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Mohammed El Azhari

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A REMARK ON CONTINUITY OF POSITIVE LINEAR FUNCTIONALS ON SEPARABLE BANACH *-ALGEBRAS

M. El Azhari

Abstract. Using a variation of the Murphy-Varopoulos Theorem, we give a new proof of the following R. J. Loy Theorem: Let A be a separable Banach $*$ -algebra with center Z such that ZA has at most countable codimension, then every positive linear functional on A is continuous.

Keywords. Banach $*$ -algebra, positive linear functional, continuity.

Mathematics Subject Classification 2010. 46K05.

If A is a $*$ -algebra, a linear functional f on A is called positive if $f(x^*x) \geq 0$ for all $x \in A$. A linear functional g on A dominates a linear functional h on A if $g-h$ is positive. Given an algebra A and vector subspaces X, Y of A , XY will denote the vector subspace of A spanned by the products xy for $x \in X$, $y \in Y$. Let $n \geq 2$, X^n denotes the vector subspace of A spanned by the products $x_1 \cdots x_n$ for $x_i \in X$ ($1 \leq i \leq n$). A Hausdorff topological space S is called a Souslin space if there is a complete separable metric space P and a continuous mapping of P onto S .

We need two theorems and a preliminary proposition.

Theorem 1 ([1, Theorem 5.5]). Let A be a complete separable metrizable topological vector space, B a vector subspace of A . If B is a Souslin space and has at most countable codimension, then B is closed and of finite codimension.

In [5, Theorem], Varopoulos proved that if A is a commutative Banach $*$ -algebra, with continuous involution, such that A^3 is closed and of finite codimension, then every positive linear functional on A is continuous. In [3, Corollary and Remark], Murphy gave another proof of this result without the assumption of continuity of the involution. In [4, Theorem 13.7], by the same methods in [3], Sinclair obtained the following improvement: Let A be a Banach $*$ -algebra with center Z . If Z^2A is closed and of finite codimension, then every positive linear functional on A is continuous. Here we show that the above result works for ZA^2 .

Theorem 2. (Variation of the Murphy-Varopoulos Theorem). Let A be a Banach $*$ -algebra with center Z . If ZA^2 is closed and of finite codimension, then every positive linear functional on A is continuous.

Proof: By hypothesis and [4, Lemma 13.6], it is sufficient to prove that every positive linear functional on A , nonzero on A^2 , dominates a continuous positive linear functional on A , nonzero on A^2 . Let f be a positive linear functional on A , nonzero on A^2 . By the Schwarz inequality, we have $|f(z^*xy)|^2 \leq f_z(xx^*)f(y^*y)$ for all $z \in Z$, $x \in A$ and $y \in A$, where $f_z(a) = f(z^*az)$ for all $a \in A$. If f_z is zero on A^2 for all $z \in Z$, then f is zero on ZA^2 and hence f is continuous on A . If there is $z \in A$ such that f_z is nonzero on A^2 , we can suppose $\|z^*z\| < 1$. By the square root lemma [4, Lemma 13.1], there is $u \in A$ such that $u^* = u$ and $2u - u^2 = z^*z$. Let $x \in A$, $(f - f_z)(x^*x) = f(x^*x - z^*x^*xz) = f(x^*x - x^*z^*zx) = f((x - ux)^*(x - ux)) \geq 0$. Thus f dominates f_z which is a continuous positive linear functional on A by [4, Corollary 13.3].

Proposition 3. Let A be an algebra with center Z . The following assertions are equivalent:

1. ZA^m has at most countable codimension for some $m \geq 1$.
2. ZA^n has at most countable codimension for all $n \geq 1$.

Proof: (1) \Rightarrow (2): We prove the implication by induction on n . A^2 and ZA have at most countable codimension since $ZA^m \subset ZA \subset A^2$. Suppose that ZA^n has at most countable codimension. We have $A = ZA^n + E$, $A = A^2 + F$, E and F are vector subspaces of A with at most countable dimension. $A = A^2 + F = (ZA^n + E)^2 + F \subset ZA^{n+1} + E^2 + F$, then $A = ZA^{n+1} + E^2 + F$. So ZA^{n+1} has at most countable codimension because $E^2 + F$ has at most countable dimension.

Proof of R. J. Loy Theorem([2, Theorem 2.1]): By Proposition 3, ZA^2 has at most countable codimension. Consider the continuous mapping $G : Z \times A \times A \rightarrow A$, $G(z, x, y) = zxy$. Since ZA^2 is the linear span of $G(Z \times A \times A)$, it follows that ZA^2 is a Souslin space. By Theorem 1, ZA^2 is closed and of finite codimension, and so we can apply Theorem 2.

Remark. The interest of the above proof is the observation that the argument for continuity in the Loy Theorem, which uses a result [2, Theorem 1.2] concerning multilinear mappings on products of separable Banach algebras, can be replaced by a variation of the well known theorem of Murphy-Varopoulos.

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Ecole Normale Supérieure
Avenue Oued Akreuch
Takaddoum, BP 5118, Rabat
Morocco

E-mail: mohammed.elazhari@yahoo.fr