p-Rank and p-groups in algebraic groups
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To cite this version:
Adrien Deloro. p-Rank and p-groups in algebraic groups. Turkish Journal of Mathematics, 2012, 36 (4), pp.578-582. 10.3906/mat-1103-52 . hal-01303309

HAL Id: hal-01303309
https://hal.science/hal-01303309
Submitted on 17 Apr 2016

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\textbf{Abstract}

A few remarks on the measures of the \( p \)-rank of a group equipped with a dimension, including the refutation of a result of Burdges and Cherlin.

\textbf{Key Words:} Algebraic groups, groups of finite Morley rank, torsion, \( p \)-groups

Groups of finite Morley rank are abstract analogues of algebraic groups; like them they bear a dimension enabling various genericity arguments. They do not come from geometry but from logic; yet the Cherlin-Zilber conjecture and related work suggest tight relationships between both aspects. My reader may thus view what follows as naive properties of algebraic groups obtained by elementary means; the word “definable” stands for “constructible”. Should he desire more on groups of finite Morley rank, [1] would provide references.

I wish to thank Éric Jaligot for his many suggestions.

A group of finite Morley rank is \( U_{\bot} p \) if it has no infinite elementary abelian \( p \)-subgroup. \( U_{\bot} p \) groups conjugate their Sylow \( p \)-subgroups [3, Theorem 4], i.e. their maximal (non-necessarily definable) \( p \)-subgroups; these are finite extensions of \( p \)-tori. Hence, for \( S \) a Sylow \( p \)-subgroup of a \( U_{\bot} p \) group, \( S^* \) is a \( p \)-torus.

Given a \( U_{\bot} p \) group, 3 measures of its Sylow \( p \)-subgroups are available. One can consider the \( \text{Prüfer} \ p \)-rank \( \text{Pr}_p(G) \), which is the number of \( \mathbb{Z}_{p^n} \) factors in a Sylow \( p \)-subgroup. One can also estimate the \textit{normal} \( p \)-rank \( n_p(G) \), which is the maximal \( p \)-rank of an elementary abelian \( p \)-group normal in a Sylow \( p \)-subgroup. Or one can simply compute the \textit{p-rank} \( m_p(G) \), which is the maximal \( p \)-rank of an elementary abelian \( p \)-subgroup. All 3 numbers are well defined by conjugacy of the Sylow \( p \)-subgroups, and \( m_p(G) \geq n_p(G) \geq \text{Pr}_p(G) \).

\begin{itemize}
    \item 2010 \textit{AMS Mathematics Subject Classification:} 20F11 (primary); 20G07, 20C11, 20D15.
\end{itemize}
1. The \( n \)-rank

**Lemma 1** If \( G \) is a connected, \( U^1_p \) group, then \( n_p(G) = \text{Pr}_p(G) \).

**Proof.** Let \( S \) be a Sylow \( p \)-subgroup of \( G \), \( V \triangleleft S \) an elementary abelian normal subgroup, and \( v \in V \). As \( V \triangleleft S \), \( v^{S^0} \subseteq V \) which is finite; by connectedness, \( S^0 \) centralizes \( v \). So \( v \in C_S(S^0) = S^0 \) by [3, Corollary 3.1], and \( V \leq S^0 \).

2. Not quite a digression

For a \( p \)-torus \( T \simeq \mathbb{Z}_{p^\infty}^d \), \( \Omega_{p^\infty}(T) \) denotes the set of elements of order at most \( p^n \).

**Fact 1** Let \( \varphi \) be an automorphism of finite order of a \( p \)-torus \( T \simeq \mathbb{Z}_{p^\infty}^d \).

1. Suppose \( \Omega_{p^d}(T) \leq C_T(\varphi) \). Then \( \varphi = \text{Id} \).

2. Suppose \( \Omega_p(T) \leq C_T(\varphi) \). If \( p = 2 \), then \( \varphi^2 = \text{Id} \). If \( p \neq 2 \), then \( \varphi = \text{Id} \).

**Proof.** This must be classical but I know no reference.

1. Up to taking a power of \( \varphi \), we may assume that \( \varphi \) has prime order \( q \). Let \( x \not\in C_T(\varphi) \) have minimal order. Then \( \varphi(x^p) = x^p \) so there is \( y \in \Omega_p(T) \setminus \{1\} \) with \( \varphi(x) = xy \). By assumption \( y \in C_T(\varphi) \), so \( x = \varphi^q(x) = xy^q \) and \( q = p \). Let \( \hat{x} \) and \( \hat{y} \) be such that \( \hat{x}^p = x \) and \( \varphi(\hat{x}) = \hat{x}\hat{y} \). Then \( \hat{y}^p = y \) so \( \hat{y} \in \Omega_{p^d}(T) \leq C_T(\varphi) \) and \( \hat{x} = \varphi^p(\hat{x}) = \hat{x}\hat{y}^p = \hat{x}y \), a contradiction.

2. Since \( \varphi \) centralizes \( \Omega_p(T) \), for \( x \in \Omega_{p^d}(T) \) there is \( y \in \Omega_p(T) \) with \( \varphi(x) = xy \); hence \( \varphi^p(x) = xy^p = x \) and \( \Omega_{p^d}(T) \leq C_T(\varphi^p) \). So \( \varphi^p = \text{Id} \); we may assume \( p \neq 2 \). Represent \( \varphi|_{\Omega_p(T)} \) by a matrix \( M \in \text{GL}_d(\mathbb{Z}/p^3\mathbb{Z}) \). As \( \Omega_p(T) \leq C_T(\varphi) \), the reduction of \( M \) modulo \( p \) is the identity: there is a matrix \( N \) with \( M = \text{Id} + pN \). Since \( \varphi^p = \text{Id} \),

\[
0 \equiv \sum_{\ell=1}^p \binom{p}{\ell} p^\ell N^\ell \equiv p^2 N + \frac{p(p-1)}{2} p^2 N^2 \quad [p^3].
\]

Since \( p \neq 2 \), \( p \) divides \( \frac{p(p-1)}{2} \), so \( p^2 N \equiv 0 [p^3] \) and \( N \equiv 0 [p] \). Hence the reduction of \( M = \text{Id} + pN \) modulo \( p^2 \) is the identity: \( M \) centralizes \( \Omega_{p^d}(T) \), and \( \varphi \) is trivial.

**Consequence** If \( p \neq 2 \), the restriction map \( \rho : \text{Aut}(T) \to \text{Aut}(\Omega_p(T)) \) kills no element of finite order. In particular if \( W \) is a finite subgroup of \( \text{Aut}(T) \) then \( W \) embeds into \( \text{Aut}(\Omega_p(T)) \simeq \text{GL}_d(\mathbb{F}_p) \). If \( p = 2 \) then \( \ker \rho|_W \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^d \).

**Proof.** The only non-immediate claim is about the rank of \( K = \ker \rho|_W \) when \( p = 2 \). Observe that \( K \) has exponent 2, so it is abelian. We go in a direction that will prove fruitful. Taking automorphism groups changes inductive limits to projective limits, so \( \text{Aut}(T) \simeq \lim\limits_{\longleftarrow} \text{Aut}((\mathbb{Z}/p^n\mathbb{Z})^d) = \lim\limits_{\longleftarrow} \text{GL}_d(\mathbb{Z}/p^n\mathbb{Z}) = \text{GL}_d(\mathbb{Z}_p) \). Hence \( K \) embeds into \( \text{GL}_d(\mathbb{Z}_2) \). Now elements of \( K \) are simultaneously diagonalizable over \( \overline{\mathbb{Q}}_2 \) with eigenvalues \( \pm 1 \), so
Let $p$ be an automorphism of order $p$ of a $p$-torus $T \simeq \mathbb{Z}_p^d$.

Fact 2 (Maschke’s Theorem) Let $T_1 \leq T$ be a $\varphi$-invariant subtorus. Then there is a $\varphi$-invariant subtorus $T_2 \leq T$ such that $T = T_1 + T_2$ and $T_1 \cap T_2 \leq \Omega_p(T_1)$.

Proof. There is a subtorus $T_0 \leq T$ with $T = T_1 \oplus T_0$. Let $\pi$ be the projection on $T_1$ along $T_0$ and $\hat{\pi} = \sum_{i=0}^{p-1} \varphi^i \pi \varphi^{-i}$. Then $\hat{\pi}$ is $\varphi$-covariant, $\Im \hat{\pi} = T_1$, and $\hat{\pi}(t_1) = pt_1$ for $t_1 \in T_1$. Take $T_2$ to be the maximal subtorus of $\ker \hat{\pi}$.

Fact 3 ($\varphi, T$ as above) If $C^{\hat{\pi}}_{\bar{\varphi}}(\varphi) = 1$ then $p - 1|d$ and $\text{Id} + \varphi + \cdots + \varphi^{p-1} = 0$.

Proof. (This again must be well known.) We may assume $p \neq 2$. Let $\tau \leq T$ be isomorphic to $\mathbb{Z}_p^\infty$, and set $\Theta = \sum_{i=0}^{p-1} \varphi^i(\tau)$; $\Theta$ is $\varphi$-invariant and $\text{Pr}_p(\Theta) \leq p$. So by Maschke’s Theorem, we may assume $d \leq p$. As in the proof of the Consequence above, let us view $\varphi$ as an element of order $p$ of $\text{GL}_d(\mathbb{Z}_p) \leq \text{GL}_d(\mathbb{Q}_p)$. By assumption, $1$ is not an eigenvalue.

The minimal polynomial $\mu$ of $\varphi$ over $\mathbb{Q}_p$ divides $X^p - 1 = (X - 1)(1 + X + \cdots + X^{p-1})$, so it divides $1 + X + \cdots + X^{p-1}$. The latter is irreducible over $\mathbb{Z}_p$ by Eisenstein’s criterion, so $\mu = 1 + X + \cdots + X^{p-1}$. But $\mu$ divides the characteristic polynomial which has degree $d$. So $p - 1 \leq d \leq p$. Over $\mathbb{Q}_p$, $\varphi$ has $p - 1$ eigenvalues, which sum to $-1$. So if $d = p$, one of them, say $j$, occurs twice: hence $1 + \text{Tr} \varphi = j \in \mathbb{Q}_p$, against $p \neq 2$. So $d = p - 1$.

Lemma 2 For $W \leq \text{Aut} \mathbb{Z}_p^d$ an elementary abelian $p$-group, $\text{rk} W \leq \frac{d}{p-1} d$.

Proof. $E = \mathbb{Q}_p^d$ is a sum of $W$-irreducible subspaces $\bigoplus_{i \in I} E_i \bigoplus_{j \in J} F_j$ with $E_i$’s the $W$-trivial lines. Since $W$ is abelian, it acts $W$-covariantly. Let $\rho_j : W \rightarrow \text{Aut}_W(F_j)$ be the restriction map, with (non-trivial) image $W_j$ and kernel $K_j$. Each $\text{End}_W(F_j)$ is a skew-field by Schur’s Lemma, so the abelian group $W_j$ of exponent $p$ has order $p$. As $C_W(E) = 1$, $W \hookrightarrow \prod_{j \in J} W/K_j$, and $\text{rk} W \leq \# J$. By Fact 3, $\dim F_j \geq p - 1$, whence $\# J \leq \frac{d}{p-1} d$.

Corollary 1 Let $G$ be a connected, $U_p^d$ group. Then $m_p(G) \leq \frac{d}{p-1} \text{Pr}_p(G)$.

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Let $p$ maximal. In Theorem 1.2.

4. Maximal abelian $p$-subgroups

**Theorem 3.1** Let $G$ be a connected, $U_p^+$ group with $m_p(G) \geq 3$. Then any maximal elementary abelian $p$-subgroup $V < G$ has $p$-rank at least 3.

The flaw in [2] lies at the bottom of page 172. On the very last line, “commutation with $v$” need not in general be “a map from $\Omega_1(T)/A$ to $A$”. Observe that in [2] Theorem 6.4 relies on Corollary 4.2, which relies on Theorem 1.2.

**Counter-Example** In $\text{PSL}_5(\mathbb{C})$ let $\Theta$ be the usual torus and $\sigma$ be the Weyl element naturally associated with the 5-cycle $(12345)$. Let $\theta \in C_\Theta(\sigma) \setminus \{1\}$. Then $(\theta, \sigma)$ does not extend to an elementary abelian 5-group of rank 3.

**Observation** Let $G$ be a connected, $U_p^+$ group, and $S \leq G$ a Sylow $p$-subgroup. Then $S$ is connected iff abelian iff nilpotent.

**Proof.** Only one claim is non-trivial; we prove it by induction on the Morley rank (read: dimension) of $G$. Suppose $S$ nilpotent; let $\omega \in S$. Then by nilpotence, $\tau = C^\omega_5(\omega) \neq 1$. By [3, Corollary 3.1], $\omega$ lies in any maximal $p$-torus of $C^\omega_5(\omega)$, so $\omega \in C^\omega_5(\tau)$. Hence $(S^\omega, \omega) \leq C^\omega_5(\tau)$. If $C^\omega_5(\tau) < G$ we are done by induction. Otherwise $\tau$ is central and we can factor by $Z^\omega(G)$, pursuing by induction.

I shall now bring my reader some comfort.

**Lemma 3** The thesis of [2, Theorem 1.2] holds for $p = 2$, and so does [2, Corollary 6.5].

**Proof.** Suppose $m_2(G) \geq 3$; clearly $\text{Pr}_2(G) \geq 2$. Let $i, j$ be 2 commuting involutions; by torality [3, Theorem 3] there is a Sylow 2-subgroup $S$ with $i \in S^\circ$ and $j \in S$.

Suppose $\text{Pr}_2(G) \geq 3$. If $j \in S^\circ$ we are done. If not, consider the map $\varphi(k) = [j, k] : \Omega_2(S^\circ) \to \Omega_2(S^\circ)$. Then $\text{im} \varphi \leq \ker \varphi$ and $\text{rk} \text{im} \varphi + \text{rk} \ker \varphi \geq 3$, so $\text{rk} \ker \varphi \geq 2$ and we are done. From now on, suppose $\text{Pr}_2(G) = 2$ (so $m_2(G) \leq 4$) and let $V = \Omega_2(S^\circ)$.

Assume first that $j \in S^\circ \setminus S^\circ$. If $j$ inverts $S^\circ$ then $(i, j) \leq (V, j)$: we are done. Otherwise $\tau = C^\omega_5(j) \neq 1$. If $i \not\in \tau$ then $(i, j) \leq (i, \Omega_2(\tau), j)$: we are done. So assume $i \in \tau \leq C^\omega(j)$. By torality [3, Theorem 3 and Corollary 3.1], $i$ lies in a 2-torus of $C^\omega(j)$ and $j$ lies in any 2-torus of $C^\omega(j)$, so $i$ and $j$ are cotoral.
So assume that \( j \in S^0 \), that is \( V = \langle i, j \rangle \). By assumption there is an elementary abelian 2-subgroup of rank 3: \( A = \langle r, s, t \rangle \leq S \); clearly \( A \cap S^0 \neq 1 \), say \( r \in V \). If \( s \) or \( t \) is in \( V \) then \( \langle i, j \rangle = V \leq A \); we are done. Suppose that \( s \) and \( t \) (hence \( st \) as well) lie in \( S \setminus S^0 \). Since \( |\text{Aut}(V)| = 6 \), one of \( s, t, st \) must centralize \( V = \langle i, j \rangle \); we are done again. \( \square \)

Here is a final word on counter-examples.

**Lemma 4** Let \( G \) be a counter-example to [2, Theorem 1.2]. Then \( \text{Pr}_p(G) = p - 1 \). In particular, [2, Theorem 1.2] also holds for \( p = 3 \).

**Proof.** By Lemma 3, \( p \geq 3 \). As \( m_p(G) \geq 3 \), one sees with Corollary 1 that \( \text{Pr}_p(G) \geq 2 \). Equality can hold only for \( p = 3 \); as there is an elementary 3-group of rank 3, there is an automorphism of order 3 fixing \( \Omega_3(\mathbb{Z}^\omega_{2\infty}) \), against Fact 1: equality cannot hold.

Hence \( \text{Pr}_p(G) \geq 3 \). Let \( V = \langle \alpha, \omega \rangle \) be a maximal abelian \( p \)-group and \( S \geq V \) a Sylow \( p \)-subgroup. By torality we may assume \( \alpha \in S^0 \), so \( \omega \in S \setminus S^0 \). If \( C^*_{S^0}(\omega) \neq 1 \) then by maximality, \( \alpha \in C^*_{S^0}(\omega) \leq C^0(\omega) \), and as in the proof of Lemma 3, \( \alpha \) and \( \omega \) are cotoral, a contradiction. Hence \( C^*_{S^0}(\omega) = 1 \). Let \( \varphi \in \text{End}_{\Omega_p(S^0)} \) map \( x \) to \( [x, \omega] \); writing \( \omega \) as an automorphism, \( \varphi(x) = \omega(x) - x \) and \( \varphi^n(x) = \sum_{i=0}^n (-1)^i \binom{n}{i} \omega^i(x) \). As \( (-1)^i (p^{-1}) \equiv 1 \ [p] \), \( \varphi^{p-1} = \text{Id} + \omega + \cdots + \omega^{p-1} \). But \( C^*_{S^0}(\omega) = 1 \), so Fact 3 applied to \( \omega \) implies \( \varphi^{p-1} = 0 \).

Since \( \ker \varphi = C_{\Omega_p(S^0)}(\omega) = \langle \alpha \rangle \), one has \( \text{rk} \Omega_p(S^0) \leq p - 1 \). \( \square \)

**References**


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Received: 01.04.2011

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