Weyl groups of small groups of finite Morley rank

Jeffrey Burdges, Adrien Deloro

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Abstract

We examine Weyl groups of minimal connected simple groups of finite Morley rank of degenerate type. We show that they are cyclic, and lift isomorphically to subgroups of the ambient group.

1 Introduction

Infinite groups of finite Morley rank have little truly geometric structure; however, their algebraic properties are remarkably reminiscent of algebraic groups. The strongest conjecture to this effect is the *Cherlin-Zilber algebraicity conjecture* which posits that an infinite simple group of finite Morley rank is a linear algebraic group over an algebraically closed field.

There are a number of partial results towards this conjecture and a complete proof in the *even & mixed type* cases [ABC08], read potentially characteristic two. In other cases, much recent work has followed three themes : various identification theorems [Bur07b, Del08a], an analysis of the minimal simple groups where Bender's method is well understood [Bur07a], and the analysis of torsion using genericity arguments [BBC07, BC08b], as well as work involving several techniques [CJ04, BCJ07, Del08b, AB08].

A major part of the combined thread is the analysis of the Weyl group $W = N(T)/C(T)$ of G associated to some maximal decent torus T. Here a *decent torus* is merely the smallest definable subgroup containing some divisible abelian torsion subgroup, such as a p-torus $\mathbb{Z}(p^\infty)^n$. One may speak of the Weyl group of G because maximal decent tori are conjugate in a group of finite Morley rank [Che05].

The main result of the present article is that the Weyl group of a minimal connected simple group of finite Morley rank is cyclic. We also show that, for $p > 2$ dividing $|W|$, the group G has no divisible p-torsion.

Theorem 4.1. *Let G be a minimal connected simple group of finite Morley rank, and let T be a decent torus of G. Then the Weyl group $W := N(T)/C(T)$*

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is cyclic, and has an isomorphic lifting to G; moreover no primes dividing $|W|$ appear in T except possibly 2.

In spirit, the proof proceeds by proving the three conclusions in the reverse order.

Our primary concern here is the case of groups of *degenerate type*, where Sylow 2-subgroups are finite, or equivalently trivial [BBC07]. The second author's thesis work covers the *odd type* case (see Fact 4.2 below), where the Sylow 2-subgroup is divisible-abelian-by-finite, and the afore mentioned classification in even & mixed types [ABC08] covers those cases.

In degenerate type, one may also extract some unlikely number theoretic consequences from this argument.

Corollary 3.10*. *Let G be a minimal connected simple group of finite Morley rank and degenerate type, and let T be a nontrivial decent torus of G. Then G interprets a bad field in characteristic p for every prime divisor p of $|W|$.*

A bad field $(k, H, +, \cdot)$ is a field k of finite Morley rank with a proper non-trivial definable subgroup H of it's multiplicative subgroup. Such fields exist in characteristic zero [BHMPW08] but are quite unlikely in positive characteristic p because then :

1. there are only finitely many primes of the form $\frac{p^n-1}{p-1}$ by [Wag97].
2. $(p^n - 1)_\pi \asymp p^{\alpha n}$ where π denotes the set of primes appearing in the bad field's multiplicative subgroup H, $(\cdot)_\pi$ denotes removing all primes but these, and $\alpha := \mathrm{rk}(H)/\,\mathrm{rk}(k)$ [HW].

These conditions are regarded as unlikely by specialists.

As an exercise, we point out that Corollary 3.10 is quite easy when p is the *minimal* prime divisor of $|W|$ and T is a maximal decent torus. First Fact 2.5 provides a Borel subgroup B containing p-unipotence. So one considers a B-minimal $A \leq U_p(B)$. If $C_B(A) < B$ then Zilber's field theorem produces the bad field, as desired. Otherwise $C_B(A) = B$ implies that $\cup B^G$ is generic in G. So then, by [BC08b, Theorem 1], B contains a maximal decent torus of G, whose Weyl group normalizes B. Finally the Weyl group element may not live either inside or outside B by Facts 2.2 and 2.6.

In §2 we expose minor adjustments of various known results, and recall some facts that will be used. §3 is devoted to analyzing Cartan subgroups $C(T)$ where T a maximal decent torus, as well as how Weyl cosets lift into elements of the group. Theorem 4.1 is proved in §4. In §5 we conclude that under the assumption that maximal decent tori are large enough and there is a non-trivial Weyl group, only finitely many Borel subgroups can contain the Carter subgroup.

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2 Literature

This section serves as general background. However, the reader should be aware that later sections freely use standard material from [BN94] without any comment.

2.1 Weyl Groups

The Weyl group of an algebraic group is the quotient of the normalizer of a maximal algebraic torus by the centralizer of the same torus. In general, groups of finite Morley rank need not contain any algebraic torus, but frequently have so-called decent tori. A *decent torus* is a divisible abelian group which is the *definable hull* of its torsion, i.e. is the intersection of all definable subgroups containing its torsion subgroup. The group $W_T := N(T)/C(T)$ is the Weyl group associated to a given torus T. Here one may use only the connected component $C^\circ(T)$ of the centralizer by the following.

Fact 2.1 ([AB08, Theorem 1]). *Let T be a decent torus of a connected group H of finite Morley rank, Then $C(T)$ is connected.*

We naturally say "the Weyl group *of G*" when T is a maximal decent torus; this is well-defined because maximal decent tori are always conjugate [Che05, Last lines].

In the present article, we use repeatedly the fact that connected solvable groups of finite Morley rank have trivial Weyl groups. This fact is originally due to Oliver Frécon, but we employ the following formulation.

Fact 2.2 ([AB08, Lemma 6.6]). *Let H be a connected solvable group of finite Morley rank and let K be a definable connected subgroup of H such that $[N_H(K) : K] < \infty$. Then $N_H(K) = K$.*

A priori, if T is a decent subtorus of another torus S, then its Weyl group W_T is just a section of W_S. But in the special case of minimal connected simple groups, one can say more thanks to the preceding fact.

Lemma 2.3. *Let G be a minimal connected simple group of finite Morley rank, and let T be a nontrivial decent torus of G. Then the Weyl group $W_T = N(T)/C(T)$ associated to T is naturally isomorphic to a subgroup of the Weyl group W of G.*

Proof. Let S be a maximal decent torus containing T, so that $W = W_S = N(S)/C(S)$. Then $S \leq C(T) \lhd N(T)$. A Frattini argument using conjugacy of maximal decent tori says that $N(T) = C(T) \cdot N_{N(T)}(S)$ and

$$W_T = N(T)/C(T) = N_{N(T)}(S)/N_{C(T)}(S).$$

Of course $C(T)$ is solvable by minimal simplicity of G and connected by Fact 2.1. So Fact 2.2 says that $N_{C(T)}(S) = C_{C(T)}(S) = C(S)$, which implies that

$$W_T = N_{N(T)}(S)/C(S) \leq N(S)/C(S) = W_S = W. \qquad \square$$

A similar argument shows that, if G is a minimal connected simple group containing a non-trivial decent torus, the Weyl group of G is isomorphic to $W_Q := N(Q)/Q$ where Q is a Carter subgroup containing the maximal decent torus T. Indeed one need not specify the Carter subgroup Q as they are all conjugate [Fré08].

2.2 Connected torsion

We shall employ numerous results about connected torsion subgroups. A connected solvable p-subgroup of a group of finite Morley rank is always a central product of a p-torus, i.e. a power of the Prüfer p-group $\mathbb{Z}(p^\infty)$, and a p-unipotent subgroup, i.e. a definable connected nilpotent p-group of bounded exponent. A number of recent results provide criteria either forcing unipotence to appear or preventing it from occurring.

Fact 2.4 ([BC08b, Theorem 3])**.** *Let G be a connected group of finite Morley rank, π a set of primes, and a any π-element of G such that $C_G^\circ(a)$ has $\pi^\perp$ type. Then a belongs to a π-torus.*

Fact 2.5 ([BC08b, Corollary 5.3])**.** *Let G be a minimal connected simple group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup in the centralizer of any r-element representing an r-element of W.*

Fact 2.6 ([AB08, Lemma 4.3])**.** *Let G be a minimal connected simple group of finite Morley rank. Let B be a Borel subgroup of G such that $U_p(B) \neq 1$ for some prime p. Then $p \not| [N_G(B) : B]$.*

The preceding facts can be used to prove that minimal connected simple groups are covered by their Borel subgroups.

Fact 2.7 ([AB08, Corollary 4.4])**.** *Let G be a minimal connected simple group of finite Morley rank. Any torsion element x of G lies inside any Borel subgroup of G which contains $C^\circ(x)$.*

We will make crucial use of the following striking consequence.

Lemma 2.8. *Let G be a minimal connected simple group of finite Morley rank, let T be a nontrivial maximal torus, and let $\bar{x} \in N(T)/C(T)$. Then any lifting of $\bar{x}$ to a torsion element $x \in N(T) \setminus C(T)$ has same order as $\bar{x}$. In other words $\langle x \rangle \cap C(T) = 1$ whenever $x \in N(T) \setminus C(T)$ is torsion.*

Proof. Suppose towards a contradiction that $x^n \in C(T)$ with $x^n \neq 1$. Fix some Borel subgroup B containing $C^\circ(x^n) \geq C^\circ(x)$. Since x is torsion, $x \in B$ by Fact 2.7. But $T \leq C^\circ(x^n) \leq B$ too, contradicting Fact 2.2. $\square$

2.3 Intersections

In general, p-unipotent subgroups are more difficult to handle than tori, but their role inside minimal connected simple groups is well understood, thanks to the following trivial alteration of [CJ04, Proposition 3.11].

Fact 2.9 ([Bur07a, Lemma 2.1]). *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups of G satisfying $U_{p_i}(B_i) \neq 1$ for some prime p_i ($i = 1, 2$). Then $F(B_1) \cap F(B_2) = 1$.*

Here the p-unipotent radical $U_p(\cdot)$ denotes the largest p-unipotent subgroup, and the Fitting subgroup $F(\cdot)$ is the largest normal nilpotent subgroup. One normally invokes this fact to say that, if B is a Borel subgroup with $U_p(B) \neq 1$, then $B \cap B^g$ is $p^\perp$ for all $g \notin N_G(B)$. Otherwise one considers a third Borel subgroup containing $C(t)$ where $t \in B \cap B^g$ has order p. Such a situation violates Fact 2.9 because $C_{U_p(B)}(t)$ and $C_{U_p(B^g)}(t)$ would be infinite.

As in [JF0n], a *maximal unipotence parameter* $\tilde{q}$ of a group B of finite Morley rank is a pair (p, r) where p is a prime number or ∞, and r an integer or ∞, such that $p = \infty \Leftrightarrow r < \infty$.

The phrase "$\tilde{q}$-subgroup" has the obvious meaning if p is prime; but "$\tilde{q}$-subgroup" means a $U_{(0,r)}$-subgroup [Bur04a] in the second case.

Fact 2.10 ([Del07, Lemme 1.9.1]). *Let G be a minimal connected simple group, and let B be a Borel subgroup of G. Also let $\tilde{q}$ be a maximal unipotence parameter of B. If A is a normal $\tilde{q}$-subgroup of B, then B is the only Borel subgroup containing A for which $\tilde{q}$ is still a maximal unipotence parameter.*

2.4 Actions

We require the following easy result which we have not located in the literature.

Lemma 2.11. *Let G be a group of finite Morley rank and $U \leq G$ be a p-unipotent subgroup. If $t \in N_G(U)$ is such that $C_U(t) \neq 1$, then $C_U(t)$ is infinite.*

In particular, if G is a minimal connected simple group and $B \leq G$ a Borel subgroup with $U_p(B) \neq 1$, then $C_{U_p(B)}(t)$ is infinite for any $t \in G$ with $C_{U_p(B)}(t) \neq 1$.

This will follow from the following two facts.

Fact 2.12 ([Bur04b, Fact 3.3]). *Let $H = KT$ be a group of finite Morley rank. Suppose that T is a solvable π-group of bounded exponent and that K is a definable abelian normal $\pi^\perp$-subgroup of H. Then $H = [H, T] \oplus C_H(T)$.*

Fact 2.13 ([BC08a, Lemma 2.5], see also [Bur04a, Fact 3.12]). *Let H be a solvable group of finite Morley rank, v a definable automorphism of H of order q^n for some prime q, and K a definable normal v-invariant connected $q^\perp$-subgroup of H. Then*

$$C_H^\circ(v \bmod K) = C_H^\circ(v)K/K.$$

The notation $C^\circ_G(v \bmod K)$ refers to the connected component of the preimage in G of $C_{G/K}(v)$.

Proof of Lemma 2.11. Let $x \in C_U(t)^\#$ and let $Z_i := Z^\circ_i(U)$ for $i \in \mathbb{N}$. As U is nilpotent and connected, there is some integer i such that $x \in Z_{i+1} \setminus Z_i$. In particular t when acting on the *connected* abelian quotient $Y := Z_{i+1}/Z_i$ centralizes $\bar{x}$.

We now write $t = uvw$ with $u, v, w \in d(t)$ such that $d(u)$ is divisible, v is a p'-element, and w is a p-element. By [Wag01, Corollary 9], there are no torsion-free definable sections of the multiplicative subgroup of a field of finite Morley rank in positive characteristic. In consequence, $Y \cdot d(u)$ must be nilpotent, that is u centralizes Y.

Notice that $v \in d(t) \leq C(\bar{x})$. By Fact 2.12, $C_Y(v) \cong Y/[Y, v]$ is connected. So $C_Y(v)$ is nontrivial because it contains $\bar{x}$, and hence $Y_1 := C_Y(v) = C_Y(uv)$ is infinite.

Eventually the p-element w acts on Y_1. As Y_1 is an infinite p-group of bounded exponent, $Y_2 := C_{Y_1}(w) \leq C_Y(uvw)$ is infinite, by [BN94, Corollary 6.20]. Hence $C_Y(t)$ is infinite. It follows from Fact 2.13 that $C_{Z_i}(t)$, and then also $C_U(t)$, are infinite.

To prove the second claim, it suffices to notice that $t \in N_G(B) = N_G(U_p(B))$ by Fact 2.9. $\square$

We will also use the following generation principle.

Fact 2.14 ([Bur04b, Fact 3.7])**.** *Let H be a solvable $q^\perp$-group of finite Morley rank. Let E be a finite elementary abelian q-group acting definably on H. Then $H = \langle C_H(E_0) : E_0 \leq E, [E : E_0] = q\rangle$.*

Another useful fact is this nilpotence criterion.

Fact 2.15 ([Wag97, Theorem 2.4.7] plus [BN94, Exercise 14 p.79])**.** *Let H be a connected solvable group of finite Morley rank with an automorphism of prime order whose centralizer in H is finite. Then H is nilpotent.*

Almost any article about Weyl groups in our context will employ its p-adic representation : $\mathrm{End}(\mathbb{Z}(p^\infty)^n)$ is the ring $M_{n\times n}(\mathbb{Z}_p)$ of $n \times n$ matrices over the p-adic integers $\mathbb{Z}_p$. For us, this isomorphism, sometimes referred to as the Tate module, will be exploited via the following argument.

Fact 2.16. *Let G be group of finite Morley rank. Let T be a p-torus and $a \in N(T) \setminus C(T)$ be a p-element. Then the Prüfer p-rank of T is at least $p - 1$.*

Proof. We may assume that a^p centralizes T. Represent a as an automorphism of $\mathrm{GL}_d(\mathbb{Z}_p)$, where d denotes the Prüfer p-rank of T, and work in $\mathrm{GL}_d(\mathbb{Q}_p)$. The minimal polynomial μ_a divides $X^p - 1 = (X - 1)(1 + X + \cdots + X^{p-1})$. Of course $a \notin C(T)$ implies $\mu_a \neq X - 1$. On the other hand $1 + X + \cdots + X^{p-1}$ is irreducible over $\mathbb{Q}_p$. So it follows that μ_a has degree at least $p - 1$. Now the characteristic polynomial has degree d and is a multiple of μ_a, whence $d \geq p - 1$. $\square$

3 Cartan subgroups

A Cartan subgroup of an algebraic group is the centralizer of a maximal torus. In this section we analyze the "Cartan subgroup" $C(T)$ of our group by analyzing representatives of the Weyl group, eventually proving the following.

Theorem 3.1 (Henri/Élie)**.** *Let G be a minimal connected simple group of degenerate type. Suppose also that G has a nontrivial Weyl group $W := N(T)/C(T)$ where T is a maximal decent torus. Then the Cartan subgroup $C(T)$ is nilpotent, and thus is a Carter subgroup of G.*

Moreover, $C(T)$ is actually a Borel subgroup if either $C(T)$ is not abelian or G has Prüfer q-rank ≥ 3 for some prime q.

Throughout this section we use the hypotheses and notation of Theorem 3.1, i.e. G is a minimal connected simple group of degenerate type with a nontrivial Weyl group $W := N(T)/C(T)$. Of course the decent torus T is necessarily nontrivial under this hypothesis.

3.1 Weylian elements

We say that a torsion element a of G is *Weylian* if it normalizes some maximal decent torus T. Also let τ denote the set of primes occurring in a maximal decent torus T. By conjugacy of decent tori [Che05], τ is just the set of primes for which G contains divisible torsion. Then

$$\tau' := \{\, p \text{ prime} \mid \mathbb{Z}(p^\infty) \text{ does not embed into } G \,\}\,.$$

Of course an element a of G is τ' iff it's (finite) order $|a|$ is a τ' number. We note that both of these properties are closed under taking powers. In §4, we will show that any element of W lifts to a Weylian τ'-element.

Lemma 3.2. *If a is a τ'-element, then there is a unique Borel subgroup B_a containing $C^\circ(a)$, and a is a product of unipotent elements of B_a.*

Proof. Let B_a be a Borel subgroup containing $C^\circ(a)$. Then a lies inside B_a by Fact 2.7. As $G \geq B$ has no divisible τ' torsion, a lies inside $\Pi_p U_p(B_a)$, proving the second part. So the first part follows from Fact 2.9. □

We preserve this B_a notation for the unique Borel containing $C^\circ(a)$, when a is τ', throughout the remainder of the article.

Lemma 3.3. *If $p \in \tau'$ divides $|W|$, then $C(T)$ is $p^\perp$.*

Proof. There is a Weylian p-element a normalizing $C(T)$ by the usual torsion lifting principle [BN94, Ex. 11 p. 93; Ex. 13c p. 72]. Clearly $U_p(C(T)) \neq 1$ since T is $\pi^\perp$ by definition (see [BN94, §6.4]). If $U_p(C(T)) \neq 1$, then $T \leq C(T) \leq B_a$ by Fact 2.9. As $a \in B_a$ too, this contradicts the fact that connected solvable groups have trivial Weyl groups (Fact 2.2). □

Lemma 3.4. *If $a \in G$ is Weylian, and $U_p(C^\circ(a)) \neq 1$ for each prime divisor p of $|a|$, then a is a τ'-element.*

Proof. Suppose that some prime $p \in \tau$ divides $|a|$. Then there is a power b of a which is a p-element. We may assume that a, and b, normalize T, by conjugacy of decent tori [Che05]. By [BN94, Corollary 6.20], there must be $t \in T^\#$ of order p such that $[b, t] = 1$.

By assumption, the Borel subgroup B containing $C^\circ(b)$ contains p-unipotence. It follows from Fact 2.6 that $b \in B$. As $b \in B \cap B^t$, $t \in N(B)$ by Fact 2.9. So again $t \in B$ using Fact 2.6.

As t has order p too, it follows that $C^\circ_{U_p(B)}(t) \neq 1$ by [BN94, Corollary 6.20]. So again $T \leq C^\circ(t) \leq B$ by Fact 2.9. Now once more we contradict the fact that connected solvable groups have trivial Weyl groups (Fact 2.2).

So there is no p-torus in T, or hence in G, as desired. $\square$

Now consider the minimal prime divisor r of $|W|$, which is nontrivial by hypothesis. By Fact 2.5, the centralizer of any r-element a representing an r-element of W contains r-unipotence. So Lemma 3.4 says :

Corollary 3.5. $r \in \tau'$.

In particular, there are Weylian τ'-elements.

In §4 we shall prove that any element of the Weyl group lifts to a Weylian τ'-element of G.

3.2 Carter subgroups

We next argue that the Cartan subgroup $C(T)$ is in fact a Carter subgroup of G.

Lemma 3.6. *If a is a Weylian τ'-element for T, then $C_{C(T)}(a)$ is trivial.*

Proof. We may assume that a has prime order p. Again let B_a be the unique Borel containing $C^\circ(a)$. Consider some $x \in C_{C(T)}(a)^\#$. By Lemma 2.11, $C_{U_p(B_a)}(x)$ is infinite. Thus $T \leq C^\circ(x) \leq B_a$ by Fact 2.9. Here again we contradict the fact that connected solvable groups have trivial Wely groups (Fact 2.2). Therefore $C_{C(T)}(a) = 1$, as desired. $\square$

Corollary 3.7. *$C(T)$ is a Carter subgroup of G.*

Proof. By Corollary 3.5, there is a Weylian τ'-element a. So $C_{C(T)}(a) = 1$ by Lemma 3.6. It follows from Fact 2.15 that $C(T)$ is nilpotent. So the result follows because $C(T)$ is almost self-normalizing by [BN94, Theorem 6.16]. $\square$

3.3 Invariance

We finally show that the Cartan subgroup $C(T)$ is a Borel subgroup in certain cases. Consider a Borel subgroup B_T containing $C(T)$.

Proposition 3.8. *B_T can be chosen to be W-invariant in either of the following cases :*

(i) if $C(T)$ is not abelian; or

(ii) if the Prüfer q-rank of T is ≥ 3 for prime q.

Proof. By [Bur07a, Proposition 4.1ii], every nilpotent subgroup of two distinct Borel subgroups is abelian. So part (i) follows from Corollary 3.7.

Assume now (ii), i.e. suppose that T has Prüfer q-rank ≥ 3. We may assume that no Borel subgroup containing $C^\circ(T)$ is abelian, as otherwise $B_T = C(T)$ is clearly W-invariant. In particular, every such Borel subgroup admits at least one unipotence parameter with $r > 0$ [Bur04b, JF0n, Theorem 2.12] For each Borel subgroup B_i containing $C(T)$, let $\tilde{q}_i$ be such a maximal unipotence parameter of B_i with $r > 0$. Choose B_T as to maximize its maximal unipotence parameter $\tilde{q}_T$ among all $\tilde{q}_i$.

Let $Y_T = [U_{\tilde{q}}(Z(F^\circ(B_T))), B_T]$. By [Fré05, JF0n, Theorem 2.18], Y_T is a homogeneous $\tilde{q}_T$-subgroup of B_T. Of course Y_T is clearly characteristic in B_T too. If $Y_T = 1$, then $U_{\tilde{q}}(Z(F^\circ(B_T))) \leq Z(B_T) \leq C(T)$, and Fact 2.10 implies that B_T is the only Borel subgroup containing $C(T)$.

So we may assume $Y_T \neq 1$. Then, by Fact 2.14, there is a subgroup $V \leq \Omega_q(T)$ of index q such that $X = C_{Y_T}(V) \neq 1$. Now X is central in $F^\circ(B_T)$ and is actually normalized by $C(T)$. As $C(T)$ contains a Carter subgroup of B_T, one has $B_T = F^\circ(B_T) \cdot C(T)$, so X is normal in B_T.

Consider a lifting $w \in N(T) \setminus C(T)$ of an element of W. As $\mathrm{pr}_q(T) \geq 3$, there is some $t \in (V \cap V^w)^\#$ and $C^\circ(t) \geq X, X^w$. Let B_t be any Borel subgroup containing $C^\circ(t) \geq C(T)$. Notice that $B_t \geq C^\circ(t) \geq X, X^w$. By maximality of $\tilde{q}_T$, it follows that $\tilde{q}_T$ is a maximal unipotence parameter of B_t. In particular, Fact 2.10 says $B_T = B_t = B_T^w$. Consequently B_T is W-invariant. ☐

Proposition 3.9. *Let a be a Weylian τ'-element for T. If a normalizes B_T, then a has no fixed point in B_T, B_T is nilpotent, and $C(T) = B_T$ is a Borel subgroup.*

Proof. We may assume that a has order p. First notice that $B_a \neq B_T$ as there is no Weyl group in a connected solvable group (Fact 2.2). So let $H := (B_T \cap B_a)^\circ$.

We claim that H is abelian. Otherwise [Bur07a, Theorems 4.3 & 4.5 (8)] say that B_a is involved in a maximal non-abelian intersection, and $F^\circ(B_a)$ is divisible. But this contradicts $U_p(B_a) \neq 1$. Hence H is abelian.

As there is no toral p-torsion in G, H must be $p^\perp$ by Fact 2.9. Consider the action of the p-element a on the definable abelian connected $p^\perp$ group H. By Fact 2.12, we may write $H = K \oplus L$ with $K = C_H(a)$ and $L = [H, a]$. Assume towards a contradiction that $K \neq 1$.

Let $A \leq U_p(B_a)$ be a B_a-minimal subgroup. We observe that $[A, a] = 1$ since $a \in U_p(B_a)$. Hence $[L, a] = L$, $[A, a] = 1$, and L normalizes A. A commutator computation proves $[L, A] = 1$.

Suppose first that $[A, K] = 1$. Then $[A, H] = 1$. So, by Fact 2.9, B_a is the only Borel subgroup containing $C^\circ(H)$. This proves that $N^\circ_{B_T}(H) = H$, whence

H is a Carter subgroup of B_T. As Carter subgroups of B_T are Carter subgroups of G, Lemma 3.6 and Corollary 3.7 imply that $K = 1 = C_H^\circ(a)$, a contradiction.

Hence $[A, K] \neq 1$. It follows that some nontrivial definable section of K embeds definably into the multiplicative subgroup of a field of characteristic p. So, by [Wag01, Corollary 9], no such section can be torsion-free. It now follows that K contains a non-trivial decent torus, say T_1. Notice that $a \in C(T_1)$ forces $C(T_1) \leq B_a$. Let $T_2 \geq T_1$ be the maximal decent torus of H. Then $N^\circ(H) \leq N^\circ(T_2) = C(T_2) \leq C(T_1) \leq B_a$. So H is a Carter subgroup of B_T, yielding the same contradiction as above.

This proves that $K = 1$. Hence $C_{B_T}^\circ(a) = K = 1$. By Fact 2.15, B_T is nilpotent. So Lemma 3.6 implies that there is no centralization at all. $\square$

We now prove Theorem 3.1.

Proof of Theorem 3.1. Let G be a minimal connected simple group of degenerate type, having a nontrivial maximal decent torus T and a nontrivial Weyl group $W = N(T)/C(T)$. By Corollary 3.7, a Cartan subgroup $C(T)$ is a Carter subgroup of G. The converse is also true because Carter subgroups are conjugate in G [Fré08].

If either $C(T)$ is abelian or T has Prüfer q-rank ≥ 3 for some prime number q, then, by Proposition 3.8, there is a W-invariant Borel subgroup containing $C(T)$. This Borel subgroup is then nilpotent by Proposition 3.9, and hence equal to $C(T)$. $\square$

3.4 Consequences

Corollary 3.10. *Let a be a Weylian τ'-element normalizing T and let B_a be the unique Borel subgroup containing $C^\circ(a)$. Then:*

- *for each prime p dividing $|a|$, B_a interprets a bad field of characteristic p;*
- *B_a contains a nontrivial decent torus;*
- *If B_a contains a q-torus of Prüfer rank 2 for some prime q, then it contains a Carter subgroup of G;*
- *B_a does not contain a q-torus of Prüfer rank ≥ 3 for any prime q.*

Proof. We may assume that a has order p. Fix a B_a-minimal subgroup A of $U_p(B_a)$. If $A \leq Z(B_a)$, then, by Fact 2.9, $B_a \cap B_a^g \neq 1$ implies $g \in N(B_a)$. Hence B_a is disjoint from its distinct conjugates, and a standard rank computation shows that B_a^G is generic in G. On the other hand, a generic element g of G contains a maximal decent torus inside its definable closure $d(g)$ by [BC08b, Theorem 1]. This implies that B_a contains a maximal decent torus T^g. But since B_a is disjoint from its distinct conjugates, B_a is the only conjugate of B_a containing T^g. In particular $a^g \in N(B_a)$.

As usual $a^g \in B_a$ by Fact 2.6, whence $a \in B_a \cap B_a^{g^{-1}}$. So, by Fact 2.9, $g \in N(B_a)$. As $T \leq d(g)$, we have $T \leq B_a$. But this contradicts $a \in B_a$ for the usual reasons (Fact 2.2).

Hence A is not central in B_a. Zilber's field theorem [BN94, Theorem 9.1] now says that B_a interprets a field of characteristic p, which is bad since the group has degenerate type. On the other hand, B_a has a nontrivial decent torus T_0 because the latter field has a locally finite model by [Wag01, Corollary 9].

Assume that B_a contains a q-torus T_q of Prüfer q-rank ≥ 2, and consider the action of T_q on A. As T_q can't embed into the multiplicative group of any field, there is $t \in T_q$ such that $A \leq C^\circ(t)$. By Fact 2.9, $C(T_q) \leq C^\circ(t) \leq B_a$. But $C(T_q)$ contains a Cartan subgroup of G, which is a Carter subgroup of G by Corollary 3.7.

Now assume that T_q has Prüfer q-rank ≥ 3; say $T_q \leq T^g$. Then by Theorem 3.1, B_a is a Carter subgroup of G, so $B_a = C(T^g)$, and we argue as in the first paragraph to get a contradiction. □

The next corollary will play a key role in §4.

Corollary 3.11. *If a is Weylian τ'-element for T and $1 \neq x \in N(T) \cap C(a)$, then x is a Weylian τ'-element and $B_x = B_a$.*

Proof. We may assume that a is a p-element as usual.

We first prove that x is torsion and not inside $C(T)$. Let $y = x^n$ be such that $y \in C(T)$. Then $y \in C_{C(T)}(a) = 1$ by Lemma 3.6. So it follows that x is torsion and not inside $C(T)$.

We may now assume that x is a q-element. If $p = q$, then $x \in B_a$ by Fact 2.7, as desired. So we may assume that $q \neq p$. It follows from uniqueness of B_a that x normalizes B_a. So Lemma 2.11 says x centralizes elements of $U_p(B_a)$. Now, by Fact 2.9, B_a is the only Borel subgroup containing $C^\circ(x)$.

In view of Lemma 3.4, it suffices to show that $C^\circ(x)$ contains q-unipotence. So suppose that $U_q(C^\circ(x)) = 1$. Then q is not the minimal prime divisor of $|W|$ by Fact 2.5, so $q \geq 5$ in particular. Moreover x is toral in G by Fact 2.4, i.e. $x \in T^g$ for some $g \in G$. So then $T^g \leq C^\circ(x) \leq B_a$.

By Fact 2.16, T^g must have Prüfer q-rank at least $q - 1 \geq 4$, contradicting Corollary 3.10. □

4 Isomorphic lifting

The main result of the paper is the following characterization of the Weyl group.

Theorem 4.1. *Let G be a minimal connected simple group of finite Morley rank, and let T be a decent torus of G. Then the Weyl group $W := N(T)/C(T)$ is cyclic, and has an isomorphic lifting to G; moreover no primes dividing $|W|$ appear in T except possibly 2.*

To prove this, we first observe that the decent torus T may be taken maximal by Lemma 2.3.

In fact, our proof needs only cover groups of *degenerate type*. So let us first reduce to this case. If G has a nontrivial 2-unipotent subgroup, then G is algebraic by [ABC08], and hence $G \cong \mathrm{PSL}_2$ where $|W| = 2$. If G has a non-trivial 2-torus, then $|W| = 1, 2, 3$ by the comments following Lemma 2.3 and the second author's thesis work:

Fact 4.2 ([Del07, see Théorème p.89 and p.91])**.** *Let G be a minimal connected simple group of finite Morley rank of odd type, and let Q be a Carter subgroup of G containing a Sylow° 2-subgroup. Then $W_Q := N(Q)/Q$ has order* 1, 2, *or* 3*.*

Moreover, if the Weyl group has order 3, then [Del07, Corollaire 5.5.15] says that $N(Q) = Q \rtimes \langle\sigma\rangle$ where σ has order 3. So by Fact 2.5, $U_3(C^\circ(\sigma)) \neq 1$, and Lemma 3.4 proves that σ is a Weylian τ'-element, which means that G has no non-trivial 3-torus.

So Theorem 4.1 holds in odd type. As such, we may assume that the Sylow 2-subgroup is finite, and hence trivial by [BBC07].

We also recall an earlier analysis, by the first author and Tuna Altınel, of the Weyl group $W := N(T)/C(T)$ in a minimal connected simple group G.

Fact 4.3 ([AB08, Proposition 5.1 & Corollary 5.7])**.** *The Weyl group W of G is a metacyclic Frobenius complement.*

A *Frobenius complement* is the stabilizer of a point in a Frobenius group; which is a transitive permutation group on a finite set, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. Such groups have a quite restrictive structure described by the following.

Fact 4.4 ([Gor80, 10.3.1 p. 339])**.** *Let W be a Frobenius complement. Then*

1. *Any subgroup of W of order pq, p and q primes, is cyclic.*
2. *Sylow p-subgroups of W are either cyclic or possibly generalized quaternion groups if $p = 2$.*

Metacyclicity follows for any Frobenius complement inside a degenerate type group [Gor80, 7.6.2 p. 258].

For us, the important fact about these groups is the following.

Lemma 4.5. *Let X be a finite group with cyclic Sylow subgroups such that any subgroup of order pq with p, q primes, is cyclic. Then any two elements of prime order commute.*

In particular any $x, y \in X$ have nontrivial powers which commute.

Proof. Since the Sylow 2-subgroups of X are cyclic, we know that X is solvable by [Gor80, 7.6.2 p. 258]. Also $F(X)$ is abelian because its Sylow 2-subgroups are cyclic. Now if $F(X) \leq Z(X)$ then $X = F(X)$ is abelian, and we are done.

So assume $F(X) \not\leq Z(X)$. Choose p prime such that the Sylow p-subgroup P of $F(X)$ is not central in X. Then X induces a nontrivial $p^\perp$-action on

$P \simeq \mathbb{Z}/p^n\mathbb{Z}$, by hypothesis. It follows that X induces a nontrivial action on $\Omega_1(P)$. So let $a \in \Omega_1(P) \setminus Z(X)$. Then $C(a) < X$.

Now $\langle a \rangle \lhd X$ so for any element $x \in X$ of prime order, $\langle a, x \rangle$ is cyclic by hypothesis. Thus, if $x, y \in G$ have prime order, then $x, y \in C(a)$, and we conclude by induction. $\square$

With the above reductions & lemma at hand, we turn to proving Theorem 4.1, by considering the following situation.

Hypothesis 4.6. Our minimal connected simple group G of degenerate type has a nontrivial maximal decent torus T with a Weyl group $W := N(T)/C(T)$ that is a metacyclic Frobenius complement.

Again τ' is the set of primes for which G, or equivalently T, has no p-torus. Let $|W| = \Pi_{i=1}^k p_i^{n_i}$ with $p_1 < \cdots < p_k$. We shall construct a sequence $(a_i)_{i=1\ldots k}$ of commuting Weylian τ'-elements such that $w := \Pi_{i=1}^k a_i$ generates W modulo $C(T)$. To restate this, there are a_i for $i = 1 \ldots k$ such that

(i). each $a_i \in N(T) \setminus C(T)$ has order $p_i^{n_i}$,

(ii). all the a_i's commute, and

(iii). G has no p_i-torus.

A priori, it doesn't matter in what order these a_i appear, but p_1 is for free, thanks to Fact 2.5 via Corollary 3.5.

Let us start. By Fact 4.4 (ii), W contains an element of order $p_1^{n_1}$. We lift this element into a p_1-element a_1 of $N(T)$. Lemma 2.8 says that a_1 has the adequate order, and Corollary 3.5 explains why $p_1 \in \tau'$. So a_1 is a Weylian τ'-element. Set $B_W := B_{a_1}$.

We now perform the induction. Assume that $a_1, \ldots, a_i$ have been constructed as claimed. Let $b_i = a_1 \ldots a_i$. So then b_i is a Weylian τ'-element. By Corollary 3.11, $B_{a_1} = \cdots = B_{a_i} = B_{b_i}$. Also for convenience let $\pi_i = \{p_1, \ldots, p_i\}$, and β_i denote the image of b_i in W.

Claim 1. *There is a p_{i+1}-element a' of $N(T) \setminus C(T)$ that commutes with a power b' of b_i.*

Proof. By Lemma 4.5, there is a p_{i+1}-element α' of W commuting with a nontrivial power β' of β_i. Say $\beta' = \beta_i^k$, and let $b' = b_i^k \neq 1$. Let a_0 be a lifting of α' into a p_{i+1}-element of $N(T) \setminus C(T)$. So far $[a_0, b']$ is in $C(T)$ but need not equal the identity. Therefore we shall adjust a_0 to get a suitable a'.

By Lemma 3.3, $C(T)$ is $\pi_i^\perp$. It follows that $\langle b' \rangle$ is a Hall π_i-subgroup of $K = C(T) \cdot \langle b' \rangle = C(T) \rtimes \langle b' \rangle$. As K has $\pi_i^\perp$-type, it conjugates its Sylow π_i-subgroups by [BC08b, Theorem 4]; as a straightforward consequence, K conjugates its Hall π_i-subgroups.

Now a' normalizes K. Let $H = K \cdot \langle a_0 \rangle$. A Frattini argument says that $H = K \cdot N_H(\langle b' \rangle)$, so there must be a p_{i+1}-element a' in $N_H(\langle b' \rangle)$. It follows that a' does not lie inside $C(T)$. Since $p_{i+1} > \pi_i$, $\langle b' \rangle$ of order π_i has no automorphisms of order p_{i+1}; hence a' must centralize b'. $\square$

Claim 2. *There is a Weylian τ'-element a_{i+1} of order $p_{i+1}^{n_{i+1}}$ such that $B_{a_{i+1}} = B_W$.*

Proof. Let a' and b' be as in Claim 1. Notice that b', being a power of b_i, is a Weylian τ'-element and $B_{b'} = B_W$. By Corollary 3.11, a' is a Weylian τ'-element and $B_{a'} = B_W$.

By Fact 4.4 (2), Sylow p_{i+1}-subgroups of W are cyclic. Let α_{i+1} be any element of W of order $p_{i+1}^{n_{i+1}}$ generating a Sylow p_{i+1}-subgroup containing the image of a'. Let a'' be a lifting of α_{i+1} into a p_{i+1}-element of $N(T) \setminus C(T)$. By Lemma 2.8, a'' has order $p_{i+1}^{n_{i+1}}$. Also, notice that $[a', a''] \in C(T)$; we'll use the same argument as in Claim 1.

Let $K = C(T) \rtimes \langle a' \rangle$ and $H = K \cdot \langle a'' \rangle \leq N(K)$. Since $C(T)$ is $p_{i+1}^{\perp}$, $\langle a' \rangle$ is a Hall p_{i+1}-subgroup of K. There must therefore be a p_{i+1}-element of $N_{N(T)}(\langle a' \rangle)$ lifting α_{i+1}. Let a_{i+1} be this element; it has order $p_{i+1}^{n_{i+1}}$, it is a Weylian τ'-element; and above all, a_{i+1} normalizes $\langle a' \rangle$, whence $B_{a_{i+1}} = B_{a'} = B_W$. □

Now a_{i+1} commutes with b_i because $B_{a_{i+1}} = B_W = B_{b_i}$ and a_{i+1} and b_i are unipotent elements of B_W of coprime order. So the induction is completed, and eventually we set $w := b_k$. Then w has order $|W|$ both in G and in W; in particular $W \simeq \langle w \rangle$ is cyclic. Besides, w is a Weylian τ'-element and $B_w = B_W$. By Corollary 3.3, $C(T)$ is $\pi_k^{\perp}$, so no prime number occurring in T can divide $|W|$.

This concludes the proof of Theorem 4.1. □

Connectedness of the Sylow p-subgroups is an immediate consequence.

Corollary 4.7. *Let G be a minimal connected simple group and $p \neq 2$ a prime. Then the maximal p-subgroups of G are connected.*

Proof. Let S be a maximal p-subgroup of G. If $d(S) = G$, then S is connected.

So we may assume $d(S) < G$. In this case, $S^\circ \leq d^\circ(S)$ which is solvable, and as usual $S^\circ = T * U$ with T a p-torus and U p-unipotent. Also $S^\circ \neq 1$ by [BBC07].

Suppose first that $U \neq 1$. Then there is a unique Borel subgroup B containing U. If $s \in S$, Fact 2.9 forces $B^s = B$, that is $s \in N(B)$. But $s \in B$ by Fact 2.6. As $S \leq B$ is a Sylow p-subgroup of the connected solvable group B, it is connected.

Suppose otherwise that $U = 1$. So $S^\circ = T$ and S is toral-by-finite. Here the result follows by Theorem 4.1. □

5 Counting Borel subgroups

We conclude by noticing that only finitely many Borel subgroups contain the Carter subgroup of a minimal connected simple group with a "large" decent torus. If there is a Weyl group W, this number is either a multiple of $|W|$, or simply 1.

Lemma 5.1. *Let G be any group of finite Morley rank. Fix a Carter subgroup Q of G and a Borel subgroup B of G containing Q. Then there are finitely many G-conjugates of B containing Q.*

Proof. Consider the set $\mathcal{S} := \{(x, y) : x \in G/N_G(Q), y \in G/N_G(B), Q^x \leq B^y\}$.

The first projection has fibers $\mathcal{S}_x = \{y \in G/N_G(B) : Q^x \leq B^y\}$. These sets are uniformly definable and have the same rank, say k.

The second projection has fibers $\mathcal{S}_y = \{x \in G/N_G(Q) : Q^x \leq B^y\}$, which are uniformly definable. Since solvable groups of finite Morley rank conjugate their Carter subgroups, $\mathcal{S}_y$ has rank $\operatorname{rk} B - \operatorname{rk} N_B(Q)$.

Putting this together, we find

$$\operatorname{rk} \mathcal{S} = \operatorname{rk} G - \operatorname{rk} N_G(Q) + k = \operatorname{rk} G - \operatorname{rk} N_G(B) + \operatorname{rk} B - \operatorname{rk} N_B(Q),$$

whence $k = 0$. $\square$

We exploit the theory of maximal intersections of Borel subgroups [Bur07a] to bound the number of conjugacy classes (see also [Del07, Proposition 5.5.3]).

Lemma 5.2. *Let G be a minimal simple connected group. Assume that for some prime number q, there is a q-torus of Prüfer-rank at least 2. Then a Carter subgroup of G is included in finitely many non-conjugate Borel subgroups.*

Proof. Let T_q be a q-torus of Prüfer q-rank $d \geq 2$, and let $k = 1 + \cdots + q^{d-1}$ be the number of lines in $\Omega_q(T_q)$. Let $t_1, \ldots, t_k$ be vectors of $\Omega_q(T_q)$ representing these lines. For $\ell = 1, \ldots, k$, fix some Borel subgroup $\beta_\ell \geq C^\circ(t_\ell)$. Also let $\tilde{q}_\ell$ be maximal unipotence parameters for β_ℓ ($\ell = 1, \ldots, k$).

Assume that there is an infinite set I and Borel subgroups B_i ($i \in I$) pairwise non-conjugate, and all containing Q. Since the B_i's are pairwise non-conjugate, we may assume that none of the B_i is conjugate to any of the β_ℓ. For each i, choose a maximal unipotence parameter $\tilde{p}_i$ for B_i.

Fix some i. Then $B_i \geq Q \geq T_q$, and Fact 2.14 implies $B_i = \langle C^\circ_{B_i}(t_\ell) : \ell = 1, \ldots, k\rangle = \langle (B_i \cap \beta_\ell)^\circ : \ell = 1, \ldots, k\rangle$. So it follows that at least 2 of the intersections $H_{i,\ell} = (B_i \cap \beta_\ell)^\circ$ are distinct and non-abelian.

So we may assume that for all $i \in I$, $H_{i,1}$ and $H_{i,2}$ are distinct and non-abelian. Since B_i is never conjugate to β_1 nor to β_2, every $H_{i,1}$ and every $H_{i,2}$ is a maximal non-abelian intersection [Bur07a, Theorem 4.3]. The same result tells us that one cannot have $\tilde{p}_i = \tilde{q}_1$ nor $\tilde{p}_i = \tilde{q}_2$.

If there are indices i and j such that $\tilde{q}_1 > \tilde{p}_i$ and $\tilde{q}_1 > \tilde{p}_j$, then [Bur07a, Lemma 3.30] conjugates B_i to B_j, a contradiction. So infinitely often, it must be the case that $\tilde{p}_i > \tilde{q}_1$, and similarly we may assume that for all $i \in I$, we have $\tilde{p}_i > \tilde{q}_2$.

In particular, [Bur07a, Lemma 3.30] conjugates β_1 to β_2. Since $\overline{r}_0(H'_{i,\ell})$ is the only value s such that $F_s(\beta_\ell)$ is not abelian [Bur07a, Theorem 4.5 (4)], we see that $r' = \overline{r}_0(H'_{i,\ell})$ does not depend on $i \in I$ nor on $\ell = 1, 2$.

So we let $Q_{r'} = U_{(0,r')}(Q)$. By [Bur07a, Lemma 3.23], this subgroup is non-trivial and central in $H_{i,\ell}$ for all $i \in I$ and $\ell = 1, 2$. Now $\langle H_{i,1}, H_{i,2}\rangle \leq C^\circ(Q_{r'})$;

recall that B_i and β_1 are the only Borel subgroups containing $H_{i,1}$ and that B_i and β_2 are the only Borel subgroups containing $H_{i,2}$.

As $H_{i,1} \neq H_{i,2}$, it follows that $\beta_1 \neq \beta_2$. Then B_i is the only Borel subgroup containing $C^\circ(Q_{r'})$. In particular $B_j = B_i$ for any $j \in I$, a contradiction. □

We note that an effective version of this result is possible, but the obvious bound does not seem useful at present.

Corollary 5.3. *Let G be a minimal connected simple group. Assume that there is a q-torus of Prüfer q-rank ≥ 2 for some prime number q, and a non-trivial Weyl group. Then the number of Borel subgroups containing a Carter subgroup of G is 1 or a multiple of $|W|$.*

Proof. This number is finite by the two preceding lemmas. By Theorem 4.1, we lift W to an isomorphic subgroup $\hat{W} \leq G$, and let $\hat{W}$ act on a G-conjugacy class of Borel subgroups containing Q. By a Frattini argument, $\hat{W}$ acts transitively on each G-conjugacy class of such Borel subgroups. If some $a \in \hat{W}$ fixes a Borel subgroup $B \geq Q$, then $B = Q$ by Proposition 3.9. So unless Q is a Borel subgroup, $\hat{W}$ acts freely on each conjugacy class. □

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