p-Rank and p-groups in algebraic groups
Adrien Deloro

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Abstract

A few remarks on the measures of the $p$-rank of a group equipped with a dimension, including the refutation of a result of Burdges and Cherlin.

Key Words: Algebraic groups, groups of finite Morley rank, torsion, $p$-groups

Groups of finite Morley rank are abstract analogues of algebraic groups; like them they bear a dimension enabling various genericity arguments. They do not come from geometry but from logic; yet the Cherlin-Zilber conjecture and related work suggest tight relationships between both aspects. My reader may thus view what follows as naive properties of algebraic groups obtained by elementary means; the word “definable” stands for “constructible”. Should he desire more on groups of finite Morley rank, [1] would provide references.

I wish to thank Éric Jaligot for his many suggestions.

A group of finite Morley rank is **connected** if it has no proper definable subgroup of finite index. One lets $H^\circ$ be the **connected component** of a definable subgroup $H$, i.e. its smallest definable connected subgroup of finite index. This extends to arbitrary $H$: $H$ is included in a smallest definable subgroup $d(H)$, one takes the connected component $d(H)^\circ$ and sets $H^\circ = H \cap d(H)^\circ$.

Throughout, $p$ will be a prime (possibly 2). A **$p$-torus** $T$ is a finite power of the Prüfer quasi-cyclic $p$-group $\mathbb{Z}_p^\infty$; $T \simeq \mathbb{Z}_p^d$ is injective among abelian groups. For $H \leq T$, $H^\circ$ consistently denotes the maximal subtorus of $H$.

A group of finite Morley rank is $U_p^\perp$ if it has no infinite elementary abelian $p$-subgroup. $U_p^\perp$ groups conjugate their Sylow $p$-subgroups [3, Theorem 4], i.e. their maximal (non-necessarily definable) $p$-subgroups; these are finite extensions of $p$-tori. Hence, for $S$ a Sylow $p$-subgroup of a $U_p^\perp$ group, $S^\circ$ is a $p$-torus.

Given a $U_p^\perp$ group, 3 measures of its Sylow $p$-subgroups are available. One can consider the **Prüfer $p$-rank** $\text{Pr}_p(G)$, which is the number of $\mathbb{Z}_p^\infty$ factors in a Sylow $p$-subgroup. One can also estimate the **normal $p$-rank** $n_p(G)$, which is the maximal $p$-rank of an elementary abelian $p$-group normal in a Sylow $p$-subgroup. Or one can simply compute the **$p$-rank** $m_p(G)$, which is the maximal $p$-rank of an elementary abelian $p$-subgroup. All 3 numbers are well defined by conjugacy of the Sylow $p$-subgroups, and $m_p(G) \geq n_p(G) \geq \text{Pr}_p(G)$.

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1. The $n$-rank

**Lemma 1** If $G$ is a connected, $U^1_p$ group, then $n_p(G) = \text{Pr}_p(G)$.

**Proof.** Let $S$ be a Sylow $p$-subgroup of $G$, $V < S$ an elementary abelian normal subgroup, and $v \in V$. As $V < S$, $v^{S^v} \subseteq V$ which is finite; by connectedness, $S^v$ centralizes $v$. So $v \in C_S(S^v) = S^v$ by [3, Corollary 3.1], and $V \leq S^v$. \hfill $\square$

2. Not quite a digression

For a $p$-torus $T \simeq \mathbb{Z}_p^{d \infty}$, $\Omega_{p^d}(T)$ denotes the set of elements of order at most $p^n$.

**Fact 1** Let $\varphi$ be an automorphism of finite order of a $p$-torus $T \simeq \mathbb{Z}_p^{d \infty}$.

1. Suppose $\Omega_{p^d}(T) \leq C_T(\varphi)$. Then $\varphi = \text{Id}$.

2. Suppose $\Omega_{p^d}(T) \leq C_T(\varphi)$. If $p = 2$, then $\varphi^2 = \text{Id}$. If $p \neq 2$, then $\varphi = \text{Id}$.

**Proof.** This must be classical but I know no reference.

1. Up to taking a power of $\varphi$, we may assume that $\varphi$ has prime order $q$. Let $x \notin C_T(\varphi)$ have minimal order. Then $\varphi(x^p) = x^p$ so there is $y \in \Omega_{p^d}(T) \setminus \{1\}$ with $\varphi(x) = xy$. By assumption $y \in C_T(\varphi)$, so $x = \varphi^q(x) = xy^q$ and $q = p$. Let $\hat{x}$ and $\hat{y}$ be such that $\hat{x}^p = x$ and $\varphi(\hat{x}) = \hat{x}\hat{y}$. Then $\hat{y}^p = y$ so $\hat{y} \in \Omega_{p^d}(T) \leq C_T(\varphi)$ and $\hat{x} = \varphi^q(\hat{x}) = \hat{x}\hat{y}^q = \hat{x}y$, a contradiction.

2. Since $\varphi$ centralizes $\Omega_{p^d}(T)$, for $x \in \Omega_{p^d}(T)$ there is $y \in \Omega_{p^d}(T)$ with $\varphi(x) = xy$; hence $\varphi^p(x) = xy^q = x$ and $\Omega_{p^d}(T) \leq C_T(\varphi^p)$. So $\varphi^p = \text{Id}$; we may assume $p \neq 2$. Represent $\varphi|_{\Omega_{p^d}(T)}$ by a matrix $M \in \text{GL}_d(\mathbb{Z}/p^3\mathbb{Z})$. As $\Omega_{p^d}(T) \leq C_T(\varphi)$, the reduction of $M$ modulo $p$ is the identity: there is a matrix $N$ with $M = \text{Id} + pN$. Since $\varphi^p = \text{Id}$,

$$0 \equiv \sum_{\ell = 1}^{p^d} \left( \begin{array}{c} p \\ \ell \end{array} \right) p^\ell N^\ell \equiv p^2 N + \frac{p(p-1)}{2} p^2 N^2 \quad [p^3].$$

Since $p \neq 2$, $p$ divides $\frac{p(p-1)}{2}$, so $p^2 N \equiv 0[p^3]$ and $N \equiv 0[p]$. Hence the reduction of $M = \text{Id} + pN$ modulo $p^2$ is the identity: $M$ centralizes $\Omega_{p^d}(T)$, and $\varphi$ is trivial. \hfill $\square$

**Consequence** If $p \neq 2$, the restriction map $\rho : \text{Aut}(T) \rightarrow \text{Aut}(\Omega_{p^d}(T))$ kills no element of finite order. In particular if $W$ is a finite subgroup of $\text{Aut}(T)$ then $W$ embeds into $\text{Aut}(\Omega_{p^d}(T)) \simeq \text{GL}_d(\mathbb{F}_p)$. If $p = 2$ then $\ker \rho|_W \rightarrow (\mathbb{Z}/2\mathbb{Z})^d$.

**Proof.** The only non-immediate claim is about the rank of $K = \ker \rho|_W$ when $p = 2$. Observe that $K$ has exponent 2, so it is abelian. We go in a direction that will prove fruitful. Taking automorphism groups changes inductive limits to projective limits, so $\text{Aut}(T) \simeq \varprojlim \text{Aut}((\mathbb{Z}/p^n\mathbb{Z})^d) = \varprojlim \text{GL}_d(\mathbb{Z}/p^n\mathbb{Z}) = \text{GL}_d(\mathbb{Z}_p)$. Hence $K$ embeds into $\text{GL}_d(\mathbb{Z}_2)$. Now elements of $K$ are simultaneously diagonalizable over $\overline{\mathbb{Q}}_2$ with eigenvalues $\pm 1$, so...
$K$ embeds into $\{\pm 1\}^d$. \hfill $\square$

We could use a similar method to get a lazy bound on $\text{rk} W$ for $W \leq \text{Aut}(T)$ an elementary abelian $p$-group; observe how we are naturally moving towards the $p$-adic representation. Anyway, embedding into $\text{GL}_d(\mathbb{F}_p)$, i.e. restricting to $\Omega_p(T)$, was too clumsy in the first place. For instance, any element of $\text{GL}_d(\mathbb{F}_p)$ comes from an element of $\text{GL}_d(\mathbb{Z}_p)$, but not necessarily from one of finite order. (The reader may check that $\text{GL}_2(\mathbb{Z}_3)$ has no element of order 5.) Representation-theoretically speaking, embedding into $\text{GL}_d(\mathbb{Z}_p)$ is more appropriate, and this is what we shall now do.

3. Bounding the $m$-rank

Let $\varphi$ be an automorphism of order $p$ of a $p$-torus $T \simeq \mathbb{Z}_p^d$.

**Fact 2** (Maschke’s Theorem) Let $T_1 \leq T$ be a $\varphi$-invariant subtorus. Then there is a $\varphi$-invariant subtorus $T_2 \leq T$ such that $T = T_1 + T_2$ and $T_1 \cap T_2 \leq \Omega_p(T_1)$.

**Proof.** There is a subtorus $T_0 \leq T$ with $T = T_1 \oplus T_0$. Let $\pi$ be the projection on $T_1$ along $T_0$ and $\hat{\pi} = \sum_{i=0}^{p-1} \varphi^i \pi \varphi^{-i}$. Then $\hat{\pi}$ is $\varphi$-covariant, $\text{im} \hat{\pi} = T_1$, and $\hat{\pi}(t_1) = pt_1$ for $t_1 \in T_1$. Take $T_2$ to be the maximal subtorus of $\text{ker} \hat{\pi}$.

**Fact 3** ($\varphi, T$ as above) If $C_{\hat{\pi}}(\varphi) = 1$ then $p - 1|d$ and $\text{Id} + \varphi + \cdots + \varphi^{p-1} = 0$.

**Proof.** (This again must be well known.) We may assume $p \neq 2$. Let $\tau \leq T$ be isomorphic to $\mathbb{Z}_p^\infty$, and set $\Theta = \sum_{i=0}^{p-1} \varphi^i(\tau)$; $\Theta$ is $\varphi$-invariant and $\text{Pr}_p(\Theta) \leq p$. So by Maschke’s Theorem, we may assume $d \leq p$. As in the proof of the Consequence above, let us view $\varphi$ as an element of order $p$ of $\text{GL}_d(\mathbb{Z}_p) \leq \text{GL}_d(\mathbb{Q}_p)$. By assumption, 1 is not an eigenvalue.

The minimal polynomial $\mu$ of $\varphi$ over $\mathbb{Q}_p$ divides $X^p - 1 = (X - 1)(1 + X + \cdots + X^{p-1})$, so it divides $1 + X + \cdots + X^{p-1}$. The latter is irreducible over $\mathbb{Z}_p$ by Eisenstein’s criterion, so $\mu = 1 + X + \cdots + X^{p-1}$. But $\mu$ divides the characteristic polynomial which has degree $d$. So $p - 1 \leq d \leq p$. Over $\mathbb{Q}_p$, $\varphi$ has $p - 1$ eigenvalues, which sum to $-1$. So if $d = p$, one of them, say $j$, occurs twice: hence $1 + \text{Tr} \varphi = j \in \mathbb{Q}_p$, against $p \neq 2$. So $d = p - 1$. \hfill $\square$

**Lemma 2** For $W \leq \text{Aut} \mathbb{Z}_p^d$ an elementary abelian $p$-group, $\text{rk} W \leq \frac{d}{p-1}d$.

**Proof.** $E = \mathbb{Q}_p^d$ is a sum of $W$-irreducible subspaces $\oplus_{\ell \in I} E_\ell \oplus \oplus_{j \in J} F_j$ with $E_\ell$’s the $W$-trivial lines. Since $W$ is abelian, it acts $W$-covariantly. Let $\rho_j : W \rightarrow \text{Aut}_W(F_j)$ be the restriction map, with (non-trivial) image $W_j$ and kernel $K_j$. Each $\text{End}_W F_j$ is a skew-field by Schur’s Lemma, so the abelian group $W_j$ of exponent $p$ has order $p$. As $C_W(E) = 1$, $W \hookrightarrow \prod_{j \in J} W/K_j$, and $\text{rk} W \leq \# J$. By Fact 3, $\dim F_j \geq p - 1$, whence $\# J \leq \frac{d}{p-1}$. \hfill $\square$

**Corollary 1** Let $G$ be a connected, $U_p^\perp$ group. Then $m_p(G) \leq \frac{p^d}{p-1} \text{Pr}_p(G)$.

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Proof. For $V \leq S$ an elementary abelian subgroup of a Sylow $p$-subgroup $S$, write $V = (V \cap S^0) \oplus W$. By [3, Corollary 3.1], $C_5(S^0) = S^0$; use Lemma 2. \hfill \Box

4. Maximal abelian $p$-subgroups

**Thesis** [2, Theorem 1.2] Let $G$ be a connected, $U_p^*$ group with $m_p(G) \geq 3$. Then any maximal elementary abelian $p$-subgroup $V < G$ has $p$-rank at least 3.

The flaw in [2] lies at the bottom of page 172. On the very last line, “commutation with $v$” need not in general be “a map from $\Omega_1(T)/A$ to $A$”. Observe that in [2] Theorem 6.4 relies on Corollary 4.2, which relies on Theorem 1.2.

**Counter-Example** In $\mathrm{PSL}_5(\mathbb{C})$ let $\Theta$ be the usual torus and $\sigma$ be the Weyl element naturally associated with the 5-cycle $(12345)$. Let $\theta \in C_\Theta(\sigma) \setminus \{1\}$. Then $(\theta, \sigma)$ does not extend to an elementary abelian 5-group of rank 3.

**Proof.** The actual computations will take place in $\mathrm{SL}_5(\mathbb{C})$. Let $\lambda = e^{\frac{2\pi i}{5}} \in \mathbb{C}$; then $Z(\mathrm{SL}_5(\mathbb{C})) = \{\lambda^k \text{Id}\}$. The matrix $s = (\delta_{j,j+1}) \in \mathrm{SL}_5(\mathbb{C})$ (equality modulo 5) reduces modulo $Z(\mathrm{SL}_5(\mathbb{C}))$ to $\sigma \in \mathrm{PSL}_5(\mathbb{C})$; conjugation by $s$ rotates coefficients of a matrix $(m_{i,j}) \in \mathrm{SL}_5(\mathbb{C})$ along the 5 (complete) diagonals. So given $\theta \in \Theta \leq \mathrm{PSL}_5(\mathbb{C})$ and a diagonal matrix $t \in \mathrm{SL}_5(\mathbb{C})$ representing it, one sees that $[\sigma, \theta] = 1$ if $t_{i,j} = \lambda^{k+i}$ for some integers $k$ and $\ell$; thus $C_\Theta(\sigma)$ has order 5. Fix $\theta \in C_\Theta(\sigma) \setminus \{1\}$. Conjugation by $t$ on $(m_{i,j})$ multiples $m_{i,j}$ by $\lambda^{j-i}$. So $C(\theta) = \Theta \rtimes \langle \sigma \rangle$, and $(\theta, \sigma)$ is maximal. \hfill \Box

The following merely serves the purpose of exposing an important method.

**Observation** Let $G$ be a connected, $U_p^*$ group, and $S \leq G$ a Sylow $p$-subgroup. Then $S$ is connected iff abelian iff nilpotent.

**Proof.** Only one claim is non-trivial; we prove it by induction on the Morley rank (read: dimension) of $G$. Suppose $S$ nilpotent; let $\omega \in S$. Then by nilpotence, $\tau = C^{S^0}_S(\omega) \neq 1$. By [3, Corollary 3.1], $\omega$ lies in any maximal $p$-torus of $C^S(\omega)$, so $\omega \in C^S(\tau)$. Hence $C^S(\omega) \leq C^S(\tau)$. If $C^S(\tau) < G$ we are done by induction. Otherwise $\tau$ is central and we can factor by $Z^0(G)$, pursuing by induction. \hfill \Box

I shall now bring my reader some comfort.

**Lemma 3** The thesis of [2, Theorem 1.2] holds for $p = 2$, and so does [2, Corollary 6.5].

**Proof.** Suppose $m_2(G) \geq 3$; clearly $\text{Pr}_2(G) \geq 2$. Let $i, j$ be 2 commuting involutions; by torality [3, Theorem 3] there is a Sylow 2-subgroup $S$ with $i \in S^0$ and $j \in S$.

Suppose $\text{Pr}_2(G) \geq 3$. If $j \in S^0$ we are done. If not, consider the map $\varphi(k) = [j, k] : \Omega_2(S^0) \to \Omega_2(S^0)$. Then $\text{rk im } \varphi \leq \text{ker } \varphi$ and $\text{rk im } \varphi + \text{rk ker } \varphi \geq 3$, so $\text{rk ker } \varphi \geq 2$ and we are done. From now on, suppose $\text{Pr}_2(G) = 2$ (so $m_2(G) \leq 4$) and let $V = \Omega_2(S^0)$.

Assume first that $j \in S \setminus S^0$. If $j$ inverts $S^0$ then $(i, j) \leq (V, j)$: we are done. Otherwise $\tau = C^{S^0}_S(j) \neq 1$. If $i \notin \tau$ then $(i, j) \leq (i, \Omega_2(\tau), j)$: we are done. So assume $i \in \tau \leq C^S(j)$. By torality [3, Theorem 3 and Corollary 3.1], $i$ lies in a 2-torus of $C^S(j)$ and $j$ lies in any 2-torus of $C^S(j)$, so $i$ and $j$ are cotoral.
So assume that \( j \in S^o \), that is \( V = \langle i, j \rangle \). By assumption there is an elementary abelian 2-subgroup of rank 3: \( A = \langle r, s, t \rangle \leq S \); clearly \( A \cap S^o \neq 1 \), say \( r \in V \). If \( s \) or \( t \) is in \( V \) then \( \langle i, j \rangle = V \leq A \): we are done. Suppose that \( s \) and \( t \) (hence \( st \) as well) lie in \( S \setminus S^o \). Since \( |\text{Aut}(V)| = 6 \), one of \( s, t, st \) must centralize \( V = \langle i, j \rangle \): we are done again.

Here is a final word on counter-examples.

**Lemma 4** Let \( G \) be a counter-example to [2, Theorem 1.2]. Then \( \Pr_p(G) = p - 1 \). In particular, [2, Theorem 1.2] also holds for \( p = 3 \).

**Proof.** By Lemma 3, \( p \geq 3 \). As \( m_p(G) \geq 3 \), one sees with Corollary 1 that \( \Pr_p(G) \geq 2 \). Equality can hold only for \( p = 3 \); as there is an elementary 3-group of rank 3, there is an automorphism of order 3 fixing \( \Omega_3(\mathbb{Z}_3^\infty) \), against Fact 1: equality cannot hold.

Hence \( \Pr_p(G) \geq 3 \). Let \( V = \langle \alpha, \omega \rangle \) be a maximal abelian \( p \)-group and \( S \geq V \) a Sylow \( p \)-subgroup. By torality we may assume \( \alpha \in S^o \), so \( \omega \in S \setminus S^o \). If \( C^o_{S^o}(\omega) \neq 1 \) then by maximality, \( \alpha \in C^o_{S^o}(\omega) \leq C^o(S) \), and as in the proof of Lemma 3, \( \alpha \) and \( \omega \) are cotoral, a contradiction. Hence \( C^o_{S^o}(\omega) = 1 \). Let \( \varphi \in \text{End} \Omega_p(S^o) \) map \( x \) to \( [x, \omega] \); writing \( \omega \) as an automorphism, \( \varphi(x) = \omega(x) - x \) and \( \varphi^n(x) = \sum_{i=0}^n (-1)^i \omega^i(x) \). As \( (-1)^i (p-1)^{-1} \equiv 1 \pmod{p} \), \( \varphi^{p-1} = \text{Id} + \omega + \cdots + \omega^{p-1} \). But \( C^o_{S^o}(\omega) = 1 \), so Fact 3 applied to \( \omega \) implies \( \varphi^{p-1} = 0 \).

Since \( \ker \varphi = C_{\Omega_p(S^o)}(\omega) = \langle \alpha \rangle \), one has \( \text{rk} \Omega_p(S^o) \leq p - 1 \).

**References**

