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# THE FIRST POSITIVE EIGENVALUE OF THE SUB-LAPLACIAN ON CR SPHERES

AMINE ARIBI AND AHMAD EL SOUFI

ABSTRACT. We prove that the first positive eigenvalue, normalized by the volume, of the sub-Laplacian associated with a strictly pseudoconvex pseudo-Hermitian structure  $\theta$  on the CR sphere  $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ , achieves its maximum when  $\theta$  is the standard contact form.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

According to a classical result of Hersch [15], given any Riemannian metric  $g$  on the 2-dimensional sphere  $\mathbb{S}^2$ , the first positive eigenvalue  $\lambda_1(g)$  of the Laplace-Beltrami operator  $\Delta_g$  satisfies the estimate

$$\lambda_1(g)A(g) \leq \lambda_1(g_0)A(g_0) \quad (1)$$

where  $g_0$  is the standard metric of  $\mathbb{S}^2$  and  $A(g)$  is the area of  $\mathbb{S}^2$  with respect to  $g$ . Moreover, the equality holds in (1) if and only if  $g$  is isometric to  $g_0$ . This result was extended to higher dimensional spheres by Ilias and the second author as follows (see [12, proposition 3.1]) : If a Riemannian metric  $g$  on the  $n$ -dimensional sphere  $\mathbb{S}^n$  is conformal to the standard metric  $g_0$ , then

$$\lambda_1(g)V(g)^{\frac{2}{n}} \leq \lambda_1(g_0)V(g_0)^{\frac{2}{n}} \quad (2)$$

where  $V(g)$  denotes the Riemannian volume of the sphere with respect to  $g$ . Again, the equality holds in (2) if and only if  $g$  is isometric to  $g_0$ .

The aim of the present paper is to establish a version of the estimate (2) for the first positive eigenvalue of the sub-Laplacian on the CR sphere  $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ . Indeed, let

$$\theta_0 = \frac{i}{2} \sum_{j=1}^{n+1} (\zeta_j d\bar{\zeta}_j - \bar{\zeta}_j d\zeta_j)$$

be the standard contact form on  $\mathbb{S}^{2n+1}$  whose kernel coincides with the Levi distribution  $H(\mathbb{S}^{2n+1}) = T\mathbb{S}^{2n+1} \cap J T\mathbb{S}^{2n+1}$ , where  $J$  is the complex structure of  $\mathbb{C}^{n+1}$ . The set  $\mathcal{P}_+(\mathbb{S}^{2n+1}) = \{f\theta_0 ; f \in C^\infty(\mathbb{S}^{2n+1}) \text{ and } f > 0\}$  contains all pseudo-Hermitian structures on  $\mathbb{S}^{2n+1}$  whose Levi form is positive definite. Given a pseudo-Hermitian structure  $\theta \in \mathcal{P}_+(\mathbb{S}^{2n+1})$ , we denote by  $\lambda_1(\theta)$  the first positive eigenvalue of the corresponding sub-Laplacian  $\Delta_\theta$ , and by  $V(\theta)$  the volume of  $\mathbb{S}^{2n+1}$  with respect to the volume form  $\psi_\theta = \frac{1}{n!2^n} \theta \wedge (d\theta)^n$  (see the next section for precise definitions). The main result of this paper is

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**Theorem 1.1.** *For every pseudo-Hermitian structure  $\theta \in \mathcal{P}_+(\mathbb{S}^{2n+1})$  we have*

$$\lambda_1(\theta)V(\theta)^{\frac{1}{n+1}} \leq \lambda_1(\theta_0)V(\theta_0)^{\frac{1}{n+1}}. \quad (3)$$

*The equality holds in (3) if and only if there exists a CR-automorphism  $\gamma$  of  $\mathbb{S}^{2n+1}$  such that  $\theta = c\gamma^*\theta_0$  for some constant  $c > 0$ , or if and only if there exist  $p \in \mathbb{S}^{2n+1}$  and  $t \geq 0$  such that*

$$\theta = \frac{c}{|\cosh t + \sinh t (\zeta, p)|^2} \theta_0,$$

*where  $(\cdot, \cdot)$  denotes the standard Hermitian product of  $\mathbb{C}^{n+1}$ .*

This result can be seen as a contribution to the program aiming to recovering the main results of spectral geometry, established for the eigenvalues of the Laplace-Beltrami operator on a compact Riemannian manifold, in the realm of CR and pseudo-Hermitian geometry. This program has motivated a lot of research in recent years and we can find significant contributions in [1, 2, 3, 4, 6, 5, 7, 8, 9, 10, 13, 14, 17, 16, 18, 19, 20, 21, 22].

## 2. PROOF OF THEOREM 1.1

Let  $M$  be a connected differentiable manifold of dimension  $2n+1 \geq 3$ . A CR structure on  $M$  is a couple  $(H(M), J)$  where  $H(M)$  is a  $2n$ -dimensional subbundle of the tangent bundle  $TM$ , the so-called Levi distribution, endowed with a pseudo-complex operator  $J$  satisfying the following integrability condition :  $\forall X, Y \in \Gamma(H(M))$ ,

$$[X, Y] - [JX, JY] \in \Gamma(H(M))$$

and

$$J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

Real hypersurfaces of  $\mathbb{C}^{n+1}$  are the most natural examples of CR manifolds. Indeed, if  $M \subset \mathbb{C}^{n+1}$  is such a hypersurface, then  $H(M) := TM \cap J(TM)$  endowed with the restriction of the standard complex structure  $J$  of  $\mathbb{C}^{n+1}$ , is a CR structure on  $M$ .

If  $(M, H(M), J)$  is an orientable CR manifold, then there exists a nontrivial 1-form  $\theta \in \Gamma(T^*M)$  such that  $\text{Ker}\theta = H(M)$ . Such a 1-form, called pseudo-Hermitian structure, is of course not unique. The set of pseudo-Hermitian structures consists in all the forms  $f\theta$ , where  $f$  is a smooth nowhere zero function on  $M$ . To each pseudo-Hermitian structure  $\theta$  we associate its Levi form  $L_\theta$  defined on  $H(M)$  by

$$L_\theta(X, Y) = -d\theta(JX, Y) = \theta([JX, Y]).$$

The integrability of  $J$  implies that  $L_\theta$  is symmetric and  $J$ -invariant. The CR manifold  $M$  is called strictly pseudoconvex if  $L_\theta$  is definite. Of course, this condition does not depend on the choice of  $\theta$  (since  $L_{f\theta} = fL_\theta$ ). In the sequel, we denote by  $\mathcal{P}_+(M)$  the set of all pseudo-Hermitian structures with positive definite Levi form. Every  $\theta \in \mathcal{P}_+(M)$  is in fact a contact form which induces on  $M$  the following volume form

$$\psi_\theta = \frac{1}{2^n n!} \theta \wedge (d\theta)^n.$$

The associated divergence  $\text{div}_\theta$  is defined, for every smooth vector field  $Z$  on  $M$ , by

$$\mathcal{L}_Z \psi_\theta = \text{div}_\theta(Z) \psi_\theta.$$

The sub-Laplacian  $\Delta_\theta$  is then defined for all  $u \in C^\infty(M)$ , by

$$\Delta_\theta u = \operatorname{div}_\theta(\nabla^H u)$$

where  $\nabla^H u \in \Gamma(H(M))$  is the horizontal vectorfield such that,  $\forall X \in H(M)$ ,  $du(X) = L_\theta(\nabla^H u, X)$ . The following integration by parts formula holds for any  $u, v \in C_0^\infty(M)$ :

$$\int_M (\Delta_\theta u) v \psi_\theta = - \int_M L_\theta(\nabla^H u, \nabla^H v) \psi_\theta.$$

Given  $\theta \in \mathcal{P}_+(M)$ , there is a unique vectorfield  $\xi$ , often called Reeb vectorfield, that satisfies  $\theta(\xi) = 1$  and  $\xi \lrcorner d\theta = 0$ . The Levi form  $L_\theta$  extends to a Riemannian metric on  $M$  (the Webster metric) given by

$$g_\theta(X, Y) = L_\theta(X^H, Y^H) + \theta(X)\theta(Y)$$

with  $X^H = X - \theta(X)\xi$ . The corresponding Laplace-Beltrami operator  $\Delta_{g_\theta}$  is related to  $\Delta_\theta$  by the following formula (see [13])

$$\Delta_\theta = \Delta_{g_\theta} - \xi^2. \quad (4)$$

The sub-Laplacian  $\Delta_\theta$  is a sub-elliptic operator of order  $1/2$ . When  $M$  is compact, it admits a self-adjoint extension to an unbounded operator of  $L^2(M, \psi_\theta)$  whose resolvent is compact (see for instance [2, Lemma 2.2]). Hence, the spectrum of  $-\Delta_\theta$  is discrete and consists of a sequence of nonnegative eigenvalues of finite multiplicity  $\{\lambda_k(\theta)\}_{k \geq 0}$  with  $\lambda_0(\theta) = 0$ . The min-max variational principle gives

$$\lambda_1(\theta) = \inf_{\int_M u \psi_\theta = 0} \frac{\int_M |\nabla^H u|_\theta^2 \psi_\theta}{\int_M u^2 \psi_\theta} \quad (5)$$

where  $|\nabla^H u|_\theta^2 = L_\theta(\nabla^H u, \nabla^H u)$ .

**2.1. The CR Sphere.** Let  $\mathbb{S}^{2n+1}$  be the unit Sphere in  $\mathbb{C}^{n+1}$

$$\mathbb{S}^{2n+1} = \left\{ \zeta = (\zeta_1, \dots, \zeta_{n+1}) \in \mathbb{C}^{n+1}; \sum_{j=1}^{n+1} |\zeta_j|^2 = 1 \right\}$$

endowed with its standard CR-structure. The restriction to  $\mathbb{S}^{2n+1}$  of the contact form

$$\theta_0 = -\frac{i}{2} \sum_{j=1}^{n+1} (\bar{\zeta}_j d\zeta_j - \zeta_j d\bar{\zeta}_j)$$

is a pseudohermitian structure whose Reeb field  $\xi = i \sum_{j=1}^{n+1} \left( \zeta_j \frac{\partial}{\partial \bar{\zeta}_j} - \bar{\zeta}_j \frac{\partial}{\partial \zeta_j} \right)$  generates the natural action of  $\mathbb{S}^1$  on  $\mathbb{S}^{2n+1}$ . Since  $d\theta_0$  is the standard Kähler form of  $\mathbb{C}^{n+1}$ , the induced Levi form on  $H(\mathbb{S}^{2n+1})$  coincides with the standard metric of the sphere.

If  $V^{p,q}$  is the space of harmonic polynomials of bi-degree  $(p, q)$  in  $\mathbb{C}^{n+1}$ , then,  $\xi$  acts on  $V^{p,q}$  as the multiplication by  $i(p - q)$  and it can be deduced, using (4), that  $V^{p,q}$  is an eigenspace of  $\Delta_{\theta_0}$  on  $\mathbb{S}^{2n+1}$  with eigenvalue  $2n(p + q) + 4pq$  (see [10, Theorem 4.1] and [22, Proposition 4.4] for details). Therefore,

$$\lambda_1(\theta_0) = 2n. \quad (6)$$

**2.2. One-parameter groups of CR-automorphisms of the sphere.** A differentiable map  $\varphi : M \rightarrow \widetilde{M}$  between two CR manifolds is a CR map if for any  $x \in M$ ,

$$d_x\varphi(H_x(M)) \subset H_{\varphi(x)}(\widetilde{M}) \quad \text{and} \quad d_x\varphi \circ J_x^M = J_{\varphi(x)}^{\widetilde{M}} \circ d_x\varphi. \quad (7)$$

A CR-automorphism of a CR manifold  $M$  is a diffeomorphism of  $M$  which is a CR map.

Let  $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{S}^{2n+1}$ . The punctured sphere  $\mathbb{S}^{2n+1} \setminus \{e_{n+1}\}$  can be identified with the boundary of the so-called Siegel domain  $\Omega_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Im } w > |z|^2\} \subset \mathbb{C}^{n+1}$  through the CR diffeomorphism  $\Phi : \mathbb{S}^{2n+1} \setminus \{e_{n+1}\} \rightarrow \partial\Omega_{n+1}$  given by

$$\Phi(\zeta) = \frac{1}{1 - \zeta_{n+1}} (\zeta_1, \dots, \zeta_n, i(1 + \zeta_{n+1}))$$

with

$$\Phi^{-1}(z, w) = \frac{1}{w + i} (2iz_1, \dots, 2iz_n, w - i)$$

For every  $t \in \mathbb{R}$ , the ‘‘dilation’’

$$H_t : \begin{array}{ccc} \partial\Omega_{n+1} & \rightarrow & \partial\Omega_{n+1} \\ (z, w) & \mapsto & (e^t z, e^{2t} w) \end{array}$$

is a CR-automorphism of  $\partial\Omega_{n+1}$ . We define  $\gamma_t : \mathbb{S}^{2n+1} \rightarrow \mathbb{S}^{2n+1}$  by  $\gamma_t(e_{n+1}) = e_{n+1}$  and,  $\forall \zeta \in \mathbb{S}^{2n+1} \setminus \{e_{n+1}\}$ ,

$$\gamma_t(\zeta) = \Phi^{-1} \circ H_t \circ \Phi(\zeta) = \frac{1}{\cosh t + \sinh t \zeta_{n+1}} (\zeta_1, \dots, \zeta_n, \sinh t + \cosh t \zeta_{n+1})$$

or

$$\gamma_t(\zeta) = \frac{1}{\cosh t + \sinh t \zeta_{n+1}} (\zeta + (\sinh t + (\cosh t - 1) \zeta_{n+1}) e_{n+1}).$$

**Lemma 2.1.** *For every  $t$ , the map  $\gamma_t$  is a CR-automorphism of  $\mathbb{S}^{2n+1}$  which satisfies*

$$(\gamma_t)^*\theta_0 = \frac{1}{|\cosh t + \sinh t \zeta_{n+1}|^2} \theta_0.$$

*Proof.* Let  $f_j(\zeta) = \frac{\zeta_j}{\cosh t + \sinh t \zeta_{n+1}}$ ,  $j \leq n$ , and  $f_{n+1}(\zeta) = \frac{\sinh t + \cosh t \zeta_{n+1}}{\cosh t + \sinh t \zeta_{n+1}}$ . Then

$$df_j = \frac{d\zeta_j}{\cosh t + \sinh t \zeta_{n+1}} - \frac{\sinh t \zeta_j d\zeta_{n+1}}{(\cosh t + \sinh t \zeta_{n+1})^2} \quad \text{and} \quad df_{n+1} = \frac{d\zeta_{n+1}}{(\cosh t + \sinh t \zeta_{n+1})^2}.$$

Therefore

$$f_j d\bar{f}_j = \frac{\zeta_j d\bar{\zeta}_j}{|\cosh t + \sinh t \zeta_{n+1}|^2} - \frac{|\zeta_j|^2 \sinh t d\bar{\zeta}_{n+1}}{|\cosh t + \sinh t \zeta_{n+1}|^2 (\cosh t + \sinh t \bar{\zeta}_{n+1})},$$

$$\bar{f}_j df_j = \frac{\bar{\zeta}_j d\zeta_j}{|\cosh t + \sinh t \zeta_{n+1}|^2} - \frac{|\zeta_j|^2 \sinh t d\zeta_{n+1}}{|\cosh t + \sinh t \zeta_{n+1}|^2 (\cosh t + \sinh t \zeta_{n+1})}$$

which gives with  $\sum_{j=1}^n |\zeta_j|^2 = 1 - |\zeta_{n+1}|^2$ ,

$$\sum_{j=1}^n (f_j d\bar{f}_j - \bar{f}_j df_j) = \frac{1}{|\cosh t + \sinh t \zeta_{n+1}|^2} \sum_{j=1}^n (\zeta_j d\bar{\zeta}_j - \bar{\zeta}_j d\zeta_j) +$$

$$\frac{(1 - |\zeta_{n+1}|^2) \sinh t}{|\cosh t + \sinh t \zeta_{n+1}|^2} \left( \frac{d\zeta_{n+1}}{\cosh t + \sinh t \zeta_{n+1}} - \frac{d\bar{\zeta}_{n+1}}{\cosh t + \sinh t \bar{\zeta}_{n+1}} \right).$$

On the other hand,

$$f_{n+1} d\bar{f}_{n+1} - \bar{f}_{n+1} df_{n+1} = \frac{1}{|\cosh t + \sinh t \zeta_{n+1}|^2} \left( \frac{\sinh t + \cosh t \zeta_{n+1}}{\cosh t + \sinh t \bar{\zeta}_{n+1}} d\bar{\zeta}_{n+1} - \frac{(\sinh t + \cosh t \bar{\zeta}_{n+1})}{\cosh t + \sinh t \zeta_{n+1}} d\zeta_{n+1} \right)$$

Now (with  $|\zeta_{n+1}|^2 = \zeta_{n+1}\bar{\zeta}_{n+1}$ ),

$$\frac{(1 - |\zeta_{n+1}|^2) \sinh t - (\sinh t + \cosh t \bar{\zeta}_{n+1})}{\cosh t + \sinh t \zeta_{n+1}} = -\bar{\zeta}_{n+1}$$

and

$$\frac{-(1 - |\zeta_{n+1}|^2) \sinh t + (\sinh t + \cosh t \zeta_{n+1})}{\cosh t + \sinh t \bar{\zeta}_{n+1}} = \zeta_{n+1}$$

Thus,

$$\sum_{j=1}^{n+1} (f_j d\bar{f}_j - \bar{f}_j df_j) = \frac{1}{|\cosh t + \sinh t \zeta_{n+1}|^2} \sum_{j=1}^{n+1} (\zeta_j d\bar{\zeta}_j - \bar{\zeta}_j d\zeta_j)$$

that is,

$$(\gamma_t)^* \theta_0 = \frac{i}{2} \sum_{j=1}^{n+1} (f_j d\bar{f}_j - \bar{f}_j df_j) = \frac{1}{|\cosh t + \sinh t \zeta_{n+1}|^2} \theta_0.$$

□

Let  $p \in \mathbb{S}^{2n+1}$  be any point of the sphere and let  $\alpha_p \in U(n+1)$  be such that  $\alpha_p(p) = e_{n+1}$ . The family  $\gamma_t^p = \alpha_p^{-1} \circ \gamma_t \circ \alpha_p$  is a 1-parameter group of CR-automorphisms of the sphere  $\mathbb{S}^{2n+1}$  with

$$\gamma_t^p(\zeta) = \frac{1}{\cosh t + \sinh t (\zeta, p)} \{ \zeta + (\sinh t + (\cosh t - 1) (\zeta, p)) p \} \quad (8)$$

and

$$(\gamma_t^p)^* \theta_0 = \frac{1}{|\cosh t + \sinh t (\zeta, p)|^2} \theta_0. \quad (9)$$

### 2.3. Preparatory lemmas.

**Lemma 2.2.** *Let  $M$  be a strictly pseudoconvex CR manifold of dimension  $2n+1$  and let  $\theta, \hat{\theta} \in \mathcal{P}_+(M)$  be two pseudo-Hermitian structures with  $\hat{\theta} = f\theta$ ,  $f \in C^\infty(M)$ . Then*

$$\psi_{\hat{\theta}} = f^{n+1} \psi_\theta \quad (10)$$

*Proof.* From  $d\hat{\theta} = f d\theta + df \wedge \theta$  we deduce, by induction,

$$(d\hat{\theta})^n = f^n (d\theta)^n + \alpha_n \wedge \theta$$

where  $\alpha_n$  is a differential form of degree  $2n-1$ . Thus,

$$\hat{\theta} \wedge (d\hat{\theta})^n = f\theta \wedge (f^n (d\theta)^n + \alpha_n \wedge \theta) = f^{n+1} \theta \wedge (d\theta)^n.$$

□

**Lemma 2.3.** *Let  $M$  be a strictly pseudoconvex CR manifold of dimension  $2m + 1$  and let  $\phi : M \rightarrow (\mathbb{S}^{2n+1}, \theta_0)$  be a CR map. Then, for every  $\theta \in \mathcal{P}_+(M)$ ,*

$$\phi^* \theta_0 = \frac{1}{2m} \left( \sum_{i=1}^{2n+2} |\nabla^H \phi_i|_\theta^2 \right) \theta \quad (11)$$

where  $\phi_1, \dots, \phi_{2n+2}$  are the Euclidean components of  $\phi$ .

*Proof.* Since  $\phi$  is a CR map, the 1-form  $\phi^* \theta_0$  vanishes on  $H(M)$  which implies that there exists  $f \in C^\infty(M)$  such that

$$\phi^* \theta_0 = f\theta. \quad (12)$$

Differentiating, we get

$$\phi^* d\theta_0 = df \wedge \theta + f d\theta.$$

Hence, for every  $X, Y \in H_x(M)$ , one has  $\phi^* d\theta_0(X, Y) = f d\theta(X, Y)$  and, using (7),

$$\begin{aligned} L_{\theta_0}(d\phi(X), d\phi(X)) &= d\theta_0(d\phi(X), J^{\mathbb{S}^{2n+1}} d\phi(X)) = d\theta_0(d\phi(X), d\phi(J^M X)) \\ &= \phi^* d\theta_0(X, J^M X) = f d\theta(X, J^M X) = f L_\theta(X, X). \end{aligned}$$

On the other hand, since  $L_{\theta_0}$  coincides with the standard inner product on  $H_{\phi(x)}(\mathbb{S}^{2n+1})$ ,

$$L_{\theta_0}(d\phi(X), d\phi(X)) = \sum_{i=1}^{2n+2} (d\phi_i(X))^2 = \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, X)^2.$$

Thus,

$$f L_\theta(X, X) = \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, X)^2.$$

Taking an  $L_\theta$ -orthonormal basis  $\{e_1, \dots, e_{2m}\}$  of  $H_x(M)$ , we get

$$2mf = \sum_{j=1}^{2m} \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, e_j)^2 = \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, \nabla^H \phi_i) = \sum_{i=1}^{2n+2} |\nabla^H \phi_i|_\theta^2$$

which implies (11), thanks to (12).  $\square$

If  $\phi : M \rightarrow \mathbb{R}^N$  is a map and  $\mu$  is a measure on  $M$ , we denote by  $\int_M \phi d\mu$  the vector  $(\int_M \phi_1 d\mu, \dots, \int_M \phi_N d\mu) \in \mathbb{R}^N$ , where  $\phi = (\phi_1, \dots, \phi_N)$ .

**Lemma 2.4.** *Let  $M$  be a compact manifold and let  $\mu$  be a measure on  $M$  such that no open set has measure zero. If  $\phi : M \rightarrow \mathbb{S}^{2n+1}$  is a non constant continuous map, then there exists a pair  $(p, t) \in \mathbb{S}^{2n+1} \times [0, +\infty)$  such that*

$$\int_M \gamma_t^p \circ \phi d\mu = 0.$$

*Proof.* The proof uses standard arguments (see [11, 15]). We consider the map

$$\begin{aligned} F : \mathbb{S}^{2n+1} \times (0, +\infty) &\rightarrow \mathbb{B}^{2n+2} \subset \mathbb{R}^{2n+2} \\ (p, t) &\mapsto \frac{1}{V} \int_M \gamma_t^p \circ \phi d\mu \end{aligned}$$

where  $V = \int_M d\mu$  and  $\mathbb{B}^{2n+2}$  is the unit Euclidean ball. Observe that (see (8)),  $\forall p \in \mathbb{S}^{2n+1}$ ,  $\gamma_0^p$  is the identity map while,  $\forall \zeta \neq -p$ ,  $\gamma_t^p(\zeta)$  tends to  $p$  as  $t \rightarrow +\infty$ . Consequently,  $F(\cdot, 0) = \frac{1}{V} \int_M \varphi d\mu$  is a constant map and  $F(\cdot, t)$  tends to the identity of  $\mathbb{S}^{2n+1}$  as

$t \rightarrow +\infty$ . Such a map  $F$  is necessarily onto which implies that the origin of  $\mathbb{R}^{2n+2}$  belongs to its image.  $\square$

**2.4. Proof Theorem 1.1.** Let  $\theta \in \mathcal{P}_+(\mathbb{S}^{2n+1})$  be a strictly pseudoconvex pseudo-Hermitian structure. We apply Lemma 2.4 to the identity map of  $\mathbb{S}^{2n+1}$  and the measure induced by  $\psi_\theta$  to obtain the existence of a pair  $(p, t) \in \mathbb{S}^{2n+1} \times [0, +\infty)$  such that

$$\int_{\mathbb{S}^{2n+1}} \gamma_t^p \psi_\theta = 0.$$

For simplicity, we write  $\gamma$  for  $\gamma_t^p$ . The Euclidean components  $\gamma_1, \dots, \gamma_{2n+2}$  of  $\gamma$  satisfy  $\int_{\mathbb{S}^{2n+1}} \gamma_j \psi_\theta = 0$ . Hence, applying the min-max principle (5), we get for every  $j \leq 2n+2$ ,

$$\lambda_1(\theta) \int_{\mathbb{S}^{2n+1}} \gamma_j^2 \psi_\theta \leq \int_{\mathbb{S}^{2n+1}} |\nabla^H \gamma_j|_\theta^2 \psi_\theta.$$

Summing up we obtain, with  $\sum_{j \leq 2n+2} \gamma_j^2 = 1$ ,

$$\lambda_1(\theta) V(\theta) \leq \sum_{j \leq 2n+2} \int_{\mathbb{S}^{2n+1}} |\nabla^H \gamma_j|_\theta^2 \psi_\theta \tag{13}$$

$$\leq 2n \left( \int_{\mathbb{S}^{2n+1}} \left( \frac{1}{2n} |\nabla^H \gamma_j|_\theta^2 \right)^{n+1} \psi_\theta \right)^{\frac{1}{n+1}} V(\theta)^{1-\frac{1}{n+1}}. \tag{14}$$

Using Lemma 2.3, we see that, since  $\gamma$  is a CR map from  $(\mathbb{S}^{2n+1}, \theta)$  to  $(\mathbb{S}^{2n+1}, \theta_0)$ ,

$$\gamma^* \theta_0 = \left( \frac{1}{2n} \sum_{j \leq 2n+2} |\nabla^H \gamma_j|_\theta^2 \right) \theta \tag{15}$$

which gives, thanks to Lemma 2.2

$$\psi_{\gamma^* \theta_0} = \left( \frac{1}{2n} \sum_{j \leq 2n+2} |\nabla^H \gamma_j|_\theta^2 \right)^{n+1} \psi_\theta.$$

Thus

$$\lambda_1(\theta) V(\theta) \leq 2n V(\gamma^* \theta_0)^{\frac{1}{n+1}} V(\theta)^{1-\frac{1}{n+1}}.$$

Since  $V(\gamma^* \theta_0) = V(\theta_0)$ , we finally get

$$\lambda_1(\theta) V(\theta)^{\frac{1}{n+1}} \leq 2n V(\theta_0)^{\frac{1}{n+1}}$$

which proves the inequality of the theorem thanks to (6).

Now, if the equality holds in (3), this means that we have equality in the Cauchy-Schwarz inequality used in (14). Thus,  $\sum_{j \leq 2n+2} |\nabla^H \gamma_j|_\theta^2$  is constant and, thanks to (15),  $\theta$  is proportional to  $\gamma^* \theta_0$  which takes the form given by (9).

Conversely, it is clear that when  $\gamma$  is a CR-automorphism of the sphere then  $\lambda_1(\gamma^* \theta_0) = \lambda_1(\theta_0) = 2n$  and  $V(\gamma^* \theta_0) = V(\theta_0)$ . Hence, the equality holds in (3).



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