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Error estimates for phaseless inverse scattering in the Born approximation at high energies*

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Abstract. We study explicit formulas for phaseless inverse scattering in the Born approximation at high energies for the Schrödinger equation with compactly supported potential in dimension $d \geq 2$. We obtain error estimates for these formulas in the configuration space.

1 Introduction

We consider the time-independent Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \quad (1)$$

where

$$v \in L^\infty(\mathbb{R}^d), \quad \text{supp } v \subset D, \quad (2)$$

where D is some fixed open bounded domain in \mathbb{R}^d .

In quantum mechanics equation (1) describes an elementary particle interacting with a macroscopic object contained in D at fixed energy E . In this setting one usually assumes that v is real-valued.

Equation (1) at fixed E can be also interpreted as the Helmholtz equation of acoustics or electrodynamics. In these frameworks the coefficient v can be complex-valued. In addition, the imaginary part of v is related to the absorption coefficient.

For equation (1) we consider the classical scattering solutions $\psi^+ = \psi^+(x, k)$, where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $k = (k_1, \dots, k_d) \in \mathbb{R}^d$, $k^2 = k_1^2 + \dots + k_d^2 = E$. These solutions ψ^+ can be specified by the following asymptotics as $|x| \rightarrow \infty$:

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + O(|x|^{-(d+1)/2}), \\ x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d, \quad k^2 = E, \quad kx &= k_1x_1 + \dots + k_dx_d, \\ c(d, |k|) &= -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \end{aligned} \quad (3)$$

for some a priori unknown f . The function f arising in (3) is defined on

$$\mathcal{M}_E = \{(k, l) \in \mathbb{R}^d \times \mathbb{R}^d : k^2 = l^2 = E\}, \quad (4)$$

*Dedicated to G. M. Henkin

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and is known as the classical scattering amplitude for equation (1).

In quantum mechanics $|f(k, l)|^2$ describes the probability density of scattering of particle with initial momentum k into direction $l/|l| \neq k/|k|$, and is known as differential scattering cross section for equation (1); see, e.g., [11, Chapter 1, Section 6].

The problem of finding ψ^+ and f from v is known as the direct scattering problem for equation (1). For solving this problem, one can use, in particular, the Lippmann-Schwinger integral equation for ψ^+ and an explicit integral formula for f , see, e.g., [5, 10, 29].

In turn, the problem of finding v from f is known as the inverse scattering problem (with phase information) and the problem of finding v from $|f|^2$ is known as the phaseless inverse scattering problem for equation (1).

There are many important results on the former inverse scattering problem with phase information; see [3, 4, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 28] and references therein. In particular, it is well known that the scattering amplitude f uniquely determines v via the Born approximation formulas at high energies:

$$\widehat{v}(k-l) = f(k, l) + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, \quad (k, l) \in \mathcal{M}_E, \quad (5)$$

$$\widehat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d, \quad (6)$$

and the inverse Fourier transform; see, e.g., [9, 28].

On the other hand, the literature for the phaseless case is much more limited; see [7, 29] and references therein for the case of the aforementioned phaseless inverse problem and see [19, 20, 21, 26, 27, 29] and references therein for the case of some similar inverse problems without phase information. In addition, it is well known that the phaseless scattering data $|f|^2$ does not determine v uniquely, even if $|f|^2$ is given completely for all positive energies. In particular, it is known that

$$\begin{aligned} f_y(k, l) &= e^{i(k-l)y} f(k, l), \\ |f_y(k, l)|^2 &= |f(k, l)|^2, \quad k, l \in \mathbb{R}^d, \quad k^2 = l^2 > 0, \end{aligned} \quad (7)$$

where f is the scattering amplitude for v and f_y is the scattering amplitude for $v_y = v(\cdot - y)$, where $y \in \mathbb{R}^d$; see [29] and references therein.

In the present work, in view of the aforementioned non-uniqueness for the problem of finding v from $|f|^2$, we consider the modified phaseless inverse scattering problem formulated below as Problem 1. Let

$$S = \{|f|^2, |f_1|^2, \dots, |f_m|^2\}, \quad (8)$$

where f is the scattering amplitude for v and f_1, \dots, f_m are the scattering amplitudes for v_1, \dots, v_m , where

$$v_j = v + w_j, \quad j = 1, \dots, m, \quad (9)$$

where w_1, \dots, w_m are additional a priori known background scatterers such that

$$w_j \in L^\infty(\mathbb{R}^d), \quad \text{supp } w_j \subset \Omega_j, \quad (10)$$

Ω_j is an open bounded domain in \mathbb{R}^d , $\Omega_j \cap D = \emptyset$,

$$w_j \neq 0, \quad w_{j_1} \neq w_{j_2} \text{ if } j_1 \neq j_2 \text{ (in } L^\infty(\mathbb{R}^d)),$$

where $j, j_1, j_2 \in \{1, \dots, m\}$. Thus, S consists of the phaseless scattering data $|f|^2, |f_1|^2, \dots, |f_m|^2$ measured sequentially, first, for the unknown scatterer v and then for v in the presence of known scatterer w_j disjoint from v for $j = 1, \dots, m$.

Actually, in the present work we continue studies of [29] on the following inverse scattering problem for equation (1):

Problem 1. *Reconstruct potential v from the phaseless scattering data S for some appropriate background scatterers w_1, \dots, w_m .*

Studies of Problem 1 in dimension $d \geq 2$ were started in [29]. In dimension $d = 1$ for $m = 1$ studies of Problem 1 were started earlier in [2], where phaseless scattering data was considered for all $E > 0$.

Actually, the key result of [29] consists in a proper extension of formula (5) for the Fourier transform \widehat{v} of v to the phaseless case of Problem 1; see Section 2.

In the present work we proceed from the aforementioned result of [29] and study related approximate reconstruction of v in the configuration space. In this connection our results consist in obtaining related error estimates in the configuration space at high energies E ; see Section 4.

In addition, results of the present work are necessary for extending the iterative algorithm of [28] to the phaseless case of Problem 1. The latter extension will be given in [1].

2 Extension of formula (5) to the phaseless case

Actually, the key result of [29] consists in the following formulas for solving Problem 1 in dimension $d \geq 2$ for $m = 2$ at high energies E :

$$\begin{pmatrix} \text{Re } \widehat{v} \\ \text{Im } \widehat{v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \text{Re } \widehat{w}_1 & \text{Im } \widehat{w}_1 \\ \text{Re } \widehat{w}_2 & \text{Im } \widehat{w}_2 \end{pmatrix}^{-1} \begin{pmatrix} |\widehat{v}_1|^2 - |\widehat{v}|^2 - |\widehat{w}_1|^2 \\ |\widehat{v}_2|^2 - |\widehat{v}|^2 - |\widehat{w}_2|^2 \end{pmatrix}, \quad (11)$$

$$|\widehat{v}_j(p)|^2 = |f_j(k, l)|^2 + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, \quad (12)$$

$$p \in \mathbb{R}^d, \quad (k, l) \in \mathcal{M}_E, \quad k - l = p, \quad j = 0, 1, 2,$$

where:

- $v_0 = v$, v_j is defined by (9), $j = 1, 2$, and $f_0 = f$, f_1, f_2 are the scattering amplitudes for v_0, v_1, v_2 , respectively;
- $\widehat{v} = \widehat{v}(p)$, $\widehat{v}_j = \widehat{v}_j(p)$, $\widehat{w}_j = \widehat{w}_j(p)$, $p \in \mathbb{R}^d$, are the Fourier transforms of v, v_j, w_j (defined as in (6));

- formula (11) is considered for all $p \in \mathbb{R}^d$ such that the determinant

$$\zeta_{\widehat{w}_1, \widehat{w}_2}(p) \stackrel{def}{=} \operatorname{Re} \widehat{w}_1(p) \operatorname{Im} \widehat{w}_2(p) - \operatorname{Im} \widehat{w}_1(p) \operatorname{Re} \widehat{w}_2(p) \neq 0. \quad (13)$$

The point is that using formulas (12) for $d \geq 2$ with

$$\begin{aligned} k &= k_E(p) = \frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \\ l &= l_E(p) = -\frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \\ |\gamma(p)| &= 1, \quad \gamma(p)p = 0, \end{aligned} \quad (14)$$

where $p \in \mathbb{R}^d$, $|p| \leq 2\sqrt{E}$, one can reconstruct $|\widehat{v}|^2$, $|\widehat{v}_1|^2$, $|\widehat{v}_2|^2$ from S at high energies for any $p \in \mathbb{R}^d$. And then using formula (11) one can reconstruct \widehat{v} completely, provided that condition (13) is fulfilled for almost all $p \in \mathbb{R}^d$.

Remark 1. Formulas (12) can be precised as formula (2.15) of [29]:

$$\begin{aligned} \left| |\widehat{v}_j(p)|^2 - |f_j(k, l)|^2 \right| &\leq c(D_j) N_j^3 E^{-\frac{1}{2}}, \\ p &= k - l, \quad (k, l) \in \mathcal{M}_E, \quad E^{\frac{1}{2}} \geq \rho(D_j, N_j), \quad j = 0, 1, 2, \end{aligned} \quad (15)$$

where $\|v_j\|_{L^\infty(D_j)} \leq N_j$, $j = 0, 1, 2$, and $D_0 = D$, $D_j = D \cup \Omega_j$, $j = 1, 2$, and constants c, ρ are given by formulas (3.10) and (3.11) in [29] (and, in particular, $\rho \geq 1$).

In addition, from the experimental point of view it seems to be, in particular, convenient to consider Problem 1 with $m = 2$ for the case when w_2 is just a translation of w_1 :

$$w_2(x) = w_1(x - y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d. \quad (16)$$

In this case

$$\widehat{w}_2(p) = e^{ipy} \widehat{w}_1(p), \quad \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = \sin(py) |\widehat{w}_1(p)|^2, \quad p \in \mathbb{R}^d. \quad (17)$$

On the level of analysis, the principal complication of (11), (12) in comparison with (5) consists in possible zeros of the determinant $\zeta_{\widehat{w}_1, \widehat{w}_2}$ of (13). For some simplest cases, we study these zeros in the next section.

3 Zeros of the determinant $\zeta_{\widehat{w}_1, \widehat{w}_2}$

Let

$$\begin{aligned} Z_{\widehat{w}_1, \widehat{w}_2} &= \{p \in \mathbb{R}^d : \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = 0\}, \\ Z_{\widehat{w}_j} &= \{p \in \mathbb{R}^d : \widehat{w}_j(p) = 0\}, \quad j = 1, 2, \end{aligned} \quad (18)$$

where ζ is defined by (13). From (13), (18) it follows that

$$Z_{\widehat{w}_1} \cup Z_{\widehat{w}_2} \subseteq Z_{\widehat{w}_1, \widehat{w}_2}. \quad (19)$$

In view of (19), in order to construct examples of w_1, w_2 such that the set $Z_{\widehat{w}_1, \widehat{w}_2}$ is as simple as possible, we use the following lemma:

Lemma 1. *Let*

$$w(x) = |x|^\nu K_\nu(|x|) \int_{\mathbb{R}^d} q(x-y)q(y) dy, \quad x \in \mathbb{R}^d, \nu > 0, \quad (20)$$

$$K_\nu(s) = \frac{\Gamma(\frac{1}{2} + \nu)}{\sqrt{\pi}} \left(\frac{2}{s}\right)^\nu \int_0^\infty \frac{\cos(st) dt}{(1+t^2)^{\frac{1}{2}+\nu}}, \quad s > 0, \quad (21)$$

$$\begin{aligned} q &\in L^\infty(\mathbb{R}^d), \quad q = \bar{q}, \quad q \neq 0 \text{ in } L^\infty(\mathbb{R}^d), \\ q(x) &= 0 \text{ if } |x| > r, \quad q(x) = q(-x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (22)$$

Then

$$\begin{aligned} w &\in C(\mathbb{R}^d), \quad w = \bar{w}, \quad w(x) = 0 \text{ if } |x| > 2r, \quad x \in \mathbb{R}^d, \\ \widehat{w}(p) &= \overline{\widehat{w}(p)} \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (23)$$

for $\beta = d + 2\nu$ and some positive constant $c_1 = c_1(q, \nu)$, where \widehat{w} is the Fourier transform of w . In addition, if $q \geq 0$, then $w \geq 0$.

We recall that K_ν defined by (21) is the modified Bessel function of the second kind and order ν . In addition, Γ denotes the gamma function.

Lemma 1 is proved in Section 8.

As a corollary of Lemma 1, functions

$$w_j(x) = w(x - T_j), \quad x \in \mathbb{R}^d, \quad T_j \in \mathbb{R}^d, \quad (24)$$

where w is constructed in Lemma 1, give us examples of w_j satisfying (10) for fixed D , Ω_j and for appropriate radius r of Lemma 1 and translations T_j of (24), and such that

$$\begin{aligned} Z_{\widehat{w}_j} &= \emptyset, \\ |\widehat{w}_j(p)| &= \widehat{w}(p) \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (25)$$

where c_1, β are the same as in (23). In addition,

$$\begin{aligned} \zeta_{\widehat{w}_1, \widehat{w}_2}(p) &= \sin(py) |\widehat{w}(p)|^2, \quad y = T_2 - T_1 \neq 0, \quad p \in \mathbb{R}^d, \\ Z_{\widehat{w}_1, \widehat{w}_2} &= \{p \in \mathbb{R}^d : \sin(py) = 0\} = \{p \in \mathbb{R}^d : py \in \pi\mathbb{Z}\}, \end{aligned} \quad (26)$$

for w_1, w_2 of (24).

As another corollary of Lemma 1, we have that

$$\begin{aligned} \text{if } w_1 &\text{ is defined as in (24) and } w_2 = iw_1, \text{ then} \\ \zeta_{\widehat{w}_1, \widehat{w}_2}(p) &= |\widehat{w}(p)|^2 \geq c_1^2(1+|p|)^{-2\beta}, \quad p \in \mathbb{R}^d, \\ Z_{\widehat{w}_1, \widehat{w}_2} &= \emptyset. \end{aligned} \quad (27)$$

We recall that complex-valued v and w_j naturally arise if we interpret equation (1) for fixed E as the Helmholtz equation of acoustics or electrodynamics.

Finally, note that

$$\begin{aligned} Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}} &= \frac{\pi}{s} \mathbb{Z}^d, \quad \text{where} \\ Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}} &= Z_{\widehat{w}_1, \widehat{w}_2} \cap Z_{\widehat{w}_1, \widehat{w}_3} \cap \dots \cap Z_{\widehat{w}_1, \widehat{w}_{d+1}}, \end{aligned} \quad (28)$$

if w_1 is defined as in (24), and

$$w_2(x) = w_1(x - se_1), \dots, w_{d+1}(x) = w_1(x - se_d), \quad (29)$$

where (e_1, \dots, e_d) is the standard basis of \mathbb{R}^d and $s > 0$.

Thus, in principle, for Problem 1 with background scatterers w_1, \dots, w_{d+1} as in (29), for each $p \in \mathbb{R}^d \setminus \frac{\pi}{s}\mathbb{Z}^d$ formulas (11), (12) can be used with appropriate w_j in place of w_2 , where $j = 2, \dots, d+1$.

4 Error estimates in the configuration space

We recall that for inverse scattering with phase information the scattering amplitude f on \mathcal{M}_E processed by (5) and the inverse Fourier transform yield the approximate reconstruction

$$u(\cdot, E) = v + O(E^{-\alpha}) \quad \text{in } L^\infty(D) \text{ as } E \rightarrow +\infty, \quad \alpha = \frac{n-d}{2n}, \quad (30)$$

if $v \in W^{n,1}(\mathbb{R}^d)$, $n > d$ (in addition to the initial assumption (2)), where $W^{n,1}(\mathbb{R}^d)$ denotes the standard Sobolev space of n -times differentiable functions in $L^1(\mathbb{R}^d)$:

$$\begin{aligned} W^{n,1}(\mathbb{R}^d) &= \{u \in L^1(\mathbb{R}^d) : \|u\|_{n,1} < \infty\}, \\ \|u\|_{n,1} &= \max_{|J| \leq n} \left\| \frac{\partial^{|J|} u}{\partial x^J} \right\|_{L^1(\mathbb{R}^d)}, \quad n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (31)$$

More precisely, the approximation $u(\cdot, E)$ in (30) is defined by

$$\begin{aligned} u(x, E) &= \int_{\mathcal{B}_r(E)} e^{-ipx} f(k_E(p), l_E(p)) dp, \quad x \in D, \\ r(E) &= 2\tau E^{\frac{\alpha}{n-d}} \quad \text{for some fixed } \tau \in (0, 1], \end{aligned} \quad (32)$$

where

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : |p| \leq r\}, \quad (33)$$

α is defined in (30), and $k_E(p), l_E(p)$ are defined as in (14) with some piecewise continuous vector-function γ on \mathbb{R}^d ; see, e.g., [28]. In addition, estimate (30) can be precised as

$$|u(x, E) - v(x)| \leq A(D, N, M, d, n, \tau) E^{-\alpha}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho(D, N), \quad (34)$$

where $\|v\|_{L^\infty(D)} \leq N$, $\|v\|_{n,1} \leq M$, ρ is the same as in (15) and the expression for A can be found in formula (3.10) of [28].

Analogs of $u(\cdot, E)$ for the phaseless case are given below in this section. In particular, related formulas depend on the zeros of determinant $\zeta_{\hat{w}_1, \hat{w}_2}$ of (13).

We consider

$$U_{\widehat{w}_1, \widehat{w}_2} = \operatorname{Re} U_{\widehat{w}_1, \widehat{w}_2} + i \operatorname{Im} U_{\widehat{w}_1, \widehat{w}_2},$$

$$\begin{pmatrix} \operatorname{Re} U_{\widehat{w}_1, \widehat{w}_2}(p, E) \\ \operatorname{Im} U_{\widehat{w}_1, \widehat{w}_2}(p, E) \end{pmatrix} = \frac{1}{2} M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) b_{\widehat{w}_1, \widehat{w}_2}(p, E), \quad (35)$$

$$M_{\widehat{w}_1, \widehat{w}_2}(p) = \begin{pmatrix} \operatorname{Re} \widehat{w}_1(p) & \operatorname{Im} \widehat{w}_1(p) \\ \operatorname{Re} \widehat{w}_2(p) & \operatorname{Im} \widehat{w}_2(p) \end{pmatrix}, \quad (36)$$

$$M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) = \frac{1}{\zeta_{\widehat{w}_1, \widehat{w}_2}(p)} \begin{pmatrix} \operatorname{Im} \widehat{w}_2(p) & -\operatorname{Im} \widehat{w}_1(p) \\ -\operatorname{Re} \widehat{w}_2(p) & \operatorname{Re} \widehat{w}_1(p) \end{pmatrix}, \quad (37)$$

$$b_{\widehat{w}_1, \widehat{w}_2}(p, E) = \begin{pmatrix} |f_1(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_1(p)|^2 \\ |f_2(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_2(p)|^2 \end{pmatrix}, \quad (38)$$

$$f(p, E) = f(k_E(p), l_E(p)), \quad f_j(p, E) = f_j(k_E(p), l_E(p)), \quad j = 1, 2, \quad (39)$$

where $\widehat{w}_1, \widehat{w}_2, f, f_1, f_2$ are the same as in (11), (12), $\zeta_{\widehat{w}_1, \widehat{w}_2}$ is defined by (13), $k_E(p), l_E(p)$ are the same as in (14), (32), and $p \in \mathcal{B}_{2\sqrt{E}}, d \geq 2$.

For Problem 1 for $d \geq 2, m = 2$, and for the case when $\zeta_{\widehat{w}_1, \widehat{w}_2}$ has no zeros (the case of (27) in Section 3) we have the following result:

Theorem 1. *Let v satisfy (2) and $v \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$. Let w_1, w_2 be the same as in (27). Let*

$$u(x, E) = \int_{\mathcal{B}_{r_1(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp, \quad x \in D, \quad (40)$$

$$r_1(E) = 2\tau E^{\frac{\alpha_1}{n-d}}, \quad \alpha_1 = \frac{n-d}{2(n+\beta)}, \quad \text{for some fixed } \tau \in (0, 1],$$

where $U_{\widehat{w}_1, \widehat{w}_2}$ is defined by (35), \mathcal{B}_r is defined by (33), β is the number of (23), (27). Then

$$\begin{aligned} u(\cdot, E) &= v + O(E^{-\alpha_1}) \quad \text{in } L^\infty(D), \quad E \rightarrow +\infty, \\ |u(x, E) - v(x)| &\leq A_1 E^{-\alpha_1}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_1, \end{aligned} \quad (41)$$

where ρ_1 and A_1 are defined in formulas (55) and (64) of Section 5.

Theorem 1 is proved in Section 5.

Next, we set

$$Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon = \{p \in \mathbb{R}^d : py \in (-\varepsilon, \varepsilon) + \pi\mathbb{Z}\}, \quad y \in \mathbb{R}^d \setminus 0, \quad 0 < \varepsilon < 1, \quad (42)$$

where $\widehat{w}_1, \widehat{w}_2$ and y are the same as in (24)–(26). One can see that $Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon$ is the open $\frac{\varepsilon}{|y|}$ -neighborhood of $Z_{\widehat{w}_1, \widehat{w}_2}$ defined in (26).

Note that

$$\begin{aligned} &\text{for any } p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon \text{ there exists} \\ &\text{the unique } z(p) \in \mathbb{Z} \text{ such that } |py - \pi z(p)| < \varepsilon. \end{aligned} \quad (43)$$

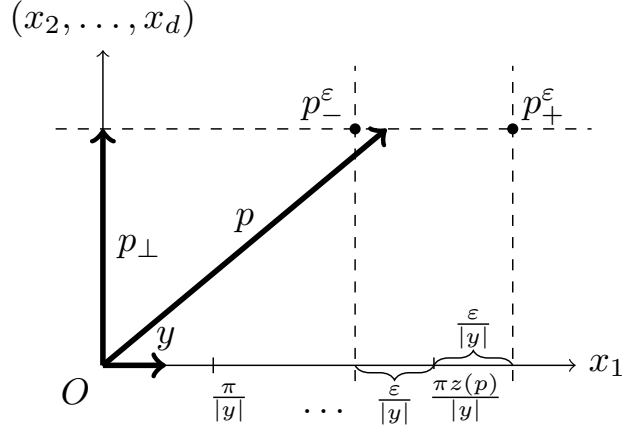


Figure 1: Vectors p , p_{\perp} , y and p_{\pm}^{ε} of formula (44)

In addition to $U_{\widehat{w}_1, \widehat{w}_2}$ of (35), we define

$$U_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}(p, E) = \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}(p_{-}^{\varepsilon}, E) + U_{\widehat{w}_1, \widehat{w}_2}(p_{+}^{\varepsilon}, E)), \quad (44)$$

$$p_{\pm}^{\varepsilon} = p_{\perp} + \pi z(p) \frac{y}{|y|^2} \pm \varepsilon \frac{y}{|y|^2}, \quad p_{\perp} = p - (py) \frac{y}{|y|^2}, \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon},$$

where $z(p)$ is the integer number of (43). The geometry of vectors p , p_{\perp} , y , p_{\pm}^{ε} is illustrated in Fig. 1 for the case when the direction of y coincides with the basis vector $e_1 = (1, 0, \dots, 0)$.

For Problem 1 for $d \geq 2$, $m = 2$, and for the case when $\zeta_{\widehat{w}_1, \widehat{w}_2}$ has zeros on hyperplanes (the case of (26) in Section 3) we have the following result:

Theorem 2. *Let v satisfy (2) and $v \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$. Let w_1, w_2 be the same as in (24)–(26). Let*

$$u(x, E) = u_1(x, E) + u_2(x, E), \quad x \in D,$$

$$u_1(x, E) = \int_{\mathcal{B}_{r_2(E)} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp,$$

$$u_2(x, E) = \int_{\mathcal{B}_{r_2(E)} \cap Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}(p, E) dp, \quad (45)$$

$$r_2(E) = 2\tau E^{\frac{\alpha_2}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_2}{2}},$$

$$\alpha_2 = \frac{n-d}{2(n+\beta+\frac{n-d}{2})}, \quad \text{for some fixed } \tau \in (0, 1],$$

where $U_{\widehat{w}_1, \widehat{w}_2}$ and $U_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}$ are defined by (35), (44), \mathcal{B}_r and $Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}$ are defined

by (33), (42), and β is the number of (23). Then

$$\begin{aligned} u(\cdot, E) &= v + O(E^{-\alpha_2}) \quad \text{in } L^\infty(D), \quad E \rightarrow +\infty, \\ |u(x, E) - v(x)| &\leq A_2 E^{-\alpha_2}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_2, \end{aligned} \quad (46)$$

where ρ_2 and A_2 are defined in formulas (65) and (85) of Section 6.

Theorem 2 is proved in Section 6.

Next, we set

$$\begin{aligned} Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon &= \mathcal{B}'_{\varepsilon/s} + \frac{\pi}{s} \mathbb{Z}^d, \quad 0 < \varepsilon < 1, \\ \mathcal{B}'_r &= \mathcal{B}_r \setminus \partial \mathcal{B}_r, \quad r > 0, \end{aligned} \quad (47)$$

where $\widehat{w}_1, \dots, \widehat{w}_{d+1}$ are the same as in (28), (29), and \mathcal{B}_r is defined by (33). One can see that $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$ is the open $\frac{\varepsilon}{s}$ -neighborhood of $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}$ defined in (28).

Note that

$$\begin{aligned} &\text{for any } p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon \text{ there exists} \\ &\text{the unique } z(p) \in \mathbb{Z}^d \text{ such that } |sp - \pi z(p)| < \varepsilon. \end{aligned} \quad (48)$$

In addition, we consider i' such that

$$\begin{aligned} i' &= i'(p, s), \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d \setminus \frac{\pi}{s} \mathbb{Z}^d, \quad s > 0, \\ i' &\text{ take values in } \{2, \dots, d+1\}, \\ |\sin(sp_{i'-1})| &\geq |\sin(sp_{i-1})| \quad \text{for all } i \in \{2, \dots, d+1\}. \end{aligned} \quad (49)$$

Let

$$\begin{aligned} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) &= U_{\widehat{w}_1, \widehat{w}_{i'}}(p, E), \quad p \in \mathbb{R}^d \setminus \frac{\pi}{s} \mathbb{Z}^d, \\ U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) &= \end{aligned} \quad (50)$$

$$\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}\left(\frac{\varepsilon}{s} \vartheta + \frac{\pi}{s} z(p), E\right) d\vartheta, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad (51)$$

$$\mathbb{S}^{d-1} = \{p \in \mathbb{R}^d : |p| = 1\}, \quad (52)$$

where $|\mathbb{S}^{d-1}|$ denotes the standard Euclidean volume of \mathbb{S}^{d-1} , $U_{\widehat{w}_1, \widehat{w}_{i'}}$ is defined in a similar way with $U_{\widehat{w}_1, \widehat{w}_2}$ of (35), $\widehat{w}_1, \dots, \widehat{w}_{d+1}$ are the same as in (28), (29), and i' is the same as in (49).

For Problem 1 for $d \geq 2$, $m = d + 1$, we also have the following result:

Theorem 3. *Let v satisfy (2) and $v \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$. Let $w_1, \dots,$*

w_{d+1} be the same as in (29). Let

$$\begin{aligned}
u(x, E) &= u_1(x, E) + u_2(x, E), \quad x \in D, \\
u_1(x, E) &= \int_{\mathcal{B}_{r_3(E)} \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^{\varepsilon_3(E)}} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) dp, \\
u_2(x, E) &= \int_{\mathcal{B}_{r_3(E)} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^{\varepsilon_3(E)}} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) dp, \\
r_3(E) &= 2\tau E^{\frac{\alpha_3}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_3}{d+1}}, \\
\alpha_3 &= \frac{n-d}{2(n+\beta+\frac{d}{d+1})}, \quad \text{for some fixed } \tau \in (0, 1],
\end{aligned} \tag{53}$$

where $U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}$ and $U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$ are defined by (50), (51), \mathcal{B}_r and $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$ are defined by (33), (47), and β is the number of (23). Then

$$\begin{aligned}
u(\cdot, E) &= v + O(E^{-\alpha_3}) \quad \text{in } L^\infty(D), \quad E \rightarrow +\infty, \\
|u(x, E) - v(x)| &\leq A_3 E^{-\alpha_3}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_3,
\end{aligned} \tag{54}$$

where ρ_3 and A_3 are defined in formulas (86) and (99) of Section 7.

Theorem 3 is proved in Section 7.

5 Proof of Theorem 1

Proposition 1. Let v satisfy (2) and w_1, w_2 be the same as in (27), $d \geq 2$. Then:

$$\begin{aligned}
|\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}(p, E)| &\leq c_2 |\widehat{w}(p)|^{-1} E^{-\frac{1}{2}} \quad \text{for } p \in \mathcal{B}_{2\sqrt{E}}, \quad E^{\frac{1}{2}} \geq \rho_1, \\
c_2 &= 2c(D_0)N_0^3 + 2c(D_1)N_1^3, \\
\rho_1 &= \max_{j=0,1} \rho(D_j, N_j),
\end{aligned} \tag{55}$$

where w is the function of (23), (24), and $c, \rho, N_j, D_j, j = 0, 1, 2$, are the same as in estimates (15).

Proposition 1 follows from formulas (11), (14), estimates (15), definitions (35)–(39), and the properties that

$$\Omega_2 = \Omega_1, \quad D_2 = D_1, \quad N_2 = N_1. \tag{56}$$

In turn, properties (56) follow from (9) for $j = 1, 2$, (10) for $j = 1$, and from the equality $w_2 = iw_1$ assumed in (27).

Next, we represent v as follows:

$$\begin{aligned} v(x) &= v^+(x, r) + v^-(x, r), \quad x \in D, \quad r > 0, \\ v^+(x, r) &= \int_{\mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp, \\ v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp. \end{aligned} \quad (57)$$

Since $v \in W^{n,1}(\mathbb{R}^d)$, $n > d$, we have

$$\begin{aligned} |v^-(x, r)| &\leq c_3 \|v\|_{n,1} r^{d-n}, \quad x \in D, \quad r > 0, \\ c_3 &= |\mathbb{S}^{d-1}| \frac{(2\pi)^{-d} d^n}{n-d}, \end{aligned} \quad (58)$$

where $\|\cdot\|_{n,1}$ is defined in (31), and $|\mathbb{S}^{d-1}|$ is the standard Euclidean volume of \mathbb{S}^{d-1} . Indeed,

$$\begin{aligned} |p_1^{k_1} \cdots p_d^{k_d} \widehat{v}(p)| &\leq (2\pi)^{-d} \|v\|_{n,1}, \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d, \\ &\text{for any } k_1, \dots, k_d \in \mathbb{N} \cup \{0\}, \quad k_1 + \cdots + k_d \leq n, \end{aligned} \quad (59)$$

assuming also that $p_j^0 = 1$. Taking an appropriate sum in (59) over all such k_1, \dots, k_d with $k_1 + \cdots + k_d = m \leq n$, we get

$$|p|^m |\widehat{v}(p)| \leq (|p_1| + \cdots + |p_d|)^m |\widehat{v}(p)| \leq (2\pi)^{-d} d^m \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \quad (60)$$

The definition of v^- of (57) and inequalities (60) for $m = n$ imply (58).

In addition, using Proposition 1 and the estimate on \widehat{w} of (23), we obtain:

$$\begin{aligned} \left| v^+(x, r) - \int_{\mathcal{B}_r} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right| &\leq c_2 E^{-\frac{1}{2}} \int_{\mathcal{B}_r} |\widehat{w}(p)|^{-1} dp \\ &\leq c_1^{-1} c_2 E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp \leq c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad c_4 = |\mathbb{S}^{d-1}| \frac{2^{d+\beta}}{d+\beta} c_1^{-1} c_2, \\ &x \in D, \quad 1 \leq r \leq 2E^{\frac{1}{2}}, \quad E^{\frac{1}{2}} \geq \rho_1. \end{aligned} \quad (61)$$

As a corollary of (57), (58), (61), we have

$$\left| v(x) - \int_{\mathcal{B}_r} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right| \leq c_3 \|v\|_{n,1} r^{d-n} + c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad (62)$$

where $x \in D$, $1 \leq r \leq 2E^{\frac{1}{2}}$, $E^{\frac{1}{2}} \geq \rho_1$. In addition, if $r = r_1(E)$, where $r_1(E)$ is defined in (40), then

$$\begin{aligned} r^{d-n} &= (2\tau)^{d-n} E^{-\alpha_1}, \\ E^{-\frac{1}{2}} r^{d+\beta} &= (2\tau)^{d+\beta} E^{-\alpha_1}. \end{aligned} \quad (63)$$

Using formulas (62) and (63) and taking into account definitions (40), we obtain

$$\begin{aligned} |u(x, E) - v(x)| &\leq A_1 E^{-\alpha_1}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_1 \\ A_1 &= A_1(D_0, D_1, N_0, N_1, M, d, n, \beta, \tau) \\ &= (2\tau)^{d-n} c_3 \|v\|_{n,1} + (2\tau)^{d+\beta} c_4, \end{aligned} \quad (64)$$

where $D_j, N_j, j = 0, 1$, are the same as in estimates (15) and $\|v\|_{n,1} \leq M$.
Theorem 1 is proved.

6 Proof of Theorem 2

Proposition 2. *Let v satisfy (2) and w_1, w_2 be the same as in (24)–(26), $d \geq 2$. Then:*

$$\begin{aligned} |\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}(p, E)| &\leq c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad E^{\frac{1}{2}} \geq \rho_2, \quad 0 < \varepsilon < 1, \\ c_5 &= \frac{\pi}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + c(D_2)N_2^3)c_1, \\ \rho_2 &= \max_{j=0,1,2} \rho(D_j, N_j), \end{aligned} \quad (65)$$

in addition, if $v \in W^{n,1}(\mathbb{R}^d)$, $n \geq 0$, then:

$$\begin{aligned} &|\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E)| \\ &\leq 2^\beta c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}} + c_6 \varepsilon (1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|)^{-n}, \\ &p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad E \geq \rho_2, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}|y|\}; \\ &c_6 = 2^n \frac{(d+1)^{n+1}}{(2\pi)^d |y|} \max_{j=1, \dots, d} \|x_j v\|_{n,1}, \end{aligned} \quad (66)$$

where $c, \rho, N_j, D_j, j = 0, 1, 2$, are the same as in estimates (15), c_1, β are the same as in Lemma 1; $z(p), p_\perp$ are defined in (43), (44), $x_j v = x_j v(x)$, and $\|\cdot\|_{n,1}$ is defined in (31).

Proof of Proposition 2. It follows from formulas (24), (26) and (36), (37) that

$$\begin{aligned} M_{\widehat{w}_1, \widehat{w}_2}(p) &= \widehat{w}(p) \begin{pmatrix} \cos(T_1 p) & \sin(T_1 p) \\ \cos(T_2 p) & \sin(T_2 p) \end{pmatrix}, \quad p \in \mathbb{R}^d, \\ M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) &= \frac{1}{\sin(py) \widehat{w}(p)} \begin{pmatrix} \sin(T_2 p) & -\sin(T_1 p) \\ -\cos(T_2 p) & \cos(T_1 p) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (67)$$

Also note that

$$|\sin(py)| \geq \frac{2\varepsilon}{\pi}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad 0 < \varepsilon < 1. \quad (68)$$

The estimate (65) follows from (11), (15), (23), (35), (38), (39) and from (67), (68).

It remains to prove (66). Using definition (44), one can write

$$\begin{aligned} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) - \widehat{v}(p) &= \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\ \varphi_1^\varepsilon(p, E) &= \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p_-, E) - \widehat{v}(p_-)) \\ &+ \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p_+, E) - \widehat{v}(p_+)), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\ \varphi_2^\varepsilon(p) &= \frac{1}{2} (\widehat{v}(p_-) + \widehat{v}(p_+)) - \widehat{v}(p), \quad p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (69)$$

Using estimate (65), formula (69) and the definitions of p_{\pm}^{ε} in (44), we get

$$\begin{aligned} |\varphi_1^{\varepsilon}(p, E)| &\leq \frac{1}{2}c_5\varepsilon^{-1}E^{-\frac{1}{2}}((1 + |p_-^{\varepsilon}|)^{\beta} + (1 + |p_+^{\varepsilon}|)^{\beta}) \\ &\leq c_5\varepsilon^{-1}(1 + |p| + 2\frac{\varepsilon}{|y|})^{\beta}E^{-\frac{1}{2}} \leq 2^{\beta}c_5\varepsilon^{-1}(1 + |p|)^{\beta}E^{-\frac{1}{2}}, \end{aligned} \quad (70)$$

for ε as in (66), $p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}$.

Next, using the definition of φ_2^{ε} in (69) and the mean value theorem, we obtain

$$|\varphi_2^{\varepsilon}(p)| \leq \frac{\varepsilon}{|y|} \max\{|\frac{y}{|y|}\nabla\hat{v}(\xi)| : \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}]\}, \quad p \in Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}, \quad (71)$$

where $[p_-^{\varepsilon}, p_+^{\varepsilon}]$ denotes the segment joining p_-^{ε} to p_+^{ε} . Here, the mean value theorem was used for $\hat{v}(\xi)$ on $[p_-^{\varepsilon}, p]$ and on $[p, p_+^{\varepsilon}]$.

Note also that

$$|\nabla\hat{v}(\xi)| \leq d \max_{j=1, \dots, d} \left| \frac{\partial\hat{v}}{\partial\xi_j}(\xi) \right|, \quad \xi = (\xi_1, \dots, \xi_d) \in [p_-^{\varepsilon}, p_+^{\varepsilon}]. \quad (72)$$

In addition, the following estimates hold:

$$\left| \frac{\partial\hat{v}}{\partial\xi_j}(\xi) \right| \leq \frac{(1+d)^n}{(2\pi)^d(1+|\xi|)^n} \|x_j v\|_{n,1}, \quad \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}], \quad j = 1, \dots, d. \quad (73)$$

Indeed, taking the sum in (60) over all $m = 0, \dots, n$ with the binomial coefficients, we get

$$\begin{aligned} (1 + |p|)^n |\hat{v}(p)| &\leq (1 + |p_1| + \dots + |p_d|)^n |\hat{v}(p)| \\ &\leq (2\pi)^{-d} (1+d)^n \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \end{aligned} \quad (74)$$

Estimates (73) follow from (74), where we replace v by $x_j v$ and use that v belongs to $W^{n,1}(\mathbb{R}^d)$ and is compactly supported.

Estimates (71)–(73) imply

$$|\varphi_2^{\varepsilon}(p)| \leq 2^{-n} c_6 \varepsilon \max\{(1 + |\xi|)^{-n} : \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}]\}, \quad p \in Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}. \quad (75)$$

Using also that

$$\xi = \tau \frac{y}{|y|} + p_{\perp}, \quad \text{where } |\tau - \frac{\pi}{|y|} z(p)| \leq \frac{\varepsilon}{|y|}, \quad \text{if } \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}], \quad (76)$$

and that $\varepsilon < |y|$, we obtain

$$\begin{aligned} |\varphi_2^{\varepsilon}(p)| &\leq 2^{-n} c_6 \varepsilon \left(1 + \frac{1}{2} \left(\frac{\pi}{|y|} |z(p)| - \frac{\varepsilon}{|y|} + |p_{\perp}| \right)\right)^{-n} \\ &\leq c_6 \left(1 + \frac{\pi}{|y|} |z(p)| + |p_{\perp}| \right)^{-n}, \quad p \in Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}. \end{aligned} \quad (77)$$

Estimate (66) follows from (70) and (77).

Proposition 2 is proved. \square

The final part of the proof of Theorem 2 is as follows. In a similar way with (57), we represent v as follows:

$$\begin{aligned}
v(x) &= v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in D, \quad r > 0, \\
v_1^+(x, r) &= \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\
v_2^+(x, r) &= \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\
v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp.
\end{aligned} \tag{78}$$

Since $v \in W^{n,1}(\mathbb{R}^d)$, estimate (58) holds.

Using estimates (65), (66), we get:

$$\begin{aligned}
& \left| v_1^+(x, r) - \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right. \\
& \left. + v_2^+(x, r) - \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) dp \right| \leq I_1 + I_2,
\end{aligned} \tag{79}$$

$$I_1 = 2^\beta c_5 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp, \tag{80}$$

$$I_2 = c_6 \varepsilon \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} \left(1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|\right)^{-n} dp, \tag{81}$$

$$x \in D, \quad 1 \leq r \leq 2E^{\frac{1}{2}}, \quad E^{\frac{1}{2}} \geq \rho_2,$$

where ρ_2 is the same as in Proposition 2. In addition:

$$I_1 \leq c_7 \varepsilon^{-1} E^{\frac{1}{2}} r^{d+\beta}, \quad c_7 = |\mathbb{S}^{d-1}| \frac{2^{d+2\beta}}{d+\beta} c_5; \tag{82}$$

$$I_2 = c_6 \varepsilon \sum_{z \in \mathbb{Z}} \int_{\left\{ \begin{array}{l} \tau^2 + p_\perp^2 \leq r^2 \\ |\tau - \frac{\pi}{|y|} z| \leq \frac{\varepsilon}{|y|} \end{array} \right\}} \left(1 + \frac{\pi}{|y|} |z| + |p_\perp|\right)^{-n} d\tau dp_\perp,$$

where $\tau \in \mathbb{R}$, $p_\perp \in \mathbb{R}^d$, $p_\perp \cdot y = 0$,

$$\begin{aligned}
I_2 &\leq 2c_6 \frac{\varepsilon^2}{|y|} \sum_{z \in \mathbb{Z}} \int_{\xi \in \mathbb{R}^{d-1}, |\xi| \leq r} \left(1 + \frac{\pi}{|y|} |z| + |\xi|\right)^{-n} d\xi \leq c_6 c_8 \varepsilon^2, \\
c_8 &= \frac{2}{|y|} \frac{|\mathbb{S}^{d-2}|}{n-d+1} \sum_{z \in \mathbb{Z}} \left(1 + \frac{\pi}{|y|} |z|\right)^{d-n-1}.
\end{aligned} \tag{83}$$

In addition, if $r = r_2(E)$, $\varepsilon = \varepsilon_2(E)$, where $r_2(E)$, $\varepsilon_2(E)$ are defined in (45), then

$$\begin{aligned}
r^{d-n} &= (2\tau)^{d-n} E^{-\alpha_2}, \\
\varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} &= (2\tau)^{d+\beta} E^{-\alpha_2}, \\
\varepsilon^2 &= E^{-\alpha_2}.
\end{aligned} \tag{84}$$

Using representation (78), estimates (58), (79), (82), (83), formulas (84) and taking into account definitions (45), we obtain

$$\begin{aligned} |u(x, E) - v(x)| &\leq A_2 E^{-\alpha_2}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_2, \\ A_2 &= A_2(D_0, D_1, D_2, N_0, N_1, N_2, M, M_1, \dots, M_d, d, n, \beta, \tau, |y|) \\ &= (2\tau)^{d+\beta} c_7 + c_6 c_8 + (2\tau)^{d-n} c_3 \|v\|_{n,1}, \end{aligned} \quad (85)$$

where D_j, N_j are the same as in estimates (15) and $\|v\|_{n,1} \leq M, \|x_j v\|_{n,1} \leq M_j$. Theorem 2 is proved.

7 Proof of Theorem 3

Proposition 3. *Let v satisfy (2) and w_1, \dots, w_{d+1} be the same as in (24), (29), $d \geq 2$. Then:*

$$\begin{aligned} |\widehat{v}(p) - U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E)| &\leq c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad E^{\frac{1}{2}} \geq \rho_3, \quad 0 < \varepsilon < 1, \\ c_9 &= \frac{\pi\sqrt{d}}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + \max_{j=2, \dots, d+1} c(D_j)N_j^3) c_1, \\ \rho_3 &= \max_{j=0, \dots, d+1} \rho(D_j, N_j), \end{aligned} \quad (86)$$

in addition, if $v \in W^{n,1}(\mathbb{R}^d)$, $n \geq 0$, then:

$$\begin{aligned} &|\widehat{v}(p) - U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E)| \\ &\leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}} + 2c_6 \varepsilon (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad E^{\frac{1}{2}} \geq \rho_3, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}s\}, \end{aligned} \quad (87)$$

where $c, \rho, D_j, N_j, j = 0, \dots, d+1$, are defined as in (15); c_1, β are the same as in Lemma 1, c_6 is the same as in Proposition 2, $z(p)$ is defined in (48) and $\|z(p)\|_2$ is the standard Euclidean norm of $z(p)$.

Proof of Proposition 3. In a similar way with formulas (67), one can write

$$\begin{aligned} M_{\widehat{w}_1, \widehat{w}_{i'}}(p) &= \widehat{w}_1(p) \begin{pmatrix} 1 & 0 \\ \cos(sp_{i'-1}) & \sin(sp_{i'-1}) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus \frac{\pi}{s} Z^d, \\ M_{\widehat{w}_1, \widehat{w}_{i'}}^{-1}(p) &= \frac{1}{\sin(sp_{i'-1}) \widehat{w}_1(p)} \begin{pmatrix} \sin(sp_{i'-1}) & 0 \\ -\cos(sp_{i'-1}) & 1 \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \end{aligned} \quad (88)$$

where $i' = i'(p, s)$ is defined in (49). Also note that

$$|\sin(sp_{i'-1})| \geq \frac{2\varepsilon}{\pi\sqrt{d}}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad 0 < \varepsilon < 1. \quad (89)$$

Estimate (86) follows from (11), (15), (23), (35), (38), (39), (49), (50) and from (88), (89).

It remains to prove (87). Using definition (51), we represent

$$\begin{aligned}
U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) - \widehat{v}(p) &= \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon \\
\varphi_1^\varepsilon(p, E) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(\eta, E) - \widehat{v}(\eta)) \Big|_{\eta = \frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)} d\vartheta, \\
\varphi_2^\varepsilon(p) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (\widehat{v}(\frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)) - \widehat{v}(p)) d\vartheta,
\end{aligned} \tag{90}$$

where $z(p)$ is defined in (48).

Using formulas (86), (90), we obtain

$$\begin{aligned}
|\varphi_1^\varepsilon(p, E)| &\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (1 + |\frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)|)^\beta d\vartheta \\
&\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (1 + |p| + 2\frac{\varepsilon}{s})^\beta d\vartheta \leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\
&\text{for } \varepsilon \text{ as in (87)}.
\end{aligned} \tag{91}$$

Next, using the definition of φ_2^ε in formula (90) and the mean value theorem, we get the following estimate:

$$|\varphi_2^\varepsilon(p)| \leq 2\frac{\varepsilon}{s} \max\{|\nabla \widehat{v}(\xi)| : \xi \in \mathbb{R}^d, |\xi - \frac{\pi}{s}z(p)| \leq \frac{\varepsilon}{s}\}, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon. \tag{92}$$

Here, the mean value theorem was used for $\widehat{v}(\xi)$ on $[p, \frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)]$, $\vartheta \in \mathbb{S}^{d-1}$. One can see that

$$\begin{aligned}
&\text{estimates (72) and (73) hold for all } \xi \in \mathbb{R}^d \\
&\text{such that } |\xi - \frac{\pi}{s}z(p)| \leq \frac{\varepsilon}{s}, \text{ where } p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon.
\end{aligned} \tag{93}$$

It follows from (92), (93) and from the upper estimate on ε of (87), that

$$\begin{aligned}
|\varphi_2^\varepsilon(p)| &\leq 2^{1-n} c_6 \varepsilon \max\{(1 + |\xi|)^{-n} : |\xi - \frac{\pi}{s}z(p)| \leq \frac{\varepsilon}{s}\} \\
&\leq 2^{1-n} c_6 \varepsilon (1 + \frac{\pi}{s} \|z(p)\|_2 - \frac{\varepsilon}{s})^{-n} \\
&\leq 2c_6 \varepsilon (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n}, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon.
\end{aligned} \tag{94}$$

Estimate (87) follows from estimates (91) and (94).

Proposition 3 is proved. \square

The final part of the proof of Theorem 3 is as follows. In a similar way with (78), we represent v as follows:

$$\begin{aligned}
v(x) &= v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in D, \quad r > 0, \\
v_1^+(x, r) &= \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\
v_2^+(x, r) &= \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\
v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp.
\end{aligned} \tag{95}$$

Since v belongs to $W^{n,1}(\mathbb{R}^d)$, estimate (58) is valid.

Using estimates (86), (87) we obtain

$$\begin{aligned}
& \left| v_1^+(x, r) - \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) dp \right. \\
& \left. + v_2^+(x, r) - \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) dp \right| \leq J_1 + J_2, \\
& J_1 = 2^\beta c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp, \\
& J_2 = 2c_6 \varepsilon \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n} dp, \\
& x \in D, 1 \leq r \leq 2E^{\frac{1}{2}}, E^{\frac{1}{2}} \geq \rho_3,
\end{aligned} \tag{96}$$

where ρ_3 is the same as in Proposition 3. In addition,

$$\begin{aligned}
& J_1 \leq c_{10} \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta}, \quad c_{10} = |\mathbb{S}^{d-1}| \frac{2^{d+\beta}}{d+\beta} c_9, \\
& J_2 \leq c_{11} \varepsilon^{d+1}, \quad c_{11} = \left(\frac{1}{s}\right)^d |\mathcal{B}_1| \sum_{z \in \mathbb{Z}^d} (1 + 2\frac{\pi}{s} \|z\|_2)^{-n},
\end{aligned} \tag{97}$$

where $|\mathcal{B}_1|$ is the standard Euclidean volume of \mathcal{B}_1 . Finally, if $r = r_3(E)$, $\varepsilon = \varepsilon_3(E)$, where $r_3(E)$, $\varepsilon_3(E)$ are defined in (53), then

$$\begin{aligned}
& r^{d-n} = (2\tau)^{d-n} E^{-\alpha_3}, \\
& \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} = (2\tau)^{d+\beta} E^{-\alpha_3}, \\
& \varepsilon^{d+1} = E^{-\alpha_3}.
\end{aligned} \tag{98}$$

Using representation (95), estimates (58), (96), (97), formulas (98) and taking into account definitions (53), we obtain

$$\begin{aligned}
& |u(x, E) - v(x)| \leq A_3 E^{-\alpha_3}, \quad x \in D, E^{\frac{1}{2}} \geq \rho_3, \\
& A_3 = A_3(D_0, \dots, D_{d+1}, N_0, \dots, N_{d+1}, M, d, n, \beta, \tau, s) \\
& = (2\tau)^{d+\beta} c_{10} + c_{11} + (2\tau)^{d-n} c_3 \|v\|_{n,1},
\end{aligned} \tag{99}$$

where $\|v\|_{n,1} \leq M$ and $D_0, \dots, D_{d+1}, N_0, \dots, N_{d+1}$ are the same as in Proposition 3.

Theorem 3 is proved.

8 Proof of Lemma 1

Note that

$$\widehat{w}(p) = \int_{\mathbb{R}^d} |\widehat{q}(\xi)|^2 \widehat{\omega}_\nu(p - \xi) d\xi, \quad p \in \mathbb{R}^d, \tag{100}$$

$$\omega_\nu(x) = |x|^\nu K_\nu(|x|), \quad x \in \mathbb{R}^d, \tag{101}$$

where \widehat{q} , $\widehat{\omega}_\nu$ are the Fourier transforms of q , ω_ν . The Fourier transform $\widehat{\omega}_\nu$ can be computed explicitly:

$$\widehat{\omega}_\nu(p) = \frac{c_{12}}{(1 + |p|^2)^{\frac{d}{2} + \nu}}, \quad c_{12} = \frac{\Gamma(\frac{d}{2} + \nu)2^{\nu-1}}{\pi^{\frac{d}{2}}}. \quad (102)$$

Indeed, formula (102) follows from the Fourier inversion theorem and the following computations:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{e^{-ipx} dp}{(1 + |p|^2)^{\frac{d}{2} + \nu}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{e^{-i|x|t} dt d\xi}{(1 + t^2 + |\xi|^2)^{\frac{d}{2} + \nu}} \\ &= |\mathbb{S}^{d-2}| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i|x|t} r^{d-2} dt dr}{(1 + t^2 + r^2)^{\frac{d}{2} + \nu}} \\ &\stackrel{r=\sqrt{1+t^2}\tau}{=} |\mathbb{S}^{d-2}| \int_{\mathbb{R}} \frac{e^{-i|x|t} dt}{(1 + t^2)^{\frac{1}{2} + \nu}} \int_0^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} \\ &= c_{12}^{-1} |x|^\nu K_\nu(|x|), \quad x \in \mathbb{R}^d. \end{aligned}$$

Here, it was used that

$$\begin{aligned} \int_0^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} &= \frac{1}{2} B\left(\frac{d-1}{2}, \nu + \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{d-1}{2})\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{d}{2} + \nu)}, \\ |\mathbb{S}^{d-2}| &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}, \end{aligned}$$

where B and Γ denote the beta and gamma functions.

Using (100), (102), we obtain the estimates

$$\begin{aligned} \widehat{w}(p) &\geq \int_{|\xi| \leq 1} \frac{c_{12} |\widehat{q}(\xi)|^2}{(1 + |p - \xi|)^{d+2\nu}} d\xi \geq \frac{c_1(q, \nu)}{(1 + |p|)^{d+2\nu}}, \quad p \in \mathbb{R}^d, \\ c_1(q, \nu) &= \frac{c_{12}}{2^{d+2\nu}} \int_{|\xi| \leq 1} |\widehat{q}(\xi)|^2 d\xi. \end{aligned} \quad (103)$$

Properties (23) follow from (20), (22), (100) and (103).

Lemma 1 is proved.

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