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► **To cite this version:**

François Laudenbach, Carlos Moraga Ferrándiz. Homoclinic bifurcation in Morse-Novikov theory a doubling phenomenon. 2016. hal-01304121v2

**HAL Id: hal-01304121**

**<https://hal.science/hal-01304121v2>**

Preprint submitted on 5 Jul 2016 (v2), last revised 20 May 2019 (v4)

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HOMOCLINIC BIFURCATION IN MORSE-NOVIKOV THEORY  
A DOUBLING PHENOMENON

FRANÇOIS LAUDENBACH & CARLOS MORAGA FERRÁNDIZ

ABSTRACT. We consider a compact manifold of dimension greater than 2 and a differential form of degree one which is closed but non-exact. This form, viewed as a multi-valued function has a gradient vector field with respect to some auxiliary Riemannian metric. According to S. Novikov's work and a complement by J.-C. Sikorav, under some genericity assumptions these data yield a complex, called today the Morse-Novikov complex. Due to the non-exactness of the form, its gradient has non-trivial dynamics in contrary to gradients of functions. In particular, it is possible that the gradient has a homoclinic orbit. The one-form being fixed, we investigate the codimension-one stratum in the space of (pseudo-) gradients formed by vector fields having one homoclinic orbit in a given homotopy class of loops. This stratum  $S$  breaks up into a left and a right part separated by a substratum. The algebraic effect on the Morse-Novikov complex of crossing  $S$  depends on the part, left or right, which is crossed. This difference makes necessary the doubling phenomenon mentioned in our title.

1. INTRODUCTION

We are given a closed connected  $n$ -dimensional manifold  $M$  with a non-zero de Rham cohomology class  $u \in H^1(M; \mathbb{R})$ . This class is represented by closed differential 1-forms of type Morse, meaning that their zeroes are non-degenerate. The set of Morse 1-forms in the class  $u$  is denoted by  $\mathcal{F}_u$ . For a Morse 1-form  $\alpha$ , the set of its zeroes will be denoted  $Z(\alpha)$ ; each zero  $p \in Z(\alpha)$  has a Morse index  $i(p)$ . Along a path in  $\mathcal{F}_u$ , the zeroes can be followed continuously and their respective indices are constant.

As in Morse theory, it is important to equip each  $\alpha \in \mathcal{F}_u$  with a *descending pseudo-gradient*  $X$  which is said to be *adapted* to  $\alpha$  (see Definition 2.1). The set of equipped Morse 1-forms  $(\alpha, X)$  with  $\alpha \in \mathcal{F}_u$  is denoted by  $\tilde{\mathcal{F}}_u$ . The Morse-Novikov theory is devoted to the understanding of this space.

With each zero  $p$  of  $\alpha$  in  $M$ , there are associated a stable and an unstable manifold. These are respectively denoted by  $W^s(p, X)$  and  $W^u(p, X)$ . Generically,  $X$  has the property that all these invariant manifolds when  $p$  runs in  $Z(\alpha)$  are mutually transverse. In that case, the pair  $(\alpha, X)$  will be called *Morse-Smale*<sup>1</sup>. In that case, if an orbit of  $X$  is a *connecting orbit* going

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2010 *Mathematics Subject Classification.* 57R99, 37B35, 37D15.

*Key words and phrases.* Closed one-form, Morse-Novikov theory, gradient, Morse-Smale, homoclinic bifurcation.

Both authors were supported by ERC Geodycon; the second author was also supported by the Japan Society for the Promotion of Science, under the "FY2013 Postdoctoral Fellowship (short-term) for North American and European Researchers" program.

<sup>1</sup>In the literature, a Morse-Smale vector field fulfills the above condition not only with respect to its set of zeroes but also to its set of periodic orbits.

from  $p$  to  $q$  then  $i(p) > i(q)$ . The  $\alpha$ -length of a connecting orbit  $\ell$ , defined by

$$(1.1) \quad \mathcal{L}(\ell) := - \int_{\ell} \alpha,$$

must be positive. Moreover, a Morse-Smale pair  $(\alpha, X)$  yields the so called *Morse-Novikov* differential  $\partial^X$  that counts the *incidences*  $\langle p, q \rangle^X$  between the zeroes of  $\alpha$  whose indices verify  $i(q) = i(p) - 1$  (see Subsection 3.1). The formulation we are going to use for this complex is due to from Jean-Claude Sikorav in his thesis [12].

In a generic one-parameter family  $(X_s)_{s \in I}$  of vector fields adapted to a given  $\alpha \in \mathcal{F}_u$ , the pair  $(\alpha, X_s)$  is Morse-Smale for every  $s$  outside  $(A \sqcup B) \subset I$ , where  $A \sqcup B$  is a countable set of *bifurcation times*. To be more precise, for each  $s \in A \sqcup B$  there is only one connecting orbit  $\ell$  along which the transversality condition  $W^u(p, X_s) \pitchfork W^s(q, X_s)$  fails and, for each  $L > 0$ , there are finite subsets  $A_L \subset A$  and  $B_L \subset B$  such that, for  $s \in A_L \sqcup B_L$ , the non-transverse connecting orbit of  $X_s$  has  $\alpha$ -length less than  $L$ . The set  $A$  – which is beyond the purposes of the present paper – consists of *annihilation times* where  $i(q) = i(p) - 1$  and, at each point of the connecting orbit  $\ell = W^u(p, X_s) \cap W^s(q, X_s)$ , there is a two-dimensional plane tangent to both  $W^u(p, X_s)$  and  $W^s(q, X_s)$ . We are concerned with *slide times*  $s \in B$ , where the loss of transversality comes from the fact that  $i(p) = i(q)$ ; in this case, the mentioned connecting orbit  $\ell$  is called a *slide orbit*. Remark that at a slide time, we can have  $p = q$  in which case  $\ell$  is a *self-slide orbit*, classically called *homoclinic orbit*.

Given a pseudo-gradient  $X$  of  $\alpha \in \mathcal{F}_u$ , a *homoclinic orbit* is a connecting orbit  $\ell$  from  $p$  to itself, where  $p$  is a zero of  $\alpha$ . This was also considered by M. Hutchings [3] (see also [9]). This orbit makes a loop in  $M$  based at  $p$  (never smooth there). In order to avoid dependency on base points, we introduce the *fundamental groupoid*  $\Pi$  of  $M$  (see Definition 3.2). Denote by  $\Pi_{\text{loop}} \subset \Pi$  the subgroupoid of homotopy classes of based loops in  $M$ . This is a topological groupoid and there is a continuous evaluation map  $\Pi_{\text{loop}} \rightarrow M$  which maps each based loop to its base point; its fibre is discrete. The homoclinic orbit  $\ell$  based in  $p$  has a homotopy class  $g := [\ell] \in \Pi_{\text{loop}}$  which will be called the  $\Pi$ -value of  $\ell$ . The canonical isomorphism  $H^1(M; \mathbb{R}) \cong \text{Hom}(\pi_1(M, p), \mathbb{R})$  yields the morphism  $g \in \pi_1(M, p) \mapsto u(g) \in \mathbb{R}$  and, if  $g$  is the  $\Pi$ -value of  $\ell$ , we have:

$$(1.2) \quad u(g) = -\mathcal{L}(\ell).$$

As a consequence,  $\ell$  is not homotopic to zero.

In what follows, the form  $\alpha$  will be kept fixed and we denote by  $\mathcal{F}_\alpha$  the fibre of the forgetful map  $\tilde{\mathcal{F}}_u \rightarrow \mathcal{F}_u$ . An element of  $\mathcal{F}_\alpha$  will be denoted by  $(\alpha, X)$  or by  $X$  only if  $\alpha$  is implied by the context. For  $g \in \Pi_{\text{loop}}$ , we consider the “stratum”  $\mathcal{S}_g \subset \mathcal{F}_\alpha$  made of elements  $(\alpha, X)$  where  $X$  has exactly one homoclinic orbit  $\ell$  whose  $\Pi$ -value is  $g$ , the field  $X$  being allowed to have another slide orbits. An element of  $\mathcal{S}_g$  will be called a *self-slide*. The stratum  $\mathcal{S}_g$  may have several connected components but, by definition of  $\Pi_{\text{loop}}$ , the element  $g$  indicates which zero  $p$  of  $\alpha$  is involved in the self-slide. In particular, with  $\mathcal{S}_g$  it is associated a Morse index which is surely distinct from 0 and  $n$  since these indices do not allow homoclinic orbits.

Generically in  $\mathcal{S}_g$ , a self-slide possesses a natural *character* which is an element in  $\{+, -\}$  and which will be defined later (see Definition 2.11). We define  $\mathcal{S}_g^+$  (resp.  $\mathcal{S}_g^-$ ) to be the open

set in  $\mathcal{S}_g$  made of elements having the mentioned character. Finally,  $\mathcal{S}_g^0$  denotes the self-slides for which the character is not defined.

**Theorem 1.1.** *For every  $g \in \Pi_{\text{loop}}$  verifying  $u(g) < 0$ , the following holds true:*

- (1) *The stratum  $\mathcal{S}_g$  is a codimension-one, co-oriented stratum of  $\mathcal{F}_\alpha$  of class  $C^\infty$ .*
- (2) *Assume  $n > 2$ . Then, the stratum  $\mathcal{S}_g^0$  is a non-empty codimension-one, co-oriented sub-stratum of class  $C^\infty$  in  $\mathcal{S}_g$ . Moreover,  $\mathcal{S}_g^0$  meets every connected component of  $\mathcal{S}_g$ .*

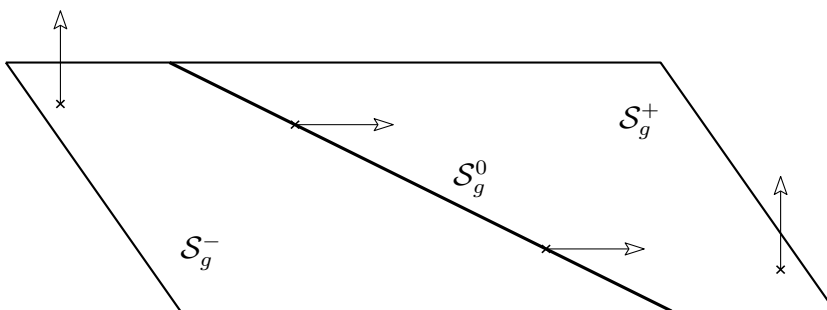


FIGURE 1. Local situation on every connected component of  $\mathcal{S}_g \subset \mathcal{F}_\alpha$  near  $\mathcal{S}_g^0$ .

Since  $\mathcal{S}_g$  is co-oriented, we can study the generic one-parameter families  $(\alpha, X_s)$  which intersect  $\mathcal{S}_g$  *positively* at  $(\alpha, X_0)$ . Let  $q \in Z(\alpha)$  be any zero whose Morse index equals  $i(p) - 1$ . When passing from  $s < 0$  to  $s > 0$ , the incidence  $\langle p, q \rangle^{X_s}$  gets multiplied by an element  $\lambda$  of the Novikov ring which does not depend on  $q$  and that we call the *self-slide factor*. It turns out that  $\lambda$  depends on the character of  $(\alpha, X_0)$ . For clarity, we give a simplified version of Theorem 3.6 right below. This theorem tells us the precise manner in which the incidence changes. This makes more precise the statement of [8, Prop. 2.2.36] whose proof was inaccurate.

**Theorem 1.2.** *The self-slide factor is given by the following formulas:*

- (1)  $\lambda = 1 + g$  if  $(\alpha, X_0) \in \mathcal{S}_g^-$  (*Polynomial type*),
- (2)  $\lambda = 1 + \sum_{j=1}^{\infty} g^j$  if  $(\alpha, X_0) \in \mathcal{S}_g^+$  (*Series type*).

Of course, in order to keep the squared differential equal to zero there are similar formulas for the change of the incidence  $\langle q, p \rangle^{X_s}$  when the Morse indices satisfy  $i(q) = i(p) + 1$ . The doubling phenomenon that has motivated our article relates the strata  $\mathcal{S}_g$  and  $\mathcal{S}_{g^2}$ . If  $\gamma : \mathbb{S}^1 \rightarrow \mathcal{F}_\alpha$  denotes a small generic loop going around the codimension-two stratum  $\mathcal{S}_g^0$ , this loop cannot avoid crossing  $\mathcal{S}_{g^2}$ :

**Theorem 1.3.** *The stratum  $\mathcal{S}_{g^2}^0$  lies in the closure of  $\mathcal{S}_{g^2}$ . Furthermore, there exists a codimension-two stratum  $\mathcal{S}_g^{0,0}$  of  $\mathcal{S}_g$ , contained in  $\mathcal{S}_g^0$ , such that  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$  is a boundary of class  $C^1$  of  $\mathcal{S}_{g^2}$ .*

The precise definition of  $\mathcal{S}_g^{0,0}$  yields a decomposition  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0} = \mathcal{S}_g^{0,-} \sqcup \mathcal{S}_g^{0,+}$  (see Definition 4.1). Locally along  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$ , the stratum  $\mathcal{S}_{g^2}$  approaches from one side of  $\mathcal{S}_g$  only. As a matter of fact,  $\mathcal{S}_{g^2}$  approaches  $\mathcal{S}_g^{0,+}$  (resp.  $\mathcal{S}_g^{0,-}$ ) from the positive (resp. negative side) of the co-oriented stratum  $\mathcal{S}_g$ . Moreover, only  $\mathcal{S}_{g^2}^+$  (the positive part of  $\mathcal{S}_{g^2}$ ) approaches  $\mathcal{S}_g^0$ . A precise statement about the latter facts is Theorem 4.2, that may be illustrated by the following figure:

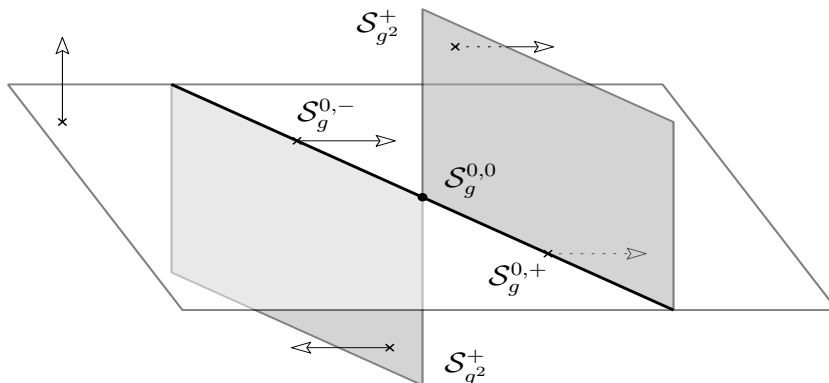


FIGURE 2. The stratum  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$  as a boundary of  $\mathcal{S}_{g^2}$ .

**Remarks 1.4.**

1) It easily seen that the total space of the fibration  $\tilde{\mathcal{F}}_u \rightarrow \mathcal{F}_u$  has also a stratification which is given in each fibre  $\mathcal{F}_\alpha$  by the union of  $\mathcal{S}_g^0 \subset \mathcal{S}_g \subset \mathcal{F}_\alpha$  where  $g$  runs in  $\Pi_{\text{loop}}$ . As a stratified fibration, it is locally trivial but it could be globally non-trivial. Indeed, a connected component of a codimension-one stratum could intersect  $\mathcal{F}_\alpha$  along some components of  $\mathcal{S}_g$  and  $\mathcal{S}_{g'}$  where  $g$  and  $g'$  are freely homotopic and based in different points.

2) Our doubling phenomenon evokes the period doubling bifurcation, also called Andronov-Hopf's bifurcation. In this aim, it would be good to know that crossing  $\mathcal{S}_g$  creates (or destroys) a periodic orbit in the free homotopy class  $g$ . Shilnikov's theorem [11] deals with this question. Unfortunately, it is not applicable here because we are going to use very non-generic Morse charts (only  $+1$  and  $-1$  as eigenvalues of the Hessian at critical points). In counterpart, such charts offer very nice advantages.

The motivation of our article comes from *pseudo-isotopy* theory for non-exact closed 1-forms. Suppose we are given  $\alpha_0$  and  $\alpha_1$  in  $\mathcal{F}_u$  without zeroes. Taking a *generic path*<sup>2</sup>  $(\alpha_s)$  from  $\alpha_0$  to  $\alpha_1$  in the cohomology class  $u$ , that is in the closure of  $\mathcal{F}_u$ , is it possible to deform  $(\alpha_s)$  – keeping  $\{\alpha_0, \alpha_1\}$  fixed – to a path of closed forms with no zeroes? In the latter case,  $\alpha_0$  and  $\alpha_1$  are said to be *isotopic*.

One way to proceed is to equip the path  $(\alpha_s)$  with a generic family  $(X_s)$  to perform a parametric Morse-Novikov theory. This approach has already been taken in the same problem for functions on compact manifolds  $N \times [0, 1]$  (see [2]).

The hope is to produce an algebraic object (such as a group<sup>3</sup>) where we could read geometrical obstructions to isotopy. Bifurcation phenomena visible in 2-parameter families should correspond to relations holding in this algebraic object. A first algebraic candidate has already been proposed in [7]. Ten algebraic relations appear on [7, Lemma 2.1]. Theorems 3.6 and 4.2 of the present article correspond to the geometric counterpart of some particular cases of one of the mentioned relations.

<sup>2</sup>A path of 1-forms in the class  $u$  is said to be generic if it avoids  $\mathcal{F}_u$  finitely many times only.

<sup>3</sup>We search for the *Steinberg group* of  $u$  viewed as a representation  $\pi_1(M) \rightarrow \mathbb{R}$ .

## 2. SELF-SLIDE STRATUM, ORIENTATION AND CHARACTER

We are focusing on self-slides despite some of the statements hold true for other slides. In the discussion below, we are given a pair  $(\alpha, X) \in \tilde{\mathcal{F}}_u$  where  $\alpha$  is a closed 1-form on  $M$  (in the cohomology class  $u$ ) equipped with an *adapted* vector field  $X$  in the sense which is now specified.

**Definition 2.1.** *For each zero  $p$  of  $X$  of index  $i(p) = i$ , Morse coordinates near  $p$  are coordinates where the form  $\alpha$  equals the differential of the standard quadratic form of index  $i$*

$$Q_i := -x_1^2 + \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2.$$

A vector field  $X$  is said to be a (negative) pseudo-gradient adapted to  $\alpha$  if the two next conditions are fulfilled:

- $\alpha(X) < 0$  outside  $Z(\alpha)$ ;
- for every  $p \in Z(\alpha)$  there are Morse coordinates about  $p$  such that  $X$  coincides with the standard descending gradient  $X_i$  of  $Q_i$ , where  $i = i(p)$ , that is:  

$$X_i = 2 \sum_{k=1}^i x_k \partial_{x_k} - 2 \sum_{k=i+1}^n x_k \partial_{x_k}.$$

Given  $(\alpha, X) \in \tilde{\mathcal{F}}_u$  and  $p \in Z(\alpha)$ , Morse coordinates about  $p$  are said to be adapted to  $(\alpha, X)$  if  $X = X_i$  in these coordinates. It will also said that  $X$  is adapted to these Morse coordinates.

**Remark 2.2.** The natural action of  $G := O(i) \times O(n - i)$  on  $\mathbb{R}^n$  keeps the pair  $(Q_i, X_i)$  invariant and  $G$  is the subgroup of  $O(i, n - i)$  preserving  $X_i$ . Actually, the simplicial group  $\mathcal{G} := \text{Diff}(Q_i, X_i)$  of germs of diffeomorphisms of  $(\mathbb{R}^n, 0)$  preserving the pair  $(Q_i, X_i)$  retracts by deformation to  $G$ . Indeed, if  $\varphi \in \mathcal{G}$ , the Alexander isotopy  $\varphi_t : x \mapsto \frac{1}{t} \varphi(tx)$  is made of elements in  $\mathcal{G}$  for every  $t \in (0, 1]$  and tends to the derivative  $\varphi'(0)x$  as  $t$  goes to 0. As a consequence, given  $(\alpha, X) \in \tilde{\mathcal{F}}_u$  and  $p \in Z(\alpha)$ , the set, modulo  $G$ , of Morse coordinates about  $p$  which  $X$  is adapted to is contractible.

**2.1. MORSE MODEL.** Given  $p \in Z(\alpha)$  of Morse index  $i$ , a *Morse model*  $\mathcal{M}_p$  with positive parameters  $(\delta, \delta^*)$  (which we do not make explicit in the notation) is diffeomorphic to the subset of  $\mathbb{R}^i \times \mathbb{R}^{n-i}$  made of pairs  $(x^-, x^+)$  such that  $Q_i(x^-, x^+) \in [-\delta^*, +\delta^*]$ ,  $|x^-|^2 |x^+|^2 \leq \delta \delta^*$  and  $\alpha|_{\mathcal{M}_p} = dQ_i$ . The bottom of  $\mathcal{M}_p$ , that is its intersection with  $\{Q_i = -\delta^*\}$  is denoted by  $\partial^- \mathcal{M}_p$ ; similarly, the top is denoted by  $\partial^+ \mathcal{M}_p$ . The rest of the boundary of  $\mathcal{M}_p$  is denoted by  $\partial^\ell \mathcal{M}_p$  and  $X_i$  is tangent to it. Notice that:

- the group  $G$  preserves  $\mathcal{M}_p$  for every parameters  $(\delta, \delta^*)$ ;
- the set of Morse models, as compact subsets of  $\mathbb{R}^n$ , is contractible.

The flow of  $X_i$  is denoted by  $(X_i^t)_{t \in \mathbb{R}}$ . The *local unstable* (resp. *local stable*) manifold is formed by the points  $x \in \mathcal{M}_p$  whose negative (resp. positive) flow line  $X_i^t(x)$  goes to  $p$  when  $t$  goes to  $-\infty$  (resp.  $+\infty$ ) without getting out of  $\mathcal{M}_p$ . Denote by  $\Sigma^-$  the  $(i - 1)$ -sphere which is formed by the points in the bottom of  $\mathcal{M}_p$  which belong to  $W_{loc}^u(p, X_i)$ ; it is called the *attaching sphere*. Similarly,  $\Sigma^+$  denotes the  $(n - i - 1)$ -sphere which is contained both in the top of  $\mathcal{M}_p$  and made of points belonging to  $W_{loc}^s(p, X_i)$ ; we call it the *co-sphere*. We will use the the two projections associated with these coordinates:

$$(2.1) \quad \pi^+ : \partial^+ \mathcal{M}_p \rightarrow \Sigma^+ \quad \text{and} \quad \pi^- : \partial^- \mathcal{M}_p \rightarrow \Sigma^-.$$

2.2. TUBE AND ORIENTATION. Let  $\alpha$  be a Morse closed 1-form on  $M$ , let  $X$  be an adapted pseudo-gradient and  $\ell$  be a homoclinic orbit of  $X$ , based at  $p \in Z(\alpha)$ . Denote by  $\underline{\ell}$  the closure of  $\ell \setminus \mathcal{M}_p$ ; it will be named the *restricted* homoclinic orbit. Its end points are denoted respectively  $a^- \in \Sigma^-$  and  $a^+ \in \Sigma^+$ . We also introduce a *compact tube*  $T$  around  $\underline{\ell}$  made of  $X$ -trajectories from  $\partial^- \mathcal{M}_p$  to  $\partial^+ \mathcal{M}_p$ . Up to time rescaling, we may suppose  $X^1(a^-) = a^+$ . On  $T$ , there are coordinates  $(x, y, v, z) \in \mathbb{R}^{i-1} \times \mathbb{R}^{n-i-1} \times [-1, 1] \times [0, 1]$  with the following properties:

- $X$  is positively colinear to  $\partial_z$ ,
- $\{z = 0\} = T \cap \partial^- \mathcal{M}_p$  and  $\{z = 1\} = T \cap \partial^+ \mathcal{M}_p$ ;
- $T \cap \Sigma^- = \{y = 0, v = 0, z = 0\}$  and  $T \cap \Sigma^+ = \{x = 0, v = 0, z = 1\}$ ;
- $\underline{\ell} = \{x = 0, y = 0, v = 0\}$ ;
- $\partial_v$  is tangent to the leaves of  $\alpha$ .

We choose the base  $\{z = 0\}$  of  $T$  to be a closed polydisc.

Orient the unstable  $W^u(p, X)$ . Thus, the stable manifold is co-oriented. Therefore, we can choose the coordinate  $v$  in the tube so that, for every  $z_0 \in [0, 1]$ , the following holds:

$$(2.2) \quad \partial_v \wedge \text{or}(W^u(p, X) \cap \{z = z_0\}) = \text{co-or}(W^s(p, X)).$$

If the orientation of  $W^u(p, X)$  is changed, then the co-orientation of  $W^s(p, X)$  is also changed and the above equation shows that the positive direction of  $v$  remains unchanged.

**Remark 2.3.** It is important to notice that (2.2) tells us nothing about the holonomy along  $\underline{\ell}$  of the foliation defined by  $X$  (see the next subsection). Therefore, for a given  $\partial_v \in T_{a^+}(\partial^+ \mathcal{M}_p)$ , the tangent vector  $\partial_v \in T_{a^-}(\partial^- \mathcal{M}_p)$  may have any position not contained in the hyperplane  $\mathbb{R}\{\partial_x, \partial_y\}$ , depending on  $X$ .

2.3. HOLONOMY AND PERTURBED HOLONOMY. Since  $T$  is compact, there exists some open neighborhood  $\mathcal{N}_X^- \subset \partial^- \mathcal{M}_p$  (depending on  $X$ ) of  $\{z = 0\} = T \cap \partial^- \mathcal{M}_p$  on which the *holonomy* along  $\underline{\ell}$  is defined from the transversal  $\partial^- \mathcal{M}_p$  to the transversal  $\partial^+ \mathcal{M}_p$ , that is: if  $\gamma(u)$ ,  $u \in [0, 1]$  is a path in  $\mathcal{N}_X^-$  starting from  $a^-$ , the solutions  $x_u(t)$  of the differential equation  $\dot{x} = X(x)$  with initial condition  $x_u(0) = \gamma(u)$  cross  $\partial^+ \mathcal{M}_p$  at a time  $\theta(u)$  depending smoothly on  $u$ . In particular,  $\theta(0) = 1$  and  $x_0(\theta(0)) = a^+$ . In this setting, the map  $\gamma(1) \mapsto x_1(\theta(1))$  defines a diffeomorphism  $H_X$  from  $\mathcal{N}_X^-$  to an open set  $\mathcal{N}_X^+ \subset \partial^+ \mathcal{M}_p$  which contains  $\{z = 1\}$ .

The existence of such holonomy diffeomorphism is an open property with respect to  $X$ . More precisely, if  $\tilde{X}$  is a close enough approximation of  $X$  in the  $C^1$ -topology, there is a *perturbed holonomy* diffeomorphism  $H_{\tilde{X}}$  from an open neighborhood  $\mathcal{N}_{\tilde{X}}^-$  in  $\partial^- \mathcal{M}_p$  of  $\{z = 0\}$  in  $\partial^- \mathcal{M}_p$  to an open neighborhood  $\mathcal{N}_{\tilde{X}}^+$  of  $\{z = 1\}$  in  $\partial^+ \mathcal{M}_p$ .

**Remark 2.4.** According to this property of the perturbed holonomy, it makes sense to speak of  $H_{\tilde{X}}^{-1}(\Sigma^+) \cap \{z = 0\}$ . It is an  $(n - i - 1)$ -disc close to the  $y$ -axis in  $\{z = 0\}$ . Similarly, it makes sense to speak of  $H_{\tilde{X}}(\Sigma^-) \cap \{z = 1\}$ . It is an  $(i - 1)$ -disc close to the  $x$ -axis in  $\{z = 1\}$ .

We now state and prove Item (1) of Theorem 1.1.

**Proposition 2.5.** *For every  $g \in \Pi_{\text{loop}}$  with  $u(g) < 0$ , the stratum  $\mathcal{S}_g$  is a  $C^\infty$  codimension-one stratum of  $\mathcal{F}_\alpha$ . This stratum has a canonical co-orientation.*

**Proof.** Let  $X_0$  be any point in  $\mathcal{S}_g$ . Let  $\ell$  denote the homoclinic orbit of  $\Pi$ -value  $g$  and let  $p$  be the involved zero of  $\alpha$ . We intend to find a regular real valued equation for  $\mathcal{S}_g$  near  $X_0$ . We recall:

- the local stability of the adapted vector fields near  $p$ ;
- the acyclicity of the space of Morse models adapted to  $X_0$  near  $p$ ; near  $\{z = 0\} \cup \{z = 1\}$ .

Therefore, the action of the group  $\text{Diff}(M)$  reduces us to consider a *slice*  $S \subset \mathcal{F}_\alpha$  such that every  $X \in S$  near  $X_0$  satisfies the following:  $X = X_0$  in a given Morse model  $\mathcal{M}_p$ .

We use the tube  $T$  and its coordinates as introduced in Subsection 2.2. The Implicit Function Theorem allows us to follow continuously, for  $X$  close to  $X_0$ , a connected component  $D(X)$  of  $W^u(p, X) \cap T \cap \{z = 1\}$  which coincides with  $\{y = 0, v = 0, z = 1\}$  when  $X = X_0$ . Let

$$(2.3) \quad p_v : \{z = 1\} \rightarrow \{v = 0, z = 1\}$$

denote the projection parallel to  $\partial_v$  onto the  $(x, y)$ -space. The image  $p_v(D(X))$  is transverse to  $\Sigma^+$ . The intersection is a point  $a^+(X)$  which depends  $C^\infty$  on  $X$ . Let  $b(X)$  be the point of  $D(X)$  which has the same coordinates as  $a^+(X)$  except the last coordinate  $v$ . Thus, the wanted equation is

$$(2.4) \quad v(b(X)) = 0.$$

This is clearly a  $C^\infty$  equation. For proving this equation is regular it is sufficient to exhibit a one-parameter family  $(X_s)$  passing through  $X_0$  and satisfying the next inequality:

$$(2.5) \quad \partial_s (v \circ H_s(a^-))|_{s=0} > 0,$$

where  $H_s$  stands for the perturbed holonomy diffeomorphism of  $X_s$  (compare Lemma 2.7). This is easy to perform by taking  $X_s$  as  $X_0$  plus a small vector field  $sg(x, y, z, v)\partial_v$  where the function  $g$  is non-negative, supported in the interior of the tube  $T$  and has a positive integral along  $\underline{\ell}$ .

After (2.2) we noticed that the positive sense of  $v$  does not depend on the chosen orientation of the unstable manifolds. Therefore, (2.5) defines a canonical co-orientation of  $\mathcal{S}_g$ .  $\square$

**Definition 2.6.** Let  $(X_s)_{s \in \mathcal{O}_p(0)}$  be a one-parameter family<sup>4</sup> of vector fields adapted to  $\alpha$  and standard in  $\mathcal{M}_p$ . It is assumed  $X_0 \in \mathcal{S}_g$ . The family  $X_s$  is said to be positively transverse to the stratum  $S_g$  if it satisfies (2.5).

Consider hence  $(X_s)$  being positively transverse. Below, we use the coordinates  $(x, y, v)$  of  $T \cap \{z = 0, 1\}$ . According to Subsection 2.3, for  $s$  close to 0 there exists an open tube  $T_s \supset T$  made of trajectories of  $X_s$  going from  $\partial^- \mathcal{M}_p$  to  $\partial^+ \mathcal{M}_p$ . That defines a *holonomy* local diffeomorphism  $H_s$  from a neighborhood of  $\{z = 0\}$  in  $\partial^- \mathcal{M}_p$  to a neighborhood of  $\{z = 1\}$  in  $\partial^+ \mathcal{M}_p$  and we have:

$$(2.6) \quad v [H_s(T_s \cap \Sigma^-) \cap \{z = 1, x = 0\}] > 0, \quad \text{when } s > 0.$$

A consequence of (2.6) for the inverse flow is the following:

$$(2.7) \quad v [H_s^{-1}(T_s \cap \Sigma^+) \cap \{z = 0, y = 0\}] < 0, \quad \text{when } s > 0.$$

<sup>4</sup>Following M. Gromov,  $\mathcal{O}_p(0)$  stands for an open interval  $(-\varepsilon, \varepsilon)$  as small as wanted.



Actually, the above inequalities are implied by the some relation between velocities which is stated below and will be useful elsewhere. In relation to (2.6), we are interested in the local solution  $x_s$  of the equation

$$(2.8) \quad p_{y,v} H_s(x, 0, 0) = 0$$

which equals 0 when  $s = 0$ ; here,  $p_{y,v}$  stands for the projection parallel to the span of  $\{\partial_y, \partial_v\}$ . And in relation to (2.7), we consider the solution  $y_s$  of the equation

$$(2.9) \quad p_{x,v} H_s^{-1}(0, y, 0) = 0$$

which equals 0 when  $s = 0$ ; here,  $p_{x,v}$  stands for the projection parallel to the span of  $\{\partial_x, \partial_v\}$ .

**Lemma 2.7.** *With the above data and notations, the following equality holds:*

$$(2.10) \quad \partial_s (v \circ H_s(x_s, 0, 0))|_{s=0} + \partial_s (v \circ H_s^{-1}(0, y_s, 0))|_{s=0} = 0.$$

*In particular, if the first term equals +1 the second equals -1.*

**Proof.** A Taylor expansion gives

$$v \circ H_s(x_s, 0, 0) = v \circ H_s(0, 0, 0) + O(s^2).$$

That follows from the fact that the velocity of  $x_s$  is a vector which is contained in the kernel of  $dv$ . Similarly, we have:

$$v \circ H_s^{-1}(0, y_s, 0) = v \circ H_s^{-1}(0, 0, 0) + O(s^2).$$

Observe that  $H_0(x, y, v) = (x, y, v)$ . Thus, derivating with respect to  $s$  at  $s = 0$  the composed map  $H_s^{-1} \circ H_s = Id$  gives:

$$\partial_s H_s^{-1}(0, 0, 0)|_{s=0} + \partial_s H_s(0, 0, 0)|_{s=0} = 0.$$

Altogether, we get the desired formula. □

**2.4. EQUATORS, SIGNED HEMISPHERES AND LATITUDES.** We introduce some useful notations. Let  $\mathbb{D}^k$ ,  $k \geq 1$ , be the closed Euclidean disc of dimension  $k$  and radius  $\delta$  (the first parameter of the Morse model), equipped with spherical coordinates  $(r, \theta) \in [0, \sqrt{\delta}] \times \mathbb{S}^{k-1}$ . Denote by

$$(2.11) \quad \partial_r^\theta \in T_0 \mathbb{D}^k \text{ the unitary vector pointing towards } \theta \in \mathbb{S}^{k-1}.$$

Suppose that we are given a preferred co-oriented hyperplane  $\Delta \subset T_0 \mathbb{D}^k$ . We obtain a *preferred co-oriented equator*  $E^\Delta \subset \mathbb{S}^{k-1}$ .

The co-orientation of  $E^\Delta$  points to an open *hemisphere* denoted by  $\mathcal{H}^+(\mathbb{S}^{k-1})$ ; the opposite hemisphere is denoted by  $\mathcal{H}^-(\mathbb{S}^{k-1})$ . We have two marked points determined by the oriented normal to  $\Delta$ : a *North pole*  $\nu \in \mathcal{H}^+(\mathbb{S}^{k-1})$  and a *South pole*  $\sigma \in \mathcal{H}^-(\mathbb{S}^{k-1})$ .

Any point  $x \in \mathbb{S}^{k-1}$  determines an angle with respect to the North pole  $\nu$ . The cosinus of this angle defines a *latitude* on  $\mathbb{S}^{k-1}$ , relative to  $\Delta$ , that we denote by:

$$(2.12) \quad \omega_\Delta : \mathbb{S}^{k-1} \rightarrow [-1, 1].$$

Clearly  $\omega_\Delta^{-1}(1) = \{\nu\}$ ,  $\omega_\Delta^{-1}(-1) = \{\sigma\}$  and  $\omega_\Delta^{-1}(0) = E^\Delta$ . Moreover, if we denote by

$$(2.13) \quad p_\Delta : T_0 \mathbb{D}^k \rightarrow \mathbb{R} \partial_r^\nu$$

the projection parallel to  $\Delta$ , the definition of cosinus reads

$$(2.14) \quad p_\Delta(\partial_r^\theta) = \omega_\Delta(\theta) \partial_r'.$$

**Proposition 2.8.** *Every  $X$  in  $\mathcal{S}_g$  defines a preferred latitude on both the attaching sphere  $\Sigma^-$  and the co-sphere  $\Sigma^+$ .*

For this aim, we use radial-multispherical coordinates  $(\phi, r, \psi) \in S^{i-1} \times [0, \sqrt{\delta}] \times S^{n-i-1}$  on each level set of  $\mathcal{M}_p$  and we recall the map

$$(2.15) \quad Desc : \partial^+ \mathcal{M}_p \setminus \Sigma^+ \rightarrow \partial^- \mathcal{M}_p \setminus \Sigma^-$$

obtained by descending the flow lines in  $\mathcal{M}_p$ . This map reads  $Id$  in these coordinates.

The preferred latitude which we are going to define on  $\Sigma^-$  and  $\Sigma^+$  will be called respectively the  $\phi$ -latitude and the  $\psi$ -latitude. We insist that these functions depend on  $X \in \mathcal{S}_g$ . We denote them by

$$(2.16) \quad \omega_\phi^X : \Sigma^- \rightarrow [-1, 1] \quad \text{and} \quad \omega_\psi^X : \Sigma^+ \rightarrow [-1, 1].$$

When the vector field is understood, these functions will be denoted  $\omega_\phi$  and  $\omega_\psi$ . We shall decorate all the data related to  $\omega_\phi$  or  $\omega_\psi$  by using the letter  $\phi$  or  $\psi$  respectively; namely, the preferred hyperplane  $\Delta^\phi$ , the preferred equator  $E^\phi \subset \Sigma^-$ , the poles  $\nu_\phi \in \mathcal{H}^+(\Sigma^-)$ ,  $\sigma_\phi \in \mathcal{H}^-(\Sigma^-)$  and so on.

**Proof.** Take any  $X$  in  $\mathcal{S}_g$  and denote by  $\ell$  its homoclinic orbit from  $p$  to itself. The end point  $a^+$  of  $\underline{\ell}$  has coordinates  $a^+ = (-, 0, \psi_0)$ ; as usual with polar coordinates, when the radius is 0 the spherical coordinate is not defined. Let

$$(2.17) \quad \pi^{\psi_0} : \partial^+ \mathcal{M}_p \rightarrow \{\psi = \psi_0\}, \quad \pi^{\psi_0}(\phi, r, \psi) = (\phi, r, \psi_0),$$

be the projection onto the meridian  $i$ -disc.

Let  $H : \{z = 0\} \rightarrow \{z = 1\}$  denote the holonomy diffeomorphism defined by the vector field  $X$  in the tube  $T$ . The image of  $T \cap \Sigma^-$  through  $H$  is a  $(i-1)$ -disc  $D \subset \partial^+ \mathcal{M}_p$ . Due to the transversality condition associated with  $\ell$ , this disc is a graph over its projection  $D_{\psi_0} := \pi^{\psi_0}(D)$  if the tube is small enough around  $\underline{\ell}$ . Then,

$$(2.18) \quad \Delta^\phi := T_{a^+} D_{\psi_0} \subset T_{a^+} \{\psi = \psi_0\} \text{ is the preferred hyperplane.}$$

As we noticed in Remark 2.3, the vector  $\partial_v \in T_{a^+} \partial^+ \mathcal{M}_p$  is neither tangent to  $\Sigma^+$  nor to  $D$ , which implies that

$$(2.19) \quad (d\pi^{\psi_0})_{a^+}(\partial_v) \text{ defines a co-orientation of } \Delta^\phi \text{ in } T_{a^+} \{\psi = \psi_0\}.$$

This provides a preferred latitude on the  $\phi$ -sphere  $\partial\{\psi = \psi_0\} = S^{i-1} \times \{\sqrt{\delta}\} \times \{\psi_0\}$ . By the canonical isomorphism

$$(2.20) \quad S^{i-1} \times \{\sqrt{\delta}\} \times \{\psi_0\} \cong S^{i-1} \times \{0\} \times \{-\} = \Sigma^-,$$

the preferred latitude on  $\partial\{\psi = \psi_0\}$  descends to some  $\phi$ -latitude which defines the claimed  $\omega_\phi^X$  of (2.16).

For the  $\psi$ -latitude on the co-sphere  $\Sigma^+$ , we do the same construction by using the reversed flow and its holonomy  $H^{-1}$ . More precisely, take the image of  $T \cap \Sigma^+$  through  $H^{-1}$ ; it is a  $(n-i-1)$ -disc  $D' \subset \partial^- \mathcal{M}_p$  centered in  $a^-$  whose spherical coordinates are  $a^- = (\phi_0, 0, -)$ . Let

$$(2.21) \quad \pi^{\phi_0} : \partial^- \mathcal{M}_p \rightarrow \{\phi = \phi_0\}, \quad \pi^{\phi_0}(\phi, r, \psi) = (\phi_0, r, \psi),$$

be the projection onto the meridian  $(n-i)$ -disc and let  $D_{\phi_0}$  be the image  $\pi^{\phi_0}(D')$ . Hence,

$$(2.22) \quad \Delta^\psi := T_{a^-} D_{\phi_0} \subset T_{a^-} \{\phi = \phi_0\} \text{ is the preferred hyperplane.}$$

Moreover,  $\partial_v \in T_{a^-} \partial^- \mathcal{M}_p$  cannot be neither tangent to  $\Sigma^-$  nor to  $D'$ , which implies that

$$(2.23) \quad (d\pi^{\phi_0})_{a^-}(\partial_v) \text{ defines a co-orientation of } \Delta^\psi \text{ in } T_{a^-} \{\phi = \phi_0\}.$$

This yields a preferred latitude on the sphere  $\partial\{\phi = \phi_0\}$ , that can be carried to  $\Sigma^+$  by means of the canonical isomorphism

$$(2.24) \quad \{\phi_0\} \times \{\sqrt{\delta}\} \times S^{n-i-1} \cong \{-\} \times \{0\} \times S^{n-i-1} = \Sigma^+.$$

This defines the claimed  $\psi$ -latitude  $\omega_\psi^X$  of (2.16).  $\square$

**2.5. HOLONOMIC FACTOR AND CHARACTER.** By construction of the  $\psi$ -latitude, the tangent space  $T_{a^-} \{\phi = \phi_0\}$  splits as  $\mathbb{R}\partial_r^{\psi} \oplus \Delta^\psi$ . This together with the decomposition of  $T_{a^+} \{\psi = \psi_0\}$  defined by the  $\phi$ -latitude, we obtain the following decompositions:

$$(2.25) \quad \begin{cases} T_{a^-} \{z = 0\} = T_{a^-} \Sigma^- \oplus (\mathbb{R}\partial_r^{\psi} \oplus \Delta^\psi), \\ T_{a^+} \{z = 1\} = (\Delta^\phi \oplus \mathbb{R}\partial_r^{\phi}) \oplus T_{a^+} \Sigma^+. \end{cases}$$

Given  $(\alpha, X) \in \mathcal{S}_g$ , recall the homoclinic orbit  $\ell$  whose  $\Pi$ -value is  $g$  and all associated objects that we introduced in Subsection 2.2: the tube  $T$ , its coordinates  $(x, y, v, z)$  and the holonomy diffeomorphism  $H : \{z = 0\} \rightarrow \{z = 1\}$ . It reads  $Id$  in the  $(x, y, v)$ -coordinates and  $H(a^-) = a^+$ .

We are free to choose the coordinates of the tube such that the unit tangent vector  $\partial_v^1 := \partial_v \in T_{a^+} \{z = 1\}$  verifies

$$(2.26) \quad \partial_v^1 = \partial_r^{\psi}.$$

The *linearized holonomy*  $T_{a^+} H^{-1}$  sends  $\partial_v^1$  onto  $\partial_v^0 := \partial_v \in T_{a^-} \{z = 0\}$ . By (2.25), the latter vector decomposes as

$$(2.27) \quad \partial_v^0 = v_x + \bar{\eta} \partial_r^{\psi} + v_y, \text{ where } v_x \in T_{a^-} \Sigma^-, v_y \in \Delta^\psi, \bar{\eta} \in \mathbb{R}.$$

As we pointed in Remark 2.3, the only restriction on the holonomy of  $X$  along  $\underline{\ell}$  is that  $\bar{\eta} \neq 0$ . Moreover, according to (2.23), the vector  $\partial_v^0$  defines the positive side of the preferred hyperplane  $\Delta^\psi$ . As a consequence,  $\bar{\eta}$  must be positive.

**Definition 2.9.** *The holonomic factor associated with  $(\alpha, X)$  is the positive real number given by*

$$(2.28) \quad \eta(X) := \frac{1}{\bar{\eta}} > 0.$$

The next subsets of  $\mathcal{S}_g$  are respectively called the  $\phi$ -axis and the  $\psi$ -axis of  $\mathcal{S}_g$ :

$$(2.29) \quad \mathcal{S}_g^\phi := \{X \in \mathcal{S}_g \mid a^+(X) \in E^\psi\} \quad \text{and} \quad \mathcal{S}_g^\psi := \{X \in \mathcal{S}_g \mid a^-(X) \in E^\phi\},$$

that we also call the *spherical axes*. Here,  $a^\pm(X)$  stand for the respective extremities of the restricted orbit  $\underline{\ell}$ , where  $\ell$  is the unique homoclinic orbit of  $X$  in the homotopy class  $g$ . Denote the intersection of the axes:

$$(2.30) \quad \mathcal{S}_g^{0,0} := \mathcal{S}_g^\phi \cap \mathcal{S}_g^\psi$$

which is empty when one axis is so.

**Remark 2.10.** When the Morse index of  $\mathcal{S}_g$  equals 1, then the  $\phi$ -equator is empty but there are still signed poles. In that case, the  $\phi$ -latitude takes only the values  $\{-1, +1\}$  and the  $\psi$ -axis is empty. When the Morse index of  $\mathcal{S}_g$  equals  $n - 1$ , then the  $\psi$ -equator and the  $\phi$ -axis are empty and the  $\psi$ -latitude is valued in  $\{-1, +1\}$ . If  $n > 2$ , these two events do not happen simultaneously. This is the reason for the dimension assumption in Theorem 1.1 (2).

We are now ready for defining the important notion of *character*.

**Definition 2.11.** Let  $\chi : \mathcal{S}_g \rightarrow \mathbb{R}$  be the character function given by:

$$(2.31) \quad \chi(X) := \eta(X) \omega_\psi^X(a^+(X)) + \omega_\phi^X(a^-(X)).$$

Let  $\mathcal{S}_g^0 := \chi^{-1}(\{0\})$ . The character of  $(\alpha, X)$  is then defined as the map

$$(2.32) \quad \varepsilon : \mathcal{S}_g \setminus \mathcal{S}_g^0 \rightarrow \{+, -\},$$

given by the sign of  $\chi$ . Thus,  $\mathcal{S}_g \setminus \mathcal{S}_g^0$  divides into  $\mathcal{S}_g^+ \sqcup \mathcal{S}_g^-$ .

By the very definition of the latitudes, it is clear that each axis intersects  $\mathcal{S}_g^0$  along  $\mathcal{S}_g^{0,0}$ , as Figure 3 suggests (compare to Figure 1).

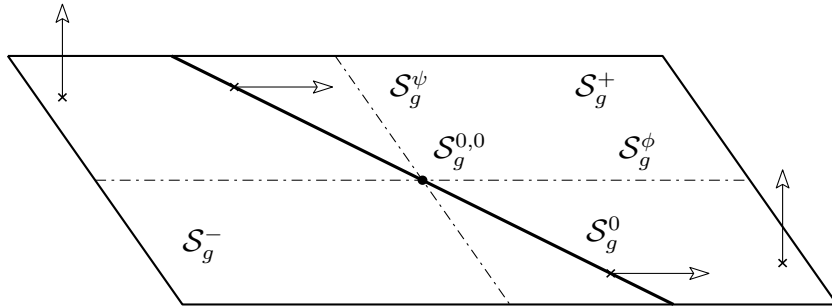


FIGURE 3. The substratum  $\mathcal{S}_g^{0,0} \subset \mathcal{S}_g^0$  as the intersection of the  $\phi$ -axis with the  $\psi$ -axis.

Below, we start giving some information about  $\mathcal{S}_g^{0,0}$  and prove Item (2) of Theorem 1.1.

**Proposition 2.12.**

- 1) The axes  $\mathcal{S}_g^\phi$  and  $\mathcal{S}_g^\psi$  are  $C^\infty$  submanifolds of codimension 1 in  $\mathcal{S}_g$ . Moreover, they intersect transversely. Hence, their intersection  $\mathcal{S}_g^{0,0}$  is a  $C^\infty$ -submanifold of codimension 2 in  $\mathcal{S}_g$ .
- 2) The stratum  $\mathcal{S}_g^0$  is a co-oriented sub-stratum of codimension 1 of class  $C^\infty$  in  $\mathcal{S}_g$ .

**Proof.**

1) First assume  $1 < i < n - 1$ , where  $i$  denotes the Morse index of  $\mathcal{S}_g$ . The equation of the  $\psi$ -axis  $\mathcal{S}_g^\psi$  in  $\mathcal{S}_g$  reads with the notations introduced in (2.16) and (2.29):

$$\omega_\phi^X(a^-(X)) = 0.$$

Let  $X_0 \in \mathcal{S}_g^\psi$ . In order to prove that the equation is regular, we have to exhibit a germ  $(X_s)_s$  of path in  $\mathcal{S}_g$  passing through  $X_0$  such that the  $s$ -derivative of  $\omega_\phi^X(a^-(X_s))$  at  $s = 0$  is non-zero. For  $s$  close to 0, let  $H_s$  be the local holonomy diffeomorphism of  $X_s$  from a neighborhood of  $\{z = 0\}$  in  $\partial^- \mathcal{M}_p$  to a neighborhood of  $\{z = 1\}$  in  $\partial^+ \mathcal{M}_p$ . Let  $a^- = (\phi_0, 0, -)$  and  $a^+ = (-, 0, \psi_0)$  be the end points of the restricted orbit  $\ell$ . We arrange that  $H_s$  keeps the  $\pi^{\psi_0}$ -projection of  $H_s(\Sigma^-)$  into the meridian  $\{\psi = \psi_0\} \subset \partial^+ \mathcal{M}_p$  independent of  $s$ . Thus, the equator  $E^\phi$  is so. Therefore, we are reduced to control the  $s$ -derivative of  $\omega_\phi(a^-(X_s))$ .

We recall that every germ of isotopy of the holonomy  $H_0$  lifts to a deformation of  $X_0$  (compare the solution of inequation (2.5)). Then, we are free to choose the holonomy so that  $s \mapsto a^-(X_s) \in \Sigma^-$  crosses  $E^\phi$  transversely at time  $s = 0$  and  $a^+(X_s)$  moves in  $\Sigma^+$  (in order to guarantee  $X_s \in \mathcal{S}_g$ ). Thus, we are done.

For a similar reason, the equation  $\omega_\psi^X(a^+(X)) = 0$  of the  $\phi$ -axis  $\mathcal{S}_g^\phi$  is regular. For showing the transversality of the two axis, we consider any  $X_{0,0} \in \mathcal{S}_g^{0,0}$ . We choose a (germ of) 2-parameter family  $X_{s,u} \in \mathcal{S}_g$  whose holonomy  $H_{s,u}$  satisfies the next conditions:

- (1) the equator  $E^\phi$  is independent of  $s$  when  $u = 0$  and  $\partial_s \omega_\phi(a^-(X_{s,0})) > 0$ ;
- (2) the equator  $E^\psi$  is independent of  $u$  when  $s = 0$  and  $\partial_u \omega_\psi(a^+(X_{0,u})) > 0$ ;
- (3) for every  $(s, u)$  close to  $(0, 0)$ , we have  $a^-(X_{s,u}) \in \Sigma^-$  and  $a^+(X_{s,u}) \in \Sigma^+$ .

Condition (3) guarantees that  $X_{s,u}$  runs in  $\mathcal{S}_g$ . Thanks to (1) and (2), the evaluation map  $(s, u) \mapsto (a^-(X_{s,u}), a^+(X_{s,u})) \in \Sigma^- \times \Sigma^+$  is transverse to the submanifold  $E^\phi \times E^\psi$ . This proves the independence of  $\partial_s X_{s,u}$  and  $\partial_u X_{s,u}$  at  $(0, 0)$ .

Suppose now  $1 < i = n - 1$ . The axis  $\mathcal{S}_g^\phi$  is empty since  $E^\psi = \emptyset$ . We only need to prove the assertion about  $\mathcal{S}_g^\psi$ . The above proof is still correct by only noticing that  $a^+(X_s)$  remains constantly equal to  $a^+ \in \{(-, 0, \nu_\psi), (-, 0, \sigma_\psi)\}$  and that  $E^\phi$  is a  $(n-3)$ -sphere inside  $\Sigma^-$ . The case  $i = 1 < n - 1$  is treated analogously. If  $n = 2$ , both axes are empty and there is nothing to prove.

2) By Definition 2.11, the equation of  $\mathcal{S}_g^0$  in  $\mathcal{S}_g$  reads

$$\chi(X) := \eta(X) \omega_\psi^X(a^+(X)) + \omega_\phi^X(a^-(X)) = 0$$

For proving that the equation is regular in any  $X_0 \in \mathcal{S}_g^0$  we have to exhibit a germ of path  $(X_s)_s$  in  $\mathcal{S}_g$  passing through  $X_0$  such that  $\partial_s \chi(X_s) > 0$ .

First, we arrange that the equators  $E^\phi$  and  $E^\psi$  do not depend on  $s$  by requiring that the holonomy  $H_s$  along the self-connecting orbit  $\ell$  of  $X_0$  fulfils the next conditions:

- for every  $s$ , there is a self-connecting orbit  $\ell_s$  (the end points of the restricted orbit  $\ell_s$  are noted  $a^-(X_s)$  and  $a^+(X_s)$ );
- the  $\pi^{\psi_0}$ -projection of  $H_s(\Sigma^-)$  into the meridian  $\{\psi = \psi_0\}$  is constant;
- the  $\pi^{\phi_0}$ -projection of  $(H_s)^{-1}(\Sigma^+)$  into the meridian  $\{\phi = \phi_0\}$  is constant.

Now, there are two cases depending on whether  $\omega_\psi(a^+(X_0))$  equals 0 or not. If  $\omega_\psi(a^+(X_0)) \neq 0$ , the germ of  $H_s$  at  $a^-(X_0)$  is chosen to be a contraction: its center is  $a^+(X_0)$  and its factor

is  $e^s$  (in the coordinate  $(x, y, v)$  of the extremity  $\{z = 1\}$  of the tube  $T$  around  $\underline{\ell}$ ). Notice that such a contraction preserves the above requirements for the constancy of the equators. Then, a calculation shows that the holonomic factor is multiplied by the same factor, which implies that  $\partial_s \chi(X_s) > 0$  since  $a^\pm(X_s)$  is constant.

Finally, we have to solve the case when  $\omega_\psi(a^+(X_0)) = 0$ . Here, we arrange the holonomy  $H_s$  so that  $\partial_s \omega_\psi(a^+(X_s)) > 0$  and  $\partial_s \omega_\phi(a^-(X_s)) = 0$ , which again implies  $\partial_s \chi(X_s) > 0$  since  $\eta(X_s) > 0$ .  $\square$

After Proposition 2.5 and Proposition 2.12 we have a complete proof of Theorem 1.1.  $\square$

**2.6. NORMALIZATION OF CROSSING PATH.** The normalization in question will be used for proving Theorem 1.2 and Theorem 1.3. The normalization is achieved by making some group act on  $M$  (actually the product of two groups). At the end of the subsection it will be proved that the stratification  $(\mathcal{S}_g, \mathcal{S}_g^0, \mathcal{S}_g^{0,0})$  is invariant under this action.

In this subsection we use notations as  $D_1(0), C_1(0)$  which will be used repeatedly in Section 3 (see Notation 3.7). Consider the image  $H_0(\Sigma^-) \subset \partial^+ \mathcal{M}_p$  by the holonomy map of  $X_0$  along its homoclinic orbit  $\ell$  in the homotopy class  $g \in \Pi_{\text{loop}}$ . Define

$$D_1(0) := H_0(\Sigma^-) \cap \{z = 1\}.$$

**Definition 2.13.**

- 1) The pseudo-gradient  $X_0 \in \mathcal{S}_g$  is said to be normalized or in normal form if  $D_1(0)$  is contained in the preferred hyperplane  $\Delta^\phi$  of the meridian  $i$ -disc  $\{\psi = \psi_0\}$ .
- 2) A crossing path  $(X_s)_s$  of  $\mathcal{S}_g$  is said to be normalized if  $X_0$  is so.

That any  $X_0 \in \mathcal{S}_g$  can be normalized by conjugation follows from the next lemma about diffeomorphisms of  $\mathcal{M}_p$  whose proof by Taylor expansion is detailed in the Appendix to [5].

**Lemma 2.14.** *Let  $K$  be a  $C^1$ -diffeomorphism of  $\partial^+ \mathcal{M}_p$  of the form  $(\phi, r, \psi) \mapsto (\phi, r, k(\phi, r, \psi))$ . It is assumed that  $k(\phi, 0, \psi) = \psi$ . Then,  $K$  uniquely extends to  $\mathcal{M}_p$  as a  $C^1$ -diffeomorphism which is the identity on both stable and unstable local manifolds and which keeps the standard vector field  $X_i$  invariant. Moreover, the extension  $\overline{K}$  is  $C^1$ -tangent to  $Id$  along the attaching sphere  $\Sigma^-$ .*

It is worth noticing that the extension cannot be  $C^2$  in general, even if  $K$  is  $C^\infty$ . This lemma can be also used by interchanging the roles of  $\partial^+ \mathcal{M}_p$  and  $\partial^- \mathcal{M}_p$  and simultaneously the roles of  $\phi$  and  $\psi$ .

**Corollary 2.15.** *Given a positive crossing path  $(X_s)_s$  of the stratum  $\mathcal{S}_g$ , there exists a  $C^1$ -diffeomorphism  $\overline{K}$  of  $M$ , isotopic to  $Id_M$  among the  $C^1$ -diffeomorphisms keeping  $\alpha$  and  $\mathcal{M}_p$  invariant, such that the crossing path  $(\overline{K}_* X_s)_s$  carried by  $\overline{K}$  is normalized. Moreover,  $\overline{K}$  may be chosen so that it preserves  $\ell$  pointwise.*

Notice that the vector field  $(\overline{K}_* X_s)$  could be  $C^0$  only. But it is integrable and the associate foliation is  $C^1$  transversely; its holonomy is changed by  $C^1$ -diffeomorphism.

**Proof.** Recall that  $D_1(0)$  is nowhere tangent to the fibres of the projection  $\pi^{\psi_0}$  to the meridian disc  $\{\psi = \psi_0\}$ . As a consequence, its projected disc  $D_{\psi_0}$  is smooth and there exists a smooth

map  $\bar{k} : D_{\psi_0} \rightarrow \Sigma^+$  such that  $D_1(0)$  reads

$$D_1(0) = \{(\phi, r, \bar{k}(\phi, r)) \mid (\phi, r) \in D_{\psi_0}\}.$$

Since its source is contractible,  $\bar{k}$  is homotopic to the constant map valued in  $\psi_0$ . By isotopy extension preserving the fibres of  $\pi^{\psi_0}$ , there exists some diffeomorphism  $K_1$  of  $\mathcal{M}_p$  of the form assumed in Lemma 2.14 which maps the given  $D_1(0)$  to  $D_{\psi_0}$ . This  $K_1$  extends to  $\mathcal{M}_p$  according to Lemma 2.14. Since  $K_1$  is isotopic to  $Id$  through diffeomorphisms of the same type, its extension  $\bar{K}_1$  to  $\mathcal{M}_p$  also extends to  $M$  with the same name. Moreover, the isotopy of  $\bar{K}_1$  to  $Id_M$  is supported in a neighborhood of  $\mathcal{M}_p$  and preserves each level set of a local primitive of  $\alpha$ . Since  $\Sigma^\pm$  are fixed by  $\bar{K}_1$ , it is easy to get that  $\ell$  is fixed by  $\bar{K}_1$ .

After having applied this  $\bar{K}_1$  we are reduced to the case where  $D_1(0)$  is contained in the meridian disc  $\{\psi = \psi_0\}$ . Decreasing the “multi-radius” of the tube  $T$  if necessary, the tangent plane  $T_m D_1(0)$  is almost orthogonal to the pole axis  $\mathbb{R}\partial_r^{\psi_0}$  in each  $m \in D_1(0)$ . This implies that, for every  $r \in (0, \sqrt{\delta})$ , the disc  $D_1(0)$  is transverse to the  $(i-1)$ -sphere of radius  $r$  in  $\{\psi = \psi_0\}$ .

Denote by  $C_1(0)$  the image of  $D_1(0)$  by  $Desc$ . It is diffeomorphic to  $S^{i-2} \times (0, 1]$  and contained in the spherical annulus  $\mathbb{A}_{\psi_0} := \{(\phi, r, \psi_0) \mid \phi \in \Sigma^-, r \in (0, \sqrt{\delta}]\}$ . By tangency of  $D_1(0)$  with the preferred hyperplane  $\Delta^\phi$ , the *end* of  $C_1(0)$  when  $r \rightarrow 0$  compactifies as the  $\phi$ -equator  $E^\phi \subset \Sigma^-$ . Moreover,  $C_1(0)$  is transverse (inside  $\mathbb{A}_{\psi_0}$ ) to the sphere  $\Sigma^- \times \{(r, \psi_0)\}$  for every  $r$ , since the corresponding assertion holds in  $\partial^+ \mathcal{M}_p$ . Thus, there is an annulus  $C_{eq} \subset E^\phi \times [0, \sqrt{\delta}] \times \{\psi_0\}$  such that  $C_1(0)$  reads as the graph of some map  $\bar{\kappa} : C_{eq} \rightarrow \Sigma^-$  valued in the complement of the poles. Then,  $\bar{\kappa}$  is homotopic to the map  $(\phi, r) \mapsto \phi$  from  $C_{eq}$  to the equator  $E^\phi$  of  $\Sigma^-$ .

By isotopy extension preserving each sphere  $\Sigma^- \times \{(r, \psi)\}$ , we have some diffeomorphism  $\bar{K}_2$  of  $\partial^- \mathcal{M}_p$  of the form  $(\phi, r, \psi) \mapsto (\kappa(\phi, r, \psi), r, \psi)$  which pushes  $C_1(0)$  to its *flat* position  $C_{eq}$  and satisfies  $\kappa(\phi, 0, \psi) = \phi$ . By applying Lemma 2.14 “up side down”,  $\bar{K}_2$  extends to  $M$  preserving  $\mathcal{M}_p$  with its standard gradient. On the upper boundary of the Morse model, this means that  $\bar{K}_2$  pushes  $D_1(0)$  to an  $(i-1)$ -disc in the hyperplane  $\Delta^\phi$  by some diffeomorphism tangent to  $Id$  in  $a_+$ . As for  $\bar{K}_1$ , this  $\bar{K}_2$  may be chosen so that  $\ell$  is fixed pointwise. The composed diffeomorphism  $\bar{K}_2 \circ \bar{K}_1$  is as desired.  $\square$

**Remark 2.16.** Corollary 2.15 holds true for a finite dimensional family. For instance, if we are given a two-dimensional germ  $(X_{s,t})_{s,t}$  adapted to the pair  $(\mathcal{S}_g, \mathcal{S}_g^0)$  in  $X_{0,0}$  – in the sense of Definition 4.1 – then each crossing path  $\gamma_t := (s \mapsto X_{s,t})$  of  $\mathcal{S}_g$  has a normalization by some  $C^1$ -diffeomorphism  $K_t$  depending continuously on  $t$  in the  $C^1$ -topology.

**Notation 2.17.** Let  $\mathcal{G}^\pm$  be the group of diffeomorphisms of  $M$  isotopic to  $Id_M$ , fixing the homoclinic orbit  $\ell$  pointwise, preserving the closed one-form  $\alpha$  and its standard gradient in  $\mathcal{M}_p$ , and having the following form:

- if  $G \in \mathcal{G}^+$ , its restriction to  $\partial^+ \mathcal{M}_p$  reads  $(\phi, r, \psi) \mapsto (\phi, r, k^+(r, \psi))$  with  $k^+(0, \psi) = \psi$ ;
- if  $G \in \mathcal{G}^-$ , its restriction to  $\partial^- \mathcal{M}_p$  reads  $(\phi, r, \psi) \mapsto (k^-(\phi, r), r, \psi)$  with  $k^-(\phi, 0) = \phi$ .

**Proposition 2.18.** *The action of the groups  $\mathcal{G}^+$  and  $\mathcal{G}^-$  on the space of pseudo-gradients of  $\alpha$  preserves the strata  $\mathcal{S}_g$ ,  $\mathcal{S}_g^0$  and  $\mathcal{S}_g^{0,0}$ .*

**Proof.** We do it for  $\mathcal{G}^+$ . Let  $G \in \mathcal{G}^+$  and  $X_0 \in \mathcal{S}_g$ . Since  $G$  fixes the homoclinic orbit  $\ell$  pointwise, the carried vector field  $G_*(X_0)$  has the same homoclinic orbit. According to the form of the restriction of  $G$  to the upper boundary of  $\mathcal{M}_p$ , the projection of  $D_1(0)$  to the meridian disc is unchanged. Therefore, the  $\phi$ -equator is preserved. Looking in the lower boundary, one derives that the  $\phi$ -latitude of  $a^-(X_0) = \Sigma^- \cap \ell$  is preserved.

Consider the disc  $D'_1(0) := H_{X_0}^{-1}(\Sigma^+)$ . Recall from Lemma 2.14 that  $G|_{\partial^-\mathcal{M}_p}$  is tangent to  $Id$  at every point of  $\Sigma^-$ . Therefore, the tangent space  $T_{a^-}D'_1(0)$  remains invariant by  $G$ . It follows that the  $\psi$ -equator is not changed, and hence, the  $\psi$ -latitude of  $a^+$  is preserved. Thus we have the invariance of the spherical axes and of their intersection  $\mathcal{S}_g^{0,0}$ .

It remains to show that the character function is invariant. We already have seen the invariance of the latitudes. The last term to control is the holonomic factor  $\eta(X_0)$  – resp.  $\eta(G_*(X_0))$  – defined in (2.27), which remains unchanged by the action of  $G$  thanks to the invariance of:

- $\partial_v^1$  by invariance of the  $\phi$ -latitude,
- $\partial_v^0$  since  $DG_{a^-} = Id$ ,
- the framing in which  $\partial_v^0$  decomposes (this framing is preserved by invariance of the  $\psi$ -latitude).

□

### 3. CHANGE IN THE MORSE-NOVIKOV COMPLEX

3.1. A GROUPOID APPROACH. A groupoid  $\mathcal{G}$  is a small category where every arrow is invertible. For every object  $p \in \mathcal{G}^0$ , we denote by  $1_p \in \text{Hom}(p, p) \subset \mathcal{G}^1$  its *identity* arrow. The maps *source* and *target*  $s, t : \mathcal{G}^1 \rightarrow \mathcal{G}^0$  are the respective components of the map  $g \in \text{Hom}(p, q) \mapsto (p, q) \in \mathcal{G}^0 \times \mathcal{G}^0$ .

**Remark 3.1.** We denote by  $\mathbb{Z}[[\mathcal{G}]]$  the set of formal series of the arrows of  $\mathcal{G}$ . An element  $\lambda \in \mathbb{Z}[[\mathcal{G}]]$  is usually written as  $\lambda = \sum_{g \in \mathcal{G}^1} n_g(\lambda)g$ , where  $n_g(\lambda) \in \mathbb{Z}$ . Define the *support* of  $\lambda$  as the set  $\text{supp}(\lambda) := \{g \in \mathcal{G}^1 \mid n_g(\lambda) \neq 0\}$ . Consider the set

$$(3.1) \quad \mathbb{Z}[\mathcal{G}] := \{\lambda \in \mathbb{Z}[[\mathcal{G}]] \mid \text{supp}(\lambda) \text{ is finite}\}.$$

Given two arrows  $g, h$  – seen as elements of  $\mathbb{Z}[\mathcal{G}]$  – the product  $gh \in \mathbb{Z}[\mathcal{G}]$  is defined by their composition in  $\mathcal{G}^1$  when  $t(g) = s(h)$  and by 0 otherwise. Extending the previous rule distributively with respect to the sum, we obtain a ring structure for  $\mathbb{Z}[\mathcal{G}]$ . Moreover, when  $\mathcal{G}^0$  is finite, the element  $1 := \sum_{p \in \mathcal{G}^0} 1_p \in \mathbb{Z}[\mathcal{G}]$  gives an *identity element* for this product. We call  $\mathbb{Z}[\mathcal{G}]$  the *groupoid ring* associated with  $\mathcal{G}$ . The next definition is classical.

**Definition 3.2.** *The fundamental groupoid  $\Pi$  of the manifold  $M$  is defined as follows: its objects are the points of  $M$  and, if  $(p, q)$  is a pair of points,  $\text{Hom}(p, q)$  is the set of homotopy classes of paths from  $p$  to  $q$ . If  $\gamma$  is a such a path, its homotopy class  $[\gamma]$  will be called the  $\Pi$ -value of  $\gamma$ .*

Given any representative  $\alpha$  of the fixed cohomology class  $u$ , we obtain a groupoid morphism

$$(3.2) \quad u_\alpha : \Pi \rightarrow \mathbb{R}, \quad g \mapsto \int_\gamma \alpha,$$



where  $g$  is the  $\Pi$ -value of a path  $\gamma$  in  $M$  and  $\mathbb{R}$  is seen as a groupoid with a single object. The restriction of any such  $u_\alpha$  to the fundamental group  $\pi_1(M, p)$  clearly coincides with the group morphism associated with  $u$ . The subgroupoid  $\Pi_{\text{loop}} \subset \Pi$  is formed with the homotopy classes of based loops in  $M$ .

We denote by  $\Pi_\alpha$  the full subcategory of  $\Pi$  whose set of objects coincides with  $Z(\alpha)$ . By Remark 3.1, when  $\alpha$  is Morse and  $Z(\alpha)$  is non-empty, we may consider the groupoid ring  $\mathbb{Z}[\Pi_\alpha]$ .

A formal series  $\lambda \in \mathbb{Z}[[\Pi_\alpha]]$  fulfills the *Novikov condition* if

$$(3.3) \quad \text{for every } L \in \mathbb{R}, \text{ the set } \text{supp}(\lambda) \cap u_\alpha^{-1}(L, +\infty) \text{ is finite.}$$

Denote by  $\Lambda_\alpha \subset \mathbb{Z}[[\Pi_\alpha]]$  the subset of formal series satisfying the Novikov condition. It can be easily checked that the product rule given in Remark 3.1 also gives a ring structure to  $\Lambda_\alpha$ , having the same identity element. We call  $\Lambda_\alpha$  the *Novikov ring associated with  $\alpha$* .

**Example 3.3.** Let  $g \in \Pi_{\text{loop}}$  with  $u(g) < 0$  – for instance, the  $\Pi$ -value of a self slide  $\ell$ . The following formal series are elements of the Novikov ring:

$$(3.4) \quad \sum_{j=1}^{\infty} g^j \quad \text{and} \quad \sum_{j=1}^{\infty} (-1)^j g^j$$

Indeed, the Novikov condition (3.3) is fulfilled since  $u_\alpha(g^j) = j \cdot u_\alpha(g)$  which goes to  $-\infty$  when  $j \rightarrow +\infty$ . Thus, they belong to the Novikov ring  $\Lambda_\alpha$ . In particular  $1 + g + g^2 + \dots$  is a unit whose inverse is  $1 - g$ .

We are going to deduce a chain complex  $(C_*(\alpha), \partial^X)$  when  $(\alpha, X)$  is a Morse-Smale pair. For a given index  $i$ , the left  $\Lambda_\alpha$ -module freely generated by  $Z_i(\alpha)$  is denoted by  $C_i(\alpha)$ . The map  $\partial_*^X : C_*(\alpha) \rightarrow C_{*-1}(\alpha)$  is then defined by:

$$(3.5) \quad \partial_*^X(p) = \sum_{q \in Z_{*-1}(\alpha)} \langle p, q \rangle^X q,$$

where the *incidence*  $\langle p, q \rangle^X$  is the algebraic count in  $\Lambda_\alpha$  which we are going to define, associated with the set  $\text{Orb}^X(p, q)$  of connecting orbits from  $p$  to  $q$ . First, we define the sign of a connecting orbit  $\ell \in \text{Orb}^X(p, q)$ . Given a point  $x \in \ell$ , the sign  $\varepsilon_\ell$  is defined by the following equation:

$$(3.6) \quad \varepsilon_\ell X(x) \wedge \text{co-or}(W^s(q, X)) = \text{or}(W^u(p, X)).$$

This definition is clearly independent of  $x \in \ell$ .

**Definition 3.4.** Given a Morse-Smale pair  $(\alpha, X)$ , the Morse-Novikov incidence associated with the data  $(p, q, X)$ ,  $p \in Z_i(\alpha)$ ,  $q \in Z_{i-1}(\alpha)$ , is defined by:

$$(3.7) \quad \langle p, q \rangle^X := \sum_{\ell \in \text{Orb}^X(p, q)} \varepsilon_\ell g_\ell \in \Lambda_\alpha,$$

where  $g_\ell$  denotes the  $\Pi$ -value of the connecting orbit  $\ell$ .

The map  $\partial^X$  as in (3.5) is indeed a differential; this can be found in [4]. The resulting  $(C_*(\alpha), \partial^X)$  is known as the Morse-Novikov complex (see [10], [12]).

We denote by  $\text{Orb}_L^X(p, q)$  the set of connecting orbits from  $p$  to  $q$  whose  $\alpha$ -length  $\mathcal{L}(\ell)$  is less than a fixed  $L > 0$ . Since these orbits verify the inequality  $u_\alpha(g_\ell) > -L$ , we are led to define a  $L$ -truncation map  $\mathcal{T}_L : \Lambda_\alpha \rightarrow \mathbb{Z}[\Pi_\alpha]$  by:

$$(3.8) \quad \mathcal{T}_L(\lambda) := \sum_{u_\alpha(g) > -L} n_g(\lambda)g.$$

Two elements  $\lambda, \mu \in \Lambda_\alpha$  are said to be  $L$ -equal if  $\mathcal{T}_L(\lambda - \mu) = 0$ , which is denoted by  $\lambda \equiv_L \mu$ .

Finally, the  $L$ -incidence is defined as follows:

$$(3.9) \quad \langle p, q \rangle_L^X := \sum_{\ell \in \text{Orb}_L^X(p, q)} \varepsilon_\ell g_\ell \in \mathbb{Z}[\Pi_\alpha].$$

Of course we have  $\mathcal{T}_L(\langle p, q \rangle^X) = \langle p, q \rangle_L^X$ .

**3.2. EFFECT OF SELF-SLIDES ON THE INCIDENCE.** Consider a generic one-parameter family of pseudo-gradients  $(X_s)_s$  adapted to the fixed Morse closed 1-form  $\alpha$  such that  $X_0 \in \mathcal{S}_g$ . By definition,  $X_0$  has a unique homoclinic orbit  $\ell$  connecting  $p \in Z(\alpha)$  to itself whose  $\Pi$ -value is  $g$ . Since  $\mathcal{S}_g^0$  has codimension one in  $\mathcal{S}_g$  (Proposition 2.12), generically the character of  $X_0$  is defined. Denote the index of  $p$  by  $i$ .

The next definition specifies some genericity conditions that will be needed to prove the theorem below. The remainder of this section is devoted to its proof and consequences.

**Definition 3.5.** *Let  $L > 0$ .*

- 1) *The pair  $(\alpha, X)$  is said to be Morse-Smale up to  $L$  if, for every pair of zeroes  $p, q \in Z(\alpha)$  and every  $X$ -orbit  $\ell$  from  $p$  to  $q$  with  $\int_\ell \alpha > -L$ , the unstable and stable manifolds,  $W^u(p, X)$  and  $W^s(q, X)$ , are transverse along  $\ell$ .*
- 2) *When  $X \in \mathcal{S}_g$  and  $u_\alpha(g) > -L$ , then  $(\alpha, X)$  is said to be almost Morse-Smale up to  $L$ , if the preceding transversality condition is fulfilled except for the unique homoclinic orbit whose  $\Pi$ -value is  $g$ .*

If  $(X_s)_s$  is a path which intersects  $\mathcal{S}_g$  transversely in  $X_0$  and if  $(\alpha, X_0)$  is almost Morse-Smale up to  $L$ , then  $(\alpha, X_s)$  is easily checked to be Morse-Smale up to  $L$  for every  $s \neq 0$  close enough to 0.

**Theorem 3.6.** *Let  $(X_s)_s$  be a path intersecting  $\mathcal{S}_g$  transversely in  $X_0$  and let  $L > -u_\alpha(g)$ . Assume  $(\alpha, X_0)$  to be almost Morse-Smale up to  $L$ , assume that its character is defined and the  $\phi$ -latitude is non-zero. Let  $q \in Z_{i-1}(\alpha)$ . The  $L$ -incidence  $\langle p, q \rangle_L^{X_s}$  is defined when  $s = 0_+$  and  $s = 0_-$ ; it is denoted by  $\langle p, q \rangle_L^\pm$  respectively. Then, we have the following.*

*When  $(X_s)_s$  intersects the stratum  $\mathcal{S}_g$  positively, the next relations hold in  $\Lambda_\alpha$ :*

- (1) *if  $X_0 \in \mathcal{S}_g^+$ , then  $\langle p, q \rangle_L^+ \equiv_L (1 + g + g^2 + g^3 + \dots) \cdot \langle p, q \rangle_L^-$ ,*
- (2) *if  $X_0 \in \mathcal{S}_g^-$ , then  $\langle p, q \rangle_L^+ \equiv_L (1 + g) \cdot \langle p, q \rangle_L^-$ .*

*When  $(X_s)_s$  intersects the stratum  $\mathcal{S}_g$  negatively, we have:*

- (1') if  $X_0 \in \mathcal{S}_g^+$ , then  $\langle p, q \rangle_L^+ \equiv_L (1 - g) \cdot \langle p, q \rangle_L^-$ ,  
 (2') if  $X_0 \in \mathcal{S}_g^-$ , then  $\langle p, q \rangle_L^+ \equiv_L (1 - g + g^2 - g^3 + \dots) \cdot \langle p, q \rangle_L^-$ .

**Proof of (1)  $\iff$  (1')  $\iff$  (2)  $\iff$  (2').** The first equivalence is obvious since  $1 - g$  is the inverse of  $1 + \sum_{j=1}^{\infty} g^j$ ; and similarly for the last equivalence.

We are left with the middle equivalence. It is obtained by changing the vector  $\partial_v$  into its opposite in the coordinates of the tube around the homoclinic orbit of  $X_0$ . This amounts to put a sign  $-$  in Formula (2.2). The latter change has three effects:

- i) It reverses the co-orientation of  $\mathcal{S}_g$ . Hence, positive and negative crossings are exchanged.
- ii) The character is changed into its opposite since the  $\phi$ - and  $\psi$ -latitudes are. Thus, both sides of  $\mathcal{S}_g^0$  are exchanged.
- iii) The homoclinic orbit becomes negative in the following sense: if the  $\phi$ -sphere is seen as the boundary of the meridian disc of  $a^+$ , the new positive hemisphere projects to the preferred hyperplane  $\Delta^\phi$  by reversing the orientation. This implies that in the algebraic counting of connecting orbits from  $p$  to  $q$  (where  $i(p) = i(q) + 1$ ) the coefficient  $g$  has to be changed to  $-g$  (see the orientation claim in Lemma 3.8).

We are left to prove the Theorem 3.6 in case (1). This will be done in Subsection 3.4.  $\square$

According to Proposition 2.18, the statement of Theorem 3.6 is invariant by the groups  $\mathcal{G}^\pm$  introduced in Notation 2.17. After Corollary 2.15, it is sufficient to consider the case where the crossing path in question is *normalized* in the sense of Definition 2.13. This assumption is done in what follows. We need some more notations and lemmas. The setting of Theorem 3.6 is still assumed.

### Notation 3.7.

- 1) The symbol  $\mathcal{O}_-$  stands for an open interval  $(-\epsilon, 0)$  whose size is not specified and which is understood as small as needed by the statement; and similarly for  $\mathcal{O}_+$ . If  $A$  is a closed subset of  $B$ ,  $\mathcal{O}_p(A)$  stands for an open neighborhood of  $A$  in  $B$  which is not specified.
- 2) Recall from Subsection 2.3 that  $H_s$  stands for the local holonomy diffeomorphism along the homoclinic orbit  $\ell$  from  $\mathcal{O}_p(\{z = 0\}) \subset \partial^- \mathcal{M}_p$  to  $\mathcal{O}_p(\{z = 1\}) \subset \partial^+ \mathcal{M}_p$ .
- 3) For  $s \in \mathcal{O}_p(0)$ , let  $D_1(s)$  denote the image  $H_s(\Sigma^-) \cap \{z = 1\}$ . Consider its projection  $\pi^{\psi_0}(D_1(s))$  onto the meridian disc  $\{\psi = \psi_0\}$  and define

$$(3.10) \quad a^+(s) := \pi^{\psi_0}(D_1(s)) \cap \mathbb{R}\partial_v^1.$$

The *crossing velocity* of the crossing path is

$$(3.11) \quad V_1 := \frac{d}{ds} a^+(s)|_{s=0}.$$

Up to a reparametrization in  $s$ , it can be assumed to be equal to 1.

- 4) For  $s \in \mathcal{O}_\pm$ , by definition of a crossing path  $D_1(s)$  avoids  $\Sigma^+$ . Therefore we are allowed to define  $C_1(s) := \text{Desc}(D_1(s)) \subset \partial^- \mathcal{M}_p$ . It is still an  $(i - 1)$ -disc.

**Lemma 3.8.** *Recall the natural projection  $\pi^- : \partial^- \mathcal{M}_p \rightarrow \Sigma^-$ . Let  $K$  be any compact disc in the open hemisphere  $\mathcal{H}^-(\Sigma^-)$  (as in Subsection 2.4). Then, for  $s \in \mathcal{O}_-$ , the disc  $C_1(s) \cap (\pi^-)^{-1}(K)$*

is a graph over  $K$  of a section of  $\pi^-$  which goes to the zero-section  $0_K$  of  $\pi^-$  in the  $C^1$ -topology when  $s$  goes to 0. Moreover,  $\pi^- : C_1(s) \rightarrow \Sigma^-$  is orientation reversing.

A similar statement holds when  $K \subset \mathcal{H}^+(\Sigma^-)$  and  $s \in \mathcal{O}_+$ , except that  $\pi^- : C_1(s) \rightarrow \Sigma^-$  is orientation preserving in that case.

**Proof.** The statement about orientation is clear after the claim about the  $C^1$ -convergence. Consider the case  $s \in \mathcal{O}_-$ , the other case being similar. Recall the normalization assumption: the disc  $D_1(0)$  is contained in the meridian disc  $\{\psi = \psi_0\}$ . Recall the projection  $\pi^{\psi_0}$  of  $\partial^+ \mathcal{M}_p$  to the meridian disc. The normalization implies that the projected discs  $\pi^{\psi_0}(D_1(s))$  tend to  $D_1(0)$  in the  $C^1$ -topology.

Recall the identification  $\partial(\{\psi = \psi_0\}) \cong \Sigma^-$  of (2.20) and think of  $K$  as a compact subset of the South hemisphere in the boundary of the meridian disc  $\{\psi = \psi_0\}$ . For every such  $K$ , the next property holds:

*For every  $s$  close enough to 0 and for every  $\phi \in K$ , the disc  $\pi^{\psi_0}(D_1(s))$  intersects the ray directed by  $\partial_r^\phi$  in one point only and transversely.*

This point is denoted by  $m_s(\phi)$ ; it is the image of some  $\tilde{m}_s(\phi) \in D_1(s)$  through  $\pi^{\psi_0}$ . We have  $m_0(\phi) = \tilde{m}_0(\phi) = a^+$ , but when  $s \neq 0$ , the point  $\tilde{m}_s(\phi)$  has well-defined multi-spherical coordinates  $(\phi, r_s(\phi), \psi_s(\phi))$  where  $r_s$  and  $\psi_s$  depend smoothly on  $s$ .

Going back to  $\partial^- \mathcal{M}_p$  by the map  $Desc$ , we see that  $C_1(s)$  is the image of a section of the projection  $\pi^-$  over  $K$ . When  $s \in \mathcal{O}_-$  goes to 0, then  $D_1(s) \cap \{(\phi, r, \psi) \in \partial^+ \mathcal{M}_p \mid \phi \in K\}$  goes to  $a^+$  in the metric sense. In particular,  $\max_{\phi \in K} \{r \mid (\phi, r, \psi) \in D_1(s)\}$  goes to 0. Therefore,  $C_1(s) \cap (\pi^-)^{-1}(K)$  goes to  $0_K$  in the  $C^0$ -topology when  $s$  goes to 0 negatively.

For the  $C^1$ -convergence, we use that  $K$  is far from the  $\phi$ -equator. Therefore, the angle in the meridian disc  $\{\psi = \psi_0\}$  between the ray  $\mathbb{R}\partial_r^\phi$  and the tangent plane to  $\pi^{\psi_0}(D_1(s))$  at  $m_s(\phi)$  is bounded from below. Including the fact that  $s \rightarrow 0_-$  implies  $r \rightarrow 0$ , it follows that the smooth functions  $r_s(\phi)$  and  $\psi_s(\phi)$  satisfy

$$\begin{cases} |dr| = O_s(r)|d\phi|, \\ |d\psi| = O_s(r)|d\phi|, \end{cases}$$

where  $O_s(r)$  stands for a quantity which is uniformly bounded with respect to  $r$  when  $s$  goes to 0. This yields the wanted  $C^1$ -convergence of the part of  $C_1(s)$  over  $K \subset \mathcal{H}^-(\Sigma^-)$  to  $0_K = K$ .  $\square$

**3.3. GEOMETRIC INTERPRETATION OF THE CHARACTER FUNCTION.** We still consider a germ of *normalized* positive crossing path  $(X_s)_s$ . Let  $D'_1(0)$  be the connected component  $W^s(p, X_0) \cap \{z = 0\}$  which contains  $a^-$ . It is an  $(n - i - 1)$ -disc which is the image of  $\Sigma^+$  by the inverse holonomy diffeomorphism  $H_0^{-1}$  along  $\ell$ . For every  $s \in \mathcal{O}p(0)$ , consider now  $D'_1(s) := H_s^{-1}(\Sigma^+) \cap \{z = 0\}$ .

Recall from Subsection 2.2 that  $\Sigma^- \cap \{z = 0\}$  is identified with the  $x$ -axis whereas  $D'_1(0)$  is identified with the  $y$ -axis. Let also  $p_v : \{z = 0\} \rightarrow \{v = 0, z = 0\}$  denote the projection parallel to  $\partial_v$  onto the  $(x, y)$ -space. When  $s \in \mathcal{O}p(0)$  goes to 0, the family  $D'_1(s)$  accumulates to the  $y$ -axis in the  $C^1$ -topology. And, according to Lemma 3.8, the family  $C_1(s) \cap \{z = 0\}$  accumulates to the  $x$ -axis in the  $C^1$ -topology if and only if  $s\omega_\phi(a^-)$  goes to  $0_+$ . In particular, when  $s\omega_\phi(a^-) > 0$  the projections  $p_v(C_1(s))$  and  $p_v(D'_1(s))$  intersect in a unique point  $b_1(s)$  and transversely. If  $s\omega_\phi(a^-)$  is negative, then  $C_1(s) \cap \{z = 0\}$  is empty.

Denote by  $c_1(s)$  and  $d'_1(s)$  the only points in  $C_1(s) \cap \{z = 0\}$  and in  $D'_1(s)$  respectively such that  $p_v(c_1(s)) = b_1(s) = p_v(d'_1(s))$ . Consider the real number

$$(3.12) \quad v_1(s) := v(c_1(s)) - v(d'_1(s)) \quad \text{for every } s \text{ such that } s \omega_\phi(a^-) \in \mathcal{O}_+.$$

This function  $v_1(s)$  depends smoothly on  $s$ . Its derivative with respect to  $s$  is denoted by  $\dot{v}_1(s)$ .

**Remark 3.9.** By construction,  $v_1(s) = 0$  implies  $c_1(s) = d'_1(s)$  which in turn implies the existence of an orbit  $\ell_s \in \text{Orb}^{X_s}(p, p)$  passing through  $c_1(s)$  such that  $[\ell_s] = g^2$ .

Lemma 3.10 will show the geometrical meaning of the character function  $\chi$  at  $X = X_0$ . For simplicity, in what follows  $a^\pm(X_0)$  will be denoted by  $a^\pm$ .

**Lemma 3.10.** *Given a normalized positive crossing path  $(X_s)_s$  whose crossing velocity defined in (3.11) equals +1, the next relation holds:*

$$(3.13) \quad \omega_\phi(a^-) \dot{v}_1(0) = \chi(X_0).$$

**Proof.** Let us study the  $v$ -coordinate of  $c_1(s)$  first. We notice that, if  $\bar{c}_1(s)$  is another point of  $C_1(s)$  depending smoothly on  $s$  and such that  $\bar{c}_1(0) = a^- = (\phi_0, 0, -)$ , we have the same velocity in  $s = 0$ :

$$(3.14) \quad \frac{d}{ds} v(\bar{c}_1(s))|_{s=0} = \frac{d}{ds} v(c_1(s))|_{s=0}.$$

Indeed,  $C_1(s)$  accumulates  $C^1$  to  $\Sigma^-$  (Lemma 3.8), then the difference  $\dot{c}_1(0) - \dot{\bar{c}}_1(0)$  is a vector in  $T_{a^-}\Sigma^-$ . We apply this remark to the point  $\bar{c}_1(s) := C_1(s) \cap \{\phi = \phi_0\}$ . We now lift that point to  $d_1(s) \in D_1(s)$  by  $Desc^{-1}$ .

Since  $Desc$  preserves the  $(\phi, r, \psi)$ -coordinates, both paths  $s \mapsto \bar{c}_1(s)$  and  $s \mapsto d_1(s)$  have the same coordinates  $(\phi(s) = \phi_0, r(s), \psi(s))$  when  $s \neq 0$ . Since  $\bar{c}_1(s) \in \{\phi = \phi_0\}$ , the vector  $\dot{d}_1(0)$  belongs to the  $(n-i)$ -plane which is span of  $\{\partial_r^{\phi_0}, T_{a^+}\Sigma^+\}$ . Let  $\hat{d}_1(0)$  be its projection to the line  $\mathbb{R}\partial_r^{\phi_0}$  in the meridian disc  $\{\psi = \psi_0\}$ . Then,

$$(3.15) \quad \hat{d}_1(0) = \rho \partial_r^{\phi_0}, \quad \text{where } \rho = \frac{d}{ds} r(s)|_{s=0}.$$

By definition of the  $\phi$ -latitude (Proposition 2.8 and (2.14)) we have:

$$(3.16) \quad p_{\Delta^\phi}(\hat{d}_1(0)) = \rho p_{\Delta^\phi}(\partial_r^{\phi_0}) = \rho \omega_\phi(a^-) \partial_r^{\psi_0}.$$

By definition, the hyperplane  $\Delta^\phi$  is tangent in  $a^+$  to the projection  $p_y(D_1(0))$ . Therefore, some calculus of Taylor expansion allows us to deduce that

$$(3.17) \quad p_{\Delta^\phi}(\hat{d}_1(0)) = \frac{d}{ds} a^+(s)|_{s=0} = +1.$$

We derive:

$$(3.18) \quad \rho = \frac{1}{\omega_\phi(a^-)}.$$

Since  $d_1(s)$  goes to  $a^+ = (-, 0, \psi_0)$  when  $s$  goes to 0 and since the radial velocity is preserved by  $Desc$ , then we have:

$$(3.19) \quad \dot{\bar{c}}_1(0) = \rho \partial_r^{\psi_0} \in T_{a^-}\{\phi = \phi_0\}.$$

Using again (2.14), but relatively to the hyperplane  $\Delta^\psi$  which defines the  $\psi$ -latitude we obtain

$$(3.20) \quad p_{\Delta^\psi}(\dot{\tilde{c}}_1(0)) = \rho \omega_\psi(a^+) \partial_r^{\nu_\psi}.$$

This together with the decomposition of  $T_{a^-} \{z = 0\}$  of (2.25) says that there are  $w_x \in T_{a^-} \Sigma^-$  and  $w_y \in \Delta^\psi$  such that

$$(3.21) \quad \dot{\tilde{c}}_1(0) = w_x + \rho \omega_\psi(a^+) \partial_r^{\nu_\psi} + w_y.$$

Consider the projection  $p_{xy} : T_{a^-} \{z = 0\} \rightarrow \mathbb{R} \partial_v^0$  parallel to the hyperplane  $T_{a^-} \Sigma^- \oplus \Delta^\psi$ . Thanks to (3.21), we have  $v(\dot{\tilde{c}}_1(0)) \partial_v^0 = p_{xy}(\dot{\tilde{c}}_1(0)) = \rho \omega_\psi(a^+) p_{xy}(\partial_r^{\nu_\psi})$ . On the other hand, (2.27) tells us that:

$$(3.22) \quad v(\partial_r^{\nu_\psi}) = \frac{1}{\bar{\eta}} = \eta.$$

Putting together (3.14), (3.18), (3.21) and (3.22) we obtain:

$$(3.23) \quad v(\dot{\tilde{c}}_1(0)) = v(\dot{\tilde{c}}_1(0)) = \rho \omega_\psi(a^+) v(\partial_r^{\nu_\psi}) = \frac{\omega_\psi(a^+)}{\omega_\phi(a^-)} \eta.$$

We come now to estimate the term  $v(\dot{d}'_1(0))$ . We apply Lemma 2.7 for comparing velocities associated with the holonomy  $H_s$  and its inverse. From the formula (3.11) we derive that  $\frac{d}{ds}(v \circ H_s)(a^-)|_{s=0} = +1$ . Then, the inverse holonomy satisfies

$$(3.24) \quad \frac{d}{ds}(v \circ H_s^{-1})(a^+)|_{s=0} = -1$$

from which it is easily derived that  $v(\dot{d}'_1(0)) = -1$ . Therefore:

$$(3.25) \quad \dot{v}_1(0) = \eta \frac{\omega_\psi(a^+)}{\omega_\phi(a^-)} + 1$$

which is a reformulation of the desired formula.  $\square$

Lemma 3.11 right below is the last tool that we need for proving Theorem 3.6. It extracts the geometrical information contained in Equation (3.13). The setting is the same as in the previous lemma. We are only looking at normalized paths  $(X_s)_s$  which cross  $\mathcal{S}_g$  positively in a point  $X_0$  whose character is positive. A similar statement holds for paths crossing at a point whose character is negative.

**Lemma 3.11.**

1) Suppose  $\dot{v}_1(0)$  and the  $\phi$ -latitude  $\omega_\phi(a^-)$  are positive. Then, for  $s \in \mathcal{O}_+$  there are sequences of non-empty  $(i-1)$ -discs  $(D_k(s))_{k>1}$  and  $(C_k(s))_{k>1}$  inductively defined from the previous  $D_1(s)$  and  $C_1(s)$  by

$$(3.26) \quad \begin{cases} D_k(s) := H_s(C_{k-1}(s)) \cap \{z = 1\} \\ C_k(s) := Desc(D_k(s)) \subset \partial^- \mathcal{M}_p \end{cases}$$

Moreover, when  $s$  goes to 0, the discs  $C_k(s)$  tend to the North hemisphere  $\mathcal{H}^+(\Sigma^-)$  in the  $C^1$ -topology, uniformly over every compact set of  $\mathcal{H}^+(\Sigma^-)$ . When  $s \in \mathcal{O}_-$ , both previous sequences are empty when  $k > 1$ .

2) If  $\dot{v}_1(0)$  and  $\omega_\phi(a^-)$  are negative, then for  $s \in \mathcal{O}_-$  the disc  $C_2(s)$  is well defined as in (3.26) and the next ones are empty. Moreover,  $C_2(s)$  tends to  $\mathcal{H}^+(\Sigma^-)$  in the  $C^1$ -topology with the reversed orientation. When  $s \in \mathcal{O}_+$ , every disc in (3.26) is empty when  $k > 1$ .

Notice that, according to Lemma 3.10 the assumption of 1) reads also: the character and  $\omega_\phi(a^-)$  are positive; and the assumption of 2) reads: the character is positive and  $\omega_\phi(a^-) < 0$ .

**Proof.** 1) When  $s \in \mathcal{O}_-$ , the disc  $C_1(s)$  does not meet the tube  $T$  around the homoclinic orbit  $\ell$ . Then  $D_2(s)$  is empty and, hence, all the further discs are so.

Assume now that  $s \in \mathcal{O}_+$ . In that case,  $C_1(s)$  goes to  $\mathcal{H}^+(\Sigma^-)$  (Lemma 3.8) and therefore meets the set  $\{z = 0\}$ . Then, the discs  $D_2(s)$  and  $C_2(s)$  defined in (3.26) are non empty. We are going to compute the position of  $C_2(s)$  with respect to  $D'_1(s)$  measured by some  $v_2(s)$  in the direction of the  $v$ -coordinate. We shall check the positivity of  $\dot{v}_2(0)$  which will allow us to pursue the induction.

Recall the projection  $\pi^{\psi_0} : \partial^+ \mathcal{M}_p \rightarrow \{\psi = \psi_0\}$  and define the spherical annulus  $A := (\pi^{\psi_0})^{-1}(\mathbb{R}\partial_v)$ . Consider the point  $\tilde{c}_1(s)$  which is the transverse intersection  $C_1(s) \cap H_s^{-1}(A)$ . By projecting to the  $v$ -axis we find a function  $v(\tilde{c}_1(s))$  which satisfies

$$(3.27) \quad \frac{d}{ds} v(\tilde{c}_1(s))|_{s=0} = \frac{d}{ds} v(c_1(s))|_{s=0}$$

Recall the definition of  $d'_1(s) \in D'_1(s)$  from (3.12). Compute the derivative  $V_2$  at  $s = 0$  of  $v[H_s(\tilde{c}_1(s))] - v[H_s(d'_1(s))]$ , which is nothing but the velocity of the projection of  $H_s(\tilde{c}_1(s)) \in D_2(s)$  onto the  $v$ -axis of  $\{z = 1\}$  at  $s = 0$ . Using  $\tilde{c}_1(0) = d'_1(0) = a^-$  and  $dH_0(a^-) = Id$  in the coordinates of the tube  $T$ , we find:

$$(3.28) \quad V_2 = \dot{v}_1(0)$$

which is positive by assumption. This  $V_2$  will play the same rôle as the crossing velocity.

Since  $V_2 > 0$ , Lemma 3.8 tells us that  $C_2(s)$  meets  $\{z = 0\}$  when  $s \in \mathcal{O}_+$ . Therefore, we choose points  $c_2(s) \in C_2(s) = Desc(D_2(s))$  and  $d'_2(s) \in D'_1(s)$  which forms the unique pair of points of the respective subsets which have the same  $p_v$ -projection. We define

$$(3.29) \quad v_2(s) := v(c_2(s)) - v(d'_2(s))$$

The computation of  $\dot{v}_2(0)$  is exactly the same except we have to replace  $V_1 = 1$  with  $V_2$ . The result is:

$$(3.30) \quad \dot{v}_2(0) = \eta \frac{\omega_\psi(a^+)}{\omega_\phi(a^-)} V_2 + 1.$$

Here, it is appearing some discussion according to the sign of  $\omega_\psi(a^+)$ :

- (i) if  $\omega_\psi(a^+)$  is positive, then  $\dot{v}_2(0)$  is larger than  $V_1 = +1$ . In that case the induction goes on with  $V_k > V_{k-1} > \dots > 1$ .
- (ii) if  $\omega_\psi(a^+)$  is negative, then  $0 < V_2 = \dot{v}_1(0) < 1$ , where the last inequality comes from (3.25). Therefore,  $\dot{v}_2(0) - 1$  is the product of two numbers<sup>5</sup> of opposite signs and whose absolute values are smaller than 1. Thus,  $\dot{v}_2(0)$  belongs to  $(0, 1)$ . Such a fact is preserved at each step of the induction.

<sup>5</sup>One of them being  $V_2$ .

The induction can be carried on.

2) Take  $s \in \mathcal{O}_-$ . The calculation yielding the equality (3.28) still holds and tells us that  $V_2$  is negative. Remark that  $H_s(d'_1(s)) \in \Sigma^+$ . As  $s < 0$ , one derives:

$$v[H_s(\tilde{c}_1(s))] = v[H_s(\tilde{c}_1(s))] - v[H_s(d'_1(s))] > 0.$$

Thus, Lemma 3.8 says that  $C_2(s)$  tends to  $\mathcal{H}^+(\Sigma^+)$  in the  $C^1$ -topology. As  $\omega_\phi(a^-) < 0$ ,  $C_2(s)$  does not meet  $\{z = 0\}$  and the next discs are empty. Concerning the orientation, we check that  $D_2(s)$  tends to  $-D_1(0)$  in  $\partial^+ \mathcal{M}_p$ . Then,  $C_2(s)$  tends to  $-\mathcal{H}^+(\Sigma^-)$ . Finally, the statement when  $s \in \mathcal{O}_+$  is clear.  $\square$

3.4. PROOF OF THEOREM 3.6 CONTINUED. We continue the proof which begins just after the statement of that theorem. After a series of equivalences, we are left to prove the case (1) of a positive crossing of the stratum  $\mathcal{S}_g$  at a point  $(\alpha, X_0)$  where the character is positive. We recall that the statement of Theorem 3.6 is preserved under the action of the groups  $\mathcal{G}^\pm$ . Therefore, we may assume that  $X_0 \in \mathcal{S}_g$  is normalized.

The element  $g \in \Pi_\alpha$  is thought of as an arrow from the set of zeroes  $Z(\alpha)$  into itself. Then  $g$  determines its origin  $p$  which is also its end point. Recall that the Morse index of  $p$  is  $i$ . We look at any zero  $q \in Z(\alpha)$  of Morse index  $i - 1$ . We have to compute the change of  $\langle p, q \rangle^X$  when  $X$  changes from  $X_{0-}$  to  $X_{0+}$  in a crossing path  $(X_s)_s$ . Taking into account that we search for an isomorphism of complexes over the Novikov ring  $\Lambda_\alpha$ , the first attempt is to look at each connecting orbit from  $p$  to  $q$  separately. But it does not work. It is useful to make some partition, adapted to  $g$ , of the set of connecting orbits.

PARTITION OF THE CONNECTING ORBITS. We may assume that each connecting orbit is the unique one in its homotopy class. In general, one would take the multiplicity into account. The equivalence relation defining the partition is the following:  $\Gamma_0 \sim \Gamma_1$  if and only if the homotopy class of  $\Gamma_1$  reads  $[\Gamma_1] = g^k[\Gamma_0]$  with  $k \in \mathbb{Z}$ .

Consider  $[\Gamma]_\sim$ , the  $\sim$ -class of a fixed connecting orbit  $\Gamma$ . Since the  $\alpha$ -lengths of connecting orbits are positive, we have  $u([\Gamma']) < 0$  for every  $\Gamma' \in [\Gamma]_\sim$ . Therefore, as  $u(g) < 0$ , there are only finitely many connecting orbits  $\Gamma' \in [\Gamma]_\sim$  verifying  $u([\Gamma']) \geq u([\Gamma])$ . Let  $\Gamma_0$  be a connecting orbit in  $[\Gamma]_\sim$  such that  $u([\Gamma_0])$  is maximal. Then, any element of  $[\Gamma]_\sim$  reads  $g^k[\Gamma_0]$  with  $k \geq 0$ .

END OF THE PROOF. Now, without loss of generality we may assume that the above partition has one  $\sim$ -class only and that the maximal element  $\Gamma$  is a positive connecting orbit. Let  $b := \Gamma \cap \Sigma^-$  and  $\Delta_s$  be the connected component of  $W^s(q, X_s) \cap \partial^- \mathcal{M}_p$  containing  $b$ , which is a  $(n - i)$ -disc intersecting transversely  $\Sigma^-$ , only at  $b$ . Since, we are looking at the change formula up to length  $L > 0$ , we may assume that the crossing path  $(X_s)_s$  is defined for  $s \in [-1, 1]$  and for every  $s \neq 0$  the pair  $(\alpha, X_s)$  is Morse-Smale up to  $L$ . Therefore, as the property of being Morse-Smale up to  $L$  is open, we are allowed to slightly move this path (in particular,  $X_0$ ) keeping the end points fixed. Thus, generically  $b$  does not lie on the equator  $E^\phi$ , and  $\Delta_s$  may be supposed to coincide with  $\Delta_0$  for every  $s$ . There are still four cases to consider where  $a^-$  stands for  $a^-(X_0)$  and  $\mathcal{H}^\pm$  stand for  $\mathcal{H}^\pm(\Sigma^-)$ :

- (a.1) The  $\phi$ -latitude  $\omega_\phi(a^-)$  is positive and  $b$  belongs to  $\mathcal{H}^+$ .
- (a.2) The  $\phi$ -latitude  $\omega_\phi(a^-)$  is positive and  $b$  belongs to  $\mathcal{H}^-$ .



- (b.1) The  $\phi$ -latitude  $\omega_\phi(a^-)$  is negative and  $b$  belongs to  $\mathcal{H}^+$ .
- (b.2) The  $\phi$ -latitude  $\omega_\phi(a^-)$  is negative and  $b$  belongs to  $\mathcal{H}^-$ .

The proof consists of applying Lemma 3.11. It is convenient to use the following definition.

### Definitions.

- 1) The positive (resp. negative) part of  $W^u(p, X_0)$  is the union of the  $X_0$ -orbits passing through the positive (resp. negative) hemisphere  $\mathcal{H}^+(\Sigma^-)$ . It will be denoted by  $W^u(p, X_0)^\pm$ .
- 2) For a given  $k > 0$ , we say that the unstable manifolds  $W^u(p, X_s)$  accumulate to  $g^k \cdot W^u(p, X_0)^\pm$  when  $s$  goes to  $0_-$  or  $0_+$  if it is true when lifting to the universal cover, that is: if  $\tilde{p}$  (resp.  $\tilde{X}_s$ ) is a lift of  $p$  (resp.  $X_s$ ), the unstable manifolds  $W^u(\tilde{p}, \tilde{X}_s)$  accumulate to  $W^u(g^k \tilde{p}, \tilde{X}_0)^\pm$ .

Here, it is worth noticing that when a point lies in the accumulation set its whole  $X_0$ -orbit is accumulated. As a consequence, Lemma 3.8 tells us that  $W^u(p, X_s)$  accumulate to  $g \cdot W^u(p, X_0)^\pm$  in the  $C^1$ -topology when  $s$  goes to  $0_\pm$ . Thanks to this topology, it makes sense to compare the orientations. The result is the following: when  $s \rightarrow 0_\pm$ , then  $W^u(p, X_s)$  accumulate to  $\pm g \cdot W^u(p, X_0)^\pm$ . Accumulation to  $g^k \cdot W^u(p, X_0)^\pm$  for some  $k > 1$  is dictated by Lemma 3.11 depending on the sign of the character and the  $\phi$ -latitude. We are now ready for proving Theorem 3.6 in each case.

Let  $\lambda_-(\Gamma)$  (resp.  $\lambda_+(\Gamma)$ ) denote the element of the Novikov ring  $\Lambda_\alpha$  which is the contribution of  $[\Gamma]_\sim$  (up to the given  $L > 0$ ) in  $\langle p, q \rangle_s^X$  when  $s < 0$  (resp.  $s > 0$ ). We have to check the next formula in each case (a.1) ... (b.2).

$$(3.31) \quad \lambda_+(\Gamma) = (1 + g + g^2 + \dots) \cdot \lambda_-(\Gamma)$$

**Case (a.1).** When  $s \rightarrow 0_-$ , the oriented unstable manifolds  $W^u(p, X_s)$  accumulate to  $-g \cdot W^u(p, X_0)^-$  and nothing else. Therefore, as  $b \in \mathcal{H}^+$ , we have  $\lambda_-(\Gamma) = [\Gamma]$ .

When  $s \rightarrow 0_+$ , then  $W^u(p, X_s)$  accumulate to  $+g^k \cdot W^u(p, X_0)^+$  for every  $k > 0$  and will intersect  $g^k \cdot \Delta_0$  transversely at a single point. Thus, we have  $\lambda_+(\Gamma) = (1 + g + g^2 + \dots) \cdot [\Gamma]$ . The change of  $\lambda_\pm(\Gamma)$  from  $s < 0$  to  $s > 0$  is well given by Formula (3.31).

**Case (a.2).** As  $b \in \mathcal{H}^-$  and taking into account the accumulation described right above, we have:  $\lambda_-(\Gamma) = (1 - g) \cdot [\Gamma]$  and  $\lambda_+(\Gamma) = [\Gamma]$ . The wanted formula is still satisfied.

**Case (b.1).** Here, the accumulation of  $W^u(p, X_s)$  is dictated by part 2) of Lemma 3.11 and the reason why formula (3.31) holds is more surprising than in the previous cases. When  $s \rightarrow 0_-$ , the manifolds  $W^u(p, X_s)$  accumulate to  $-g \cdot W^u(p, X_0)^-$  and to  $-g^2 \cdot W^u(p, X_0)^+$  and then nothing else. When  $s \rightarrow 0_+$ , the manifolds  $W^u(p, X_s)$  accumulate to  $+g \cdot W^u(p, X_0)^+$  and nothing else.

As  $b \in \mathcal{H}^+$ , we have  $\lambda_-(\Gamma) = (1 - g^2) \cdot [\Gamma]$  and  $\lambda_+(\Gamma) = (1 + g) \cdot [\Gamma]$ . Formula (3.31) is right since the identity  $(1 + g + g^2 + \dots)(1 - g^2) = 1 + g$  holds in the Novikov ring.

**Case (b.2).** Accumulation is as right above. One derives that  $\lambda_-(\Gamma) = (1 - g) \cdot [\Gamma]$  and  $\lambda_+(\Gamma) = [\Gamma]$ . The wanted formula is still satisfied.

The proof of Theorem 3.6 is now complete. □

## 4. PROOF OF THEOREM 1.3

4.1. NOTATIONS AND STATEMENT. In this section, we state and prove the refined version of Theorem 1.3 which is given right below after specifying some definition and notations. In the whole section  $X_0$  is an element in the codimension-one stratum  $\mathcal{S}_g^0 \subset \mathcal{S}_g$ , the co-oriented locus where the character function  $\chi$  vanishes. It is assumed that  $(\alpha, X_0)$  does not belong to  $\mathcal{S}_g^{0,0}$ , the locus where both  $\phi$ - and  $\psi$ -latitudes vanish.

**Definition 4.1.**

- 1) Let  $\mathbb{R}_+$  (resp.  $\mathbb{R}_-$ ) be the set of positive (resp. negative) real numbers. The open set  $\mathcal{S}_g^{0,\pm} \subset \mathcal{S}_g^0$  is defined by the sign of the  $\phi$ -latitude, that is,  $\omega_\phi(X_0) \in \mathbb{R}_\pm$ .
- 2) Let  $X_0 \in \mathcal{S}_g^0$ . Let  $(\mathcal{D}(s, t) := X_{s,t})$  be a germ in  $X_0$  of a two-parameter family of vector fields of pseudo-gradients of  $\alpha$ . It is assumed that  $\mathcal{D}$  passes through  $X_0$  when  $(s, t) = (0, 0)$ . This germ is said to be adapted to the pair  $(\mathcal{S}_g, \mathcal{S}_g^0)$  if the next conditions are fulfilled:

- (1) The one-parameter family  $(\mathcal{D}(0, t))_t$  is contained in  $\mathcal{S}_g$ , transverse to  $\mathcal{S}_g^0$  and  $\frac{\partial \mathcal{D}}{\partial t}(0, 0)$  points towards  $\mathcal{S}_g^+$ .
- (2) The partial derivative  $\frac{\partial \mathcal{D}}{\partial s}(0, 0)$  is transverse to  $\mathcal{S}_g$  and points towards to its positive side.

In particular,  $\mathcal{D}$  is transverse to  $\mathcal{S}_g^0$ .

**Theorem 4.2.** Let  $\mathcal{D}$  be a germ of 2-disc transverse to  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$  and adapted to the pair  $(\mathcal{S}_g, \mathcal{S}_g^0)$ . Then  $\mathcal{D}$  intersects  $\mathcal{S}_{g^2}$  transversely and the trace on  $\mathcal{D}$  of the strata  $(\mathcal{F}_\alpha, \mathcal{S}_g \cup \mathcal{S}_{g^2}^+, \mathcal{S}_g^{0,\pm})$  is  $C^1$ -diffeomorphic to

$$(\mathbb{R}^2, \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}_\pm, \{(0, 0)\}).$$

Moreover, the natural co-orientation of  $\mathcal{S}_{g^2}$  restricts to the natural co-orientation of  $\mathcal{S}_g^0$  in  $\mathcal{S}_g$  or to its opposite depending upon  $\mathcal{S}_{g^2}$  approaches  $\mathcal{S}_g^{0,+}$  or  $\mathcal{S}_g^{0,-}$  respectively.

We first prove Theorem 4.2 for particular germs  $\mathcal{D}$  which we call *elementary*. These germs consist of one-parameter family of positive normalized crossing paths of  $\mathcal{S}_g$  in the sense of Definition 2.13.

4.2. ELEMENTARY CROSSING PATH.<sup>6</sup> Let  $(X_s)_s$  be a normalized positive crossing path of  $\mathcal{S}_g$ . After the normalization we are allowed to prescribe more special dynamics of  $X_s$ ; the perturbed holonomy will be specified near the respective homoclinic orbit. This is the flavor of what *elementary* means.

Consider the spherical annulus  $\mathbb{A}_{\psi_0} := \Sigma^- \times (0, \sqrt{\delta}) \times \{\psi_0\} \subseteq \partial^- \mathcal{M}_p$ . Assume the  $\psi$ -latitude of  $a^+$  different from zero. Therefore, the preimage  $D'_1(s)$  of  $\Sigma^+$  by the holonomy diffeomorphism  $H_s$  is transverse to  $\mathbb{A}_{\psi_0}$ . Call  $b(s)$  the intersection point  $D'_1(s) \cap \mathbb{A}_{\psi_0}$  when the intersection is non-empty; it is the case either when  $s < 0$  or  $s > 0$  but not in both cases.

**Definition 4.3.** The path  $(X_s)_s$  is said to be elementary if the next conditions are fulfilled.

- (1) The disc  $D_1(s)$  moves in the meridian disc  $\{\psi = \psi_0\}$  parallelly to the the hyperplane  $\Delta^\phi$ .

<sup>6</sup>This terminology is inspired by that which J. Cerf used in his study of the stratification of smooth functions [1].

- (2) We have  $\frac{da^+(s)}{ds} = 1$  for every  $s$ .
- (3) The point  $b(s)$  runs on the ray  $\{(\phi_0, r, \psi_0) \mid r \geq 0\}$  for  $s$  close to 0 positively or negatively if the  $\psi$ -latitude  $\omega_\psi(a^+(X_0))$  is negative or positive respectively.
- (4) The velocity of  $b(s)$  is

$$(4.1) \quad \frac{db(s)}{ds} = -\frac{1}{\eta \omega_\psi(a^+)}.$$

Here,  $\eta$  stands for the holonomic factor of  $X_0$  (Definition 2.9).

This definition makes sense only when the  $\psi$ -latitude of  $a^+(X_0)$  is not 0, that is, when  $X_0$  does not lie on the  $\phi$ -axis  $\mathcal{S}_g^\phi$ , (see (2.29)). This is always the case when  $X_0$  belongs to  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$ .

**Lemma 4.4.** *Let  $X_0 \in \mathcal{S}_g \setminus \mathcal{S}_g^\phi$  be a pseudo-gradient of  $\alpha$  in normal form. Then there exists a germ of elementary path  $(X_s)$  passing through  $X_0$  and depending smoothly on  $s$  in the  $C^1$ -topology.*

**Proof.** Recall the tube  $T$  with coordinates  $(x, y, v, z)$  around the restricted homoclinic orbit  $\underline{\ell}$  of  $X_0$ . The holonomy  $H_0$  is defined on a neighborhood of  $\{z = 0\}$  in  $\partial^- \mathcal{M}_p$  to a neighborhood of  $\{z = 1\}$  in  $\partial^+ \mathcal{M}_p$ . Up to an isotopy of  $M$  fixing  $\mathcal{M}_p$ , a one-parameter perturbation  $X_s$  of  $X_0$  is determined by its perturbed holonomy  $H_s$ . This holonomy can be decomposed as  $H_s = H_0 \circ K_s$  where  $K_s$  is a diffeomorphism of  $\partial^- \mathcal{M}_p$  supported in the interior of this manifold with boundary and  $C^1$ -close to  $Id$ .

In order to satisfy conditions (1) and (2) of Definition 4.3, we first choose  $a^+(s)$  and  $D_1(s)$  before constructing  $H_s$ . Denote  $a^+(s)$  the point in  $\{\psi = \psi_0\}$  moving on the oriented pole axis with velocity +1 and such that  $a^+(0) = a^+$ . Let  $D_1(s)$  be the parallel disc to  $D_1(0)$  passing through  $a^+(s)$ . Take  $K_s$ , smooth with respect to  $s$ , such that:

$$(4.2) \quad K_s(\Sigma^- \cap \{z = 0\}) = H_0^{-1}(D_1(s))$$

Apart from equality (4.2), the restriction  $K_s|_{\Sigma^-}$  is not imposed. For instance, we may choose  $K_s(0, 0, 0, 0) = (0, 0, s, 0) = H_0^{-1}(a^+(s))$ . This implies the next constraint on  $K_s^{-1}$ , namely:

$$(4.3) \quad \frac{\partial}{\partial s}(v \circ K_s^{-1})(0, 0, 0, 0)|_{s=0} = -1$$

Apart from this velocity constraint in  $s = 0$ , we are free for defining  $K_s^{-1}|_{H_0^{-1}(\Sigma^+) \cap \{z=0\}}$ , that is the value of  $K_s^{-1}(0, y, 0, 0)$ . In particular, we can move  $b(s)$  with the radial velocity which is prescribed in Definition 4.3. Indeed, according to the formula (2.27) relating the  $v$ -coordinates of the unit vectors  $\partial_r^{\psi_0}$  and  $\partial_r^{\psi}$  and using (2.14), we get  $v(\dot{b}(0)) = -1$ . □

Clearly, this lemma holds with parameters, for instance when the data is a one-parameter family in  $\mathcal{S}_g \setminus \mathcal{S}_g^\phi$ . Then, the next corollary follows.

**Corollary 4.5.** *Let  $X_{0,0} \in \mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$  and let  $\gamma(t) = (X_{0,t})_t$  be a germ of path in  $\mathcal{S}_g$  passing through  $X_{0,0}$  and crossing  $\mathcal{S}_g^0$  transversely, such that  $\frac{\partial \gamma}{\partial t}(0)$  points towards  $\mathcal{S}_g^+$ . Then, there exists a two-parameter family  $\mathcal{D} = (X_{s,t})$  of pseudo-gradients of  $\alpha$  adapted to  $(\mathcal{S}_g, \mathcal{S}_g^0)$  such that, for every  $t$  close to 0, the path  $s \mapsto X_{s,t}$  is elementary. Moreover, there are such  $X_{s,t}$  which are*

smooth with respect to the parameters in the  $C^1$ -topology.

**Definition 4.6.** Let  $\mathcal{D}$  be a 2-disc transverse to  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$  and adapted to the pair  $(\mathcal{S}_g, \mathcal{S}_g^0)$ . We say that  $\mathcal{D}$  is elementary if it is made of a one-parameter family of elementary crossing paths as in Corollary 4.5.

**Proof of Theorem 4.2.**

First, we prove the theorem in the particular case where the transverse disc  $\mathcal{D}$  is elementary. Even in this particular case the proof is slightly different depending on where the base point  $X_{0,0}$  lies either in  $\mathcal{S}_g^{0,-}$  or  $\mathcal{S}_g^{0,+}$ . In each case, the proof has three items:

- (1) What is the trace on  $\mathcal{D}$  of  $\mathcal{S}_{g^2}$ ? Is there a non-empty trace of  $\mathcal{S}_{g^k}$  for  $k \neq 1$  or  $2$ ?
- (2) Is  $\mathcal{D}$  transverse to  $\mathcal{S}_{g^2}$ ? How is the positive co-orientation of  $\mathcal{S}_{g^2}$ ?
- (3) Which part  $\mathcal{S}_{g^2}^+$  or  $\mathcal{S}_{g^2}^-$  is intersected by  $\mathcal{D}$ ?

**Case  $X_{0,0} \in \mathcal{S}_g^{0,-}$ .** In other words,  $a^-(X_{0,0})$  has a negative  $\phi$ -latitude.

(1) The pseudo-gradient  $X_{0,t}$  has a homoclinic orbit  $\ell_t$  based in  $p$  and the  $\phi$ -latitude of  $a^-(X_{0,t})$  lies in  $[-1, 0)$  for every  $t \in \mathcal{O}p(0)$ . Denote  $\phi_t$  the spherical coordinate of  $a^-(X_{0,t})$ . We use the tube  $T$  around  $\ell_0$  and its extremities:  $\{z = 0\} \subset \partial^- \mathcal{M}_p$  and  $\{z = 1\} \subset \partial^+ \mathcal{M}_p$ .

For simplicity, we more specify the path  $(X_{0,t})_t$  by adding some assumptions (the discussion is similar with the other cases of latitudes by using other specifications<sup>7</sup>):

- (i) The  $\phi$ -latitude  $\omega_\phi(a^-(X_{0,0}))$  is not equal to  $-1$  and the  $\phi$ -equator of  $X_{0,t}$  is fixed.
- (ii) The point  $a^+(X_{0,t}) = \Sigma^+ \cap \ell_t$  and the  $\psi$ -equator of  $X_{0,t}$  are fixed.
- (iii) The holonomic factor  $\eta(X_{0,t})$  remains constant and is denoted by  $\eta$ .

Notice that (i) allows one to take  $(X_{0,t})_t$  positively transverse to  $\mathcal{S}_g^0$  satisfying (ii) and (iii). More precisely, one makes the  $\phi$ -coordinate  $\phi_t$  of  $a^-(X_{0,t})$  vary on  $t$  by increasing the  $\phi$ -latitude.

Denote the spherical coordinate of  $a^+(X_{0,t})$  by  $\psi_0$ , independent of  $t$ . In this setting, as the paths  $s \mapsto X_{s,t}$  are elementary the discs  $D_1(s, t) \subset \{z = 1\}$  depend only on  $s$  and are denoted by  $D_1(s)$ . For every  $s \neq 0$ , their images by the descent map are discs  $C_1(s)$  contained in the spherical annulus  $\mathbb{A}_{\psi_0} := \{(\phi, r, \psi_0)\}$ . When  $s$  goes to  $0_-$ , according to Lemma 3.8 the discs  $C_1(s)$  accumulate to the negative hemisphere  $\mathcal{H}^-(\Sigma^-)$ .

Since  $s \mapsto X_{s,t}$  is elementary and  $\omega_\psi(\psi_0) > 0$ , the disc  $D'_1(s, t)$ , preimage in  $\{z = 0\}$  of  $\Sigma^+$  by the respective perturbed holonomy, intersects  $\mathbb{A}_{\psi_0}$  in one point  $b(s, t)$  when  $s \leq 0$  and nowhere when  $s > 0$ , according to Definition 4.3. When  $t$  is constant,  $b(s, t)$  moves on the ray  $\{(\phi_t, r, \psi_0) \mid r \geq 0\}$  and its velocity is given by the formula in Definition 4.3. According to Remark 3.9, we have:

$$(4.4) \quad X_{s,t} \in \mathcal{S}_{g^2} \quad \text{if and only if} \quad b(s, t) \in C_1(s).$$

Denote by  $c_1(s, t)$  the intersection point of  $C_1(s)$  with the meridian disc  $\{\phi = \phi_t\}$ . When  $t$  is constant,  $c_1(s, t)$  also moves on the ray  $\{(\phi_t, r, \psi_0) \mid r \geq 0\}$  and its radial velocity is the same

<sup>7</sup>If  $\omega_\phi(a^-(X_{0,0})) = -1$ , one makes  $\eta(X_{0,t})$  vary increasingly with  $t$ . Since  $\omega_\psi(a^+(X_{0,0}))$  must be positive,  $\frac{\partial X}{\partial t}(X_{0,t}) > 0$ .

as the one of its lift through  $Desc$  in  $D_1(s) \subset \partial M_p^+$ . Therefore,

$$(4.5) \quad \frac{\partial c_1(s, t)}{\partial s} = \frac{1}{\omega_\phi(\phi_t)}$$

As  $X_{0,0}$  belongs to  $\mathcal{S}_g^0$ , that is  $\chi(X_{0,0}) = 0$ , the curves  $b(s, 0)$  and  $c_1(s, 0)$ , defined for  $s < 0$ , have the same radial velocities. Since both tend to  $a^-(X_{0,0})$  on the same ray when  $s$  goes to  $0_-$ , we have  $b(s, 0) = c_1(s, 0)$  for every  $s$ . Then, Equation (4.4) tells us that  $X_{s,0} \in \mathcal{S}_{g^2}$  for every  $s$  close to 0 negatively.

For  $t \neq 0$  and  $s < 0$ , the radial velocities of  $c_1(s, t)$  and  $b(s, t)$  are distinct while their limits when  $s$  goes to 0 coincide. Therefore, Equation (4.4) is never satisfied. When  $s > 0$ , the discs  $C_1(s)$  accumulate to the positive hemisphere  $\mathcal{H}^+(\Sigma^-)$ . There is no chance for  $C_1(s)$  to intersect  $D_1'(s, t)$  which is far from any point in  $\mathcal{H}^+(\Sigma^-)$ .

What about  $\mathcal{S}_{g^k}$ ? If  $k \leq 0$ , we have  $u(g^k) \geq 0$  and there is no homoclinic orbit in the homotopy class  $g^k$ . When  $k > 2$ , we have to discuss the successive passages of the unstable manifold  $W^u(p, X_{s,t})$  in  $\partial^+ \mathcal{M}_p$ , more precisely in  $\{z = 0\}$ .

According to Lemma 3.11, if  $t > 0$ , that is  $\chi(X_{0,t}) > 0$ , only the discs  $C_2(s, t)$  of the second passage are non-empty, but they accumulate to the positive hemisphere  $\mathcal{H}^+(\Sigma^-)$ . Therefore, no further passage could give rise to a homoclinic orbit.

If  $t < 0$ , one is able to see that there are infinitely many passages in  $\{z = 0\}$ . But, by velocity considerations  $C_k(s, t)$  never meet  $D_1'(s, t)$ . We do not give more details here because this is similar to the symmetric case  $X_{0,0} \in \mathcal{S}_g^{0,+}$  and  $t > 0$  where the analysis of velocities will be completely performed. Thus, the first item of case  $X_{0,0} \in \mathcal{S}_g^{0,-}$  is proved.

(2) The reason for transversality to  $\mathcal{S}_{g^2}$  relies again on some computations of velocity. Define for  $s \leq 0$ :

$$\delta(s, t) := v(c_1(s, t)) - v(b_1(s, t)) \quad \text{and} \quad V(t) := \left. \frac{\partial \delta(s, t)}{\partial s} \right|_{s=0}.$$

Despite the points are not the same, this velocity  $V(t)$  at  $s = 0$  is easily checked to be the same as the velocity computed in Lemma 3.10. Then, for every  $t$  close to 0 we have:

$$(4.6) \quad V(t) = \eta \frac{\omega_\psi(\psi_0)}{\omega_\phi(\phi_t)} + 1 \quad \text{which implies} \quad \frac{dV(t)}{dt} < 0.$$

By definition of the character function, we have  $V(0) = 0$  which implies  $V(t) < 0$  for  $t > 0$ . Define  $V(s, t) := \partial_s \delta(s, t)$ . By construction of  $(X_{s,t})$ , we have  $V(s, 0) = 0$  for every  $s < 0$  close to 0. According to (4.6), the second partial derivative  $\partial_{ts}^2 \delta(s, t)$  is negative for every  $(s, t)$  close to  $(0, 0)$  with  $s \leq 0$  (here we use the smoothness with respect to the parameters<sup>8</sup>). By integrating in the variable  $s$  from  $s_0 < 0$  to 0 and noticing that  $\delta(0, t) = 0$ , we get:

$$(4.7) \quad \frac{\partial \delta}{\partial t}(s_0, t) = -\frac{\partial}{\partial t} \left( \int_{s_0}^0 \frac{\partial \delta}{\partial s}(s, t) ds \right) = - \left( \int_{s_0}^0 \partial_{st}^2 \delta(s, t) ds \right) > 0.$$

For  $t = 0$ , this is exactly the transversality of  $\mathcal{D}$  to  $\mathcal{S}_{g^2}$  at  $X_{s_0,0}$ .

<sup>8</sup>The vector fields in a normalized crossing path are not smooth with respect to the space variable. Their holonomy is  $C^1$  only. Nevertheless, as the  $C^1$ -maps (of degree zero)  $\partial^- \mathcal{M}_p \rightarrow \partial^+ \mathcal{M}_p$  form a Banach manifold it makes sense to consider a smooth family of such holonomies.

We are now looking at orientation. Take  $s_0 < 0$  such that  $b(s_0, 0)$  lies in  $\{z = 0\}$ . It belongs to a homoclinic orbit  $\ell'$  in the homotopy class  $g^2$ . There is a tube  $T'$  around  $\ell'$  with coordinates  $(x', y', v', z')$ . The  $y'$ -axis is contained in  $D'_1(s_0, 0)$  and is given a co-orientation which follows from the co-orientation of  $D'_1(0, 0)$  by continuity. The  $x'$ -axis is contained in  $C_1(s_0)$ . Its projection to the  $x$ -axis is orientation reversing (Lemma 3.8). Therefore:

$$(4.8) \quad v(\partial_{v'}) < 0.$$

By (4.7) we have:  $\frac{\partial}{\partial t}[v(c_1(s_0, t)) - v(b_1(s_0, t))]|_{t=0} = \frac{\partial \delta}{\partial t}(s_0, 0) > 0$ . By replacing  $v$  with  $v'$  in the last inequality, we get:

$$\frac{\partial}{\partial t}[v'(c_1(s, t)) - v'(b_1(s, t))]|_{t=0} < 0.$$

This translates the fact that  $\partial_t$  points to the negative side of  $\mathcal{S}_{g^2}$  for  $s < 0$  while for  $s = 0$ ,  $\partial_t$  defines the positive co-orientation of  $\mathcal{S}_g^0$  in  $\mathcal{S}_g$ .

(3) Let  $L > 0$  be a length. Consider a small circle  $\gamma \subset \mathcal{D}$  centered at the origin of the coordinates  $(s, t)$  and turning positively with respect to the orientation given by these coordinates. If the radius of  $\gamma$  is small enough<sup>9</sup> and if  $X_{0,0}$  is generic,  $\gamma$  avoids all codimension-one strata in  $\mathcal{F}_\alpha$  except:

- $\mathcal{S}_g$  which is crossed once in  $\mathcal{S}_g^-$  positively, and once in  $\mathcal{S}_g^+$  negatively,
- $\mathcal{S}_{g^2}$  which is crossed once positively according to the above discussion.

The product of the self-slide factors should equal 1 up to  $L$  in the Novikov ring. The self-slide factor of the a small arc of  $\gamma$  crossing  $\mathcal{S}_{g^2}$  is for now unknown; call it  $m(g)$ . This (commutative) product is

$$(1 + g) \cdot (1 + g + g^2 + \dots)^{-1} \cdot m(g) \equiv_L 1.$$

Then,  $m(g) \equiv_L (1 - g^2)^{-1}$ , that can only happen if the crossing of  $\mathcal{S}_{g^2}$  takes place in  $\mathcal{S}_{g^2}^+$ . The proof of Theorem 4.2 is complete in the case  $X_{0,0} \in \mathcal{S}_{g^2}^{0,-}$ .

**Case  $X_{0,0} \in \mathcal{S}_g^{0,+}$ .** In other words,  $a^-(X_{0,0})$  has a positive  $\phi$ -latitude.

(1) The discussion is led in the same manner as in the previous case, with same notation and simplified setting. We only mention the main differences. Here,  $a^-(X_{0,0})$  belongs to the positive hemisphere of  $\Sigma^-$  while  $\psi_0$  belongs to the negative hemisphere of  $\Sigma^+$ . The discs  $C_1(s)$  intersect  $\{z = 0\}$  only when  $s > 0$ . Therefore, for  $s < 0$  there is no chance for meeting  $\mathcal{S}_{g^k}$ , for any  $k \neq 0$ .

According to Lemma 3.11 there are infinitely many passages  $C_k(s)$ ,  $k \geq 1$ ,  $s > 0$  of  $W^u(p, X_{s,t})$  in  $\partial^- \mathcal{M}_p$  meeting  $\{z = 0\}$ . Recall that the  $(i - 1)$ -discs  $C_k(s)$  do not depend on  $t$ . The fact that  $X_{s,t}$  belongs to  $\mathcal{S}_{g^2}$  if and only if  $s > 0$ ,  $t = 0$  is proved exactly as in the previous case.

Then, we are left to show that for every  $k > 2$ ,  $\mathcal{S}_{g^k}$  does not intersect  $\mathcal{D}$ . Here it is important to think of  $\mathcal{D}$  as a germ because for a given representative this result is not true; when  $k$  increases, the domain of the representative has to be restricted. Let  $C_k(s, t)$  denote the  $(i - 1)$ -disc in  $\partial^- \mathcal{M}_p$  corresponding to the  $k$ -th passage of the unstable manifold  $W^u(p, X_{s,t})$  (see Lemma 3.11); let  $D'_1(s, t)$  denote the  $(n - i - 1)$ -disc corresponding to the first passage of the

<sup>9</sup>More  $L$  is large, more this radius has to be small.

stable manifold  $W^s(p, X_{s,t})$ . Observe that  $\mathcal{D}$  intersects  $\mathcal{S}_{g^{k+1}}$  if and only if, for  $(s, t)$  close to  $(0, 0)$ ,  $C_k(s, t)$  intersects  $D'_1(s, t)$ . This translates in the next equation:

$$(4.9) \quad c_k(s, t) = d'_k(s, t)$$

where  $c_k(s, t)$  and  $d'_k(s, t)$  are the only two points of  $\{z = 0\}$  lying respectively on  $C_k(s, t)$  and  $D'_1(s, t)$  which have the same  $(x, y)$ -coordinates. Then, the above equation becomes:

$$(4.10) \quad v(c_k(s, t)) = v(d'_k(s, t)).$$

When  $s$  goes to 0, these two points go to the same point  $a^-(X_{s,t}) \in \Sigma^-$ . By computations done in the proof of Lemma 3.11, we know that:

$$(4.11) \quad \frac{\partial}{\partial s} [v(c_k(s, t)) - v(d'_k(s, t))]_{|s=0} \neq 0.$$

It follows that, for  $s$  close to 0 (closeness depending on  $t$ ), the equation (4.10) cannot be fulfilled. Theorem 4.2 is now proved for elementary 2-discs as in Definition 4.6.

Let  $\mathcal{D}$  be an elementary 2-disc centered at  $X_{0,0}$ . Let  $X_{0,0}$  run in a neighborhood  $U \subset \mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$ . This move extends to a move of  $\mathcal{D}$  among elementary discs. Limit ourselves to the  $C^1$ -topology in order that the space of vector fields is a Banach manifold. Then, we have a  $C^1$ -map

$$F : U \times [-1, +1]^2 \rightarrow \mathcal{F}_\alpha, \quad (X, s, t) \mapsto F(X, s, t),$$

such that  $F(X, 0, 0) = X$ ,  $F(X, 0, t)$  belongs to  $\mathcal{S}_g$  and  $\text{span}\{\partial_s F, \partial_t F\}$  is transverse to  $\mathcal{S}_g \setminus \mathcal{S}_g^{0,0}$ . Here, we notice that the perturbed holonomy  $H_{F(X,s,t)}$  depends linearly on  $(s, t)$  near  $a^-(X_{0,0}) \in \partial^- \mathcal{M}_p$  and the extension by partition of unity makes  $F$  depend smoothly on  $(s, t)$  for the  $C^1$ -topology of vector fields.

The Inverse Function Theorem is available and states that after a suitable restriction (that we do not denote)  $F$  is a diffeomorphism onto its image  $\mathcal{N}$  which is an open set in  $\mathcal{F}_\alpha$  ( $C^1$ -completed). Therefore  $\mathcal{N}$  has a product structure and a projection  $P : \mathcal{N} \rightarrow [-1, +1]^2$  such that, for every  $X \in \mathcal{N}$ , the next equivalences hold:

$$(4.12) \quad \begin{cases} X \in \mathcal{S}_g & \iff (s \circ P)(X) = 0 \\ X \in \mathcal{S}_g^0 & \iff P(X) = (0, 0) \\ X \in \mathcal{S}_{g^2} & \iff (t \circ P)(X) = 0 \text{ and } (s \circ P)(X) \in \mathbb{R}_\pm \text{ depending on } X_{0,0} \in \mathcal{S}_g^{0,\pm}. \end{cases}$$

Let  $\mathcal{D}'$  be any germ of two-parameter family centered in  $X_{0,0}$  transverse to  $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$  and contained in  $\mathcal{N}$ . Its projection  $P \circ \mathcal{D}'$  is submersive. The equivalences (4.12) finish the proof of Theorem 4.2. □

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