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On Saturated Uniformly A-convex Algebras

M. El Azhari

Department of Mathematics, Ecole Normale Supérieure, Rabat, Morocco.
E-mail: mohammed.elazhari@yahoo.fr

Abstract. Following ideas of A. C. Cochran, we give a suitable definition of a saturated uniformly A-convex algebra. In the m-convex case, such algebra is a uniform topological one.

Keywords. Uniformly A-convex Algebra; Saturated Algebra; Uniform Topological Algebra.

1. PRELIMINARIES

A locally convex algebra is a complex algebra with a locally convex topology for which the multiplication is separately continuous. A locally m-convex algebra is a locally convex algebra whose topology is defined by a family of submultiplicative seminorms. A uniform seminorm on an algebra E is a seminorm p such that $p(x^2) = p(x)^2$ for all $x \in E$. Such a seminorm is submultiplicative. A uniform topological algebra is a locally convex algebra whose topology is defined by a family of uniform seminorms. A uniform normed algebra is a normed algebra $(E, \|\cdot\|)$ such that $\|x^2\| = \|x\|^2$ for all $x \in E$. A locally convex algebra E is uniformly A-convex if its topology is defined by a family $\{p_\alpha, \alpha \in \Lambda\}$ of seminorms with the property that for $x \in E$, there is a positive constant r_x such that $p_\alpha(xy) \leq r_x p_\alpha(y)$ and $p_\alpha(yx) \leq r_x p_\alpha(y)$ for all $\alpha \in \Lambda$ and $y \in E$. Let E be a locally convex algebra. Denote by $M^*(E)$ the set of all nonzero multiplicative linear functionals on E . Denote by $M(E)$ the space of all nonzero continuous multiplicative linear functionals on E , topologized via the weak topology, it is called the carrier space of E .

2. RESULTS

Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a Hausdorff commutative uniformly A-convex algebra with unit e . The family $\{p_\alpha, \alpha \in \Lambda\}$ of seminorms can be chosen such that $p_\alpha(e) = 1$ for all $\alpha \in \Lambda$. For $x \in E$, let $\|x\| = \sup[\sup\{p_\alpha(xy), p_\alpha(y) \leq 1\} : \alpha \in \Lambda] = \inf\{r_x > 0, p_\alpha(xy) \leq r_x p_\alpha(y) \text{ for all } \alpha \in \Lambda, y \in E\}$. By [1, Lemma 3.2], $\|\cdot\|$ is a submultiplicative norm on E for which $p_\alpha(x) \leq \|x\|$ for all $\alpha \in \Lambda$ and $x \in E$. Let M be the carrier space of $(E, (p_\alpha)_{\alpha \in \Lambda})$ and let M_n be the carrier space of $(E, \|\cdot\|)$, $M \subset M_n$. Since $(E, \|\cdot\|)$ is a commutative normed algebra with unit,

M_n is nonempty. M may be empty [2]. For the sequel, we assume that M is nonempty. For $\alpha \in \Lambda$ and $m \in M$, Cochran [1] has defined the extended real number $t_\alpha(m) = \sup\{|m(x)|, p_\alpha(x) \leq 1\}$ and the map $\phi_\alpha : M \rightarrow R$ by $\phi_\alpha(m) = t_\alpha(m)^{-1}$ if $t_\alpha(m) < \infty$ and $\phi_\alpha(m) = 0$ otherwise.

Proposition 2.1.

- (1) $t_\alpha(m) \geq 1$ and $0 \leq \phi_\alpha(m) \leq 1$ for all $\alpha \in \Lambda$ and $m \in M$;
- (2) $t_\alpha(m) < \infty$ if, and only if, m is continuous for p_α ;
- (3) If $t_\alpha(m) < \infty$, then $|m(x)| \leq t_\alpha(m) p_\alpha(x)$ for all $x \in E$.

Proof. (1) Since $m(e) = 1$ and $p_\alpha(e) = 1$, it follows that $t_\alpha(m) \geq 1$, hence $\phi_\alpha(m) = t_\alpha(m)^{-1} \leq 1$.

(2) m is continuous for p_α if, and only if, m is bounded on $\{x \in E, p_\alpha(x) \leq 1\}$, i.e. $t_\alpha(m) < \infty$.

(3) Let $x \in E$ and $\varepsilon > 0$, $|m((p_\alpha(x) + \varepsilon)^{-1} x)| \leq t_\alpha(m)$ since $p_\alpha((p_\alpha(x) + \varepsilon)^{-1} x) \leq 1$, thus $|m(x)| \leq t_\alpha(m) (p_\alpha(x) + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $|m(x)| \leq t_\alpha(m) p_\alpha(x)$.

Let $C_b(M)$ be the algebra of all complex continuous bounded functions on M , with the topology defined by the family $\{\hat{p}_\alpha, \alpha \in \Lambda\}$ of seminorms, where $\hat{p}_\alpha(f) = \sup\{\phi_\alpha(m)|f(m)|, m \in M\}$ for all $f \in C_b(M)$. For $\alpha \in \Lambda$, define $M_\alpha = \{m \in M, t_\alpha(m) < \infty\}$. If M_α is empty, $\phi_\alpha(m) = 0$ for all $m \in M$, so $\hat{p}_\alpha(f) = 0$ for all $f \in C_b(M)$. If M_α is nonempty, $\hat{p}_\alpha(f) = \sup\{t_\alpha(m)^{-1}|f(m)|, m \in M_\alpha\}$. If $\{p_\alpha, \alpha \in \Lambda\}$ is a directed family of seminorms, then $M = \bigcup_{\alpha \in \Lambda} M_\alpha$, so there exists $\alpha \in \Lambda$ such that M_α is nonempty. The following result is due to Cochran [1], the proof is given for completeness.

Proposition 2.2 [1]. Let $G : E \rightarrow C(M)$ be the Gelfand map. Then

- (1) $|\hat{x}(m)| \leq \|x\|$ for all $x \in E$ and $m \in M$, so $G(E) = \hat{E} \subset C_b(M)$;
- (2) $\hat{p}_\alpha(\hat{x}) \leq p_\alpha(x)$ for all $\alpha \in \Lambda$ and $x \in E$.

Proof. (1) Let $m \in M \subset M_n$. Since $(E, \|\cdot\|)$ is a normed algebra and $m \in M_n$, it follows that $|\hat{x}(m)| = |m(x)| \leq \|x\|$ for all $x \in E$. Consequently, $\hat{E} \subset C_b(M)$.

(2) Let $\alpha \in \Lambda$. If M_α is empty, $\hat{p}_\alpha(\hat{x}) = 0 \leq p_\alpha(x)$ for all $x \in E$. If M_α is nonempty, we have $|m(x)| \leq t_\alpha(m) p_\alpha(x)$ for all $m \in M_\alpha$ and $x \in E$, i.e. $t_\alpha(m)^{-1}|m(x)| \leq p_\alpha(x)$ for all $m \in M_\alpha$ and $x \in E$. Thus $\hat{p}_\alpha(\hat{x}) = \sup\{t_\alpha(m)^{-1}|m(x)|, m \in M_\alpha\} \leq p_\alpha(x)$ for all $x \in E$.

Let $\alpha \in \Lambda$ and let F_α be the topological dual of (E, p_α) . For $f \in F_\alpha$, put $t_\alpha(f) = \sup\{|f(x)|, p_\alpha(x) \leq 1\}$, t_α is a norm on F_α . If M_α is nonempty, M_α is usually topologized via the weak topology. But for the following proposition, we consider on M_α the topology τ_α induced by the topology of t_α . The map $M_\alpha \rightarrow R, m \rightarrow t_\alpha(m)$, is continuous for τ_α . Since (M_α, τ_α) is a metric space, it

is a completely regular space, we denote by M_α^β the Stone-Ćech compactification of M_α .

Proposition 2.3. The following assertions are equivalent:

- (1) If $p_\alpha(x) = 1$ for some $\alpha \in \Lambda$ and $x \in E$, then $\hat{p}_\alpha(\hat{x}) = 1$;
- (2) $\hat{p}_\alpha(\hat{x}) = p_\alpha(x)$ for all $\alpha \in \Lambda$ and $x \in E$;
- (3) For $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$, M_α is nonempty and the continuous extension of the map $M_\alpha \rightarrow R, m \rightarrow t_\alpha(m)^{-1}|m(x)|$, to M_α^β is equal to 1 at some $m_0 \in M_\alpha^\beta$.

Proof. (1) \Rightarrow (2) : Let $\alpha \in \Lambda$ and $x \in E$. If $p_\alpha(x) = 0$, then $0 \leq p_\alpha(\hat{x}) \leq p_\alpha(x) = 0$ by Proposition 2.2, so $\hat{p}_\alpha(\hat{x}) = 0$. If $p_\alpha(x) \neq 0$, $\hat{p}_\alpha(p_\alpha(x)^{-1}\hat{x}) = 1$ since $p_\alpha(p_\alpha(x)^{-1}x) = 1$, hence $\hat{p}_\alpha(\hat{x}) = p_\alpha(x)$.

(2) \Rightarrow (3) : Let $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$. M_α is nonempty since $1 = p_\alpha(x) = \hat{p}_\alpha(\hat{x}) \neq 0$. Let m_1, m_2 in M_α , $|m_1(x) - m_2(x)| = |(m_1 - m_2)(x)| \leq t_\alpha(m_1 - m_2)p_\alpha(x) = t_\alpha(m_1 - m_2)$, then the map $M_\alpha \rightarrow R, m \rightarrow m(x)$, is continuous. Consequently, the map $\varphi_\alpha : M_\alpha \rightarrow R, \varphi_\alpha(m) = t_\alpha(m)^{-1}|m(x)|$, is continuous. φ_α is also bounded on M_α since $\hat{p}_\alpha(\hat{x}) = \sup\{t_\alpha(m)^{-1}|m(x)|, m \in M_\alpha\} = p_\alpha(x) = 1$. Therefore φ_α has a continuous extension φ_α^β to M_α^β . Since $t_\alpha(m)^{-1}|m(x)| \leq 1$ for all $m \in M_\alpha$ and M_α is dense in M_α^β , it follows that $\varphi_\alpha^\beta(m) \leq 1$ for all $m \in M_\alpha^\beta$. We have $1 = \sup\{t_\alpha(m)^{-1}|m(x)|, m \in M_\alpha\} \leq \sup\{\varphi_\alpha^\beta(m), m \in M_\alpha^\beta\} \leq 1$, hence $\sup\{\varphi_\alpha^\beta(m), m \in M_\alpha^\beta\} = 1$. Since M_α^β is compact, there exists $m_0 \in M_\alpha^\beta$ such that $\varphi_\alpha^\beta(m_0) = 1$.

(3) \Rightarrow (1) : Let $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$. If $\hat{p}_\alpha(\hat{x}) = \sup\{t_\alpha(m)^{-1}|m(x)|, m \in M_\alpha\} = s < 1$, $t_\alpha(m)^{-1}|m(x)| \leq s < 1$ for all $m \in M_\alpha$, then $\varphi_\alpha^\beta(m) \leq s < 1$ for all $m \in M_\alpha^\beta$, this contradicts (3). Therefore $\hat{p}_\alpha(\hat{x}) \geq 1$, so $1 \leq \hat{p}_\alpha(\hat{x}) \leq p_\alpha(x) = 1$ i.e. $\hat{p}_\alpha(\hat{x}) = 1$.

We say that E is a Cochran algebra if $\hat{p}_\alpha(\hat{x}) = p_\alpha(x)$ for all $\alpha \in \Lambda$ and $x \in E$.

Proposition 2.4. If E is a Cochran algebra, then the Gelfand map G from $(E, (p_\alpha)_{\alpha \in \Lambda})$ onto $(\hat{E}, (\hat{p}_\alpha)_{\alpha \in \Lambda})$ is an algebraic and topological isomorphism.

Proof. Let $x \in E$. If $\hat{x} = 0$, then $\hat{p}_\alpha(\hat{x}) = p_\alpha(x) = 0$ for all $\alpha \in \Lambda$, so $x = 0$ since E is Hausdorff. Consequently, G is an algebraic isomorphism. Since $\hat{p}_\alpha(\hat{x}) = p_\alpha(x)$ for all $\alpha \in \Lambda$ and $x \in E$, G is a topological isomorphism.

Proposition 2.5. Let E be a Cochran algebra whose carrier space M is equicontinuous. Then E is a uniform normed algebra.

Proof. For $x \in E$, let $q(x) = \sup\{|m(x)|, m \in M\}$. Since M is equicontinuous, the map q is a continuous seminorm on E . Let $\alpha \in \Lambda$ and $x \in E, p_\alpha(x) =$

$\hat{p}_\alpha(\hat{x}) = \sup\{\phi_\alpha(m)|m(x)|, m \in M\} \leq q(x)$ since $0 \leq \phi_\alpha(m) \leq 1$ for all $m \in M$. Let $x \in E$ such that $q(x) = 0$, since E is Hausdorff and $p_\alpha(x) \leq q(x) = 0$ for all $\alpha \in \Lambda$, it follows that $x = 0$. q is a continuous uniform norm on E such that $p_\alpha(x) \leq q(x)$ for all $\alpha \in \Lambda$ and $x \in E$, then the topology of E can be defined by the uniform norm q .

Corollary 2.1. Let E be a Cochran algebra. If E is barrelled or a Q-algebra, then E is a uniform normed algebra.

Proof. (1) Assume that E is barrelled. By Proposition 2.2, $|m(x)| \leq \|x\|$ for all $x \in E$ and $m \in M$. Therefore M is bounded for the weak topology, so M is equicontinuous.

(2) Assume that E is a Q-algebra. By [3, Proposition II.7.1], every topological Q-algebra has an equicontinuous carrier space.

The saturated uniformly A-convex algebras were introduced by Cochran [1] as a subclass of the class of Cochran algebras.

Definition 2.1 [1]. E is saturated if for each $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$, there exists $m_0 \in M$ such that $m_0(x) = \sup\{|m(y)|, p_\alpha(y) \leq 1\}$ for some $m \in M$. (Then $\hat{p}_\alpha(\hat{x}) = 1 = p_\alpha(x)$).

Oudadess [2] has proved that the class of complete saturated algebras (in the sense of Definition 2.1) is empty. His proof uses the completeness of the algebra. Here we give a simple proof in the general case.

Proposition 2.6. The class of saturated algebras (in the sense of Definition 2.1) is empty.

Proof. Let $\alpha \in \Lambda$, we have $p_\alpha(-e) = p_\alpha(e) = 1$. Since the algebra is saturated, there exist m_0, m in M such that $-1 = m_0(-e) = \sup\{|m(y)|, p_\alpha(y) \leq 1\} \geq 0$, which is absurd.

We think that the definition of a saturated algebra should be as follows:

Definition 2.2. E is saturated if for each $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$, there exists $m_0 \in M$ such that $|m_0(x)| = \sup\{|m_0(y)|, p_\alpha(y) \leq 1\}$, i.e. $m_0 \in M_\alpha$ and $t_\alpha(m_0)^{-1}|m_0(x)| = 1$.

Proposition 2.7. If E is saturated, then E is a Cochran algebra.

Proof. Let $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$. By hypothesis, M_α is nonempty and the map $M_\alpha \rightarrow R, m \rightarrow t_\alpha(m)^{-1}|m(x)|$, is equal to 1 at some $m_0 \in M$. Therefore E is a Cochran algebra by Proposition 2.3.

If p_α is submultiplicative, $N_\alpha = \{x \in E, p_\alpha(x) = 0\}$ is an ideal in E and the quotient algebra E/N_α is a normed algebra with the norm $\|x_\alpha\|_\alpha = p_\alpha(x)$, $x_\alpha = x + N_\alpha$. Let E_α be the completion of E/N_α , E_α is a commutative Banach algebra with unit.

Proposition 2.8. If p_α is submultiplicative, then

- (1) M_α is a nonempty compact space;
- (2) $t_\alpha(m) = 1$ for all $m \in M_\alpha$.

Proof. (1) M_α is homeomorphic to $M(E_\alpha)$ by [4, Proposition 7.5], then M_α is a nonempty compact space.

(2) Let $m \in M_\alpha, |m(x)| \leq t_\alpha(m)p_\alpha(x)$ for all $x \in E$, then N_α is included in $\text{Ker}(m)$. We may define a multiplicative linear functional m_α on E/N_α by $m_\alpha(x_\alpha) = m(x)$. Since E/N_α is a normed algebra and $m_\alpha \in M(E/N_\alpha), |m_\alpha(x_\alpha)| \leq \|x_\alpha\|_\alpha$ for all $x \in E$. Further, as E/N_α is unital, $1 = \sup\{|m_\alpha(x_\alpha)|, \|x_\alpha\|_\alpha \leq 1\} = \sup\{|m(x)|, p_\alpha(x) \leq 1\} = t_\alpha(m)$.

Proposition 2.9. If p_α is submultiplicative for every $\alpha \in \Lambda$, then the following assertions are equivalent:

- (1) E is saturated;
- (2) E is a Cochran algebra;
- (3) $(E, (p_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra.

Proof. (1) \Rightarrow (2) : By Proposition 2.7.

(2) \Rightarrow (3) : Let $\alpha \in \Lambda$ and $x \in E, p_\alpha(x) = \hat{p}_\alpha(\hat{x}) = \sup\{t_\alpha(m)^{-1}|m(x)|, m \in M_\alpha\} = \sup\{|m(x)|, m \in M_\alpha\}$ by Proposition 2.8. Thus p_α is a uniform seminorm.

(3) \Rightarrow (1) : Let $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$. Since E_α is a uniform Banach algebra, $1 = p_\alpha(x) = \|x_\alpha\|_\alpha = \sup\{|g(x_\alpha)|, g \in M(E_\alpha)\} = \sup\{|m(x)|, m \in M_\alpha\}$ by [4, proposition 7.5]. Since M_α is compact, there exists $m_0 \in M_\alpha$ such that $|m_0(x)| = 1 = t_\alpha(m_0)$ by Proposition 2.8, hence E is saturated.

We give an example of a saturated algebra which is not m-convex.

Example. Let $C_b(R)$ be the algebra of all complex continuous bounded functions on R . Let Λ be the subset of $C_b(R)$ defined by $\phi \in \Lambda$ if $\phi(x) > 0$ for all $x \in R, \sup\{|\phi(x)|, x \in R\} = 1$ and ϕ vanishes at infinity. We endow $C_b(R)$ with the topology determined by the family $\{p_\phi, \phi \in \Lambda\}$ of seminorms, where $p_\phi(f) = \sup\{|f(x)\phi(x)|, x \in R\}$. $C_b(R)$ is a commutative complete uniformly A-convex algebra with unit $u(u(x) = 1$ for all $x \in R)$ such that $p_\phi(u) = 1$ for all $\phi \in \Lambda$. $(C_b(R), (p_\phi)_{\phi \in \Lambda})$ is not m-convex. Let $f \in C_b(R)$ and $\phi \in \Lambda$ such that $p_\phi(f) = 1$. Since ϕ vanishes at infinity, there exists $x_0 \in R$ such that $|f(x_0)\phi(x_0)| = 1$. Show that $|f(x_0)| = \sup\{|g(x_0)|, p_\phi(g) \leq 1\}$. The proof is due to Beddaa [5], it is given here for completeness. We have $|f(x_0)| \leq \sup\{|g(x_0)|, p_\phi(g) \leq 1\}$ since $p_\phi(f) = 1$. If $p_\phi(g) \leq 1 = |f(x_0)\phi(x_0)|$, then

$|g(x_0)\phi(x_0)| \leq |f(x_0)\phi(x_0)|$, so $|g(x_0)| \leq |f(x_0)|$ since $\phi(x_0) > 0$. We have shown that $|\delta_{x_0}(f)| = \sup\{|\delta_{x_0}(g)|, p_\phi(g) \leq 1\}$, where $\delta_{x_0} : C_b(R) \rightarrow C$, $\delta_{x_0}(h) = h(x_0)$, is a nonzero continuous multiplicative linear functional on $C_b(R)$. Thus $(C_b(R), (p_\phi)_{\phi \in \Lambda})$ is saturated.

Oudadess [2] has introduced the following definition:

Definition 2.3 [2]. E is v -saturated if for all $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$, there exist m_0, m in $M^* = M^*(E)$ such that $|m_0(x)| = \sup\{|m(y)|, p_\alpha(y) \leq 1\}$.

We have three remarks about this definition.

1. If E is v -saturated, for $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$, there exist m_0, m in M^* such that $\sup\{|m(x)|, p_\alpha(y) \leq 1\} = |m_0(x)| < \infty$, then M is nonempty since $m \in M$, so the fact of replacing M by M^* is not justified.

2. Oudadess [2] did not use anywhere the sup-property of his definition. He only used the following deduced property: for all $\alpha \in \Lambda$ and $x \in E$ such that $p_\alpha(x) = 1$, there exists $m_0 \in M^*$ such that $|m_0(x)| \geq 1$.

3. In [2, Theorem 5.4], Oudadess claimed that if E is complete and v -saturated, then the Gelfand map G from $(E, (p_\alpha)_{\alpha \in \Lambda})$ onto $(\hat{E}, (\hat{p}_\alpha)_{\alpha \in \Lambda})$ is an algebraic and topological isomorphism. For the proof, he used Theorem 3.5 of [1], but this theorem allows us only to conclude that G is a continuous algebraic isomorphism.

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