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An analog of Chang inversion formula for weighted Radon transforms in multidimensions^{*}

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Abstract. We study weighted Radon transforms in multidimensions. Our results consist in introducing an analog of Chang approximate inversion formula for such transforms and in describing all weights for which this formula is exact.

Keywords: weighted Radon transforms, inversion formulas

1 Introduction

We consider the weighted Radon transforms R_W defined by the formula

$$R_W f(s, \theta) \stackrel{\text{def}}{=} \int_{x\theta=s} W(x, \theta) f(x) dx, \quad (1)$$
$$(s, \theta) \in \mathbb{R} \times \mathbb{S}^{n-1}, x \in \mathbb{R}^n, n \geq 2,$$

where $W = W(x, \theta)$ is the weight, $f = f(x)$ is a test function; see e.g. [1]. Such transforms arise in many domains of pure and applied mathematics; see e.g. [4], [5], [6], [8], [10], [12]. In the present work we assume that

$$W \text{ is complex - valued,}$$
$$W \in C(\mathbb{R}^n \times \mathbb{S}^{n-1}) \cap L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1}), \quad (2)$$
$$w_0(x) \stackrel{\text{def}}{=} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} W(x, \theta) d\theta \neq 0, x \in \mathbb{R}^n,$$

where $d\theta$ is the element of standard measure on \mathbb{S}^{n-1} , $|\mathbb{S}^{n-1}|$ is the standard measure of \mathbb{S}^{n-1} .

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If $W \equiv 1$, then $R = R_W$ is the classical Radon transform in \mathbb{R}^n ; see for example [6], [9], [11], [15]. Explicit inversion formulas for R were given for the first time in [15].

In dimension $n = 2$, the transforms R_W are also known as weighted ray transforms on the plane; see e.g. [10], [12]. For several important cases of W satisfying (2) for $d = 2$, explicit (and exact) inversion formulas for R_W were obtained in [2], [7], [13], [14], [16].

On the other hand, it seems that no explicit inversion formulas for R_W were given yet in the literature under assumptions (2) for $n \geq 3$, if $W \neq w_0$.

The main results of the present work consist in introducing an analog of Chang approximate (but explicit) inversion formula for R_W under assumptions (2), for $n \geq 3$, and in describing all W for which this formula is exact. These results are presented in the next section.

2 Main Results

We consider the following approximate inversion formulas for R_W under assumptions (2) in dimension $n \geq 2$:

$$f_{appr}(x) \stackrel{def}{=} \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1}w_0(x)} \int_{\mathbb{S}^{n-1}} \mathbb{H} [R_W f]^{(n-1)}(x\theta, \theta) d\theta, \quad (3)$$

$x \in \mathbb{R}^n$, n is even,

$$f_{appr}(x) \stackrel{def}{=} \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}w_0(x)} \int_{\mathbb{S}^{n-1}} [R_W f]^{(n-1)}(x\theta, \theta) d\theta, \quad (4)$$

$x \in \mathbb{R}^n$, n is odd,

$$[R_W f]^{(n-1)}(s, \theta) = \frac{d^{n-1}}{ds^{n-1}} R_W f(s, \theta), \quad s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}, \quad (5)$$

$$\mathbb{H}\phi(s) \stackrel{def}{=} \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\phi(t)}{s-t} dt, \quad s \in \mathbb{R}. \quad (6)$$

For $W \equiv 1$ formulas (3), (4) are exact, i.e. $f_{appr} = f$, and are known as the classical Radon inversion formulas, going back to [15].

As a corollary of the classical Radon inversion formulas and definition (1), formulas (3), (4) for $W \equiv w_0$ are also exact.

Formula (3) for $n = 2$ is known as Chang approximate inversion formula for weighted Radon transforms on the plane. This explicit but approximate inversion formula was suggested for the first time in [3] for the case when

$$W(x, \theta) = \exp(-Da(x, \theta^\perp)), \quad (7)$$

$$Da(x, \theta^\perp) = \int_0^{+\infty} a(x + t\theta^\perp) dt, \quad (8)$$

where a is a non-negative sufficiently regular function on \mathbb{R}^2 with compact support, and $\theta = (\theta_1, \theta_2) \in \mathbb{S}^{n-1}$, $\theta^\perp = (\theta_2, -\theta_1)$. We recall that R_W for W given by (7), (8) is known as attenuated Radon transform on the plane and arises, in particular, in the single photon emission tomography (SPECT). In this case an explicit and simultaneously exact inversion formula for R_W was obtained for the first time in [13].

We emphasize that formulas (3), (4) are approximate, in general. In addition, the following result holds:

Theorem 1. *Let W satisfy (2). Let f_{appr} be defined by (3), (4) in terms of $R_W f$ and w_0 , $n \geq 2$. Then $f_{appr} = f$ (in the sense of distributions) on \mathbb{R}^n for all $f \in C_0(\mathbb{R}^n)$ if and only if*

$$W(x, \theta) - w_0(x) \equiv w_0(x) - W(x, -\theta), \quad x \in \mathbb{R}^n, \theta \in \mathbb{S}^{n-1}. \quad (9)$$

Here $C_0(\mathbb{R}^n)$ denotes the space of all continuous compactly supported functions on \mathbb{R}^n .

The result of Theorem 1 for $n = 2$ was obtained for the first time in [14]. Theorem 1 in the general case is proved in Section 3.

If W satisfy (2), $f \in C_0(\mathbb{R}^n)$, but the the symmetry condition (9) does not hold, i.e.

$$w_0(x) \neq \frac{1}{2} (W(x, \theta) + W(x, -\theta)), \quad \text{for some } x \in \mathbb{R}^n, \theta \in \mathbb{S}^{n-1},$$

then (3), (4) can be considered as approximate formulas for finding f from $R_W f$.

3 Proof of Theorem 1

For W satisfying (2) we also consider its symmetrization defined by

$$W_s(x, \theta) \stackrel{def}{=} \frac{1}{2} (W(x, \theta) + W(x, -\theta)), \quad x \in \mathbb{R}^n, \theta \in \mathbb{S}^{n-1}. \quad (10)$$

Using definitions (1), (10) we obtain

$$R_{W_s} f(s, \theta) = \frac{1}{2} (R_W f(s, \theta) + R_W f(-s, -\theta)). \quad (11)$$

In addition, if W satisfies (9) then

$$W_s(x, \theta) = w_0(x), \quad x \in \mathbb{R}^n, \theta \in \mathbb{S}^{n-1}. \quad (12)$$

3.1 Proof of sufficiency

The sufficiency of symmetry (9) follows from formulas (3), (4) for the exact case with $W \equiv w_0$, the identities

$$f_{appr}(x) = \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1}w_0(x)} \int_{\mathbb{S}^{n-1}} \mathbb{H}[R_{W_s}f]^{(n-1)}(x\theta, \theta)d\theta, \quad (13)$$

for even n ,

$$f_{appr}(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}w_0(x)} \int_{\mathbb{S}^{n-1}} [R_{W_s}f]^{(n-1)}(x\theta, \theta)d\theta, \quad (14)$$

for odd n ,

and from the identities (11), (12).

In turn, (13) follows from the identities

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \mathbb{H}[R_W f]^{(n-1)}(x\theta, \theta)d\theta \\ &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} \left(\mathbb{H}[R_W f]^{(n-1)}(x\theta, \theta) + \mathbb{H}[R_W f]^{(n-1)}(-x\theta, -\theta) \right) d\theta \\ &= \int_{\mathbb{S}^{n-1}} \mathbb{H}[R_{W_s} f]^{(n-1)}(x\theta, \theta)d\theta. \end{aligned} \quad (15)$$

In addition, the second of the identities of (15) follows from the identities:

$$\begin{aligned} \mathbb{H}[R_{W_s} f]^{(n-1)}(s, \theta) &= \frac{1}{2\pi} p.v. \int_{\mathbb{R}} \frac{1}{s-t} \times \\ & \times \frac{d^{n-1}}{dt^{n-1}} \left[R_W f(t, \theta) + R_W f(-t, -\theta) \right] dt \\ &= \frac{1}{2} \mathbb{H}[R_W f]^{(n-1)}(s, \theta) + \frac{(-1)^{n-1}}{2\pi} p.v. \int_{\mathbb{R}} \frac{[R_W f]^{(n-1)}(-t, -\theta)}{s-t} dt; \quad (16) \\ \frac{(-1)^{n-1}}{\pi} p.v. \int_{\mathbb{R}} \frac{[R_W f]^{(n-1)}(-t, -\theta)}{s-t} dt &= -\frac{(-1)^{n-1}}{\pi} p.v. \int_{\mathbb{R}} \frac{[R_W f]^{(n-1)}(t, -\theta)}{-s-t} dt \\ &= (-1)^n \mathbb{H}[R_W f]^{(n-1)}(-s, -\theta) = \mathbb{H}[R_W f]^{(n-1)}(-s, -\theta). \quad (17) \end{aligned}$$

This concludes the proof of sufficiency for n even.

Finally, (14) follows from the identities

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} [R_W f]^{(n-1)}(x\theta, \theta) d\theta \\ &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} \left([R_W f]^{(n-1)}(x\theta, \theta) + [R_W f]^{(n-1)}(-x\theta, -\theta) \right) d\theta, \end{aligned} \quad (18)$$

$$\begin{aligned} [R_{W_s} f]^{(n-1)}(t, \theta) &= \frac{1}{2} \frac{d^{n-1}}{dt^{n-1}} \left[[R_W f](t, \theta) + [R_W f](-t, -\theta) \right] \\ &= \frac{1}{2} \left[[R_W f]^{(n-1)}(t, \theta) + (-1)^{n-1} [R_W f]^{(n-1)}(-t, -\theta) \right] \\ &= \frac{1}{2} \left[[R_W f]^{(n-1)}(t, \theta) + [R_W f]^{(n-1)}(-t, -\theta) \right]. \end{aligned} \quad (19)$$

This concludes the proof of sufficiency for odd n .

3.2 Proof of necessity

Using that $f_{appr} = f$ for all $f \in C_0(\mathbb{R}^n)$ and using formulas (3), (4) for the exact case $W \equiv w_0$, we obtain

$$\int_{\mathbb{S}^{n-1}} \left(\mathbb{H} [R_W f]^{(n-1)}(x\theta, \theta) - \mathbb{H} [R_{w_0} f]^{(n-1)}(x\theta, \theta) \right) d\theta = 0 \quad (20)$$

on \mathbb{R}^n for even n ,

$$\int_{\mathbb{S}^{n-1}} [R_W f - R_{w_0} f]^{(n-1)}(x\theta, \theta) d\theta = 0 \quad (21)$$

on \mathbb{R}^n for odd n ,

for all $f \in C_0(\mathbb{R}^n)$.

Identities (15), (18), (19), (20), (21) imply the identities

$$\int_{\mathbb{S}^{n-1}} \left(\mathbb{H} [R_{W_s} f]^{(n-1)}(x\theta, \theta) - \mathbb{H} [R_{w_0} f]^{(n-1)}(x\theta, \theta) \right) d\theta = 0 \quad (22)$$

on \mathbb{R}^n for even n ,

$$\int_{\mathbb{S}^{n-1}} [R_{W_s} f - R_{w_0} f]^{(n-1)}(x\theta, \theta) d\theta = 0 \quad (23)$$

on \mathbb{R}^n for odd n ,

for all $f \in C_0(\mathbb{R}^n)$.

The necessity of symmetry (9) follows from the identities (22), (23) and the following lemmas:

Lemma 1. Let (22), (23) be valid for fixed $f \in C_0(\mathbb{R}^n)$ and W satisfying (2), $n \geq 2$. Then

$$R_{W_s} f = R_{w_0} f. \quad (24)$$

Lemma 2. Let (24) be valid for all $f \in C_0(\mathbb{R}^n)$ and fixed W satisfying (2), $n \geq 2$. Then

$$W_s = w_0. \quad (25)$$

Lemmas 1 and 2 are proved in Sections 4 and 5.

4 Proof of Lemma 1

We will use the following formulas

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{i\xi x} \int_{\mathbb{S}^{n-1}} g(x\theta, \theta) d\theta dx \\ &= \frac{\sqrt{2\pi}}{|\xi|^{n-1}} \left(\hat{g} \left(|\xi|, \frac{\xi}{|\xi|} \right) + \hat{g} \left(-|\xi|, -\frac{\xi}{|\xi|} \right) \right), \end{aligned} \quad (26)$$

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{i\xi x} \int_{\mathbb{S}^{n-1}} g^{(n-1)}(x\theta, \theta) d\theta dx = \int_{\mathbb{R}^n} e^{i\xi x} \int_{\mathbb{S}^{n-1}} (\theta \nabla_x)^{n-1} g(x\theta, \theta) d\theta dx \\ &= (-i)^{n-1} \sqrt{2\pi} \left(\hat{g} \left(|\xi|, \frac{\xi}{|\xi|} \right) + (-1)^{n-1} \hat{g} \left(-|\xi|, -\frac{\xi}{|\xi|} \right) \right), \end{aligned} \quad (27)$$

$$\hat{g}(\tau, \theta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau s} g(s, \theta) ds, \quad \tau \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}, \quad (28)$$

where $g \in C(\mathbb{S}^{n-1}, L^2(\mathbb{R}))$, $\xi \in \mathbb{R}^n$. The validity of formulas (26), (27) (in the sense of distributions) follows from Theorem 1.4 of [12].

4.1 The case of odd n

Using identity (11) we get

$$g(s, \theta) = g(-s, -\theta), \quad \text{for all } s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}, \quad (29)$$

where

$$g(s, \theta) = [R_{W_s} f(s, \theta) - R_{w_0} f(s, \theta)]. \quad (30)$$

From (29), we obtain the same symmetry for the Fourier transform $\hat{g}(\cdot, \theta)$ of $g(\cdot, \theta)$:

$$\begin{aligned} \hat{g}(t, \theta) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s, \theta) e^{its} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(-s, -\theta) e^{i(-s)(-t)} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s, -\theta) e^{-its} ds = \hat{g}(-t, -\theta), \quad t \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}. \end{aligned} \quad (31)$$

For odd n , from identities (23), (27) it follows that

$$\hat{g}\left(|p|, \frac{p}{|p|}\right) + \hat{g}\left(-|p|, -\frac{p}{|p|}\right) = 0 \text{ in } L_{loc}^2(\mathbb{R}^n). \quad (32)$$

Using (31), (32) we obtain

$$\begin{cases} \hat{g}\left(|p|, \frac{p}{|p|}\right) = 0, \\ \hat{g}\left(-|p|, -\frac{p}{|p|}\right) = 0 \end{cases} \Leftrightarrow \hat{g} = 0 \Leftrightarrow g = 0. \quad (33)$$

Formula (24) for odd n follows from (30), (33).

4.2 The case of even n

We consider

$$g(s, \theta) = \mathbb{H}[R_{W_s}f - R_{w_0}f](s, \theta), \quad s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}, \quad (34)$$

arising in (22). Using the identity

$$\begin{aligned} \mathbb{H}[R_{W_s}f - R_{w_0}f](-s, -\theta) &= \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s}f(t, -\theta) - R_{w_0}f(t, -\theta)}{-s - t} dt \\ &= \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s}f(-t, -\theta) - R_{w_0}f(-t, -\theta)}{-s + t} dt \\ &= -\frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s}f(t, \theta) - R_{w_0}f(t, \theta)}{s - t} dt = -\mathbb{H}[R_{W_s}f - R_{w_0}f](s, \theta), \end{aligned} \quad (35)$$

we obtain

$$g(s, \theta) = -g(-s, -\theta), \quad \text{for all } s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}. \quad (36)$$

From (36), similarly with (31), we obtain the same symmetry for the Fourier transform $\hat{g}(\cdot, \theta)$ of $g(\cdot, \theta)$:

$$\hat{g}(t, \theta) = -\hat{g}(-t, -\theta), \quad t \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}. \quad (37)$$

For n even, from the property of the Hilbert transform

$$\mathbb{H}[\phi^{(k)}] = (\mathbb{H}[\phi])^{(k)}, \quad \phi \in C_0^k(\mathbb{R}),$$

where this identity holds in the sense of distributions if $\phi \in C_0(\mathbb{R})$, and identities (22), (27) it follows that

$$\hat{g}\left(|p|, \frac{p}{|p|}\right) - \hat{g}\left(-|p|, -\frac{p}{|p|}\right) = 0 \text{ in } L_{loc}^2(\mathbb{R}^n). \quad (38)$$

Using (37), (38) we again obtain (33) but already for even n . Due to (33), (34) we have

$$\mathbb{H}[R_{W_s}f - R_{w_0}f] = 0. \quad (39)$$

Formula (24) for even n follows from (39), invertibility of the Hilbert transform on L^p , $p > 1$ and the fact that $R_W f \in C_0(\mathbb{R} \times \mathbb{S}^{n-1})$.

Lemma 1 is proved.

5 Proof of Lemma 2

Suppose that

$$W_s(y, \theta) - w_0(y) = z \neq 0 \quad (40)$$

for some $y \in \mathbb{R}^n$, $\theta \in \mathbb{S}^{n-1}$, $z \in \mathbb{C}$. Since W satisfies (2), then for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\forall y' : |y' - y| < \delta \rightarrow |W_s(y', \theta) - w_0(y') - z| < \varepsilon, \quad (41)$$

for fixed y, θ .

Let $f \in C_0(\mathbb{R}^n)$, $f \geq 0$ and satisfies the conditions

$$f(y') \equiv 1, y' \in B_{\delta/2}(y), \quad (42)$$

$$\text{supp } f \subset B_{\delta}(y), \quad (43)$$

where $B_{\delta}(y)$ is the open ball with radius δ , centered at y , $\delta = \delta(\varepsilon)$, $0 < \varepsilon < |z|$. It suffices to show that

$$|R_{W_s}f(y\theta, \theta) - R_{w_0}f(y\theta, \theta)| > 0, \quad (44)$$

which contradicts the condition of the lemma.

The identity (44) follows from the formulas

$$\begin{aligned} |R_{W_s}f(y\theta, \theta) - R_{w_0}f(y\theta, \theta)| &= \left| \int_{x\theta=y\theta} f(x)(W_s(x, \theta) - w_0(x))dx \right| \\ &= \left| \int_{x\theta=y\theta} f(x)(W_s(x, \theta) - w_0(x) - z)dx + z \int_{x\theta=y\theta} f(x)dx \right| \\ &\geq |z| \int_{x\theta=y\theta} f(x)dx - \int_{x\theta=y\theta} f(x)|W_s(x, \theta) - w_0(x) - z|dx \\ &\geq (|z| - \varepsilon) \int_{x\theta=y\theta} f(x)dx > 0, \text{ for } 0 < \varepsilon < |z|. \end{aligned} \quad (45)$$

Lemma 2 is proved.

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