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# Error estimates for phaseless inverse scattering in the Born approximation at high energies\*

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**Abstract.** We study explicit formulas for phaseless inverse scattering in the Born approximation at high energies for the Schrödinger equation with compactly supported potential in dimension  $d \geq 2$ . We obtain error estimates for these formulas in the configuration space.

## 1 Introduction

We consider the time-independent Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \quad (1)$$

where

$$v \in L^\infty(\mathbb{R}^d), \quad \text{supp } v \subset D, \quad (2)$$

where  $D$  is some fixed open bounded domain in  $\mathbb{R}^d$ .

In quantum mechanics equation (1) describes an elementary particle interacting with a macroscopic object contained in  $D$  at fixed energy  $E$ . In this setting one usually assumes that  $v$  is real-valued.

Equation (1) at fixed  $E$  can be also interpreted as the Helmholtz equation of acoustics or electrodynamics. In these frameworks the coefficient  $v$  can be complex-valued. In addition, the imaginary part of  $v$  is related to the absorption coefficient.

For equation (1) we consider the classical scattering solutions  $\psi^+ = \psi^+(x, k)$ , where  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ ,  $k = (k_1, \dots, k_d) \in \mathbb{R}^d$ ,  $k^2 = k_1^2 + \dots + k_d^2 = E$ . These solutions  $\psi^+$  can be specified by the following asymptotics as  $|x| \rightarrow \infty$ :

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + O(|x|^{-(d+1)/2}), \\ x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d, \quad k^2 = E, \quad kx &= k_1x_1 + \dots + k_dx_d, \\ c(d, |k|) &= -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \end{aligned} \quad (3)$$

for some a priori unknown  $f$ . The function  $f$  arising in (3) is defined on

$$\mathcal{M}_E = \{(k, l) \in \mathbb{R}^d \times \mathbb{R}^d : k^2 = l^2 = E\}, \quad (4)$$

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\*Dedicated to G. M. Henkin

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and is known as the classical scattering amplitude for equation (1).

In quantum mechanics  $|f(k, l)|^2$  describes the probability density of scattering of particle with initial momentum  $k$  into direction  $l/|l| \neq k/|k|$ , and is known as differential scattering cross section for equation (1); see, e.g., [11, Chapter 1, Section 6].

The problem of finding  $\psi^+$  and  $f$  from  $v$  is known as the direct scattering problem for equation (1). For solving this problem, one can use, in particular, the Lippmann-Schwinger integral equation for  $\psi^+$  and an explicit integral formula for  $f$ , see, e.g., [5, 10, 29].

In turn, the problem of finding  $v$  from  $f$  is known as the inverse scattering problem (with phase information) and the problem of finding  $v$  from  $|f|^2$  is known as the phaseless inverse scattering problem for equation (1).

There are many important results on the former inverse scattering problem with phase information; see [3, 4, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 28] and references therein. In particular, it is well known that the scattering amplitude  $f$  uniquely determines  $v$  via the Born approximation formulas at high energies:

$$\widehat{v}(k-l) = f(k, l) + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, \quad (k, l) \in \mathcal{M}_E, \quad (5)$$

$$\widehat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d, \quad (6)$$

and the inverse Fourier transform; see, e.g., [9, 28].

On the other hand, the literature for the phaseless case is much more limited; see [7, 29] and references therein for the case of the aforementioned phaseless inverse problem and see [19, 20, 21, 26, 27, 29] and references therein for the case of some similar inverse problems without phase information. In addition, it is well known that the phaseless scattering data  $|f|^2$  does not determine  $v$  uniquely, even if  $|f|^2$  is given completely for all positive energies. In particular, it is known that

$$\begin{aligned} f_y(k, l) &= e^{i(k-l)y} f(k, l), \\ |f_y(k, l)|^2 &= |f(k, l)|^2, \quad k, l \in \mathbb{R}^d, \quad k^2 = l^2 > 0, \end{aligned} \quad (7)$$

where  $f$  is the scattering amplitude for  $v$  and  $f_y$  is the scattering amplitude for  $v_y = v(\cdot - y)$ , where  $y \in \mathbb{R}^d$ ; see [29] and references therein.

In the present work, in view of the aforementioned non-uniqueness for the problem of finding  $v$  from  $|f|^2$ , we consider the modified phaseless inverse scattering problem formulated below as Problem 1. Let

$$S = \{|f|^2, |f_1|^2, \dots, |f_m|^2\}, \quad (8)$$

where  $f$  is the scattering amplitude for  $v$  and  $f_1, \dots, f_m$  are the scattering amplitudes for  $v_1, \dots, v_m$ , where

$$v_j = v + w_j, \quad j = 1, \dots, m, \quad (9)$$

where  $w_1, \dots, w_m$  are additional a priori known background scatterers such that

$$w_j \in L^\infty(\mathbb{R}^d), \quad \text{supp } w_j \subset \Omega_j, \quad (10)$$

$\Omega_j$  is an open bounded domain in  $\mathbb{R}^d$ ,  $\Omega_j \cap D = \emptyset$ ,

$$w_j \neq 0, \quad w_{j_1} \neq w_{j_2} \text{ if } j_1 \neq j_2 \text{ (in } L^\infty(\mathbb{R}^d)),$$

where  $j, j_1, j_2 \in \{1, \dots, m\}$ . Thus,  $S$  consists of the phaseless scattering data  $|f|^2, |f_1|^2, \dots, |f_m|^2$  measured sequentially, first, for the unknown scatterer  $v$  and then for  $v$  in the presence of known scatterer  $w_j$  disjoint from  $v$  for  $j = 1, \dots, m$ .

Actually, in the present work we continue studies of [29] on the following inverse scattering problem for equation (1):

**Problem 1.** *Reconstruct potential  $v$  from the phaseless scattering data  $S$  for some appropriate background scatterers  $w_1, \dots, w_m$ .*

Studies of Problem 1 in dimension  $d \geq 2$  were started in [29]. In dimension  $d = 1$  for  $m = 1$  studies of Problem 1 were started earlier in [2], where phaseless scattering data was considered for all  $E > 0$ .

Actually, the key result of [29] consists in a proper extension of formula (5) for the Fourier transform  $\widehat{v}$  of  $v$  to the phaseless case of Problem 1; see Section 2.

In the present work we proceed from the aforementioned result of [29] and study related approximate reconstruction of  $v$  in the configuration space. In this connection our results consist in obtaining related error estimates in the configuration space at high energies  $E$ ; see Section 4.

In addition, results of the present work are necessary for extending the iterative algorithm of [28] to the phaseless case of Problem 1. The latter extension will be given in [1].

## 2 Extension of formula (5) to the phaseless case

Actually, the key result of [29] consists in the following formulas for solving Problem 1 in dimension  $d \geq 2$  for  $m = 2$  at high energies  $E$ :

$$\begin{pmatrix} \text{Re } \widehat{v} \\ \text{Im } \widehat{v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \text{Re } \widehat{w}_1 & \text{Im } \widehat{w}_1 \\ \text{Re } \widehat{w}_2 & \text{Im } \widehat{w}_2 \end{pmatrix}^{-1} \begin{pmatrix} |\widehat{v}_1|^2 - |\widehat{v}|^2 - |\widehat{w}_1|^2 \\ |\widehat{v}_2|^2 - |\widehat{v}|^2 - |\widehat{w}_2|^2 \end{pmatrix}, \quad (11)$$

$$|\widehat{v}_j(p)|^2 = |f_j(k, l)|^2 + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, \quad (12)$$

$$p \in \mathbb{R}^d, \quad (k, l) \in \mathcal{M}_E, \quad k - l = p, \quad j = 0, 1, 2,$$

where:

- $v_0 = v$ ,  $v_j$  is defined by (9),  $j = 1, 2$ , and  $f_0 = f$ ,  $f_1, f_2$  are the scattering amplitudes for  $v_0, v_1, v_2$ , respectively;
- $\widehat{v} = \widehat{v}(p)$ ,  $\widehat{v}_j = \widehat{v}_j(p)$ ,  $\widehat{w}_j = \widehat{w}_j(p)$ ,  $p \in \mathbb{R}^d$ , are the Fourier transforms of  $v, v_j, w_j$  (defined as in (6));

- formula (11) is considered for all  $p \in \mathbb{R}^d$  such that the determinant

$$\zeta_{\widehat{w}_1, \widehat{w}_2}(p) \stackrel{def}{=} \operatorname{Re} \widehat{w}_1(p) \operatorname{Im} \widehat{w}_2(p) - \operatorname{Im} \widehat{w}_1(p) \operatorname{Re} \widehat{w}_2(p) \neq 0. \quad (13)$$

The point is that using formulas (12) for  $d \geq 2$  with

$$\begin{aligned} k &= k_E(p) = \frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \\ l &= l_E(p) = -\frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \\ |\gamma(p)| &= 1, \quad \gamma(p)p = 0, \end{aligned} \quad (14)$$

where  $p \in \mathbb{R}^d$ ,  $|p| \leq 2\sqrt{E}$ , one can reconstruct  $|\widehat{v}|^2$ ,  $|\widehat{v}_1|^2$ ,  $|\widehat{v}_2|^2$  from  $S$  at high energies for any  $p \in \mathbb{R}^d$ . And then using formula (11) one can reconstruct  $\widehat{v}$  completely, provided that condition (13) is fulfilled for almost all  $p \in \mathbb{R}^d$ .

**Remark 1.** Formulas (12) can be precised as formula (2.15) of [29]:

$$\begin{aligned} \left| |\widehat{v}_j(p)|^2 - |f_j(k, l)|^2 \right| &\leq c(D_j) N_j^3 E^{-\frac{1}{2}}, \\ p &= k - l, \quad (k, l) \in \mathcal{M}_E, \quad E^{\frac{1}{2}} \geq \rho(D_j, N_j), \quad j = 0, 1, 2, \end{aligned} \quad (15)$$

where  $\|v_j\|_{L^\infty(D_j)} \leq N_j$ ,  $j = 0, 1, 2$ , and  $D_0 = D$ ,  $D_j = D \cup \Omega_j$ ,  $j = 1, 2$ , and constants  $c, \rho$  are given by formulas (3.10) and (3.11) in [29] (and, in particular,  $\rho \geq 1$ ).

In addition, from the experimental point of view it seems to be, in particular, convenient to consider Problem 1 with  $m = 2$  for the case when  $w_2$  is just a translation of  $w_1$ :

$$w_2(x) = w_1(x - y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d. \quad (16)$$

In this case

$$\widehat{w}_2(p) = e^{ipy} \widehat{w}_1(p), \quad \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = \sin(py) |\widehat{w}_1(p)|^2, \quad p \in \mathbb{R}^d. \quad (17)$$

On the level of analysis, the principal complication of (11), (12) in comparison with (5) consists in possible zeros of the determinant  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  of (13). For some simplest cases, we study these zeros in the next section.

### 3 Zeros of the determinant $\zeta_{\widehat{w}_1, \widehat{w}_2}$

Let

$$\begin{aligned} Z_{\widehat{w}_1, \widehat{w}_2} &= \{p \in \mathbb{R}^d : \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = 0\}, \\ Z_{\widehat{w}_j} &= \{p \in \mathbb{R}^d : \widehat{w}_j(p) = 0\}, \quad j = 1, 2, \end{aligned} \quad (18)$$

where  $\zeta$  is defined by (13). From (13), (18) it follows that

$$Z_{\widehat{w}_1} \cup Z_{\widehat{w}_2} \subseteq Z_{\widehat{w}_1, \widehat{w}_2}. \quad (19)$$

In view of (19), in order to construct examples of  $w_1, w_2$  such that the set  $Z_{\widehat{w}_1, \widehat{w}_2}$  is as simple as possible, we use the following lemma:

**Lemma 1.** *Let*

$$w(x) = |x|^\nu K_\nu(|x|) \int_{\mathbb{R}^d} q(x-y)q(y) dy, \quad x \in \mathbb{R}^d, \nu > 0, \quad (20)$$

$$K_\nu(s) = \frac{\Gamma(\frac{1}{2} + \nu)}{\sqrt{\pi}} \left(\frac{2}{s}\right)^\nu \int_0^\infty \frac{\cos(st) dt}{(1+t^2)^{\frac{1}{2}+\nu}}, \quad s > 0, \quad (21)$$

$$\begin{aligned} q &\in L^\infty(\mathbb{R}^d), \quad q = \bar{q}, \quad q \neq 0 \text{ in } L^\infty(\mathbb{R}^d), \\ q(x) &= 0 \text{ if } |x| > r, \quad q(x) = q(-x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (22)$$

*Then*

$$\begin{aligned} w &\in C(\mathbb{R}^d), \quad w = \bar{w}, \quad w(x) = 0 \text{ if } |x| > 2r, \quad x \in \mathbb{R}^d, \\ \widehat{w}(p) &= \overline{\widehat{w}(p)} \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (23)$$

for  $\beta = d + 2\nu$  and some positive constant  $c_1 = c_1(q, \nu)$ , where  $\widehat{w}$  is the Fourier transform of  $w$ . In addition, if  $q \geq 0$ , then  $w \geq 0$ .

We recall that  $K_\nu$  defined by (21) is the modified Bessel function of the second kind and order  $\nu$ . In addition,  $\Gamma$  denotes the gamma function.

Lemma 1 is proved in Section 8.

As a corollary of Lemma 1, functions

$$w_j(x) = w(x - T_j), \quad x \in \mathbb{R}^d, \quad T_j \in \mathbb{R}^d, \quad (24)$$

where  $w$  is constructed in Lemma 1, give us examples of  $w_j$  satisfying (10) for fixed  $D$ ,  $\Omega_j$  and for appropriate radius  $r$  of Lemma 1 and translations  $T_j$  of (24), and such that

$$\begin{aligned} Z_{\widehat{w}_j} &= \emptyset, \\ |\widehat{w}_j(p)| &= \widehat{w}(p) \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (25)$$

where  $c_1, \beta$  are the same as in (23). In addition,

$$\begin{aligned} \zeta_{\widehat{w}_1, \widehat{w}_2}(p) &= \sin(py) |\widehat{w}(p)|^2, \quad y = T_2 - T_1 \neq 0, \quad p \in \mathbb{R}^d, \\ Z_{\widehat{w}_1, \widehat{w}_2} &= \{p \in \mathbb{R}^d : \sin(py) = 0\} = \{p \in \mathbb{R}^d : py \in \pi\mathbb{Z}\}, \end{aligned} \quad (26)$$

for  $w_1, w_2$  of (24).

As another corollary of Lemma 1, we have that

$$\begin{aligned} \text{if } w_1 &\text{ is defined as in (24) and } w_2 = iw_1, \text{ then} \\ \zeta_{\widehat{w}_1, \widehat{w}_2}(p) &= |\widehat{w}(p)|^2 \geq c_1^2(1+|p|)^{-2\beta}, \quad p \in \mathbb{R}^d, \\ Z_{\widehat{w}_1, \widehat{w}_2} &= \emptyset. \end{aligned} \quad (27)$$

We recall that complex-valued  $v$  and  $w_j$  naturally arise if we interpret equation (1) for fixed  $E$  as the Helmholtz equation of acoustics or electrodynamics.

Finally, note that

$$\begin{aligned} Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}} &= \frac{\pi}{s} \mathbb{Z}^d, \quad \text{where} \\ Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}} &= Z_{\widehat{w}_1, \widehat{w}_2} \cap Z_{\widehat{w}_1, \widehat{w}_3} \cap \dots \cap Z_{\widehat{w}_1, \widehat{w}_{d+1}}, \end{aligned} \quad (28)$$

if  $w_1$  is defined as in (24), and

$$w_2(x) = w_1(x - se_1), \dots, w_{d+1}(x) = w_1(x - se_d), \quad (29)$$

where  $(e_1, \dots, e_d)$  is the standard basis of  $\mathbb{R}^d$  and  $s > 0$ .

Thus, in principle, for Problem 1 with background scatterers  $w_1, \dots, w_{d+1}$  as in (29), for each  $p \in \mathbb{R}^d \setminus \frac{\pi}{s}\mathbb{Z}^d$  formulas (11), (12) can be used with appropriate  $w_j$  in place of  $w_2$ , where  $j = 2, \dots, d+1$ .

## 4 Error estimates in the configuration space

We recall that for inverse scattering with phase information the scattering amplitude  $f$  on  $\mathcal{M}_E$  processed by (5) and the inverse Fourier transform yield the approximate reconstruction

$$u(\cdot, E) = v + O(E^{-\alpha}) \quad \text{in } L^\infty(D) \text{ as } E \rightarrow +\infty, \quad \alpha = \frac{n-d}{2n}, \quad (30)$$

if  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n > d$  (in addition to the initial assumption (2)), where  $W^{n,1}(\mathbb{R}^d)$  denotes the standard Sobolev space of  $n$ -times differentiable functions in  $L^1(\mathbb{R}^d)$ :

$$\begin{aligned} W^{n,1}(\mathbb{R}^d) &= \{u \in L^1(\mathbb{R}^d) : \|u\|_{n,1} < \infty\}, \\ \|u\|_{n,1} &= \max_{|J| \leq n} \left\| \frac{\partial^{|J|} u}{\partial x^J} \right\|_{L^1(\mathbb{R}^d)}, \quad n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (31)$$

More precisely, the approximation  $u(\cdot, E)$  in (30) is defined by

$$\begin{aligned} u(x, E) &= \int_{\mathcal{B}_r(E)} e^{-ipx} f(k_E(p), l_E(p)) dp, \quad x \in D, \\ r(E) &= 2\tau E^{\frac{\alpha}{n-d}} \quad \text{for some fixed } \tau \in (0, 1], \end{aligned} \quad (32)$$

where

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : |p| \leq r\}, \quad (33)$$

$\alpha$  is defined in (30), and  $k_E(p), l_E(p)$  are defined as in (14) with some piecewise continuous vector-function  $\gamma$  on  $\mathbb{R}^d$ ; see, e.g., [28]. In addition, estimate (30) can be precised as

$$|u(x, E) - v(x)| \leq A(D, N, M, d, n, \tau) E^{-\alpha}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho(D, N), \quad (34)$$

where  $\|v\|_{L^\infty(D)} \leq N$ ,  $\|v\|_{n,1} \leq M$ ,  $\rho$  is the same as in (15) and the expression for  $A$  can be found in formula (3.10) of [28].

Analogs of  $u(\cdot, E)$  for the phaseless case are given below in this section. In particular, related formulas depend on the zeros of determinant  $\zeta_{\hat{w}_1, \hat{w}_2}$  of (13).

We consider

$$U_{\widehat{w}_1, \widehat{w}_2} = \operatorname{Re} U_{\widehat{w}_1, \widehat{w}_2} + i \operatorname{Im} U_{\widehat{w}_1, \widehat{w}_2},$$

$$\begin{pmatrix} \operatorname{Re} U_{\widehat{w}_1, \widehat{w}_2}(p, E) \\ \operatorname{Im} U_{\widehat{w}_1, \widehat{w}_2}(p, E) \end{pmatrix} = \frac{1}{2} M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) b_{\widehat{w}_1, \widehat{w}_2}(p, E), \quad (35)$$

$$M_{\widehat{w}_1, \widehat{w}_2}(p) = \begin{pmatrix} \operatorname{Re} \widehat{w}_1(p) & \operatorname{Im} \widehat{w}_1(p) \\ \operatorname{Re} \widehat{w}_2(p) & \operatorname{Im} \widehat{w}_2(p) \end{pmatrix}, \quad (36)$$

$$M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) = \frac{1}{\zeta_{\widehat{w}_1, \widehat{w}_2}(p)} \begin{pmatrix} \operatorname{Im} \widehat{w}_2(p) & -\operatorname{Im} \widehat{w}_1(p) \\ -\operatorname{Re} \widehat{w}_2(p) & \operatorname{Re} \widehat{w}_1(p) \end{pmatrix}, \quad (37)$$

$$b_{\widehat{w}_1, \widehat{w}_2}(p, E) = \begin{pmatrix} |f_1(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_1(p)|^2 \\ |f_2(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_2(p)|^2 \end{pmatrix}, \quad (38)$$

$$f(p, E) = f(k_E(p), l_E(p)), \quad f_j(p, E) = f_j(k_E(p), l_E(p)), \quad j = 1, 2, \quad (39)$$

where  $\widehat{w}_1, \widehat{w}_2, f, f_1, f_2$  are the same as in (11), (12),  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  is defined by (13),  $k_E(p), l_E(p)$  are the same as in (14), (32), and  $p \in \mathcal{B}_{2\sqrt{E}}, d \geq 2$ .

For Problem 1 for  $d \geq 2, m = 2$ , and for the case when  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  has no zeros (the case of (27) in Section 3) we have the following result:

**Theorem 1.** *Let  $v$  satisfy (2) and  $v \in W^{n,1}(\mathbb{R}^d)$  for some  $n > d$ . Let  $w_1, w_2$  be the same as in (27). Let*

$$u(x, E) = \int_{\mathcal{B}_{r_1(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp, \quad x \in D, \quad (40)$$

$$r_1(E) = 2\tau E^{\frac{\alpha_1}{n-d}}, \quad \alpha_1 = \frac{n-d}{2(n+\beta)}, \quad \text{for some fixed } \tau \in (0, 1],$$

where  $U_{\widehat{w}_1, \widehat{w}_2}$  is defined by (35),  $\mathcal{B}_r$  is defined by (33),  $\beta$  is the number of (23), (27). Then

$$\begin{aligned} u(\cdot, E) &= v + O(E^{-\alpha_1}) \quad \text{in } L^\infty(D), \quad E \rightarrow +\infty, \\ |u(x, E) - v(x)| &\leq A_1 E^{-\alpha_1}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_1, \end{aligned} \quad (41)$$

where  $\rho_1$  and  $A_1$  are defined in formulas (55) and (64) of Section 5.

Theorem 1 is proved in Section 5.

Next, we set

$$Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon = \{p \in \mathbb{R}^d : py \in (-\varepsilon, \varepsilon) + \pi\mathbb{Z}\}, \quad y \in \mathbb{R}^d \setminus 0, \quad 0 < \varepsilon < 1, \quad (42)$$

where  $\widehat{w}_1, \widehat{w}_2$  and  $y$  are the same as in (24)–(26). One can see that  $Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon$  is the open  $\frac{\varepsilon}{|y|}$ -neighborhood of  $Z_{\widehat{w}_1, \widehat{w}_2}$  defined in (26).

Note that

$$\begin{aligned} &\text{for any } p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon \text{ there exists} \\ &\text{the unique } z(p) \in \mathbb{Z} \text{ such that } |py - \pi z(p)| < \varepsilon. \end{aligned} \quad (43)$$



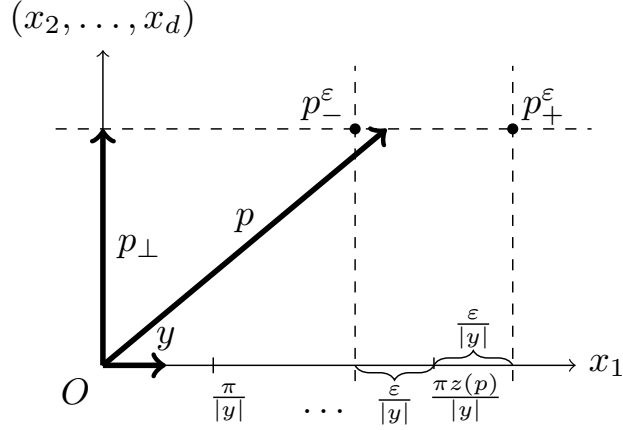


Figure 1: Vectors  $p$ ,  $p_{\perp}$ ,  $y$  and  $p_{\pm}^{\varepsilon}$  of formula (44)

In addition to  $U_{\widehat{w}_1, \widehat{w}_2}$  of (35), we define

$$U_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}(p, E) = \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}(p_-^{\varepsilon}, E) + U_{\widehat{w}_1, \widehat{w}_2}(p_+^{\varepsilon}, E)), \quad (44)$$

$$p_{\pm}^{\varepsilon} = p_{\perp} + \pi z(p) \frac{y}{|y|^2} \pm \varepsilon \frac{y}{|y|^2}, \quad p_{\perp} = p - (py) \frac{y}{|y|^2}, \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon},$$

where  $z(p)$  is the integer number of (43). The geometry of vectors  $p$ ,  $p_{\perp}$ ,  $y$ ,  $p_{\pm}^{\varepsilon}$  is illustrated in Fig. 1 for the case when the direction of  $y$  coincides with the basis vector  $e_1 = (1, 0, \dots, 0)$ .

For Problem 1 for  $d \geq 2$ ,  $m = 2$ , and for the case when  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  has zeros on hyperplanes (the case of (26) in Section 3) we have the following result:

**Theorem 2.** *Let  $v$  satisfy (2) and  $v \in W^{n,1}(\mathbb{R}^d)$  for some  $n > d$ . Let  $w_1, w_2$  be the same as in (24)–(26). Let*

$$u(x, E) = u_1(x, E) + u_2(x, E), \quad x \in D,$$

$$u_1(x, E) = \int_{\mathcal{B}_{r_2(E)} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp,$$

$$u_2(x, E) = \int_{\mathcal{B}_{r_2(E)} \cap Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}(p, E) dp, \quad (45)$$

$$r_2(E) = 2\tau E^{\frac{\alpha_2}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_2}{2}},$$

$$\alpha_2 = \frac{n-d}{2(n+\beta+\frac{n-d}{2})}, \quad \text{for some fixed } \tau \in (0, 1],$$

where  $U_{\widehat{w}_1, \widehat{w}_2}$  and  $U_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}$  are defined by (35), (44),  $\mathcal{B}_r$  and  $Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon}$  are defined

by (33), (42), and  $\beta$  is the number of (23). Then

$$\begin{aligned} u(\cdot, E) &= v + O(E^{-\alpha_2}) \quad \text{in } L^\infty(D), \quad E \rightarrow +\infty, \\ |u(x, E) - v(x)| &\leq A_2 E^{-\alpha_2}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_2, \end{aligned} \quad (46)$$

where  $\rho_2$  and  $A_2$  are defined in formulas (65) and (85) of Section 6.

Theorem 2 is proved in Section 6.

Next, we set

$$\begin{aligned} Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon &= \mathcal{B}'_{\varepsilon/s} + \frac{\pi}{s} \mathbb{Z}^d, \quad 0 < \varepsilon < 1, \\ \mathcal{B}'_r &= \mathcal{B}_r \setminus \partial \mathcal{B}_r, \quad r > 0, \end{aligned} \quad (47)$$

where  $\widehat{w}_1, \dots, \widehat{w}_{d+1}$  are the same as in (28), (29), and  $\mathcal{B}_r$  is defined by (33). One can see that  $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$  is the open  $\frac{\varepsilon}{s}$ -neighborhood of  $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}$  defined in (28).

Note that

$$\begin{aligned} &\text{for any } p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon \text{ there exists} \\ &\text{the unique } z(p) \in \mathbb{Z}^d \text{ such that } |sp - \pi z(p)| < \varepsilon. \end{aligned} \quad (48)$$

In addition, we consider  $i'$  such that

$$\begin{aligned} i' &= i'(p, s), \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d \setminus \frac{\pi}{s} \mathbb{Z}^d, \quad s > 0, \\ i' &\text{ take values in } \{2, \dots, d+1\}, \\ |\sin(sp_{i'-1})| &\geq |\sin(sp_{i-1})| \quad \text{for all } i \in \{2, \dots, d+1\}. \end{aligned} \quad (49)$$

Let

$$\begin{aligned} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) &= U_{\widehat{w}_1, \widehat{w}_{i'}}(p, E), \quad p \in \mathbb{R}^d \setminus \frac{\pi}{s} \mathbb{Z}^d, \\ U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) &= \end{aligned} \quad (50)$$

$$\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}\left(\frac{\varepsilon}{s} \vartheta + \frac{\pi}{s} z(p), E\right) d\vartheta, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad (51)$$

$$\mathbb{S}^{d-1} = \{p \in \mathbb{R}^d : |p| = 1\}, \quad (52)$$

where  $|\mathbb{S}^{d-1}|$  denotes the standard Euclidean volume of  $\mathbb{S}^{d-1}$ ,  $U_{\widehat{w}_1, \widehat{w}_{i'}}$  is defined in a similar way with  $U_{\widehat{w}_1, \widehat{w}_2}$  of (35),  $\widehat{w}_1, \dots, \widehat{w}_{d+1}$  are the same as in (28), (29), and  $i'$  is the same as in (49).

For Problem 1 for  $d \geq 2$ ,  $m = d + 1$ , we also have the following result:

**Theorem 3.** *Let  $v$  satisfy (2) and  $v \in W^{n,1}(\mathbb{R}^d)$  for some  $n > d$ . Let  $w_1, \dots,$*

$w_{d+1}$  be the same as in (29). Let

$$\begin{aligned}
u(x, E) &= u_1(x, E) + u_2(x, E), \quad x \in D, \\
u_1(x, E) &= \int_{\mathcal{B}_{r_3(E)} \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^{\varepsilon_3(E)}} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) dp, \\
u_2(x, E) &= \int_{\mathcal{B}_{r_3(E)} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^{\varepsilon_3(E)}} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) dp, \\
r_3(E) &= 2\tau E^{\frac{\alpha_3}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_3}{d+1}}, \\
\alpha_3 &= \frac{n-d}{2(n+\beta+\frac{d}{d+1})}, \quad \text{for some fixed } \tau \in (0, 1],
\end{aligned} \tag{53}$$

where  $U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}$  and  $U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$  are defined by (50), (51),  $\mathcal{B}_r$  and  $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$  are defined by (33), (47), and  $\beta$  is the number of (23). Then

$$\begin{aligned}
u(\cdot, E) &= v + O(E^{-\alpha_3}) \quad \text{in } L^\infty(D), \quad E \rightarrow +\infty, \\
|u(x, E) - v(x)| &\leq A_3 E^{-\alpha_3}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_3,
\end{aligned} \tag{54}$$

where  $\rho_3$  and  $A_3$  are defined in formulas (86) and (99) of Section 7.

Theorem 3 is proved in Section 7.

## 5 Proof of Theorem 1

**Proposition 1.** Let  $v$  satisfy (2) and  $w_1, w_2$  be the same as in (27),  $d \geq 2$ . Then:

$$\begin{aligned}
|\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}(p, E)| &\leq c_2 |\widehat{w}(p)|^{-1} E^{-\frac{1}{2}} \quad \text{for } p \in \mathcal{B}_{2\sqrt{E}}, \quad E^{\frac{1}{2}} \geq \rho_1, \\
c_2 &= 2c(D_0)N_0^3 + 2c(D_1)N_1^3, \\
\rho_1 &= \max_{j=0,1} \rho(D_j, N_j),
\end{aligned} \tag{55}$$

where  $w$  is the function of (23), (24), and  $c, \rho, N_j, D_j, j = 0, 1, 2$ , are the same as in estimates (15).

Proposition 1 follows from formulas (11), (14), estimates (15), definitions (35)–(39), and the properties that

$$\Omega_2 = \Omega_1, \quad D_2 = D_1, \quad N_2 = N_1. \tag{56}$$

In turn, properties (56) follow from (9) for  $j = 1, 2$ , (10) for  $j = 1$ , and from the equality  $w_2 = iw_1$  assumed in (27).

Next, we represent  $v$  as follows:

$$\begin{aligned} v(x) &= v^+(x, r) + v^-(x, r), \quad x \in D, \quad r > 0, \\ v^+(x, r) &= \int_{\mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp, \\ v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp. \end{aligned} \quad (57)$$

Since  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n > d$ , we have

$$\begin{aligned} |v^-(x, r)| &\leq c_3 \|v\|_{n,1} r^{d-n}, \quad x \in D, \quad r > 0, \\ c_3 &= |\mathbb{S}^{d-1}| \frac{(2\pi)^{-d} d^n}{n-d}, \end{aligned} \quad (58)$$

where  $\|\cdot\|_{n,1}$  is defined in (31), and  $|\mathbb{S}^{d-1}|$  is the standard Euclidean volume of  $\mathbb{S}^{d-1}$ . Indeed,

$$\begin{aligned} |p_1^{k_1} \cdots p_d^{k_d} \widehat{v}(p)| &\leq (2\pi)^{-d} \|v\|_{n,1}, \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d, \\ &\text{for any } k_1, \dots, k_d \in \mathbb{N} \cup \{0\}, \quad k_1 + \cdots + k_d \leq n, \end{aligned} \quad (59)$$

assuming also that  $p_j^0 = 1$ . Taking an appropriate sum in (59) over all such  $k_1, \dots, k_d$  with  $k_1 + \cdots + k_d = m \leq n$ , we get

$$|p|^m |\widehat{v}(p)| \leq (|p_1| + \cdots + |p_d|)^m |\widehat{v}(p)| \leq (2\pi)^{-d} d^m \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \quad (60)$$

The definition of  $v^-$  of (57) and inequalities (60) for  $m = n$  imply (58).

In addition, using Proposition 1 and the estimate on  $\widehat{w}$  of (23), we obtain:

$$\begin{aligned} \left| v^+(x, r) - \int_{\mathcal{B}_r} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right| &\leq c_2 E^{-\frac{1}{2}} \int_{\mathcal{B}_r} |\widehat{w}(p)|^{-1} dp \\ &\leq c_1^{-1} c_2 E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp \leq c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad c_4 = |\mathbb{S}^{d-1}| \frac{2^{d+\beta}}{d+\beta} c_1^{-1} c_2, \\ &x \in D, \quad 1 \leq r \leq 2E^{\frac{1}{2}}, \quad E^{\frac{1}{2}} \geq \rho_1. \end{aligned} \quad (61)$$

As a corollary of (57), (58), (61), we have

$$\left| v(x) - \int_{\mathcal{B}_r} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right| \leq c_3 \|v\|_{n,1} r^{d-n} + c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad (62)$$

where  $x \in D$ ,  $1 \leq r \leq 2E^{\frac{1}{2}}$ ,  $E^{\frac{1}{2}} \geq \rho_1$ . In addition, if  $r = r_1(E)$ , where  $r_1(E)$  is defined in (40), then

$$\begin{aligned} r^{d-n} &= (2\tau)^{d-n} E^{-\alpha_1}, \\ E^{-\frac{1}{2}} r^{d+\beta} &= (2\tau)^{d+\beta} E^{-\alpha_1}. \end{aligned} \quad (63)$$

Using formulas (62) and (63) and taking into account definitions (40), we obtain

$$\begin{aligned} |u(x, E) - v(x)| &\leq A_1 E^{-\alpha_1}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_1 \\ A_1 &= A_1(D_0, D_1, N_0, N_1, M, d, n, \beta, \tau) \\ &= (2\tau)^{d-n} c_3 \|v\|_{n,1} + (2\tau)^{d+\beta} c_4, \end{aligned} \quad (64)$$

where  $D_j, N_j, j = 0, 1$ , are the same as in estimates (15) and  $\|v\|_{n,1} \leq M$ .  
Theorem 1 is proved.

## 6 Proof of Theorem 2

**Proposition 2.** *Let  $v$  satisfy (2) and  $w_1, w_2$  be the same as in (24)–(26),  $d \geq 2$ . Then:*

$$\begin{aligned} |\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}(p, E)| &\leq c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad E^{\frac{1}{2}} \geq \rho_2, \quad 0 < \varepsilon < 1, \\ c_5 &= \frac{\pi}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + c(D_2)N_2^3)c_1, \\ \rho_2 &= \max_{j=0,1,2} \rho(D_j, N_j), \end{aligned} \quad (65)$$

in addition, if  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n \geq 0$ , then:

$$\begin{aligned} &|\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E)| \\ &\leq 2^\beta c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}} + c_6 \varepsilon (1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|)^{-n}, \\ &p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad E \geq \rho_2, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}|y|\}; \\ &c_6 = 2^n \frac{(d+1)^{n+1}}{(2\pi)^d |y|} \max_{j=1, \dots, d} \|x_j v\|_{n,1}, \end{aligned} \quad (66)$$

where  $c, \rho, N_j, D_j, j = 0, 1, 2$ , are the same as in estimates (15),  $c_1, \beta$  are the same as in Lemma 1;  $z(p), p_\perp$  are defined in (43), (44),  $x_j v = x_j v(x)$ , and  $\|\cdot\|_{n,1}$  is defined in (31).

*Proof of Proposition 2.* It follows from formulas (24), (26) and (36), (37) that

$$\begin{aligned} M_{\widehat{w}_1, \widehat{w}_2}(p) &= \widehat{w}(p) \begin{pmatrix} \cos(T_1 p) & \sin(T_1 p) \\ \cos(T_2 p) & \sin(T_2 p) \end{pmatrix}, \quad p \in \mathbb{R}^d, \\ M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) &= \frac{1}{\sin(py) \widehat{w}(p)} \begin{pmatrix} \sin(T_2 p) & -\sin(T_1 p) \\ -\cos(T_2 p) & \cos(T_1 p) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (67)$$

Also note that

$$|\sin(py)| \geq \frac{2\varepsilon}{\pi}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad 0 < \varepsilon < 1. \quad (68)$$

The estimate (65) follows from (11), (15), (23), (35), (38), (39) and from (67), (68).

It remains to prove (66). Using definition (44), one can write

$$\begin{aligned} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) - \widehat{v}(p) &= \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\ \varphi_1^\varepsilon(p, E) &= \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p_-, E) - \widehat{v}(p_-)) \\ &+ \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p_+, E) - \widehat{v}(p_+)), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\ \varphi_2^\varepsilon(p) &= \frac{1}{2} (\widehat{v}(p_-) + \widehat{v}(p_+)) - \widehat{v}(p), \quad p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (69)$$

Using estimate (65), formula (69) and the definitions of  $p_{\pm}^{\varepsilon}$  in (44), we get

$$\begin{aligned} |\varphi_1^{\varepsilon}(p, E)| &\leq \frac{1}{2}c_5\varepsilon^{-1}E^{-\frac{1}{2}}((1 + |p_-^{\varepsilon}|)^{\beta} + (1 + |p_+^{\varepsilon}|)^{\beta}) \\ &\leq c_5\varepsilon^{-1}(1 + |p| + 2\frac{\varepsilon}{|y|})^{\beta}E^{-\frac{1}{2}} \leq 2^{\beta}c_5\varepsilon^{-1}(1 + |p|)^{\beta}E^{-\frac{1}{2}}, \\ &\text{for } \varepsilon \text{ as in (66), } p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}. \end{aligned} \quad (70)$$

Next, using the definition of  $\varphi_2^{\varepsilon}$  in (69) and the mean value theorem, we obtain

$$|\varphi_2^{\varepsilon}(p)| \leq \frac{\varepsilon}{|y|} \max\{|\frac{y}{|y|}\nabla\hat{v}(\xi)| : \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}]\}, \quad p \in Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}, \quad (71)$$

where  $[p_-^{\varepsilon}, p_+^{\varepsilon}]$  denotes the segment joining  $p_-^{\varepsilon}$  to  $p_+^{\varepsilon}$ . Here, the mean value theorem was used for  $\hat{v}(\xi)$  on  $[p_-^{\varepsilon}, p]$  and on  $[p, p_+^{\varepsilon}]$ .

Note also that

$$|\nabla\hat{v}(\xi)| \leq d \max_{j=1, \dots, d} \left| \frac{\partial\hat{v}}{\partial\xi_j}(\xi) \right|, \quad \xi = (\xi_1, \dots, \xi_d) \in [p_-^{\varepsilon}, p_+^{\varepsilon}]. \quad (72)$$

In addition, the following estimates hold:

$$\left| \frac{\partial\hat{v}}{\partial\xi_j}(\xi) \right| \leq \frac{(1+d)^n}{(2\pi)^d(1+|\xi|)^n} \|x_j v\|_{n,1}, \quad \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}], \quad j = 1, \dots, d. \quad (73)$$

Indeed, taking the sum in (60) over all  $m = 0, \dots, n$  with the binomial coefficients, we get

$$\begin{aligned} (1 + |p|)^n |\hat{v}(p)| &\leq (1 + |p_1| + \dots + |p_d|)^n |\hat{v}(p)| \\ &\leq (2\pi)^{-d} (1+d)^n \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \end{aligned} \quad (74)$$

Estimates (73) follow from (74), where we replace  $v$  by  $x_j v$  and use that  $v$  belongs to  $W^{n,1}(\mathbb{R}^d)$  and is compactly supported.

Estimates (71)–(73) imply

$$|\varphi_2^{\varepsilon}(p)| \leq 2^{-n} c_6 \varepsilon \max\{(1 + |\xi|)^{-n} : \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}]\}, \quad p \in Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}. \quad (75)$$

Using also that

$$\xi = \tau \frac{y}{|y|} + p_{\perp}, \quad \text{where } |\tau - \frac{\pi}{|y|} z(p)| \leq \frac{\varepsilon}{|y|}, \quad \text{if } \xi \in [p_-^{\varepsilon}, p_+^{\varepsilon}], \quad (76)$$

and that  $\varepsilon < |y|$ , we obtain

$$\begin{aligned} |\varphi_2^{\varepsilon}(p)| &\leq 2^{-n} c_6 \varepsilon \left(1 + \frac{1}{2} \left(\frac{\pi}{|y|} |z(p)| - \frac{\varepsilon}{|y|} + |p_{\perp}| \right)\right)^{-n} \\ &\leq c_6 \left(1 + \frac{\pi}{|y|} |z(p)| + |p_{\perp}| \right)^{-n}, \quad p \in Z_{\hat{w}_1, \hat{w}_2}^{\varepsilon}. \end{aligned} \quad (77)$$

Estimate (66) follows from (70) and (77).

Proposition 2 is proved.  $\square$

The final part of the proof of Theorem 2 is as follows. In a similar way with (57), we represent  $v$  as follows:

$$\begin{aligned}
v(x) &= v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in D, \quad r > 0, \\
v_1^+(x, r) &= \int_{\mathcal{B}_r \setminus Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} e^{-ipx} \hat{v}(p) dp, \\
v_2^+(x, r) &= \int_{\mathcal{B}_r \cap Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} e^{-ipx} \hat{v}(p) dp, \\
v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \hat{v}(p) dp.
\end{aligned} \tag{78}$$

Since  $v \in W^{n,1}(\mathbb{R}^d)$ , estimate (58) holds.

Using estimates (65), (66), we get:

$$\begin{aligned}
& \left| v_1^+(x, r) - \int_{\mathcal{B}_r \setminus Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} e^{-ipx} U_{\hat{w}_1, \hat{w}_2}(p, E) dp \right. \\
& \left. + v_2^+(x, r) - \int_{\mathcal{B}_r \cap Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} e^{-ipx} U_{\hat{w}_1, \hat{w}_2}^\varepsilon(p, E) dp \right| \leq I_1 + I_2,
\end{aligned} \tag{79}$$

$$I_1 = 2^\beta c_5 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp, \tag{80}$$

$$I_2 = c_6 \varepsilon \int_{\mathcal{B}_r \cap Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} \left(1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|\right)^{-n} dp, \tag{81}$$

$$x \in D, \quad 1 \leq r \leq 2E^{\frac{1}{2}}, \quad E^{\frac{1}{2}} \geq \rho_2,$$

where  $\rho_2$  is the same as in Proposition 2. In addition:

$$I_1 \leq c_7 \varepsilon^{-1} E^{\frac{1}{2}} r^{d+\beta}, \quad c_7 = |\mathbb{S}^{d-1}| \frac{2^{d+2\beta}}{d+\beta} c_5; \tag{82}$$

$$I_2 = c_6 \varepsilon \sum_{z \in \mathbb{Z}} \int_{\left\{ \begin{array}{l} \tau^2 + p_\perp^2 \leq r^2 \\ |\tau - \frac{\pi}{|y|} z| \leq \frac{\varepsilon}{|y|} \end{array} \right\}} \left(1 + \frac{\pi}{|y|} |z| + |p_\perp|\right)^{-n} d\tau dp_\perp,$$

where  $\tau \in \mathbb{R}$ ,  $p_\perp \in \mathbb{R}^d$ ,  $p_\perp \cdot y = 0$ ,

$$\begin{aligned}
I_2 &\leq 2c_6 \frac{\varepsilon^2}{|y|} \sum_{z \in \mathbb{Z}} \int_{\xi \in \mathbb{R}^{d-1}, |\xi| \leq r} \left(1 + \frac{\pi}{|y|} |z| + |\xi|\right)^{-n} d\xi \leq c_6 c_8 \varepsilon^2, \\
c_8 &= \frac{2}{|y|} \frac{|\mathbb{S}^{d-2}|}{n-d+1} \sum_{z \in \mathbb{Z}} \left(1 + \frac{\pi}{|y|} |z|\right)^{d-n-1}.
\end{aligned} \tag{83}$$

In addition, if  $r = r_2(E)$ ,  $\varepsilon = \varepsilon_2(E)$ , where  $r_2(E)$ ,  $\varepsilon_2(E)$  are defined in (45), then

$$\begin{aligned}
r^{d-n} &= (2\tau)^{d-n} E^{-\alpha_2}, \\
\varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} &= (2\tau)^{d+\beta} E^{-\alpha_2}, \\
\varepsilon^2 &= E^{-\alpha_2}.
\end{aligned} \tag{84}$$

Using representation (78), estimates (58), (79), (82), (83), formulas (84) and taking into account definitions (45), we obtain

$$\begin{aligned} |u(x, E) - v(x)| &\leq A_2 E^{-\alpha_2}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_2, \\ A_2 &= A_2(D_0, D_1, D_2, N_0, N_1, N_2, M, M_1, \dots, M_d, d, n, \beta, \tau, |y|) \\ &= (2\tau)^{d+\beta} c_7 + c_6 c_8 + (2\tau)^{d-n} c_3 \|v\|_{n,1}, \end{aligned} \quad (85)$$

where  $D_j, N_j$  are the same as in estimates (15) and  $\|v\|_{n,1} \leq M, \|x_j v\|_{n,1} \leq M_j$ . Theorem 2 is proved.

## 7 Proof of Theorem 3

**Proposition 3.** *Let  $v$  satisfy (2) and  $w_1, \dots, w_{d+1}$  be the same as in (24), (29),  $d \geq 2$ . Then:*

$$\begin{aligned} |\widehat{v}(p) - U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E)| &\leq c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad E^{\frac{1}{2}} \geq \rho_3, \quad 0 < \varepsilon < 1, \\ c_9 &= \frac{\pi\sqrt{d}}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + \max_{j=2, \dots, d+1} c(D_j)N_j^3) c_1, \\ \rho_3 &= \max_{j=0, \dots, d+1} \rho(D_j, N_j), \end{aligned} \quad (86)$$

in addition, if  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n \geq 0$ , then:

$$\begin{aligned} &|\widehat{v}(p) - U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E)| \\ &\leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}} + 2c_6 \varepsilon (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad E^{\frac{1}{2}} \geq \rho_3, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}s\}, \end{aligned} \quad (87)$$

where  $c, \rho, D_j, N_j, j = 0, \dots, d+1$ , are defined as in (15);  $c_1, \beta$  are the same as in Lemma 1,  $c_6$  is the same as in Proposition 2,  $z(p)$  is defined in (48) and  $\|z(p)\|_2$  is the standard Euclidean norm of  $z(p)$ .

*Proof of Proposition 3.* In a similar way with formulas (67), one can write

$$\begin{aligned} M_{\widehat{w}_1, \widehat{w}_{i'}}(p) &= \widehat{w}_1(p) \begin{pmatrix} 1 & 0 \\ \cos(sp_{i'-1}) & \sin(sp_{i'-1}) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus \frac{\pi}{s} Z^d, \\ M_{\widehat{w}_1, \widehat{w}_{i'}}^{-1}(p) &= \frac{1}{\sin(sp_{i'-1}) \widehat{w}_1(p)} \begin{pmatrix} \sin(sp_{i'-1}) & 0 \\ -\cos(sp_{i'-1}) & 1 \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \end{aligned} \quad (88)$$

where  $i' = i'(p, s)$  is defined in (49). Also note that

$$|\sin(sp_{i'-1})| \geq \frac{2\varepsilon}{\pi\sqrt{d}}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad 0 < \varepsilon < 1. \quad (89)$$

Estimate (86) follows from (11), (15), (23), (35), (38), (39), (49), (50) and from (88), (89).



It remains to prove (87). Using definition (51), we represent

$$\begin{aligned}
U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) - \widehat{v}(p) &= \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon \\
\varphi_1^\varepsilon(p, E) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(\eta, E) - \widehat{v}(\eta)) \Big|_{\eta = \frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)} d\vartheta, \\
\varphi_2^\varepsilon(p) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (\widehat{v}(\frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)) - \widehat{v}(p)) d\vartheta,
\end{aligned} \tag{90}$$

where  $z(p)$  is defined in (48).

Using formulas (86), (90), we obtain

$$\begin{aligned}
|\varphi_1^\varepsilon(p, E)| &\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (1 + |\frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)|)^\beta d\vartheta \\
&\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (1 + |p| + 2\frac{\varepsilon}{s})^\beta d\vartheta \leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\
&\text{for } \varepsilon \text{ as in (87)}.
\end{aligned} \tag{91}$$

Next, using the definition of  $\varphi_2^\varepsilon$  in formula (90) and the mean value theorem, we get the following estimate:

$$|\varphi_2^\varepsilon(p)| \leq 2\frac{\varepsilon}{s} \max\{|\nabla \widehat{v}(\xi)| : \xi \in \mathbb{R}^d, |\xi - \frac{\pi}{s}z(p)| \leq \frac{\varepsilon}{s}\}, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon. \tag{92}$$

Here, the mean value theorem was used for  $\widehat{v}(\xi)$  on  $[p, \frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p)]$ ,  $\vartheta \in \mathbb{S}^{d-1}$ . One can see that

$$\begin{aligned}
&\text{estimates (72) and (73) hold for all } \xi \in \mathbb{R}^d \\
&\text{such that } |\xi - \frac{\pi}{s}z(p)| \leq \frac{\varepsilon}{s}, \text{ where } p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon.
\end{aligned} \tag{93}$$

It follows from (92), (93) and from the upper estimate on  $\varepsilon$  of (87), that

$$\begin{aligned}
|\varphi_2^\varepsilon(p)| &\leq 2^{1-n} c_6 \varepsilon \max\{(1 + |\xi|)^{-n} : |\xi - \frac{\pi}{s}z(p)| \leq \frac{\varepsilon}{s}\} \\
&\leq 2^{1-n} c_6 \varepsilon (1 + \frac{\pi}{s} \|z(p)\|_2 - \frac{\varepsilon}{s})^{-n} \\
&\leq 2c_6 \varepsilon (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n}, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon.
\end{aligned} \tag{94}$$

Estimate (87) follows from estimates (91) and (94).

Proposition 3 is proved.  $\square$

The final part of the proof of Theorem 3 is as follows. In a similar way with (78), we represent  $v$  as follows:

$$\begin{aligned}
v(x) &= v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in D, \quad r > 0, \\
v_1^+(x, r) &= \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\
v_2^+(x, r) &= \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\
v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp.
\end{aligned} \tag{95}$$

Since  $v$  belongs to  $W^{n,1}(\mathbb{R}^d)$ , estimate (58) is valid.

Using estimates (86), (87) we obtain

$$\begin{aligned}
& \left| v_1^+(x, r) - \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) dp \right. \\
& \left. + v_2^+(x, r) - \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) dp \right| \leq J_1 + J_2, \\
& J_1 = 2^\beta c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp, \\
& J_2 = 2c_6 \varepsilon \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n} dp, \\
& x \in D, 1 \leq r \leq 2E^{\frac{1}{2}}, E^{\frac{1}{2}} \geq \rho_3,
\end{aligned} \tag{96}$$

where  $\rho_3$  is the same as in Proposition 3. In addition,

$$\begin{aligned}
& J_1 \leq c_{10} \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta}, \quad c_{10} = |\mathbb{S}^{d-1}| \frac{2^{d+\beta}}{d+\beta} c_9, \\
& J_2 \leq c_{11} \varepsilon^{d+1}, \quad c_{11} = \left(\frac{1}{s}\right)^d |\mathcal{B}_1| \sum_{z \in \mathbb{Z}^d} (1 + 2\frac{\pi}{s} \|z\|_2)^{-n},
\end{aligned} \tag{97}$$

where  $|\mathcal{B}_1|$  is the standard Euclidean volume of  $\mathcal{B}_1$ . Finally, if  $r = r_3(E)$ ,  $\varepsilon = \varepsilon_3(E)$ , where  $r_3(E)$ ,  $\varepsilon_3(E)$  are defined in (53), then

$$\begin{aligned}
& r^{d-n} = (2\tau)^{d-n} E^{-\alpha_3}, \\
& \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} = (2\tau)^{d+\beta} E^{-\alpha_3}, \\
& \varepsilon^{d+1} = E^{-\alpha_3}.
\end{aligned} \tag{98}$$

Using representation (95), estimates (58), (96), (97), formulas (98) and taking into account definitions (53), we obtain

$$\begin{aligned}
& |u(x, E) - v(x)| \leq A_3 E^{-\alpha_3}, \quad x \in D, E^{\frac{1}{2}} \geq \rho_3, \\
& A_3 = A_3(D_0, \dots, D_{d+1}, N_0, \dots, N_{d+1}, M, d, n, \beta, \tau, s) \\
& = (2\tau)^{d+\beta} c_{10} + c_{11} + (2\tau)^{d-n} c_3 \|v\|_{n,1},
\end{aligned} \tag{99}$$

where  $\|v\|_{n,1} \leq M$  and  $D_0, \dots, D_{d+1}, N_0, \dots, N_{d+1}$  are the same as in Proposition 3.

Theorem 3 is proved.

## 8 Proof of Lemma 1

Note that

$$\widehat{w}(p) = \int_{\mathbb{R}^d} |\widehat{q}(\xi)|^2 \widehat{\omega}_\nu(p - \xi) d\xi, \quad p \in \mathbb{R}^d, \tag{100}$$

$$\omega_\nu(x) = |x|^\nu K_\nu(|x|), \quad x \in \mathbb{R}^d, \tag{101}$$

where  $\widehat{q}$ ,  $\widehat{\omega}_\nu$  are the Fourier transforms of  $q$ ,  $\omega_\nu$ . The Fourier transform  $\widehat{\omega}_\nu$  can be computed explicitly:

$$\widehat{\omega}_\nu(p) = \frac{c_{12}}{(1 + |p|^2)^{\frac{d}{2} + \nu}}, \quad c_{12} = \frac{\Gamma(\frac{d}{2} + \nu)2^{\nu-1}}{\pi^{\frac{d}{2}}}. \quad (102)$$

Indeed, formula (102) follows from the Fourier inversion theorem and the following computations:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{e^{-ipx} dp}{(1 + |p|^2)^{\frac{d}{2} + \nu}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{e^{-i|x|t} dt d\xi}{(1 + t^2 + |\xi|^2)^{\frac{d}{2} + \nu}} \\ &= |\mathbb{S}^{d-2}| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i|x|t} r^{d-2} dt dr}{(1 + t^2 + r^2)^{\frac{d}{2} + \nu}} \\ &\stackrel{r=\sqrt{1+t^2}\tau}{=} |\mathbb{S}^{d-2}| \int_{\mathbb{R}} \frac{e^{-i|x|t} dt}{(1 + t^2)^{\frac{1}{2} + \nu}} \int_0^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} \\ &= c_{12}^{-1} |x|^\nu K_\nu(|x|), \quad x \in \mathbb{R}^d. \end{aligned}$$

Here, it was used that

$$\begin{aligned} \int_0^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} &= \frac{1}{2} B\left(\frac{d-1}{2}, \nu + \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{d-1}{2})\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{d}{2} + \nu)}, \\ |\mathbb{S}^{d-2}| &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}, \end{aligned}$$

where  $B$  and  $\Gamma$  denote the beta and gamma functions.

Using (100), (102), we obtain the estimates

$$\begin{aligned} \widehat{w}(p) &\geq \int_{|\xi| \leq 1} \frac{c_{12} |\widehat{q}(\xi)|^2}{(1 + |p - \xi|)^{d+2\nu}} d\xi \geq \frac{c_1(q, \nu)}{(1 + |p|)^{d+2\nu}}, \quad p \in \mathbb{R}^d, \\ c_1(q, \nu) &= \frac{c_{12}}{2^{d+2\nu}} \int_{|\xi| \leq 1} |\widehat{q}(\xi)|^2 d\xi. \end{aligned} \quad (103)$$

Properties (23) follow from (20), (22), (100) and (103).

Lemma 1 is proved.

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## References

- [1] A. D. Agaltsov, T. Hohage, and R. G. Novikov, “An iterative approach to monochromatic phaseless inverse scattering,” in preparation.

- [2] T. Aktosun and P. E. Sacks, “Inverse problems on the line without phase information,” *Inverse Problems*, vol. 14, pp. 211–224, 1998.
- [3] N. V. Alexeenko, V. A. Burov, and O. D. Rumyantseva, “Solution of the three-dimensional acoustical inverse scattering problem. The modified Novikov algorithm,” *Acoustical Physics*, vol. 54, no. 3, pp. 407–419, 2008.
- [4] J. A. Barceló, C. Castro, and J. M. Reyes, “Numerical approximation of the potential in the two-dimensional inverse scattering problem,” *Inverse Problems*, vol. 32, no. 1, 2016, 015006 (19pp).
- [5] F. A. Berezin and M. A. Shubin, *The Schrödinger equation*, ser. Mathematics and its applications. Dordrecht: Kluwer Academic, 1991, vol. 66.
- [6] V. A. Burov, N. V. Alekseenko, and O. D. Rumyantseva, “Multifrequency generalization of the Novikov algorithm for the two-dimensional inverse scattering problem,” *Acoustical Physics*, vol. 56, no. 6, pp. 843–856, 2009.
- [7] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory*, 2nd ed. Berlin: Springer, 1989.
- [8] G. Eskin, *Lectures on linear partial differential equations*, ser. Graduate studies in mathematics. American mathematical society, 2011, vol. 123.
- [9] L. D. Faddeev, “Uniqueness of the solution of the inverse scattering problem,” *Vest. Leningrad Univ.*, vol. 7, pp. 126–130, 1956, (in Russian).
- [10] —, “Inverse problem of quantum scattering theory. II.” *Journal of Soviet Mathematics*, vol. 5, no. 3, pp. 334–396, 1976.
- [11] L. D. Faddeev and S. P. Merkuriev, *Quantum scattering theory for several particle systems*, ser. Mathematical physics and applied mathematics. Dordrecht: Kluwer Academic, 1993, vol. 11.
- [12] A. A. Gonchar, G. M. Henkin, and N. N. Novikova, “Multipoint Padé approximants in an inverse Sturm-Liouville problem,” *Math. USSR-Sb.*, vol. 73, no. 2, pp. 479–489, 1992.
- [13] P. G. Grinevich, “The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy,” *Russian Math. Surveys*, vol. 55, no. 6, pp. 1015–1083, 2000.
- [14] P. Hähner and T. Hohage, “New stability estimates for the inverse acoustic inhomogeneous medium problem and applications,” *SIAM J. Math. Anal.*, vol. 33, no. 3, pp. 670–685, 2001.
- [15] G. M. Henkin and R. G. Novikov, “The  $\bar{\partial}$ -equation in the multidimensional inverse scattering problem,” *Russian Mathematical Surveys*, vol. 42, no. 3, pp. 109–180, 1987.

- [16] —, “A multidimensional inverse problem in quantum and acoustic scattering,” *Inv. Problems*, vol. 4, pp. 103–121, 1988.
- [17] M. I. Isayev, “Exponential instability in the inverse scattering problem on the energy interval,” *Funct. Anal. Appl.*, vol. 47, no. 3, pp. 187–194, 2013.
- [18] M. I. Isayev and R. G. Novikov, “New global stability estimates for monochromatic inverse acoustic scattering,” *SIAM J. Math. Analysis*, vol. 43, no. 5, pp. 1495–1504, 2013.
- [19] M. V. Klibanov, “Phaseless inverse scattering problems in three dimensions,” *SIAM J. Appl. Math.*, vol. 74, pp. 392–410, 2014.
- [20] M. V. Klibanov and V. G. Romanov, “Reconstruction procedures for two inverse scattering problems without the phase information,” *SIAM J. Appl. Math.*, vol. 76, no. 1, pp. 178–196, 2016.
- [21] —, “Two reconstruction procedures for a 3D phaseless inverse scattering problem for the generalized Helmholtz equation,” *Inverse Problems*, vol. 32, no. 1, 2016, 015005 (16pp).
- [22] R. G. Novikov, “Multidimensional inverse spectral problem for the equation  $-\Delta\psi + (v(x) - Eu(x))\psi = 0$ ,” *Funct. Anal. Appl.*, vol. 22, pp. 263–272, 1988.
- [23] —, “Rapidly converging approximation in inverse quantum scattering in dimension 2,” *Physics Letters A*, vol. 238, pp. 73–78, 1998.
- [24] —, “The  $\bar{\delta}$ -approach to approximate inverse scattering at fixed energy in three dimensions,” *Int. Math. Res. Pap.*, no. 6, pp. 287–349, 2005.
- [25] —, “Approximate Lipschitz stability for non-overdetermined inverse scattering at fixed energy,” *J. Inverse Ill-Posed Probl.*, vol. 26, no. 6, pp. 813–823, 2013.
- [26] —, “Inverse scattering without phase information,” in *Séminaire Laurent Schwartz — EDP et applications*. Ecole Polytechnique, 2014-2015, exp. no. 16 (13pp).
- [27] —, “Formulas for phase recovering from phaseless scattering data at fixed frequency,” *Bull. Sci. Math.*, vol. 139, no. 8, pp. 923–936, 2015.
- [28] —, “An iterative approach to non-overdetermined inverse scattering at fixed energy,” *Sbornik: Mathematics*, vol. 206, no. 1, pp. 120–134, 2015.
- [29] —, “Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions,” *The Journal of Geometric Analysis*, vol. 26, no. 1, pp. 346–359, 2016.