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Mohammed El Azhari

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## A REMARK ON CONTINUITY OF POSITIVE LINEAR FUNCTIONALS ON SEPARABLE BANACH \*-ALGEBRAS

M. El Azhari

**Abstract.** Using a variation of the Murphy-Varopoulos Theorem, we give a new proof of the following R. J. Loy Theorem: Let  $A$  be a separable Banach  $*$ -algebra with center  $Z$  such that  $ZA$  has at most countable codimension, then every positive linear functional on  $A$  is continuous.

*Keywords.* Banach  $*$ -algebra, positive linear functional, continuity.

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If  $A$  is a  $*$ -algebra, a linear functional  $f$  on  $A$  is called positive if  $f(x^*x) \geq 0$  for all  $x \in A$ . A linear functional  $g$  on  $A$  dominates a linear functional  $h$  on  $A$  if  $g-h$  is positive. Given an algebra  $A$  and vector subspaces  $X, Y$  of  $A$ ,  $XY$  will denote the vector subspace of  $A$  spanned by the products  $xy$  for  $x \in X$ ,  $y \in Y$ . Let  $n \geq 2$ ,  $X^n$  denotes the vector subspace of  $A$  spanned by the products  $x_1 \cdots x_n$  for  $x_i \in X$  ( $1 \leq i \leq n$ ). A Hausdorff topological space  $S$  is called a Souslin space if there is a complete separable metric space  $P$  and a continuous mapping of  $P$  onto  $S$ .

We need two theorems and a preliminary proposition.

**Theorem 1** ([1, Theorem 5.5]). Let  $A$  be a complete separable metrizable topological vector space,  $B$  a vector subspace of  $A$ . If  $B$  is a Souslin space and has at most countable codimension, then  $B$  is closed and of finite codimension.

In [5, Theorem], Varopoulos proved that if  $A$  is a commutative Banach  $*$ -algebra, with continuous involution, such that  $A^3$  is closed and of finite codimension, then every positive linear functional on  $A$  is continuous. In [3, Corollary and Remark], Murphy gave another proof of this result without the assumption of continuity of the involution. In [4, Theorem 13.7], by the same methods in [3], Sinclair obtained the following improvement: Let  $A$  be a Banach  $*$ -algebra with center  $Z$ . If  $Z^2A$  is closed and of finite codimension, then every positive linear functional on  $A$  is continuous. Here we show that the above result works for  $ZA^2$ .

**Theorem 2.** (Variation of the Murphy-Varopoulos Theorem). Let  $A$  be a Banach  $*$ -algebra with center  $Z$ . If  $ZA^2$  is closed and of finite codimension, then every positive linear functional on  $A$  is continuous.

**Proof:** By hypothesis and [4, Lemma 13.6], it is sufficient to prove that every positive linear functional on  $A$ , nonzero on  $A^2$ , dominates a continuous positive linear functional on  $A$ , nonzero on  $A^2$ . Let  $f$  be a positive linear functional on  $A$ , nonzero on  $A^2$ . By the Schwarz inequality, we have  $|f(z^*xy)|^2 \leq f_z(xx^*)f(y^*y)$  for all  $z \in Z$ ,  $x \in A$  and  $y \in A$ , where  $f_z(a) = f(z^*az)$  for all  $a \in A$ . If  $f_z$  is zero on  $A^2$  for all  $z \in Z$ , then  $f$  is zero on  $ZA^2$  and hence  $f$  is continuous on  $A$ . If there is  $z \in A$  such that  $f_z$  is nonzero on  $A^2$ , we can suppose  $\|z^*z\| < 1$ . By the square root lemma [4, Lemma 13.1], there is  $u \in A$  such that  $u^* = u$  and  $2u - u^2 = z^*z$ . Let  $x \in A$ ,  $(f - f_z)(x^*x) = f(x^*x - z^*x^*xz) = f(x^*x - x^*z^*zx) = f((x - ux)^*(x - ux)) \geq 0$ . Thus  $f$  dominates  $f_z$  which is a continuous positive linear functional on  $A$  by [4, Corollary 13.3].

**Proposition 3.** Let  $A$  be an algebra with center  $Z$ . The following assertions are equivalent:

1.  $ZA^m$  has at most countable codimension for some  $m \geq 1$ .
2.  $ZA^n$  has at most countable codimension for all  $n \geq 1$ .

**Proof:** (1) $\Rightarrow$ (2): We prove the implication by induction on  $n$ .  $A^2$  and  $ZA$  have at most countable codimension since  $ZA^m \subset ZA \subset A^2$ . Suppose that  $ZA^n$  has at most countable codimension. We have  $A = ZA^n + E$ ,  $A = A^2 + F$ ,  $E$  and  $F$  are vector subspaces of  $A$  with at most countable dimension.  $A = A^2 + F = (ZA^n + E)^2 + F \subset ZA^{n+1} + E^2 + F$ , then  $A = ZA^{n+1} + E^2 + F$ . So  $ZA^{n+1}$  has at most countable codimension because  $E^2 + F$  has at most countable dimension.

**Proof of R. J. Loy Theorem**([2, Theorem 2.1]): By Proposition 3,  $ZA^2$  has at most countable codimension. Consider the continuous mapping  $G : Z \times A \times A \rightarrow A$ ,  $G(z, x, y) = zxy$ . Since  $ZA^2$  is the linear span of  $G(Z \times A \times A)$ , it follows that  $ZA^2$  is a Souslin space. By Theorem 1,  $ZA^2$  is closed and of finite codimension, and so we can apply Theorem 2.

**Remark.** The interest of the above proof is the observation that the argument for continuity in the Loy Theorem, which uses a result [2, Theorem 1.2] concerning multilinear mappings on products of separable Banach algebras, can be replaced by a variation of the well known theorem of Murphy-Varopoulos.

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Ecole Normale Supérieure  
Avenue Oued Akreuch  
Takaddoum, BP 5118, Rabat  
Morocco

E-mail: mohammed.elazhari@yahoo.fr