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A REAL SEMINORM WITH SQUARE PROPERTY IS SUBMULTIPLICATIVE

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A seminorm with square property on a real associative algebra is submultiplicative.

Key words: real seminorm, seminorm with square property, submultiplicative seminorm.

1. INTRODUCTION

It was proved in [3] that every complex seminorm with square property on a commutative algebra is submultiplicative, and it was posed the problem whether this result holds in a noncommutative algebra. This problem was answered in the particular case of Banach algebras [4] and fully resolved in [2], [5] and [7]. The result of [3] holds in the real case. But the results of [2], [4], [5] and [7] don't hold in the real case since we use the Hirschfeld-Zelazko Theorem [6], or its locally bounded version [2], which are not valid in the real case. Using a functional representation theorem [1, Theorem 1], we show that every real seminorm with square property is submultiplicative.

2. PRELIMINARIES

Let A be an associative algebra over the field $K = \mathbb{R}$ or \mathbb{C} . A seminorm on A is a function $p: A \rightarrow [0, \infty)$ satisfying $p(a+b) \leq p(a)+p(b)$ for all a, b in A and $p(ka) = |k|p(a)$ for all a in A and k in K . p is a complex seminorm if $K = \mathbb{C}$ and p is a real seminorm if $K = \mathbb{R}$. p is submultiplicative if $p(ab) \leq p(a)p(b)$ for all a, b in A . p satisfies the square property if $p(a^2) = p(a)^2$ for all a in A . Let A be a real algebra and let a be any element of A . The spectrum $sp(a)$ of a in A is defined to be equal to the spectrum of a as an element of the complexification of A . If A is unital, then $sp(a) = \{s + it \in \mathbb{C}, (a - se)^2 + t^2e \notin A^{-1}\}$ for all a in A , where e is the unit of A and A^{-1} is the set of all invertible elements in A . Let $(A, \|\cdot\|)$ be a real normed algebra, the limit $r(a) = \lim \|a^n\|^{1/n}$ exists for each a in A . If A is complete, then $r(a) = \sup \{|v|, v \in sp(a)\}$ for every a in A .

3. RESULTS

Let $(A, \|\cdot\|)$ be a real Banach algebra with unit such that $\|a\| \leq m r(a)$ for some positive constant m and all a in A . Let $X(A)$ be the set of all nonzero multiplicative linear functionals from A into the noncommutative algebra H of quaternions. For a in A and x in $X(A)$, put $J(a)(x) = x(a)$. For a in A , $J(a): X(A) \rightarrow H$ is a map from $X(A)$ into H . $X(A)$ is endowed with the topology generated by $J(a)$, $a \in A$, that is the weakest topology such that all the functions $J(a)$, $a \in A$, are continuous. By [1, Theorem 1], $X(A)$ is a nonempty compact space and the map $J: A \rightarrow C(X(A), H)$, $a \rightarrow J(a)$, is an isomorphism (into), where $C(X(A), H)$ is the real algebra of all continuous functions from $X(A)$ into H .

Proposition 3.1. (1) An element a is invertible in A if and only if $J(a)$ is invertible in $C(X(A), H)$, and (2) $sp(a) = sp(J(a))$ for all a in A .

Proof. (1) The direct implication is obvious. Conversely, there exists g in $C(X(A), H)$ such that $J(a)g = gJ(a) = 1$ i.e. $x(a)g(x) = g(x)x(a) = 1$ for all x in $X(A)$. Let T be a nonzero irreducible representation of A , by the proof of [1, Theorem 1] there exists $S: T(A) \rightarrow H$ an isomorphism (into). Since $SoT \in X(A)$ and $0 \neq SoT(a) = S(T(a))$, it follows that $T(a) \neq 0$. If $a \notin A^{-1}$, there exists a maximal right ideal M containing aA . Let L be the canonical representation of A on A/M which is nonzero and irreducible, also $L(a) = 0$ since aA is included in M , contradiction. Then $aA = A$ and by the same $Aa = A$. There exist b, c in A such that $ab = ca = e$ (e is the unit of A). We have $c = c(ab) = (ca)b = b$, so a is invertible in A .

(2) $s + it \in sp(a)$ iff $(a - se)^2 + t^2e \notin A^{-1}$

iff $J((a - se)^2 + t^2e) \notin C(X(A), H)^{-1}$ by (1)

iff $(J(a) - sJ(e))^2 + t^2J(e) \notin C(X(A), H)^{-1}$

iff $s + it \in sp(J(a))$.

Theorem 3.2. Let A be a real associative algebra. Then every seminorm with square property on A is submultiplicative.

Proof. If A is commutative, see [3, Theorem 1]. If A is noncommutative, let p be a seminorm with square property on A . By [5] or [7], there exists $m > 0$ such that $p(ab) \leq mp(a)p(b)$ for all a, b in A . $\text{Ker}(p)$ is a two sided ideal in A , the norm $|\cdot|$ on the quotient algebra $A/\text{Ker}(p)$, defined by $|a + \text{Ker}(p)| = p(a)$ is a norm with square property on $A/\text{Ker}(p)$. Define $\|a + \text{Ker}(p)\| = m|a + \text{Ker}(p)|$ for all a in A . Let a, b in A , $\|ab + \text{Ker}(p)\| = m|ab + \text{Ker}(p)| \leq m^2|a + \text{Ker}(p)||b + \text{Ker}(p)| = \|a + \text{Ker}(p)\|\|b + \text{Ker}(p)\|$. $(A/\text{Ker}(p), \|\cdot\|)$ is a real normed algebra. Let a in A , $\|a^2 + \text{Ker}(p)\| = m|a^2 + \text{Ker}(p)| = m|a + \text{Ker}(p)|^2 = m^{-1}(m|a + \text{Ker}(p)|)^2 = m^{-1}\|a + \text{Ker}(p)\|^2$. The completion B of $(A/\text{Ker}(p), \|\cdot\|)$ satisfies also the property $\|b^2\| = m^{-1}\|b\|^2$ for all b in B , and consequently $\|b^{2^n}\|^{2^{-n}} = m^{2^n-1}\|b\|$ for all b in B and n in \mathbb{N}^* , then $r(b) = m^{-1}\|b\|$ i.e. $\|b\| = m r(b)$. We consider two cases.

B is unital: By [1, Theorem 1], $X(B)$ is a nonempty compact space and the map $J: B \rightarrow C(X(B), H)$ is an isomorphism (into). $C(X(B), H)$ is a real Banach algebra with unit under the supnorm $\|\cdot\|_s$. By

Proposition 3.1, $r(b) = r(J(b))$ for all b in B . Let b in B , $\|b\| = m r(b) = m r(J(b)) = m \|J(b)\|_s$ since the supnorm satisfies the square property. Then $\|J(b)\|_s = m^{-1} \|b\| = |b|$ for all b in $A/\text{Ker}(p)$, so $|\cdot|$ is submultiplicative on $A/\text{Ker}(p)$ i.e. p is submultiplicative.

B is not unital: Let B_1 be the algebra obtained from B by adjoining the unit. By the same proof of [6, Lemma 2], there exists a norm N on B_1 such that

- (i) (B_1, N) is a real Banach algebra with unit
- (ii) $N(b) \leq m^3 r_B(b)$ for all b in B_1
- (iii) N and $\|\cdot\|$ are equivalent on B .

By [1, Theorem 1], $X(B_1)$ is a nonempty compact space and the map $J: B_1 \rightarrow C(X(B_1), H)$ is an isomorphism (into). Let b in B , $\|b\| = m r_B(b) = m r_{B_1}(b)$ by (iii)

$$= m r(J(b)) \text{ by Proposition 3.1}$$

$$= m \|J(b)\|_s \text{ by the square property of the supnorm.}$$

Then $\|J(b)\|_s = m^{-1} \|b\| = |b|$ for all b in $A/\text{Ker}(p)$, so $|\cdot|$ is submultiplicative on $A/\text{Ker}(p)$ i.e. p is submultiplicative.

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