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UP VERSUS NP

FRANK VEGA

ABSTRACT. We consider two new complexity classes that are called equivalent-P and equivalent-UP which have a close relation to the P versus NP problem. The class equivalent-P contains those languages that are ordered-pairs of instances of two specific problems in P, such that the elements of each ordered-pair have the same solution, which means, the same certificate. The class equivalent-UP has almost the same definition, but in each case we define the pair of languages explicitly in UP. In addition, we define the class double-NP as the set of languages that contain each instance of another language in NP, but in a double way, that is, in form of a pair with two identical instances. We demonstrate that double-NP is a subset of equivalent-P. Moreover, we prove that equivalent-UP is a subset of UP. In this way, we show not only that every problem in double-NP can be reformulated as the pairs of elements of two languages in P which have the same certificate, but we also show $UP = NP$.

INTRODUCTION

P versus NP is a major unsolved problem in computer science [4]. This problem was introduced in 1971 by Stephen Cook [1]. It is considered by many to be the most important open problem in the field [4]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US\$1,000,000 prize for the first correct solution [4].

In 1936, Turing developed his theoretical computational model [1]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation. A deterministic Turing machine has only one next action for each step defined in its program or transition function [10]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [10].

Another huge advance in the last century has been the definition of a complexity class. A language over an alphabet is any set of strings made up of symbols from that alphabet [3]. A complexity class is a set of problems, which are represented as a language, grouped by measures such as the running time, memory, etc [3].

In the computational complexity theory, the class P contains those languages that can be decided in polynomial time by a deterministic Turing machine [7]. The class NP consists in those languages that can be decided in polynomial time by a nondeterministic Turing machine [7].

The biggest open question in theoretical computer science concerns the relationship between these classes:

Is P equal to NP ?

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In 2002, a poll of 100 researchers showed that 61 believed that the answer was not, 9 believed that the answer was yes, and 22 were unsure; 8 believed the question may be independent of the currently accepted axioms and so impossible to prove or disprove [6].

Another major complexity class is UP . The class UP has all the languages that are decided in polynomial time by a nondeterministic Turing machines with at most one accepting computation for each input [12]. It is obvious that $P \subseteq UP \subseteq NP$ [10]. Whether $P = UP$ is another fundamental question that it is as important as it is unresolved [10]. All efforts to solve the P versus UP problem have failed [10]. Nevertheless, we prove $UP = NP$.

1. THEORETICAL NOTIONS

Let Σ be a finite alphabet with at least two elements, and let Σ^* be the set of finite strings over Σ [1]. A Turing machine M has an associated input alphabet Σ [1]. For each string w in Σ^* there is a computation associated with M on input w [1]. We say that M accepts w if this computation terminates in the accepting state [1]. Note that M fails to accept w either if this computation ends in the rejecting state, or if the computation fails to terminate [1].

The language accepted by a Turing machine M , denoted $L(M)$, has an associated alphabet Σ and is defined by

$$L(M) = \{w \in \Sigma^* : M \text{ accepts } w\}.$$

We denote by $t_M(w)$ the number of steps in the computation of M on input w [1]. For $n \in \mathbb{N}$ we denote by $T_M(n)$ the worst case run time of M ; that is

$$T_M(n) = \max\{t_M(w) : w \in \Sigma^n\}$$

where Σ^n is the set of all strings over Σ of length n [1]. We say that M runs in polynomial time if there exists k such that for all n , $T_M(n) \leq n^k + k$ [1].

Definition 1.1. *A language L is in class P if $L = L(M)$ for some deterministic Turing machine M which runs in polynomial time [1].*

We state the complexity class NP using the following definition.

Definition 1.2. *A verifier for a language L is a deterministic Turing machine M , where*

$$L = \{w : M \text{ accepts } \langle w, c \rangle \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w , so a polynomial time verifier runs in polynomial time in the length of w [11]. A verifier uses additional information, represented by the symbol c , to verify that a string w is a member of L . This information is called certificate.

Observe that, for polynomial time verifiers, the certificate is polynomially bounded by the length of w , because that is all the verifier can access in its time bound [11].

Definition 1.3. *NP is the class of languages that have polynomial time verifiers [11].*

In addition, we can define another complexity class called UP .

Definition 1.4. A language L is in UP if every instance of L with a given certificate can be verified by a polynomial time verifier, and this verifier machine only accepts at most one certificate for each problem instance [8]. More formally, a language L belongs to UP if there exists a polynomial time verifier M and a constant c such that

- if $x \in L$, then there exists a unique certificate y with $|y| = O(|x|^c)$ such that $M(x, y) = \text{“yes”}$,
- if $x \notin L$, there is no certificate y with $|y| = O(|x|^c)$ such that $M(x, y) = \text{“yes”}$ [8].

A function $f : \Sigma^* \rightarrow \Sigma^*$ is a polynomial time computable function if some deterministic Turing machine M , on every input w , halts in polynomial time with just $f(w)$ on its tape [11]. Let $\{0, 1\}^*$ be the infinite set of binary strings, we say that a language $L_1 \subseteq \{0, 1\}^*$ is polynomial time reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_p L_2$, if there exists a polynomial time computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$,

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

An important complexity class is NP -complete [7]. A language $L \subseteq \{0, 1\}^*$ is NP -complete if

- (1) $L \in NP$, and
- (2) $L' \leq_p L$ for every $L' \in NP$.

Furthermore, if L is a language such that $L' \leq_p L$ for some $L' \in NP$ -complete, then L is in NP -hard [3]. Moreover, if $L \in NP$, then $L \in NP$ -complete [3]. If any single NP -complete problem can be solved in polynomial time, then every NP problem has a polynomial time algorithm [3]. No polynomial time algorithm has yet been discovered for any NP -complete problem [4].

A principal NP -complete problem is SAT [5]. An instance of SAT is a Boolean formula ϕ which is composed of

- (1) Boolean variables: x_1, x_2, \dots, x_n ;
- (2) Boolean connectives: Any Boolean function with one or two inputs and one output, such as \wedge (AND), \vee (OR), \neg (NOT), \Rightarrow (implication), \Leftrightarrow (if and only if);
- (3) and parentheses.

A truth assignment for a Boolean formula ϕ is a set of values for the variables in ϕ . A satisfying truth assignment is a truth assignment that causes ϕ to be evaluated as true. A formula with a satisfying truth assignment is a satisfiable formula. The problem SAT asks whether a given Boolean formula is satisfiable [5].

Another NP -complete language is $3CNF$ satisfiability, or $3SAT$ [3]. We define $3CNF$ satisfiability using the following terms. A literal in a Boolean formula is an occurrence of a variable or its negation [3]. A Boolean formula is in conjunctive normal form, or CNF , if it is expressed as an AND of clauses, each of which is the OR of one or more literals [3]. A Boolean formula is in 3-conjunctive normal form or $3CNF$, if each clause has exactly three distinct literals [3].

For example, the Boolean formula

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

is in $3CNF$. The first of its three clauses is $(x_1 \vee \neg x_1 \vee \neg x_2)$, which contains the three literals x_1 , $\neg x_1$, and $\neg x_2$. In $3SAT$, it is asked whether a given Boolean formula ϕ in $3CNF$ is satisfiable.

It can be demonstrated that many problems belong to NP -complete using a polynomial time reduction from $3SAT$ [5]. For example, the well-known problem $1-IN-3\ 3SAT$ which is defined as follows: Given a Boolean formula ϕ in $3CNF$, is there a truth assignment such that each clause in ϕ has exactly one true literal?

Another special case is the class of problems where each clause contains XOR (i.e. exclusive or) rather than (plain) OR operators. This is in P , since a XOR - SAT formula can also be viewed as a system of linear equations mod 2, and can be solved in cubic time by Gaussian elimination [9]. We represent the XOR function inside a Boolean formula as \oplus . The problem XOR - $3SAT$ is similar to XOR - SAT , but the clauses in the Boolean formula have exactly three distinct literals. Since $a \oplus b \oplus c$ is evaluated as true if and only if exactly 1 or 3 members of $\{a, b, c\}$ are true, then each solution of the problem $1-IN-3\ 3SAT$ for a given $3CNF$ formula is also a solution of the problem XOR - $3SAT$ and in turn each solution of XOR - $3SAT$ is a solution of $3SAT$.

In addition, a Boolean formula is in 2-conjunctive normal form, or $2CNF$, if it is in CNF and each clause has exactly two distinct literals. There is a well-known problem called $2SAT$. In $2SAT$, it is asked whether a given Boolean formula ϕ in $2CNF$ is satisfiable. This language is in P [2].

2. CLASS EQUIVALENT-P

Definition 2.1. We say that a language L belongs to $\equiv P$ if there exist two languages $L_1 \in P$ and $L_2 \in P$ and two deterministic Turing machines M_1 and M_2 , where M_1 and M_2 are the polynomial time verifiers of L_1 and L_2 respectively, such that

$$L = \{(x, y) : \exists z \text{ such that } M_1(x, z) = \text{“yes” and } M_2(y, z) = \text{“yes”}\}.$$

We call the complexity class $\equiv P$ as “equivalent- P ”. We represent this language L in $\equiv P$ as (L_1, L_2) .

A logarithmic space transducer is a Turing machine with a read-only input tape, a write-only output tape, and a read/write work tape [11]. The work tapes must contain at most $O(\log n)$ symbols [11]. A logarithmic space transducer M computes a function $f : \Sigma^* \rightarrow \Sigma^*$, where $f(w)$ is the string remaining on the output tape after M halts when it is started with w on its input tape [11]. We call f a logarithmic space computable function [11]. We say that a language $L_1 \subseteq \{0, 1\}^*$ is logarithmic space reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_l L_2$, if there exists a logarithmic space computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$,

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

The logarithmic space reduction is frequently used for P and below [10]. There is a different kind of reduction for $\equiv P$: The e -reduction.

Definition 2.2. Given two languages L_1 and L_2 , where the instances of L_1 and L_2 are ordered-pairs of binary strings, we say that the language L_1 is e -reducible to the language L_2 , written $L_1 \leq_{\equiv} L_2$, if there exist two logarithmic space computable

functions $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and $g : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$ and $y \in \{0, 1\}^*$,

$$(x, y) \in L_1 \text{ if and only if } (f(x), g(y)) \in L_2.$$

Lemma 2.3. *The e -reduction is a logarithmic space reduction.*

Proof. We can construct a logarithmic space transducer M that computes an arbitrary e -reduction and receives as input an ordered-pair of string $\langle x, y \rangle$ and outputs $\langle f(x), g(y) \rangle$ where f and g are the two logarithmic space computable functions of this e -reduction. Suppose we use a delimiter symbol for the strings x and y . For example, let's take the blank symbol \sqcup as delimiter [10]. Since f is a logarithmic space computable function, then we can simulate f on M using a logarithmic amount of space in its read/write work tape. In the meantime, M is printing the string $f(x)$ to the output without wander off after the symbol \sqcup that separates x from y on the input tape. When the simulation of f halts, then M starts to simulate g from the other string y . At the same time, M outputs the result of $g(y)$ using only a logarithmic space in its work tape, since g is also a logarithmic space computable function. Finally, we obtain the output $\langle f(x), g(y) \rangle$ from the input $\langle x, y \rangle$ using the logarithmic space transducer M . Since we take an arbitrary e -reduction, then we prove the e -reduction is also a logarithmic space reduction. \square

Theorem 2.4. *If $A \leq_{\equiv} B$ and $B \leq_{\equiv} C$, then $A \leq_{\equiv} C$.*

Proof. This is because of the logarithmic space reduction is transitive [10]. \square

We say that a complexity class C is closed under reductions if, whenever L_1 is reducible to L_2 and $L_2 \in C$, then $L_1 \in C$ [10].

Theorem 2.5. *$\equiv P$ is closed under reductions.*

Proof. Let L and L' be two arbitrary languages, where their instances are ordered-pairs of binary strings. Suppose that $L \leq_{\equiv} L'$ where L' is in $\equiv P$. We will show that L is in $\equiv P$ too. By definition of $\equiv P$, there are two languages $L'_1 \in P$ and $L'_2 \in P$, such that for each $(v, w) \in L'$ we have that $v \in L'_1$ and $w \in L'_2$. Moreover, there are two deterministic Turing machines M'_1 and M'_2 which are the polynomial time verifiers of L'_1 and L'_2 respectively. For each $(v, w) \in L'$, there will be a succinct certificate z , such that $M'_1(v, z) = \text{"yes"}$ and $M'_2(w, z) = \text{"yes"}$. Besides, by definition of e -reduction, there are two logarithmic space computable functions $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and $g : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$ and $y \in \{0, 1\}^*$,

$$(x, y) \in L \text{ if and only if } (f(x), g(y)) \in L'.$$

Based on this preliminary information, we can support that there exist two languages $L_1 \in P$ and $L_2 \in P$, such that for each $(x, y) \in L$ we have that $x \in L_1$ and $y \in L_2$. Indeed, we can define L_1 and L_2 as the strings $f^{-1}(v)$ and $g^{-1}(w)$, such that $f^{-1}(v) \in L_1$ and $g^{-1}(w) \in L_2$ if and only if $v \in L'_1$ and $w \in L'_2$. Certainly, for all $x \in \{0, 1\}^*$ and $y \in \{0, 1\}^*$, we can decide in polynomial time whether $x \in L_1$ or $y \in L_2$ just verifying that $f(x) \in L'_1$ or $g(y) \in L'_2$ respectively, since $L'_1 \in P$, $L'_2 \in P$, and $SPACE(\log n) \subseteq P$ [10].

Furthermore, there are two deterministic Turing machines M_1 and M_2 which are the polynomial time verifiers of L_1 and L_2 respectively. For each $(x, y) \in L$, there will be a succinct certificate z such that $M_1(x, z) = \text{"yes"}$ and $M_2(y, z) = \text{"yes"}$. Indeed, we can know whether $M_1(x, z) = \text{"yes"}$ and $M_2(y, z) = \text{"yes"}$ just verifying

whether $M'_1(f(x), z) = \text{"yes"}$ and $M'_2(g(y), z) = \text{"yes"}$. Certainly, for every triple of strings (x, y, z) , we can define the polynomial time computation of the verifiers M_1 and M_2 as $M_1(x, z) = M'_1(f(x), z)$ and $M_2(y, z) = M'_2(g(y), z)$, since we can evaluate $f(x)$ and $g(y)$ in polynomial time because of $SPACE(\log n) \subseteq P$ [10]. In addition, $\min(|f(x)|, |g(y)|)$ is polynomially bounded by $\min(|x|, |y|)$ where $|\dots|$ is the string length function, due to the logarithmic space transducers of f and g cannot output an exponential amount of symbols in relation to the size of the input. Consequently, $|z|$ is polynomially bounded by $\min(|x|, |y|)$, because $|z|$ will be polynomially bounded by $\min(|f(x)|, |g(y)|)$. Hence, we have just proved the necessary properties to state that L is in $\equiv P$. \square

3. CLASS DOUBLE-NP

Most of the transformations that have been used in proving *NP-completeness* are also logarithmic space transformations [5]. Thus the class of languages that are “logarithmic space complete for *NP*” is at least a large subclass of the *NP-complete* problems [5]. We define a complexity class which has a close relation to this property.

Definition 3.1. *We say that a language L belongs to $2NP$ if there exists a language $L' \in NP$, such that*

$$L = \{(x, x) : x \in L'\}.$$

*We call the complexity class $2NP$ as “double-*NP*”. We represent this language L in $2NP$ as (L', L') .*

We define the completeness of $2NP$ using the *e-reduction*.

Definition 3.2. *A language $L \subseteq \{0, 1\}^*$ is $2NP$ -complete if*

- (1) $L \in 2NP$, and
- (2) $L' \leq_{\equiv} L$ for every $L' \in 2NP$.

Furthermore, if L is a language such that $L' \leq_{\equiv} L$ for some $L' \in 2NP$ -complete, then L is in $2NP$ -hard. Moreover, if $L \in 2NP$, then $L \in 2NP$ -complete. These definitions are based on the result of Theorem 2.4.

We define $2\text{-}3SAT$ as follows,

$$2\text{-}3SAT = \{(\phi, \phi) : \phi \in 1\text{-}IN\text{-}3\ 3SAT\}.$$

Theorem 3.3. $2\text{-}3SAT \in 2NP$ -complete.

Proof. $2\text{-}3SAT$ is in $2NP$, because $1\text{-}IN\text{-}3\ 3SAT$ is in NP . We know the language $1\text{-}IN\text{-}3\ 3SAT$ has been defined in *NP-complete* using only logarithmic space reductions [5]. Certainly, we can reduce every instance of a language in NP into SAT , and this other instance of SAT into $3SAT$, and finally, this one in $3SAT$ into $1\text{-}IN\text{-}3\ 3SAT$ just using in each case a reduction in logarithmic space [5]. Since the logarithmic space reduction is transitive, then we obtain $2\text{-}3SAT \in 2NP$ -complete, because $2\text{-}3SAT$ contains the elements of $1\text{-}IN\text{-}3\ 3SAT$ but in a duplicated way. \square

Definition 3.4. $3XOR\text{-}2SAT$ is a problem in $\equiv P$, where every instance (ψ, φ) is an ordered-pair of Boolean formulas, such that if $(\psi, \varphi) \in 3XOR\text{-}2SAT$, then $\psi \in XOR\text{-}3SAT$ and $\varphi \in 2SAT$. According to the definition of $\equiv P$, this language is the ordered-pairs of elements of $XOR\text{-}3SAT$ and $2SAT$ such that they will have the same satisfying truth assignment using the same variables. We define the

polynomial time verifiers of *XOR-3SAT* and *2SAT* as the two deterministic Turing machines M_1 and M_2 , such that M_1 accepts every element of *XOR-3SAT* with each satisfying truth assignment and M_2 accepts every element of *2SAT* with each satisfying truth assignment.

Theorem 3.5. *3XOR-2SAT* \in *2NP-hard*.

Proof. Given an arbitrary Boolean formula ϕ in *3CNF* of m clauses, we iterate for $i = 1, 2, \dots, m$ over each clause $c_i = (x \vee y \vee z)$ in ϕ , where x, y and z are literals, just creating the following formulas,

$$Q_i = (x \oplus y \oplus z)$$

$$P_i = (\neg x \vee \neg y) \wedge (\neg y \vee \neg z) \wedge (\neg x \vee \neg z).$$

Since Q_i is evaluated as true if and only if exactly 1 or 3 members of $\{x, y, z\}$ are true and P_i is evaluated as true if and only if exactly 1 or 0 members of $\{x, y, z\}$ are true, then we obtain the clause c_i has exactly one true literal if and only if both formulas Q_i and P_i have the same satisfying truth assignment.

Hence, we can construct the Boolean formulas ψ and φ as the conjunction of Q_i or P_i for every clause c_i in ϕ , that is, $\psi = Q_1 \wedge \dots \wedge Q_m$ and $\varphi = P_1 \wedge \dots \wedge P_m$. Finally, we obtain that,

$$(\phi, \phi) \in 2\text{-3SAT} \text{ if and only if } (\psi, \varphi) \in 3\text{XOR-2SAT}.$$

Moreover, there are two logarithmic space computable functions $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and $g : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that $f(\langle \phi \rangle) = \langle \psi \rangle$ and $g(\langle \phi \rangle) = \langle \varphi \rangle$. Indeed, we only need a logarithmic space to analyze every time each clause c_i in the instance ϕ and generate Q_i or P_i to the output. Then, we have proved that $2\text{-3SAT} \leq_{\equiv} 3\text{XOR-2SAT}$. Consequently, we obtain that $3\text{XOR-2SAT} \in 2\text{NP-hard}$. \square

Theorem 3.6. $2\text{NP} \subseteq_{\equiv} P$.

Proof. Since 3XOR-2SAT is hard for 2NP , thus all language in 2NP reduce to $\equiv P$. Since $\equiv P$ is closed under reductions, it follows that $2\text{NP} \subseteq_{\equiv} P$. \square

4. CLASS EQUIVALENT-UP

Definition 4.1. We say that a language L belongs to $\equiv UP$ if there exist two languages $L_1 \in UP$ and $L_2 \in UP$ and two deterministic Turing machines M_1 and M_2 , where M_1 and M_2 are the polynomial time verifiers of L_1 and L_2 respectively, such that

$$L = \{(x, y) : \exists z \text{ such that } M_1(x, z) = \text{“yes” and } M_2(y, z) = \text{“yes”}\}.$$

We call the complexity class $\equiv UP$ as “equivalent-UP”. We represent this language L in $\equiv UP$ as (L_1, L_2) .

Theorem 4.2. $\equiv UP \subseteq UP$.

Proof. Let's take an arbitrary language L in $\equiv UP$ which is defined from two languages $L_1 \in UP$ and $L_2 \in UP$ and two deterministic Turing machines M_1 and M_2 , such that $L = (L_1, L_2)$ where M_1 and M_2 are the polynomial time verifiers of L_1 and L_2 respectively. We are going to show there is a polynomial time verifier M , such that M can verify every instance (x, y) of L with a given certificate and M only accepts at most one certificate for each problem instance [8]. More formally, we will show there exists a polynomial time verifier M and a constant c such that

if $(x, y) \in L$, then there exists a unique certificate z with $|z| = O(|(x, y)|^c)$ such that $M(x, y, z) = \text{"yes"}$,

if $(x, y) \notin L$, there is no certificate z with $|z| = O(|(x, y)|^c)$ such that $M(x, y, z) = \text{"yes"}$ [8].

Since $UP \subseteq NP$, then each certificate z for every element $x \in L_1$ or $y \in L_2$ can be polynomially bounded through a single constant d in this way: $|z| < |x|^d$ or $|z| < |y|^d$ where $|\dots|$ is the string length function. Hence, a certificate z for every element $(x, y) \in L$ can be polynomially bounded through a single constant c in this way: $|z| < |(x, y)|^c$. Now, we could define M in the following way: On input (x, y) and a polynomially bounded certificate z , we obtain $M(x, y, z) = \text{"yes"}$ if and only if $M_1(x, z) = \text{"yes"}$ and $M_2(y, z) = \text{"yes"}$.

For every instance x of L_1 , a certificate z that proves the membership of x in L_1 would be unique because of $L_1 \in UP$ [8]. The same happens for every instance y in L_2 : If there exists a certificate z that proves $y \in L_2$, then z will be unique [8]. If $x \notin L_1$ or $y \notin L_2$, then there will not be another possible certificate for the instance x or y . Certainly, for every instance x or y , there will be at most one certificate that proves the membership of x to L_1 or y to L_2 . Hence, for every instance (x, y) of L , there will be at most one polynomially bounded string z such that $M_1(x, z) = \text{"yes"}$ and $M_2(y, z) = \text{"yes"}$ if and only if $(x, y) \in L$. Since the Turing machines M_1 and M_2 are deterministic and the certificate z is unique, then we can support that M only accepts at most one certificate for every instance (x, y) of L and verifies whether $(x, y) \in L$. Consequently, we obtain that $L \in UP$. Since we took arbitrarily a language $L \in UP$, then we can deduce $UP \subseteq UP$. \square

Lemma 4.3. $UP \subseteq UP$.

Proof. Since $UP \subseteq UP$, then we can support that $UP \subseteq UP$ as a direct consequence of the Definitions 2.1 and 4.1 [10]. \square

Theorem 4.4. $UP = NP$.

Proof. As result of applying the Theorems 3.6 and 4.2 with the Lemma 4.3, then we can deduce $2NP \subseteq UP$. Alternatively, we can reduce in logarithmic space every language in $L \in NP$ to another language $(L, L) \in 2NP$ just using a logarithmic space transducer that copies the content on its input tape to the output twice. Since UP is closed under logarithmic space reductions and every language in $2NP$ is in UP , then every language in NP will be in UP too [10]. Consequently, we obtain that $NP \subseteq UP$. However, we already know that $UP \subseteq NP$ [10]. Since we have that $NP \subseteq UP$ and $UP \subseteq NP$, then $UP = NP$ [3]. \square

CONCLUSIONS

There is a previous known result which states that $P = UP$ if and only if there are no one-way functions [10]. Indeed, for many years it has been accepted the P versus UP question as the correct complexity context for the discussion of the cryptography and one-way functions [10]. For that reason, the proof of Theorem 4.4 negates this current idea and also the belief that $UP = NP$ is a very unlikely event. In addition, this demonstration might be a shortcut to prove $P = NP$, because if somebody proves that $P = UP$, then he will be proving, at the same time, the outstanding and difficult P versus NP problem [4]. Furthermore, if we obtain a possible proof of $P \neq NP$, then this work would also contribute to show $P \neq UP$.

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