural to state the bound (2.10) in the limit  $N \rightarrow \infty$ , i.e. for solutions of the full renormalized SPDE rather than stating a uniform-in- $N$  bound over the Galerkin approximations. In the one-dimensional case, such a result was obtained in [6, Proposition 3.4], using various a priori estimates available for  $d = 1$ . Using the technique from [9] it is straightforward to show that the approximate solutions  $\phi_N$  (say all starting in  $\phi_-$ ) and the corresponding stopping times  $\tau_N$  converge in probability as  $N \rightarrow \infty$ . One can then pass to the limit in the expectations  $\mathbb{E}^{\phi_-}[\tau_N]$  (for  $N \rightarrow \infty$  keeping  $\varepsilon$  fixed) provided the random variables  $\tau_N$  are uniformly integrable, which is implied, for example, by a bound of the type  $\sup_N \mathbb{E}^{\phi_-}[\tau_N^p] < \infty$  for some  $p > 1$ . We expect that a much stronger bound of the type  $\sup_N \mathbb{E}^{\phi_-}[\exp(\lambda\tau_N)] < \infty$  for  $\lambda$  small enough can be obtained, using the results derived in [31].

**Remark 2.8.** The error term in  $\sqrt{\varepsilon}$  in the upper bound for the expected transition time is due to our using less sharp approximations in the Laplace asymptotics. It is in principle possible, as was done in the one-dimensional case in [11], to derive further terms in the asymptotic expansion. In particular it is expected that the leading error term has order  $\varepsilon$ .

### 3 Some potential theory

When considering the spectral Galerkin approximations (2.3), it will sometimes be convenient to use Fourier variables  $z_k$  defined by

$$\phi_N(t, x) = \sum_{k \in \mathbb{Z}^2: |k| \leq N} z_k(t) e_k(x). \quad (3.1)$$

In order to ensure that  $\phi_N$  is real-valued, the coefficients  $z_k$  are chosen to take values in

$$\{(z_k) \in \mathbb{C}^{(-N, \dots, N)^2} : z_{-k} = \overline{z_k} \text{ for all } k\} \quad (3.2)$$

which we identify with  $\mathbb{R}^{(2N+1)^2}$  throughout. In particular, we will always interpret gradients and integration with respect to Lebesgue measure  $dz$  in terms of this identification. The spectral Galerkin approximation with cut-off  $N$  is equivalent to the system of Itô SDEs

$$dz(t) = -\nabla V_N(z(t)) dt + \sqrt{2\varepsilon} dW_t, \quad (3.3)$$

where the potential, obtained by evaluating (2.4) in  $\phi_N$ , is given by

$$V_N(z) = \frac{1}{2} \sum_{|k| \leq N} \lambda_k |z_k|^2 + \frac{1}{4} \left( \frac{1}{L^2} \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ |k_i| \leq N}} z_{k_1} z_{k_2} z_{k_3} z_{k_4} - 6\varepsilon C_N \sum_{|k| \leq N} |z_k|^2 + 3L^2 \varepsilon^2 C_N^2 \right). \quad (3.4)$$

Arguments based on potential theory (see [8] and [7, Corollary 7.30]) show that for any finite  $N$  one has the relation

$$\mathbb{E}^{\mu_{A,B}}[\tau_B] = \frac{1}{\text{cap}_A(B)} \int_{B^c} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz. \quad (3.5)$$

Here  $h_{A,B}(z)$  is the *committor function* (or *equilibrium potential*)

$$h_{A,B}(z) = \mathbb{P}^z\{\tau_A < \tau_B\}, \quad (3.6)$$

where  $\tau_A$  denotes the first-hitting time of a set  $A \subset \mathbb{R}^{(2N+1)^2}$ . The term  $\text{cap}_A(B)$  is the so-called *capacity*, which admits several equivalent expressions:

$$\text{cap}_A(B) = \varepsilon \int_{(A \cup B)^c} \|\nabla h_{A,B}(z)\|^2 e^{-V_N(z)/\varepsilon} dz \quad (3.7)$$

$$= \varepsilon \inf_{h \in \mathcal{H}_{A,B}} \int_{(A \cup B)^c} \|\nabla h\|^2 e^{-V_N(z)/\varepsilon} dz \quad (3.8)$$

$$= \int_{\partial A} e^{-V_N(z)/\varepsilon} \rho_{A,B}(dz), \quad (3.9)$$

where  $\mathcal{H}_{A,B}$  is the set of functions  $h \in H^1$  such that  $h = 1$  in  $A$  and  $h = 0$  in  $B$ , and  $\rho_{A,B}(dz)$  is a measure concentrated on  $\partial A$ , called the *equilibrium measure*. The density of this measure (with respect to the surface measure on  $\partial A$ ) is equal to the exterior normal derivative of the committor function  $h_{A,B}$ . Finally,  $\mu_{A,B}$  is the probability measure on  $\partial A$  obtained by normalising  $\rho_{A,B} e^{-V/\varepsilon}$ :

$$\mu_{A,B}(dz) = \frac{1}{\text{cap}_A(B)} e^{-V_N(z)/\varepsilon} \rho_{A,B}(dz). \quad (3.10)$$

The following symmetry argument allows us to link the expected transition time to the partition function of the system.

**Lemma 3.1.** *If  $A$  and  $B$  are symmetric with respect to the plane  $z_0 = 0$  then*

$$\int_{B^c} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz = \frac{1}{2} \int_{\mathbb{R}^{(2N+1)^2}} e^{-V_N(z)/\varepsilon} dz =: \frac{1}{2} \mathcal{Z}_N(\varepsilon). \quad (3.11)$$

PROOF: Consider the reflection  $S$  given by

$$S(z_0, z_\perp) = (-z_0, z_\perp).$$

The potential  $V_N$  satisfies the symmetry

$$V_N(Sz) = V_N(z)$$

which implies

$$\int_{\{z_0 < 0\}} e^{-V_N(z)/\varepsilon} dz = \int_{\{z_0 > 0\}} e^{-V_N(z)/\varepsilon} dz = \frac{1}{2} \mathcal{Z}_N(\varepsilon). \quad (3.12)$$

Assuming we choose  $A$  and  $B$  such that  $B = SA$ , the committor satisfies

$$h_{A,B}(z) = h_{B,A}(Sz).$$

In addition, we always have

$$h_{A,B}(z) = 1 - h_{B,A}(z).$$

Now observe that we have

$$\begin{aligned} & \int_{\mathbb{R}^{(2N+1)^2}} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz \\ &= \int_{\{z_0 < 0\}} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz + \int_{\{z_0 > 0\}} (1 - h_{B,A}(z)) e^{-V_N(z)/\varepsilon} dz \\ &= \int_{\{z_0 < 0\}} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz + \int_{\{z_0 > 0\}} e^{-V_N(z)/\varepsilon} dz - \int_{\{z_0 < 0\}} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz \\ &= \int_{\{z_0 > 0\}} e^{-V_N(z)/\varepsilon} dz, \end{aligned}$$

and the conclusion follows from (3.12).  $\square$

As a consequence of (3.11), we can rewrite (3.5) in the form

$$\mathbb{E}^{\mu_{A,B}}[\tau_B] = \frac{1}{2 \operatorname{cap}_A(B)} \mathcal{Z}_N(\varepsilon). \quad (3.13)$$

Note that this relation can also be written as

$$\frac{1}{\mathbb{E}^{\mu_{A,B}}[\tau_B]} = 2 \mathbb{E}^{\pi_N(\varepsilon)}[\varepsilon \|\nabla h_{A,B}\|^2], \quad (3.14)$$

where  $\pi_N(\varepsilon)$  is the probability measure on  $\mathbb{R}^{(2N+1)^2}$  with density  $\mathcal{Z}_N(\varepsilon)^{-1} e^{-V_N(z)/\varepsilon} dz$ .

The result will follow if we are able to prove the estimate

$$\operatorname{cap}_A(B) = \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0 < |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} [1 + r(\varepsilon)] \quad (3.15)$$

on the capacity with  $-c_- \sqrt{\varepsilon} \leq r(\varepsilon) \leq c_+ \varepsilon$ , as well as an estimate on the partition function of the form

$$\frac{1}{2} \mathcal{Z}_N(\varepsilon) = \prod_{|k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k + 3}} e^{L^2/4\varepsilon} e^{3L^2 C_N/2} [1 - r(\varepsilon)]. \quad (3.16)$$

Here the exponent  $L^2/4$  is the value of the original potential (1.4) at the stable stationary solution  $\phi_-$ , while the term  $3L^2 C_N/2$  is due to the renormalisation, which makes the potential well deeper and deeper as  $N \rightarrow \infty$ . Note that owing to the expression (2.6) of  $C_N$ , this is exactly the extra term transforming the divergent Fredholm determinant into a converging Carleman–Fredholm determinant.

## 4 Lower bound on the expected transition time

We will start by deriving a lower bound on the expected transition time, i.e., we will prove the upper bound on the capacity (3.15) (in Section 4.1) and the lower bound on the partition function (3.16) (in Section 4.2). These bounds are somewhat simpler to obtain than the matching lower and upper bound in Section 5.

For the upper bound on  $\operatorname{cap}_A(B)$  we choose a specific test-function  $h_+$  (defined in (4.1)) and plug it into the variational characterisation (3.9) of the capacity. The specific form of  $h_+$  is similar to the one used in [8]. It is given by (an approximation to) the explicit minimiser of the one-dimensional version of (3.9) in the  $z_0$  direction and chosen to be constant in all transversal directions. Then an explicit calculation with this test function leads to an upper bound on the capacity which consists of the desired pre-factor multiplied by the Gaussian expectation of the exponential of a negative fourth Wick power (see equation (4.4) below). Here the renormalisation introduces a difficulty, because the Wick power is not bounded from below uniformly in  $N$ . In order to analyse this expression, we invoke techniques from constructive Quantum Field Theory, more precisely, the Nelson argument (see [25]). This is implemented in two steps in Proposition 4.1 and Proposition 4.2 leading to the desired bound.

In order to derive the lower bound on the  $\mathcal{Z}_N(\varepsilon)$  we use the symmetry of the system to reduce to the integration over a half space. We then recentre the field around the minimiser 1 and perform the corresponding change of coordinates in the expression for  $V_N$ . At this point we use the binomial identity for Wick powers (see (A.3)) as well as the

transformation rules for changing the “mass” in a Wick power (see (A.5)), resulting again in a Gaussian expectation of the exponential of a quartic Wick polynomial. This time the Gaussian covariance changes with respect to Section 4.1 and the Wick polynomial is defined with respect to this modified measure and also has a non-trivial cubic term. The “renormalisation” factor  $e^{3L^2C_N/2}$  in (3.16) appears at this point, because the change of Gaussian measure is given by a quadratic Wick power which produces an extra term in the Gaussian normalisation constant. Finally, the Gaussian expectation is treated with a relatively simple argument using Jensen’s inequality in Proposition 4.4.

#### 4.1 Upper bound on the capacity

One can obtain an upper bound on the capacity by inserting any test function in the right-hand side of (3.8). Let  $\delta > 0$  be a small constant and define

$$h_+(z) = \begin{cases} 1 & \text{if } z_0 \leq -\delta, \\ \frac{\int_{z_0}^{\delta} e^{-|\lambda_0|t^2/2\varepsilon} dt}{\int_{-\delta}^{\delta} e^{-|\lambda_0|t^2/2\varepsilon} dt} & \text{if } -\delta < z_0 < \delta, \\ 0 & \text{if } z_0 \geq \delta. \end{cases} \quad (4.1)$$

Although  $|\lambda_0| = 1$ , we will keep  $\lambda_0$  in the notation as it allows to keep track of its influence on the result. Observe that

$$\|\nabla h_+(z)\|^2 = \begin{cases} \frac{e^{-|\lambda_0|z_0^2/\varepsilon}}{\left(\int_{-\delta}^{\delta} e^{-|\lambda_0|t^2/2\varepsilon} dt\right)^2} & \text{if } -\delta < z_0 < \delta, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Note that

$$\left(\int_{-\delta}^{\delta} e^{-|\lambda_0|t^2/2\varepsilon} dt\right)^2 = \frac{2\pi\varepsilon}{|\lambda_0|} [1 + \mathcal{O}(e^{-\delta^2/2\varepsilon})], \quad (4.3)$$

where here and throughout the paper, the notation  $f(\varepsilon) = \mathcal{O}(g(\varepsilon))$  indicates that there exists  $\varepsilon_0 > 0$  such that  $|f(\varepsilon)|$  is bounded by a constant times  $|g(\varepsilon)|$  uniformly in  $\varepsilon \in (0, \varepsilon_0)$ . Note that here the parameter  $\delta$  is small but fixed. Inserting (4.3) in (3.8) we get

$$\begin{aligned} \text{cap}_A(B) &\leq \frac{|\lambda_0|}{2\pi} \int_{\mathbb{R}^{(2N+1)^2}} e^{-[V_N(z) + |\lambda_0|z_0^2]/\varepsilon} dz [1 + \mathcal{O}(e^{-\delta^2/2\varepsilon})] \\ &= \frac{|\lambda_0|}{2\pi} \varepsilon^{\frac{1}{2}(2N+1)^2} \int_{\mathbb{R}^{(2N+1)^2}} e^{-[V_N(\sqrt{\varepsilon}y)/\varepsilon + |\lambda_0|y_0^2]} dy [1 + \mathcal{O}(e^{-\delta^2/2\varepsilon})]. \end{aligned} \quad (4.4)$$

Using the scaling  $z = \sqrt{\varepsilon}y$  in (3.4), which amounts to working with the scaled field

$$\phi_N = \sqrt{\varepsilon} \hat{\phi}_N, \quad (4.5)$$

shows that the exponent can be written in the form

$$\frac{1}{\varepsilon} [V_N(\sqrt{\varepsilon}y) + \varepsilon|\lambda_0|y_0^2] = g_N(y) + \varepsilon w_N(y), \quad (4.6)$$

where

$$\begin{aligned} g_N(y) &= \frac{1}{2}|\lambda_0|y_0^2 + \frac{1}{2} \sum_{0 < |k| \leq N} \lambda_k |y_k|^2, \\ w_N(y) &= \frac{1}{4} \int_{\mathbb{T}^2} (\hat{\phi}_N^4(x) - 6C_N \hat{\phi}_N^2(x) + 3C_N^2) dx. \end{aligned} \quad (4.7)$$

The quadratic form  $g_N$  allows us to define a Gaussian probability measure  $d\gamma(y) = \mathcal{N}^{-1} e^{-g(y)} dy$ , with normalisation

$$\mathcal{N} = \sqrt{\frac{2\pi}{|\lambda_0|}} \prod_{k \neq 0, |k| \leq N} \sqrt{\frac{2\pi}{\lambda_k}}. \quad (4.8)$$

We can thus rewrite the upper bound (4.4) in the form

$$\text{cap}_A(B) \leq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{k \neq 0, |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \mathbb{E}^\gamma [e^{-\varepsilon w_N}] [1 + \mathcal{O}(e^{-\delta^2/2\varepsilon})]. \quad (4.9)$$

The term  $\mathbb{E}^\gamma [e^{-\varepsilon w_N}]$  can be estimated using the Gaussian calculus developed in Appendix A. Indeed, the law of the field  $\hat{\phi}_N$  under  $\gamma$  is exactly as described there. Furthermore,  $C_N = \mathbb{E}^\gamma[\hat{\phi}_N(x)^2]$  for each  $x \in \mathbb{T}^2$  so that the term  $w_N(y)$ , defined in (4.7), can be rewritten as the integral over the fourth Wick power of  $\hat{\phi}$  with respect to this Gaussian measure, that is,

$$w_N = \frac{1}{4} \int_{\mathbb{T}^2} H_4(\hat{\phi}_N, C_N) dx, \quad (4.10)$$

(where  $H_4(X, C) = X^4 - 6CX^2 + 3C^2$ , see (A.1) for the definition of the Hermite polynomial  $H_n$ ). In particular  $w_N$  has zero mean under the Gaussian measure  $d\gamma$  and according to (A.13) all its stochastic moments are uniformly bounded in  $N$ .

We now derive a uniform-in- $N$  bound on  $\mathbb{E}^\gamma [e^{-w_N}]$  following a classical argument due to Nelson (see e.g. [16, Sec 8.6] or [10, Sec. 4]).

**Proposition 4.1.** *There exists a constant  $K$ , independent of  $N$ , such that*

$$\mathbb{E}^\gamma [e^{-w_N}] \leq K. \quad (4.11)$$

PROOF: First note that the definition (A.2) of Hermite polynomials implies for any  $M \in \mathbb{N}$

$$H_4(\hat{\phi}_M, C_M) = (\hat{\phi}_M^2(x) - 3C_M)^2 - 6C_M^2,$$

so that

$$w_M \geq -\frac{3}{2}L^2C_M^2 =: -D_M.$$

Since  $\mathbb{E}^\gamma [e^{-w_N} 1_{\{w_N \geq 0\}}] \leq \mathbb{P}^\gamma \{w_N \geq 0\} \leq 1$ , it is sufficient to bound

$$\begin{aligned} \mathbb{E}^\gamma [e^{-w_N} 1_{\{w_N < 0\}}] &= 1 + \int_0^\infty e^t \mathbb{P}^\gamma \{-w_N > t\} dt \\ &\leq e + \int_1^\infty e^t \mathbb{P}^\gamma \{-w_N > t\} dt. \end{aligned}$$

If  $t \geq D_N$ , then  $\mathbb{P}^\gamma\{-w_N > t\} = 0$ , otherwise we have for any  $M$

$$\begin{aligned}\mathbb{P}^\gamma\{-w_N > t\} &\leq \mathbb{P}^\gamma\{w_M - w_N > t - D_M\} \\ &\leq \mathbb{P}^\gamma\{|w_M - w_N|^{p(t)} > |t - D_M|^{p(t)}\},\end{aligned}$$

for any choice of  $p(t) \in 2\mathbb{N}$ . We apply this inequality for  $M = M(t)$  satisfying

$$t - D_{M(t)} \geq 1, \quad (4.12)$$

which implies that  $M(t) < N$ .

Then we get by Markov's inequality and Lemma A.4 combined with (A.11) (cf. (A.14))

$$\begin{aligned}\mathbb{P}^\gamma\{-w_N > t\} &\leq \mathbb{E}^\gamma[|w_{M(t)} - w_N|^{p(t)}] \\ &\leq \text{const} (p(t) - 1)^{p(t)} \mathbb{E}^\gamma[|w_{M(t)} - w_N|^{2\gamma p(t)/2}] \\ &\leq \text{const} \frac{(p(t) - 1)^{p(t)}}{M(t)^{(1-\eta)p(t)/2}}\end{aligned}$$

for any  $\eta > 0$ . The condition (4.12) on  $M(t)$  imposes that  $\log M(t)$  grows at most as  $t^{1/2}$ . Choosing for instance  $p(t) \sim t^\beta$  for  $\beta > \frac{1}{2}$ , since

$$\log(e^t \mathbb{P}^\gamma\{-w_N > t\}) \leq t + \beta t^\beta \log t - c(1 - \eta)t^{\beta+1/2}$$

for a constant  $c > 0$  depending only on  $L$ , we obtain a convergent integral.  $\square$

This a priori estimate can now quite easily be turned into a sharper estimate. Indeed, we have the following improvement.

**Proposition 4.2.** *We have*

$$\mathbb{E}^\gamma[e^{-\varepsilon w_N}] = 1 + \mathcal{O}(\varepsilon), \quad (4.13)$$

where the remainder is bounded uniformly in  $N$ .

PROOF: Introduce the sets

$$\Omega_+ = \{\hat{\phi}_N: w_N > 0\}$$

and  $\Omega_- = \Omega_+^c$ . Then we have

$$\mathbb{E}^\gamma[e^{-\varepsilon w_N} 1_{\Omega_+}] = \mathbb{P}^\gamma(\Omega_+) + \mathbb{E}^\gamma[(e^{-\varepsilon w_N} - 1)1_{\Omega_+}].$$

Note that on  $\Omega_+$ , we have  $-\varepsilon|w_N| \leq e^{-\varepsilon w_N} - 1 \leq 0$ , so that

$$\mathbb{P}^\gamma(\Omega_+) - \varepsilon \mathbb{E}^\gamma[|w_N|1_{\Omega_+}] \leq \mathbb{E}^\gamma[e^{-\varepsilon w_N} 1_{\Omega_+}] \leq \mathbb{P}^\gamma(\Omega_+).$$

Since  $U_{4,N}$  has finite variance bounded uniformly in  $N$ , we know by Cauchy–Schwarz that  $\mathbb{E}^\gamma[|U_{4,N}|]$  is bounded uniformly in  $N$ . Similarly, we have

$$\mathbb{E}^\gamma[e^{-\varepsilon w_N} 1_{\Omega_-}] = \mathbb{P}^\gamma(\Omega_-) + \mathbb{E}^\gamma[(e^{-\varepsilon w_N} - 1)1_{\Omega_-}].$$

This time, we use that on  $\Omega_-$ , one has  $0 \leq e^{-\varepsilon w_N} - 1 \leq \varepsilon|w_N|e^{-\varepsilon w_N}$ . Thus by Cauchy–Schwarz,

$$\begin{aligned}0 \leq \mathbb{E}^\gamma[(e^{-\varepsilon w_N} - 1)1_{\Omega_-}] &\leq \varepsilon \mathbb{E}^\gamma[|w_N|e^{-\varepsilon w_N} 1_{\Omega_-}] \\ &\leq \varepsilon \left( \mathbb{E}^\gamma[e^{-2\varepsilon w_N} 1_{\Omega_-}] \mathbb{E}^\gamma[|w_N|^2 1_{\Omega_-}] \right)^{1/2}\end{aligned}$$

The term  $\mathbb{E}^\gamma[|w_N|^2]$  is bounded uniformly in  $N$  as before, while the term  $\mathbb{E}^\gamma[e^{-2\varepsilon w_N}]$  is bounded uniformly in  $N$  for  $\varepsilon \leq 2$  by Proposition 4.1. Summing the two estimates, we get the result.  $\square$

Substituting this estimate in (4.9), we immediately get the following upper bound on the capacity.

**Corollary 4.3.** *There exists a constant  $c_+$ , uniform in  $\varepsilon$  and  $N$ , such that the capacity satisfies the upper bound*

$$\text{cap}_A(B) \leq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{k \neq 0, |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} [1 + c_+\varepsilon]. \quad (4.14)$$

## 4.2 Lower bound on the partition function

By symmetry, cf. (3.12), the partition function can be computed using the relation

$$\frac{1}{2} \mathcal{Z}_N(\varepsilon) = \int_{\Omega'_+} e^{-V_N(z)/\varepsilon} dz, \quad \Omega'_+ = \{z_0 > 0\}. \quad (4.15)$$

A lower bound on  $\mathcal{Z}_N(\varepsilon)$  can be obtained quite directly from Jensen's inequality. It will be convenient to shift coordinates to the positive stable stationary solution of the deterministic equation (without the normalisation). That is, we set

$$\phi_N(x) = 1 + \sqrt{\varepsilon} \hat{\phi}_{N,+}(x), \quad (4.16)$$

with the Fourier decomposition

$$\hat{\phi}_{N,+}(x) = \sum_{|k| \leq N} y_k e_k(x). \quad (4.17)$$

Let  $\hat{\Omega}'_+ = \{y_0 > -1/\sqrt{\varepsilon}\}$  denote the image of  $\Omega'_+$  under this transformation. Substituting in (2.4) and using the relation (A.3) yields the following expression for the potential:

$$\begin{aligned} V_N^+(y) &:= \frac{1}{\varepsilon} V_N[1 + \sqrt{\varepsilon} \hat{\phi}_{N,+}(x)] \\ &= -\frac{L^2}{4\varepsilon} + \frac{1}{2} \int_{\mathbb{T}^2} \left( \|\nabla \hat{\phi}_{N,+}(x)\|^2 - \hat{\phi}_{N,+}^2(x) + 3H_2(\hat{\phi}_{N,+}, C_N) \right) dx \\ &\quad + \frac{1}{4} \int_{\mathbb{T}^2} \left( 4\sqrt{\varepsilon} H_3(\hat{\phi}_{N,+}(x), C_N) + \varepsilon H_4(\hat{\phi}_{N,+}(x), C_N) \right) dx. \end{aligned} \quad (4.18)$$

Now the relevant Gaussian measure  $\gamma_+$  is defined by the quadratic form

$$\begin{aligned} g_{N,+}(y) &= \frac{1}{2} \int_{\mathbb{T}^2} \left( \|\nabla \hat{\phi}_{N,+}(x)\|^2 - \hat{\phi}_{N,+}^2(x) + 3\hat{\phi}_{N,+}^2(x) \right) dx \\ &= \frac{1}{2} \sum_{0 < |k| \leq N} (\lambda_k + 3) |y_k|^2. \end{aligned} \quad (4.19)$$

Observe that a term  $-\frac{3}{2} C_N L^2$  appears owing to the Hermite polynomial  $3H_2(\hat{\phi}_{N,+}, C_N)$ . It is precisely this term which is ultimately responsible for the renormalisation of the pre-factor. To bound expectations of the terms appearing in the last line of (4.18) it is convenient to rewrite them as Wick powers with respect to the Gaussian measure defined by  $g_{N,+}$ . The associated renormalisation constant is

$$C_{N,+} = \frac{1}{L^2} \sum_{0 < |k| \leq N} \frac{1}{\lambda_k + 3}. \quad (4.20)$$

Observe in particular that

$$C_N - C_{N,+} = \frac{1}{L^2} \sum_{0 < |k| \leq N} \frac{3}{|\lambda_k|(\lambda_k + 3)} \quad (4.21)$$

is bounded uniformly in  $N$ . Using the relation (A.5) that allows to transform Hermite polynomials with respect to different constants we get

$$\begin{aligned} \sqrt{\varepsilon} H_3(\hat{\phi}_{N,+}, C_N) &= \sqrt{\varepsilon} H_3(\hat{\phi}_{N,+}, C_{N,+}) - 3\sqrt{\varepsilon}(C_N - C_{N,+})\hat{\phi}_{N,+} \\ \frac{\varepsilon}{4} H_4(\hat{\phi}_{N,+}, C_N) &= \frac{\varepsilon}{4} H_4(\hat{\phi}_{N,+}, C_{N,+}) - \frac{3}{2}\varepsilon(C_N - C_{N,+})H_2(\hat{\phi}_{N,+}, C_N) \\ &\quad + \frac{3}{4}\varepsilon(C_N - C_{N,+})^2. \end{aligned} \quad (4.22)$$

Now we define the random variables

$$U_{n,N}^+ = \int_{\mathbb{T}^2} : \hat{\phi}_{N,+}^n(x) : dx = \int_{\mathbb{T}^2} H_n(\hat{\phi}_{N,+}(x), C_{N,+}) dx \quad (4.23)$$

which have zero mean under  $\gamma_+$  as well as a variance bounded uniformly in  $N$ . Substituting (4.22) in (4.18), we get

$$V_N^+(y) = q + g_{N,+}(y) + w_{N,+}(y), \quad (4.24)$$

where

$$\begin{aligned} q &= -\frac{L^2}{4\varepsilon} - \frac{3}{2}L^2C_N + \frac{3}{4}L^2\varepsilon(C_N - C_{N,+})^2 \\ w_{N,+}(y) &= \sqrt{\varepsilon}U_{3,N}^+ + \frac{1}{4}\varepsilon U_{4,N}^+ - 3(C_N - C_{N,+})\left(\frac{\varepsilon}{2}U_{2,N}^+ + \sqrt{\varepsilon}U_{1,N}^+\right). \end{aligned} \quad (4.25)$$

It follows by a similar argument as in the previous section that

$$\frac{1}{2}\mathcal{Z}_N(\varepsilon) = \prod_{|k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k + 3}} e^{-q} \mathbb{E}^{\gamma_+} \left[ e^{-w_{N,+}} 1_{\hat{\Omega}'_+} \right]. \quad (4.26)$$

**Proposition 4.4.** *There exists a constant  $c_-$ , independent of  $N$  and  $\varepsilon$ , such that*

$$\mathbb{E}^{\gamma_+} \left[ e^{-w_{N,+}} 1_{\hat{\Omega}'_+} \right] \geq 1 - e^{-c_-/\varepsilon}. \quad (4.27)$$

PROOF: Recall that  $w_{N,+}$  has zero expectation under  $\gamma_+$ . Jensen's inequality yields

$$\begin{aligned} \mathbb{E}^{\gamma_+} \left[ e^{-w_{N,+}} 1_{\hat{\Omega}'_+} \right] &= \mathbb{P}^{\gamma_+}(\hat{\Omega}'_+) \mathbb{E}^{\gamma_+} \left[ e^{-w_{N,+}} \mid \hat{\Omega}'_+ \right] \\ &\geq \mathbb{P}^{\gamma_+}(\hat{\Omega}'_+) e^{-\mathbb{E}^{\gamma_+} [w_{N,+} \mid \hat{\Omega}'_+]} \\ &= \mathbb{P}^{\gamma_+}(\hat{\Omega}'_+) e^{-\mathbb{E}^{\gamma_+} [w_{N,+} 1_{\hat{\Omega}'_+}] / \mathbb{P}^{\gamma_+}(\hat{\Omega}'_+)}. \end{aligned}$$

Note that the event  $\hat{\Omega}'_+$  simply says that the first marginal of the Gaussian distribution  $\gamma_+$  is larger than a constant. Since this marginal is a one-dimensional Gaussian distribution, centred at the positive stationary solution, standard tail estimates show that there is a constant  $c_0 > 0$  such that

$$\mathbb{P}^{\gamma_+}(\hat{\Omega}'_+) \geq 1 - e^{-c_0/\varepsilon}.$$



Furthermore, there is a constant  $K$  such that uniformly in  $N$  and for  $n = 1, 2, 3, 4$

$$\begin{aligned} |\mathbb{E}^{\gamma^+}[U_{n,N}^+ 1_{\hat{\Omega}'_+}]| &= |\mathbb{E}^{\gamma^+}[U_{n,N}^+ 1_{(\hat{\Omega}'_+)^c}]| \leq \mathbb{E}^{\gamma^+}[|U_{n,N}^+| 1_{(\hat{\Omega}'_+)^c}] \\ &\leq \mathbb{E}^{\gamma^+}[(U_{n,N}^+)^2]^{1/2} \mathbb{P}^{\gamma^+}((\hat{\Omega}'_+)^c)^{1/2} \leq K e^{-c_0/2\varepsilon}. \end{aligned}$$

It thus follows that there exists a constant  $c_1 > 0$  such that

$$\mathbb{E}^{\gamma^+}[w_{N,+} 1_{\hat{\Omega}'_+}] \geq -c_1 e^{-c_0/2\varepsilon},$$

which yields the required estimate  $\mathbb{E}^{\gamma^+}[e^{-w_{N,+}} 1_{\hat{\Omega}'_+}] \geq 1 - e^{-c_-/\varepsilon}$ .  $\square$

Combining this result with (4.26) and Corollary 4.3, we finally obtain the following lower bound on the expected transition times.

**Proposition 4.5.** *There exists a constant  $C_-$ , uniform in  $N$  and  $\varepsilon$ , such that*

$$\mathbb{E}^{\mu_{A,B}}[\tau_B] \geq 2\pi \left( \frac{e^{3/|\lambda_0|}}{|\lambda_0|(\lambda_0 + 3)} \prod_{0 < |k| \leq N} \left[ \frac{e^{3/\lambda_k}}{1 + 3/\lambda_k} \right] \right)^{1/2} e^{L^2/4\varepsilon} [1 - C_- \varepsilon] \quad (4.28)$$

holds for all  $N \geq 1$ .

PROOF: Plugging (4.27) into (4.26), using the upper bound (4.14) on the capacity and substituting in (3.13), we obtain

$$\mathbb{E}^{\mu_{A,B}}[\tau_B] \geq \sqrt{\frac{2\pi}{|\lambda_0|\varepsilon}} \sqrt{\frac{2\pi\varepsilon}{\lambda_0 + 3}} \prod_{0 < |k| \leq N} \sqrt{\frac{\lambda_k}{\lambda_k + 3}} e^{L^2/4\varepsilon} e^{3L^2 C_N/2} [1 + \mathcal{O}(\varepsilon)].$$

Using the fact that

$$\frac{3}{2} L^2 C_N = \frac{3}{2} \left( \frac{1}{|\lambda_0|} + \sum_{0 < |k| \leq N} \frac{1}{\lambda_k} \right)$$

yields the result.  $\square$

## 5 Upper bound on the expected transition time

In this section we derive the upper bound on the expected transition time, i.e., we will prove the upper bound on the partition function (3.16) (in Section 5.2) and the lower bound on the capacity (3.15) (in Section 5.3).

Inspired by [5], we decompose the field  $\phi$  into its mean (given by the zeroth Fourier coefficient  $z_0$ ) and its (rescaled) transversal fluctuations, see (5.1). In fact, in Proposition 3.2 in [5] it was observed that in a similar system the potential  $V_N$  could be bounded from below by a function which only depends on the mean and a uniformly convex function in the transversal direction. In Lemma 5.3 we obtain a similar bound in our setting (but we only state it as a  $z_0$ -dependent lower bound, which is all we need). The point here is that although this lower bound diverges logarithmically as  $N \rightarrow \infty$ , it does not become worse as  $\varepsilon \rightarrow 0$  and thus permits to mimic the Nelson argument in Proposition 5.4 to obtain a bound on the integral over the transversal directions in the partition function, which does not depend too badly on  $z_0$  and  $\varepsilon$ . Once this a priori bound is established we rewrite this transversal integral once more, this time with respect to the Gaussian reference measure

$g_{N,\perp}(z_0, y_\perp)$  (in the terminology of Quantum Field Theory this amounts to a  $z_0$ -dependent change of “mass”) to obtain a sharp upper bound on the integral over the transversal directions (in Proposition 5.7).

The argument for the lower bound on the capacity is similar to [8]. Using the characterisation (3.9) of  $\text{cap}_A(B)$  the lower bound can be obtained by solving a one-dimensional variational problem in the  $z_0$  direction (see (5.26) below). It is here where the assumption that the sets  $A$  and  $B$  are not too small enters.

Finally, in Section 5.4 it remains to treat the integral in the  $z_0$  direction. Combining the bounds of the previous two sections one obtains an upper bound on the ratio  $\frac{\mathcal{Z}_N(\varepsilon)}{2 \text{cap}_A(B)}$  in terms of an integral in  $z_0$  over a function which depends on  $z_0$  but not on  $N$ . This then permits to apply standard one-dimensional Laplace asymptotic to conclude the argument.

## 5.1 Longitudinal-transversal decomposition of the potential

We denote the rescaled fluctuating part of the Fourier expansion (3.1) by

$$\hat{\phi}_{N,\perp}(x) = \hat{\phi}_N(x) - \frac{z_0}{\sqrt{\varepsilon}L} = \sum_{0 < |k| \leq N} y_k e_k(x). \quad (5.1)$$

Note in particular the Parseval identity

$$\int_{\mathbb{T}^2} \hat{\phi}_{N,\perp}^2(x) dx = \sum_{0 < |k| \leq N} |y_k|^2. \quad (5.2)$$

Similarly to (4.18) the potential can be written in the form

$$\begin{aligned} \frac{1}{\varepsilon} V_N(z_0, y_\perp) &= \frac{1}{\varepsilon} q(z_0) + g_{N,\perp}(z_0, y_\perp) + \frac{1}{4} \int_{\mathbb{T}^2} \frac{6z_0^2}{L^2} H_2(\hat{\phi}_{N,\perp}(x), C_N) dx \\ &+ \frac{1}{4} \int_{\mathbb{T}^2} \left( \frac{4z_0}{L} \sqrt{\varepsilon} H_3(\hat{\phi}_{N,\perp}(x), C_N) + \varepsilon H_4(\hat{\phi}_{N,\perp}(x), C_N) \right) dx, \end{aligned} \quad (5.3)$$

where this time

$$\begin{aligned} q(z_0) &= \frac{1}{4L^2} z_0^4 - \frac{1}{2} |\lambda_0| z_0^2, \\ g_{N,\perp}(z_0, y_\perp) &= \frac{1}{2} \sum_{0 < |k| \leq N} \lambda_k |y_k|^2. \end{aligned} \quad (5.4)$$

Here we have used the fact that by assumption  $\int_{\mathbb{T}^2} \hat{\phi}_{N,\perp}(x) dx = 0$ , so that the corresponding term drops. The quadratic form  $g_{N,\perp}$  defines a Gaussian measure  $\gamma_0^\perp$  with normalisation

$$\mathcal{N}_0^\perp = \prod_{0 < |k| \leq N} \sqrt{\frac{2\pi}{\lambda_k}}. \quad (5.5)$$

The associated renormalisation constant is given by

$$\mathbb{E}^{\gamma_0^\perp} [\hat{\phi}_{N,\perp}^2(x)] = \frac{1}{L^2} \sum_{0 < |k| \leq N} \frac{1}{\lambda_k} =: C_N^\perp = C_N - \frac{1}{L^2}. \quad (5.6)$$

As before, we are interested in the Wick powers  $H_n(\hat{\phi}_{N,\perp}, C_N^\perp)$  with respect to this measure and set

$$U_{n,N}^\perp = \int_{\mathbb{T}^2} H_n(\hat{\phi}_{N,\perp}, C_N^\perp) dx, \quad U_n^\perp = \lim_{N \rightarrow \infty} U_{n,N}^\perp. \quad (5.7)$$

By construction, these random variables have (under the measure  $\gamma_0^\perp$ ) zero mean and a variance bounded uniformly in  $N$ . Furthermore we see that

$$\int_{\mathbb{T}^2} H_3(\hat{\phi}_{N,\perp}(x), C_N) dx = U_{3,N}^\perp, \quad (5.8)$$

owing to the fact that  $\hat{\phi}_{N,\perp}$  has zero mean, and

$$\int_{\mathbb{T}^2} H_4(\hat{\phi}_{N,\perp}(x), C_N) dx = U_{4,N}^\perp - \frac{6}{L^2} U_{2,N}^\perp + \frac{3}{L^2}. \quad (5.9)$$

The following expression for the potential then follows immediately from (5.3) and (A.5).

**Proposition 5.1.** *The potential can be decomposed as*

$$\frac{1}{\varepsilon} V_N(z_0, y_\perp) = \frac{1}{\varepsilon} q(z_0) + q_1(z_0, \varepsilon) + g_{N,\perp}(y_\perp) + w_{N,\perp}(z_0, y_\perp), \quad (5.10)$$

where  $q(z_0)$  and  $g_{N,\perp}(y_\perp)$  are given in (5.4), and

$$\begin{aligned} q_1(z_0, \varepsilon) &= -\frac{3z_0^2}{2L^2} + \frac{3\varepsilon}{4L^2}, \\ w_{N,\perp}(z_0, y_\perp) &= \frac{3(z_0^2 - \varepsilon)}{2L^2} U_{2,N}^\perp + \frac{z_0}{L} \sqrt{\varepsilon} U_{3,N}^\perp + \frac{1}{4} \varepsilon U_{4,N}^\perp. \end{aligned} \quad (5.11)$$

## 5.2 Upper bound on the partition function

In order to obtain an upper bound on  $\mathcal{Z}_N(\varepsilon)$ , we will first perform the integration over the fluctuating modes  $y_\perp$ , and then the integration over the mean value  $z_0$ . The basic observation is the following rewriting of  $\mathcal{Z}_N(\varepsilon)$ .

**Proposition 5.2.** *The partition function is given by the integral*

$$\mathcal{Z}_N(\varepsilon) = \int_{-\infty}^{\infty} e^{-q(z_0)/\varepsilon} g(z_0, \varepsilon) dz_0, \quad (5.12)$$

where

$$g(z_0, \varepsilon) = e^{-q_1(z_0, \varepsilon)} \prod_{0 < |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \mathbb{E}_{\gamma_0^\perp} [e^{-w_{N,\perp}(z_0, \cdot)}]. \quad (5.13)$$

By standard, one-dimensional Laplace asymptotics, we expect the integral (5.12) to be close to  $2\sqrt{\pi\varepsilon} e^{L^2/4\varepsilon} g(L, \varepsilon)$ . As  $g$  depends strongly on  $N$ , one has to use some care when performing the Laplace asymptotics. The solution to this difficulty is to not carry out the Laplace asymptotics directly for  $\mathcal{Z}_N(\varepsilon)$ , but instead for the ratio  $\mathcal{Z}_N(\varepsilon)/\text{cap}_A(B)$ , see Section 5.4 below. Our aim is thus to bound the expectation in (5.13). In order to apply a Nelson estimate, we will need a lower bound on  $w_{N,\perp}(z_0, y_\perp)$ . In fact, for later use (see Proposition 5.7), we will derive a lower bound for the slightly more general quantity

$$w_{N,\perp}^{(\mu)}(z_0, y_\perp) = \frac{3z_0^2}{2L^2} U_{2,N}^\perp + \mu \left( -\frac{3\varepsilon}{2L^2} U_{2,N}^\perp + \frac{z_0}{L} \sqrt{\varepsilon} U_{3,N}^\perp + \frac{1}{4} \varepsilon U_{4,N}^\perp \right), \quad (5.14)$$

where  $\mu$  is a real parameter. Note in particular that  $w_{N,\perp}^{(1)}(z_0, y_\perp) = w_{N,\perp}(z_0, y_\perp)$ . The proof of the following simple but useful lower bound is inspired by Proposition 3.2 in [5].

**Lemma 5.3.** For any  $N \in \mathbb{N}$ ,  $z_0 \in \mathbb{R}$  and  $\mu \in (0, \frac{3}{2})$ ,

$$w_{N,\perp}^{(\mu)}(z_0, y_\perp) \geq -D_N(z_0, \mu, \varepsilon), \quad (5.15)$$

where

$$D_N(z_0, \mu, \varepsilon) = \frac{3}{2} z_0^2 C_N + \frac{3}{4} \mu \varepsilon C_N^2 L^2 \left( \frac{3}{1 - 2\mu/3} - 1 \right). \quad (5.16)$$

PROOF: Using the definition (5.7) of Wick powers, we see that

$$\begin{aligned} w_{N,\perp}^{(\mu)}(z_0, y_\perp) &= \frac{1}{4} \int_{\mathbb{T}^2} \hat{\phi}_{N,\perp}^2(x) \left[ \mu \varepsilon \hat{\phi}_{N,\perp}^2(x) + 4\mu \frac{z_0}{L} \sqrt{\varepsilon} \hat{\phi}_{N,\perp}(x) + 6 \frac{z_0^2}{L^2} - 6\mu \varepsilon C_N \right] dx \\ &\quad - \frac{3}{2} (z_0^2 - \mu \varepsilon) C_N^\perp + \frac{3}{4} \mu \varepsilon (C_N^\perp)^2 L^2, \end{aligned} \quad (5.17)$$

where we have used the fact that  $C_N^\perp + \frac{1}{L^2} = C_N$  as well as  $\int_{\mathbb{T}^2} \hat{\phi}_{N,\perp}(x) dx = 0$ . A completion-of-squares argument shows that the term in square brackets in (5.17) is bounded below by

$$\mu \varepsilon \left( 1 - \frac{2}{3} \mu \right) \hat{\phi}_{N,\perp}^2(x) - 6\mu \varepsilon C_N.$$

Performing a second completion of squares shows that the integral in (5.17) is bounded below by

$$\frac{\mu \varepsilon}{4} \int_{\mathbb{T}^2} \left[ \hat{\phi}_{N,\perp}^4(x) \left( 1 - \frac{2}{3} \mu \right) - 6C_N \hat{\phi}_{N,\perp}^2(x) \right] dx \geq -\frac{9\mu \varepsilon C_N^2 L^2}{4(1 - 2\mu/3)}.$$

The result follows, bounding  $C_N^\perp$  above by  $C_N$ .  $\square$

We are now in a position to imitate the proof of Proposition 4.1, to show the following upper bound.

**Proposition 5.4.** *There exist constants  $M(\mu)$  and  $\varepsilon_0(\mu)$ , uniform in  $N$ ,  $\varepsilon$  and  $z_0$ , such that*

$$\mathbb{E}^{\gamma_0^\perp} \left[ e^{-w_{N,\perp}^{(\mu)}(z_0, \cdot)} \right] \leq M(\mu) \left[ 1 + \sqrt{\varepsilon} e^{M(\mu) z_0^2 \log(1+z_0^2)/\sqrt{\varepsilon}} \right] \quad (5.18)$$

holds for any  $\mu \in (0, \frac{3}{2})$  and all  $\varepsilon < \varepsilon_0(\mu)$ .

PROOF: We will give the proof for  $w_\perp^{(\mu)} = \lim_{N \rightarrow \infty} w_{N,\perp}^{(\mu)}(z_0, \cdot)$ , since the same proof applies for any finite  $N$ . To be able to apply the integration-by-parts formula

$$\begin{aligned} \mathbb{E}^{\gamma_0^\perp} \left[ e^{-w_\perp^{(\mu)}} \right] &\leq e + \int_1^\infty e^t \mathbb{P}^{\gamma_0^\perp} \{ -w_\perp^{(\mu)} > t \} dt \\ &= e \left[ 1 + \int_0^\infty e^t \mathbb{P}^{\gamma_0^\perp} \{ -w_\perp^{(\mu)} > 1 + t \} dt \right], \end{aligned}$$

we have to estimate  $\mathbb{P}^{\gamma_0^\perp} \{ -w_\perp^{(\mu)} > t \}$  when  $t \geq 1$ . For any such  $t$ , we pick an  $N(t) \in \mathbb{N}$  such that

$$t - D_{N(t)}(z_0, \mu, \varepsilon) \geq 1.$$

Note that by (5.16), there exists a constant  $M_0(\mu)$ , uniform in  $N$ ,  $\varepsilon$  and  $z_0$ , such that

$$D_N(z_0, \mu, \varepsilon) \leq M_0(\mu) \left[ \varepsilon (\log N)^2 + z_0^2 \log N \right].$$

The condition on  $N(t)$  is thus satisfied if we impose the condition

$$\varepsilon(\log N(t))^2 + z_0^2 \log N(t) \leq \frac{t-1}{M_0(\mu)}. \quad (5.19)$$

By Lemma 5.3 and the above condition, we have

$$\mathbb{P}\gamma_0^\perp\{-w_1^{(\mu)} > t\} \leq \mathbb{P}\gamma_0^\perp\{|w_{N(t),\perp}^{(\mu)} - w_1^{(\mu)}| > 1\}.$$

Now observe (c.f. (5.14)) that

$$w_{N(t),\perp}^{(\mu)} - w_1^{(\mu)} = \sum_{j=2}^4 a_j(z_0, \varepsilon)(U_{j,N(t)}^\perp - U_j^\perp),$$

with

$$a_2 = \frac{3}{2L^2}(z_0^2 - \mu\varepsilon), \quad a_3 = \frac{z_0}{L}\sqrt{\varepsilon}, \quad a_4 = \frac{1}{4}\varepsilon.$$

It follows that

$$\mathbb{P}\gamma_0^\perp\{|w_{N(t),\perp}^{(\mu)} - w_1^{(\mu)}| > 1\} \leq \sum_{j=2}^4 P_j, \quad P_j = \mathbb{P}\gamma_0^\perp\left\{|a_j||U_{j,N(t)}^\perp - U_j^\perp| > \frac{1}{3}\right\}.$$

For any choice of  $p_j(t) \in 2\mathbb{N}$ , we have by Markov's inequality

$$P_j \leq |3a_j(z_0, \varepsilon)|^{p_j(t)} E_j, \quad E_j = \mathbb{E}\gamma_0^\perp[|U_{j,N(t)}^\perp - U_j^\perp|^{p_j(t)}]$$

where by Nelson's estimate (A.12)

$$E_j \leq \text{const}(p_j(t) - 1)^{jp_j(t)/2} \mathbb{E}\gamma_0^\perp[|U_{j,N(t)}^\perp - U_j^\perp|^2]^{p_j(t)/2} \leq \text{const} \frac{(p_j(t) - 1)^{jp_j(t)/2}}{N(t)^{p_j(t)}}.$$

A possible choice is to take (where  $a \wedge b := \min\{a, b\}$  and  $\lfloor a \rfloor$  is the integer part of  $a$ )

$$p_j(t) = 2 \left\lfloor (t-1)^{1/2} \wedge \frac{z_0^2}{\sqrt{\varepsilon}} \right\rfloor$$

$$\log N(t) = \frac{1}{M_1} \left\lfloor \left( \frac{t-1}{\varepsilon} \right)^{1/2} \wedge \frac{t-1}{z_0^2} \right\rfloor,$$

with  $M_1$  large enough to satisfy (5.19). Indeed, this yields  $\log(N(t)^{p_j(t)}) \simeq c(t-1)/\sqrt{\varepsilon}$  for some  $c = c(\mu) > 0$ , and thus

$$\mathbb{P}\gamma_0^\perp\{-w_1^{(\mu)} > t\} \leq \text{const} e^{c' \log(1+z_0^2)z_0^2/\sqrt{\varepsilon}} e^{-c(t-1)/\sqrt{\varepsilon}}$$

for some  $c' > 0$ , where the first exponential is due to the term  $|3a_2|^{p_2(t)}$ . This shows that

$$\int_0^\infty e^t \mathbb{P}\gamma_0^\perp\{-w_1^{(\mu)} > 1+t\} dt \leq \text{const} e^{c' \log(1+z_0^2)z_0^2/\sqrt{\varepsilon}} \left(\frac{c}{\sqrt{\varepsilon}} - 1\right)^{-1}$$

if  $\varepsilon < c^2$ . Substituting in the integration-by-parts formula proves the claim.  $\square$

Our aim is now to sharpen this bound by applying a similar trick as in the proof of Proposition 4.2. To this end, it will be convenient to work with Gaussian measures  $\gamma_{z_0}^\perp$ , defined by the quadratic form

$$g_{N,\perp,z_0}(y_\perp) = \sum_{0 < |k| \leq N} \left[ \lambda_k + \frac{3z_0^2}{L^2} \right] |y_k|^2. \quad (5.20)$$

The following result allows converting between expectations with respect to  $\gamma_0^\perp$  and  $\gamma_{z_0}^\perp$ .

**Lemma 5.5.** *For any random variable  $X = X(y_\perp)$  integrable with respect to  $\gamma_0^\perp$ ,*

$$\mathbb{E}^{\gamma_0^\perp} [X] = K(z_0) \mathbb{E}^{\gamma_{z_0}^\perp} \left[ X e^{3z_0^2 U_{2,N}^\perp / 2L^2} \right], \quad (5.21)$$

where

$$K(z_0) = \left[ \prod_{0 < |k| \leq N} \frac{e^{3z_0^2/L^2 \lambda_k}}{1 + 3z_0^2/L^2 \lambda_k} \right]^{1/2}.$$

PROOF: This follows from a short computation, writing out explicitly the density of  $\gamma_0^\perp$  and expressing  $\sum_k |y_k|^2$  in terms of  $U_{2,N}^\perp$ .  $\square$

**Remark 5.6.** Writing  $\zeta_k = 3z_0^2/L^2 \lambda_k$  and using the fact that the Taylor series of  $\log(1 + \zeta_k)$  is alternating, we obtain

$$2 \log K(z_0) = \sum_{0 < |k| \leq N} [\zeta_k - \log(1 + \zeta_k)] \leq \frac{1}{2} \sum_{0 < |k| \leq N} \zeta_k^2 \leq \text{const } z_0^4. \quad (5.22)$$

This shows that  $K(z_0) \leq e^{M_1 z_0^4}$  for some constant  $M_1$ , independent of  $N$  and  $z_0$ .

We can now state the sharper bound on the expectation of  $e^{-w_{N,\perp}}$ .

**Proposition 5.7.** *There exists a constant  $M > 0$ , uniform in  $N$ ,  $\varepsilon$  and  $z_0$ , such that*

$$\mathbb{E}^{\gamma_0^\perp} [e^{-w_{N,\perp}(z_0, \cdot)}] \leq K(z_0) \left[ 1 + M \sqrt{\varepsilon} (1 + |z_0|) (1 + \sqrt{\varepsilon} e^{M z_0^2 \log(1+z_0^2)/\sqrt{\varepsilon}}) \right]. \quad (5.23)$$

PROOF: To lighten the notation, we drop the argument  $(z_0, \cdot)$  of  $w_{N,\perp}$ . By Lemma 5.5, we have

$$\mathbb{E}^{\gamma_0^\perp} [e^{-w_{N,\perp}}] = K(z_0) \mathbb{E}^{\gamma_{z_0}^\perp} [e^{-\hat{w}_{N,\perp}}]$$

where

$$\hat{w}_{N,\perp} = -\frac{3\varepsilon}{2L^2} U_{2,N}^\perp + \frac{z_0}{L} \sqrt{\varepsilon} U_{3,N}^\perp + \frac{1}{4} \varepsilon U_{4,N}^\perp.$$

As in the proof of Proposition 4.2, we write

$$\begin{aligned} \mathbb{E}^{\gamma_{z_0}^\perp} [e^{-\hat{w}_{N,\perp}}] &\leq 1 + \mathbb{E}^{\gamma_{z_0}^\perp} [(e^{-\hat{w}_{N,\perp}} - 1) 1_{\{\hat{w}_{N,\perp} < 0\}}] \\ &\leq 1 + \mathbb{E}^{\gamma_{z_0}^\perp} [|\hat{w}_{N,\perp}| e^{-\hat{w}_{N,\perp}} 1_{\{\hat{w}_{N,\perp} < 0\}}] \\ &\leq 1 + \mathbb{E}^{\gamma_{z_0}^\perp} [(\hat{w}_{N,\perp})^p]^{1/p} \mathbb{E}^{\gamma_{z_0}^\perp} [e^{-q\hat{w}_{N,\perp}}]^{1/q}. \end{aligned}$$

In the last line, we have used Hölder's inequality, and  $p, q \geq 1$  are Hölder conjugates. It follows from (A.13) that

$$\mathbb{E}^{\gamma_{z_0}^\perp} [(\hat{w}_{N,\perp})^p]^{1/p} \leq \text{const } \sqrt{\varepsilon} (1 + |z_0|).$$

Furthermore, another application of Lemma 5.5 yields

$$\begin{aligned}\mathbb{E} \gamma_{z_0}^\perp [e^{-q\hat{w}_{N,\perp}}] &= \frac{1}{K(z_0)} \mathbb{E} \gamma_0^\perp [e^{-q\hat{w}_{N,\perp} - 3z_0^2 U_{2,N}^\perp / 2L^2}] \\ &= \frac{1}{K(z_0)} \mathbb{E} \gamma_0^\perp [e^{-w_{N,\perp}^{(q)}}].\end{aligned}$$

Applying Proposition 5.4 for some  $q \in (1, \frac{3}{2})$  and combining the different estimates yields the result.  $\square$

### 5.3 Lower bound on the capacity

Assume that  $A = -I \times A_\perp$  and  $B = I \times A_\perp$  where  $\pm I = \pm[1 - \delta, 1 + \delta]$  (with  $0 < \delta < 1$ ) are small intervals around the two stationary solutions. Let  $D = J \times D_\perp$  where  $J = [-\rho, \rho]$  is an interval joining  $-I$  and  $I$ , with  $\rho = 1 - \delta$ , and  $D_\perp \subset A_\perp$  (see Figure 1 – when working with Fourier variables, coordinates are scaled by a factor  $L$ ). Then we have

$$\begin{aligned}\text{cap}_A(B) &\geq \varepsilon \int_D \left| \frac{\partial h_{A,B}(z)}{\partial z_0} \right|^2 e^{-V_N(z)/\varepsilon} dz \\ &\geq \varepsilon \int_{D_\perp} \left[ \inf_{f: f(-\rho)=1, f(\rho)=0} \int_{-\rho}^\rho e^{-V_N(z_0, z_\perp)/\varepsilon} f'(z_0)^2 dz_0 \right] dz_\perp.\end{aligned}\quad (5.24)$$

Writing the Euler–Lagrange equations, it is easy to see that the minimiser for the term in brackets is such that

$$f'(z_0) = \frac{e^{V_N(z_0, z_\perp)/\varepsilon}}{\int_{-\rho}^\rho e^{V_N(y, z_\perp)/\varepsilon} dy}.\quad (5.25)$$

This yields the lower bound

$$\text{cap}_A(B) \geq \varepsilon \int_{D_\perp} \frac{1}{\int_{-\rho}^\rho e^{V_N(z_0, z_\perp)/\varepsilon} dz_0} dz_\perp.\quad (5.26)$$

**Proposition 5.8.** *There exists a constant  $c_- > 0$ , uniform in  $\varepsilon$  and  $N$ , such that*

$$\text{cap}_A(B) \geq \sqrt{\frac{\varepsilon|\lambda_0|}{2\pi}} \prod_{0 < |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} [\mathbb{P} \gamma_0^\perp(\hat{D}_\perp) - c_- \sqrt{\varepsilon}],\quad (5.27)$$

where  $\hat{D}_\perp = D_\perp / \sqrt{\varepsilon}$ .

PROOF: We start by obtaining a lower bound on  $V_N$  in which  $z_0$  is decoupled from the transverse coordinates. Using the expression (5.10) obtained in Proposition 5.1 and the elementary inequality  $2|ab| \leq (a^2/c + cb^2)$  for  $c > 0$ , we obtain

$$\frac{1}{\varepsilon} V_N(z_0, y_\perp) \leq \frac{1}{\varepsilon} q(z_0) + q_1(z_0, \varepsilon) + \frac{z_0^2}{2L^2} + \frac{3z_0^4}{4L^2\sqrt{\varepsilon}} + g_{N,\perp}(y_\perp) + \sqrt{\varepsilon} R(y_\perp, \varepsilon),$$

where

$$R(y_\perp, \varepsilon) = \frac{3}{4L^2} (U_{2,N}^\perp)^2 + \frac{1}{2} \sqrt{\varepsilon} (U_{3,N}^\perp)^2 - \frac{3\sqrt{\varepsilon}}{2L^2} U_{2,N}^\perp + \frac{1}{4} \sqrt{\varepsilon} U_{4,N}^\perp.$$

Substituting in (5.26) (and taking into account the scaling  $z_\perp = \sqrt{\varepsilon}y_\perp$ ) yields

$$\text{cap}_A(B) \geq \frac{\varepsilon}{\mathcal{J}} \prod_{0 < |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \mathbb{E}^{\gamma_0^\perp} [e^{-\sqrt{\varepsilon}R} 1_{\hat{D}_\perp}],$$

where

$$\mathcal{J} = \int_{-\rho}^{\rho} \exp\left\{\frac{1}{\varepsilon}q(z_0) + q_1(z_0, \varepsilon) + \frac{z_0^2}{2L^2} + \frac{3z_0^4}{4L^2\sqrt{\varepsilon}}\right\} dz_0.$$

Since  $q(z_0)$  has a quadratic maximum on  $[-\rho, \rho]$  at 0, standard one-dimensional Laplace asymptotics (see for instance [27, Chapter 3, Theorems 7.1 and 8.1]) show that

$$\mathcal{J} = \sqrt{\frac{2\pi\varepsilon}{|\lambda_0|}} [1 + \mathcal{O}(\sqrt{\varepsilon})].$$

Furthermore, Jensen's inequality implies that

$$\begin{aligned} \mathbb{E}^{\gamma_0^\perp} [e^{-\sqrt{\varepsilon}R} 1_{\hat{D}_\perp}] &= \mathbb{E}^{\gamma_0^\perp} [e^{-\sqrt{\varepsilon}R} \mid \hat{D}_\perp] \mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c) \\ &\geq e^{-\sqrt{\varepsilon}\mathbb{E}^{\gamma_0^\perp}[R \mid \hat{D}_\perp]} \mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c) \\ &\geq \left(1 - \sqrt{\varepsilon} \frac{\mathbb{E}^{\gamma_0^\perp}[|R|]}{\mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c)}\right) \mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c). \end{aligned}$$

Since  $\mathbb{E}^{\gamma_0^\perp}[|R|]$  is bounded uniformly by (A.13), the result follows.  $\square$

The lower bound on the capacity is thus complete, provided we take  $D_\perp$  large enough to capture almost all the mass of  $\gamma_0^\perp$ . A possible choice is as follows.

**Lemma 5.9.** *Assume*

$$D_\perp \supset \prod_{0 < |k| \leq N} [-a_k, a_k] \quad \text{with } a_k = \sqrt{\frac{4\varepsilon \log(\varepsilon^{-1})[1 + \log \lambda_k]}{\lambda_k}}. \quad (5.28)$$

Then for sufficiently small  $\varepsilon$ , one has

$$\mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c) = \mathcal{O}(\varepsilon). \quad (5.29)$$

PROOF: Standard Gaussian tail estimates show that

$$\begin{aligned} \mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c) &\leq \sum_{0 < |k| \leq N} 2e^{-a_k^2 \lambda_k / 2} \\ &= 2 \sum_{0 < |k| \leq N} \exp\{-2 \log(\varepsilon^{-1})[1 + \log \lambda_k]\} \\ &\leq 2\varepsilon^2 \sum_{0 < |k| \leq N} \lambda_k^{-2 \log(\varepsilon^{-1})}. \end{aligned}$$

The last sum is bounded uniformly in  $N$  if  $\varepsilon \leq e^{-1}$ .  $\square$

Note that if  $D_\perp$  is a ball in a Sobolev space as specified in Definition 2.1, then it indeed satisfies (5.28).



## 5.4 Laplace asymptotics and transition times

Combining the results from the last two subsections, we finally obtain the following upper bound on the expected transition time.

**Proposition 5.10.** *There exists a constant  $C_+$ , uniform in  $N$  and  $\varepsilon$ , such that*

$$\mathbb{E}^{\mu_{A,B}}[\tau_B] \leq 2\pi \left( \frac{e^{3/|\lambda_0|}}{|\lambda_0|(\lambda_0 + 3)} \prod_{0 < |k| \leq N} \left[ \frac{e^{3/\lambda_k}}{1 + 3/\lambda_k} \right] \right)^{1/2} e^{L^2/4\varepsilon} [1 + C_+ \sqrt{\varepsilon}] \quad (5.30)$$

holds for all  $N \geq 1$ .

PROOF: It follows from Proposition 5.2, Proposition 5.7 and Proposition 5.8 that

$$\frac{\mathcal{Z}_N(\varepsilon)}{2 \operatorname{cap}_A(B)} \leq \sqrt{\frac{2\pi}{\varepsilon|\lambda_0|}} \int_0^\infty e^{-q(z_0)/\varepsilon} \hat{g}(z_0, \varepsilon) dz_0,$$

where

$$\hat{g}(z_0, \varepsilon) \leq e^{-q_1(z_0, \varepsilon)} K(z_0) \left[ 1 + M' \sqrt{\varepsilon} (1 + |z_0|) (1 + \sqrt{\varepsilon} e^{Mz_0^2 \log(1+z_0^2)/\sqrt{\varepsilon}}) \right]$$

for a constant  $M' \geq M$ . In particular, we have

$$\hat{g}(L, 0) \leq e^{3/2} K(L) = \left( e^{3/|\lambda_0|} \prod_{0 < |k| \leq N} \left[ \frac{e^{3/\lambda_k}}{1 + 3/\lambda_k} \right] \right)^{1/2}.$$

Since  $q$  reaches its minimum  $-L^2/4$  on  $\mathbb{R}_+$  in  $z_0 = L$ , writing

$$\frac{1}{\sqrt{\varepsilon}} \int_0^\infty e^{-[q(z_0) - q(L)]/\varepsilon} \hat{g}(z_0, \varepsilon) dz_0 \leq \mathcal{I}_0 + \sqrt{\varepsilon} \mathcal{I}_1 + \varepsilon \mathcal{I}_2$$

and applying one-dimensional Laplace asymptotics, we obtain

$$\mathcal{I}_0 \leq \sqrt{\pi} \hat{g}(L, 0) (1 + C_+ \sqrt{\varepsilon}),$$

for the leading term, while  $\mathcal{I}_1$  and  $\mathcal{I}_2$  are bounded. □

## A Hermite Polynomials and Wick Powers

In this appendix we recall some well-known facts about Hermite polynomials  $H_n = H_n(X, C)$  we use throughout the article. Recall that they are defined recursively by setting

$$\begin{cases} H_0 = 1, \\ H_n = XH_{n-1} - C \partial_X H_{n-1} \end{cases} \quad n \in \mathbb{N}. \quad (A.1)$$

In particular, we have

$$\begin{aligned} H_1(X, C) &= X & H_2(X, C) &= X^2 - C \\ H_3(X, C) &= X^3 - 3CX & H_4(X, C) &= X^4 - 6CX^2 + 3C^2. \end{aligned} \quad (A.2)$$

The following binomial identity for Hermite polynomials is well-known.

**Lemma A.1** (Binomial formula for Hermite polynomials, [9, Lem. 3.1]). *We have for any  $n \in \mathbb{N}$  and  $X, v, C \in \mathbb{R}$*

$$H_n(X + v, C) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(X, C) v^k. \quad (\text{A.3})$$

We will mostly be interested in Hermite polynomials of centered Gaussian random variables  $X$  and we will typically choose  $C = \mathbb{E}[X^2]$ . In this case the random variable  $H_n(X, \mathbb{E}[X^2])$  is sometimes referred to as the  $n$ -th Wick power of  $X$  and denoted by  $:X^n:$ . The following identity is one of the key properties of Wick powers.

**Lemma A.2** ([26, Lemma 1.1.1]). *Let  $X, Y$  be centered jointly Gaussian random variables. Then*

$$\mathbb{E}[:X^n: (:Y^m:)] = \begin{cases} \mathbb{E}[XY]^n & \text{if } n = m \\ 0 & \text{else.} \end{cases} \quad (\text{A.4})$$

Note that this implies in particular, that for  $n \geq 1$  we have  $\mathbb{E}[:X^n:] = 0$ . In some computations it is convenient for us to change the value of the constant  $C$  appearing in  $H_n$ . This will be relevant when changing the Gaussian reference measure. The following transformation rule, valid for any real  $X, C, \bar{C}$  is easy to check:

$$\begin{aligned} H_1(X, C) &= H_1(X, \bar{C}) \\ H_2(X, C) &= H_2(X, \bar{C}) - (C - \bar{C}) \\ H_3(X, C) &= H_3(X, \bar{C}) - 3(C - \bar{C})H_1(X, \bar{C}) \\ H_4(X, C) &= H_4(X, \bar{C}) - 6(C - \bar{C})H_2(X, \bar{C}) + 3(C - \bar{C})^2. \end{aligned} \quad (\text{A.5})$$

We now use (A.4) to derive some classical facts about (spectral Galerkin approximations of) the two-dimensional massive Gaussian free field and its Wick powers. For any  $N$  and for  $x \in \mathbb{T}^2 = \mathbb{R}^2 / (L\mathbb{Z})^2$  we consider the random field

$$\phi_N(x) = \sum_{|k| \leq N} \frac{z_k}{\sqrt{|\lambda_k + m^2|}} e_k(x), \quad (\text{A.6})$$

where the Fourier basis functions  $e_k(x)$  and the eigenvalues  $\lambda_k$  of  $(-\Delta + 1)$  are defined as in Section 2. The  $z_k$  are complex-valued Gaussian random variables which are independent up to the constraint  $z_k = \bar{z}_{-k}$  which makes  $\phi_N$  a real-valued field and which satisfy

$$\mathbb{E}[z_k z_{-\ell}] = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{else.} \end{cases} \quad (\text{A.7})$$

Note that due to the constraint  $z_k = \bar{z}_{-k}$  these  $(2N + 1)^2$  dependent complex-valued Gaussian random variables can be represented in terms of  $(2N + 1)^2$  independent real-valued random variables. The mass  $m^2 \geq 0$  is a parameter of the model, which in our case only takes either the value 0 or the value 3.

For fixed  $x$  we get

$$\mathbb{E}[\phi_N(x)^2] = \sum_{|k| \leq N} \frac{1}{|\lambda_k + m^2|} =: C_N. \quad (\text{A.8})$$

Note that  $C_N$  diverges logarithmically, which suggests that the random variables  $\phi_N(x)$  for a fixed  $x$  do not converge to a meaningful limit as  $N$  goes to  $\infty$ . However, it is well-known that for any test-function  $\psi$  the random variables  $\int \phi_N(x) \psi(x) dx$  converge in  $L^2$

(with respect to probability) to a Gaussian limiting random variable. We will not make use of this general fact, but only use that the integrals of  $\phi_N(x)$  as well as its Wick powers  $:\phi_N^n(x): = H_n(\phi_N(x), C_N)$  have a uniformly-in- $N$  bounded variance. To see this we write for  $M > N$

$$\begin{aligned}
& \mathbb{E}\left[\int_{\mathbb{T}^2} :\phi_M^n(x): dx \int_{\mathbb{T}^2} :\phi_N^n(y): dy\right] \\
&= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \mathbb{E}[:\phi_M^n(x): :\phi_N^n(y):] dx dy = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \mathbb{E}[\phi_M(x)\phi_N(y)]^n dx dy \\
&= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left(\frac{1}{L} \sum_{|k|\leq N} \frac{e_k(x-y)}{|\lambda_k + m^2|}\right)^n dx dy = L^{2-n} \int_{\mathbb{T}^2} \left(\sum_{|k|\leq N} \frac{e_k(x)}{|\lambda_k + m^2|}\right)^n dx \\
&= L^{2-2n} \sum_{\substack{k_1+k_2+\dots+k_n=0 \\ |k_i|\leq N}} \frac{1}{|\lambda_{k_1} + m^2|} \cdots \frac{1}{|\lambda_{k_n} + m^2|}. \tag{A.9}
\end{aligned}$$

This calculation has several immediate consequences. First of all we can conclude that as announced above the variances of  $\int_{\mathbb{T}^2} :\phi_N^n(x): dx$  are uniformly bounded as  $N$  goes to  $\infty$ :

$$\sup_N \mathbb{E}\left[\left(\int_{\mathbb{T}^2} :\phi_N^n(x): dx\right)^2\right] = L^{2-2n} \sum_{\substack{k_1+k_2+\dots+k_n=0 \\ k_i \in \mathbb{Z}^2}} \frac{1}{|\lambda_{k_1} + m^2|} \cdots \frac{1}{|\lambda_{k_n} + m^2|} < \infty. \tag{A.10}$$

Indeed, the convergence of this sum can be checked easily (e.g. as in [32, Lem. 3.10]). Furthermore, we get for  $M > N$

$$\begin{aligned}
& \mathbb{E}\left[\left(\int_{\mathbb{T}^2} :\phi_M^n(x): dx - \int_{\mathbb{T}^2} :\phi_N^n(x): dx\right)^2\right] \\
&= L^{4-2n} \sum_{\substack{k_1+k_2+\dots+k_n=0 \\ |k_i|\leq M}} \frac{1}{|\lambda_{k_1} + m^2|} \cdots \frac{1}{|\lambda_{k_n} + m^2|} - L^{4-2n} \sum_{\substack{k_1+k_2+\dots+k_n=0 \\ |k_i|\leq N}} \frac{1}{|\lambda_{k_1} + m^2|} \cdots \frac{1}{|\lambda_{k_n} + m^2|} \\
&\leq C_{n,L} \frac{(\log N)^{n-2}}{N^2}, \tag{A.11}
\end{aligned}$$

for a constant  $C_{n,L}$  which depends on  $n, L$  but not on  $N, M$ .

Finally, we recall the definition of Wiener chaos which in this finite dimensional context is the following:

**Definition A.3.** For  $n \in \mathbb{N}_0$  the  $n$ -th (inhomogeneous) Wiener chaos generated by the random variables  $(z_k)_{|k|\leq N}$  is the vector space of real-valued random variables  $X$  which can be written as polynomials of degree at most  $n$  in the finitely many random variables  $z_k$ .

As stated this definition depends on the number of independent Gaussians used to define the Wiener chaos. However, the following classical and important estimate holds true uniformly in that number. See e.g. [10, Thm. 4.1] for a direct proof. This Theorem can also be deduced immediately from the hyper-contractivity of the Ornstein–Uhlenbeck semigroup [26, Thm. 1.4.1].

**Lemma A.4** (Equivalence of moments). *Let  $X$  be a random variable, belonging to the  $n$ -th inhomogeneous Wiener chaos. Then for any  $p \geq 1$  one has*

$$\mathbb{E}[X^{2p}]^{\frac{1}{2p}} \leq C_n (2p-1)^{\frac{n}{2}} \mathbb{E}[X^2]^{\frac{1}{2}} \tag{A.12}$$

where  $C_n$  only depends on  $n$ .

**Remark A.5.** The  $n$ -th *homogeneous* Wiener chaos is defined as the orthogonal complement (with respect to the  $L^2$  scalar product) of the  $n - 1$ -st inhomogeneous Wiener chaos in the  $n$ -th inhomogeneous Wiener chaos. If in the previous Lemma,  $X$  takes values in the homogeneous Wiener chaos, then the estimate holds true with constant  $C_n = 1$ .

Now combining Lemma A.4 with (A.10) we obtain for  $p \geq 1$  that

$$\sup_N \mathbb{E} \left[ \left( \int_{\mathbb{T}^2} : \phi_N^n(x) : dx \right)^{2p} \right] < \infty \quad (\text{A.13})$$

and combining Lemma A.4 with (A.11) we see that for  $M > N$  and any  $p \geq 1$

$$\mathbb{E} \left[ \left( \int_{\mathbb{T}^2} : \phi_M^n(x) : dx - \int_{\mathbb{T}^2} : \phi_N^n(x) : dx \right)^{2p} \right]^{\frac{1}{p}} \leq C_{n,L} (2p - 1) \frac{(\log N)^{n-2}}{N} \quad (\text{A.14})$$

for a constant  $C_{n,L}$  which depends on  $n, L$  but not on  $p, M, N$ .

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