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# ON A CONJECTURE CONCERNING SOME AUTOMATIC CONTINUITY THEOREMS

M.El Azhari

**ABSTRACT.** Let  $A$  and  $B$  be commutative locally convex algebras with unit.  $A$  is assumed to be a uniform topological algebra. Let  $\Phi$  be an injective homomorphism from  $A$  to  $B$ . Under additional assumptions, we characterize the continuity of the homomorphism  $\Phi^{-1}/\text{Im}\Phi$  by the fact that the radical (or strong radical) of the closure of  $\text{Im}\Phi$  has only zero as a common point with  $\text{Im}\Phi$ . This gives an answer to a conjecture concerning some automatic continuity theorems on uniform topological algebras.

1. **INTRODUCTION.** Let  $A$  and  $B$  be commutative locally convex algebras with unit.  $A$  is assumed to be a uniform topological algebra. Let  $\Phi$  be an injective homomorphism from  $A$  to  $B$ . Under which conditions is  $\Phi^{-1}/\text{Im}\Phi$  continuous?

Under additional assumptions such as:

(1)  $A$  is weakly regular and functionally continuous,  $B$  an lmc algebra, and  $(\text{Im}\Phi)^{-}$  (the closure of  $\text{Im}\Phi$ ) is a semisimple  $Q$ -algebra; or

(2)  $A$  is weakly  $\sigma^*$ -compact-regular, and  $(\text{Im}\Phi)^{-}$  is a strongly semisimple  $Q$ -algebra;

it is shown in [5] that  $\Phi^{-1}/\text{Im}\Phi$  is continuous, which improves earlier results by Bedaa, Bhatt and Oudadess ([2]).

The following examples show that the hypothesis  $(\text{Im}\Phi)^{-}$  is a  $Q$ -algebra in (1) and (2) cannot be omitted.

**Example 1.** Let  $A = C[0,1]$  be the algebra of all complex continuous functions on the closed unit interval  $[0,1]$ .  $A$  is a uniform Banach algebra under the supnorm. Since  $M(A)$  is homeomorphic to  $[0,1]$ , it follows that  $A$  is weakly regular. Consider  $B = C[0,1]$ . For any countable compact subset  $K$  of  $[0,1]$ , and  $f \in B$ , we put  $p_K(f) = \sup\{|f(x)|, x \in K\}$ .  $B$  is a complete uT-algebra under the system  $(p_K)_K$ . Consider  $\Phi: A \rightarrow B$ ,  $\Phi(f) = f$ . Then  $(\text{Im}\Phi)^{-} = B$  is semisimple but not a  $Q$ -algebra. Clearly  $\Phi^{-1}/\text{Im}\Phi$  is not continuous.

**Example 2.** Let  $A = C_b(\mathbb{R})$  be the algebra of all complex continuous bounded functions on the real line.  $A$  is a uniform Banach algebra under the supnorm.  $A$  is weakly  $\sigma^*$ -compact-regular [2, Remark (4)]. Let  $B = C(\mathbb{R})$  be the algebra of all complex continuous functions on  $\mathbb{R}$ , with the compact-open topology. Consider  $\Phi: A \rightarrow B$ ,  $\Phi(f) = f$ . Then  $(\text{Im}\Phi)^{-} = C(\mathbb{R})$  is strongly semisimple but not a  $Q$ -algebra. Clearly  $\Phi^{-1}/\text{Im}\Phi$  is not continuous.

In [2], the authors conjectured that the semisimplicity of  $(\text{Im}\Phi)^{-}$  in (1) (and strong semisimplicity of  $(\text{Im}\Phi)^{-}$  in (2)) can be omitted. According to the proofs in [2] and [5], the

semisimplicity of  $(\text{Im}\Phi)^-$  in (1) can be replaced by  $\text{Im}\Phi \cap R((\text{Im}\Phi)^-) = \{0\}$ , and the strong semisimplicity of  $(\text{Im}\Phi)^-$  in (2) can be replaced by  $\text{Im}\Phi \cap \text{SR}((\text{Im}\Phi)^-) = \{0\}$ .

In this paper, we show that if  $A$  is weakly regular and functionally continuous,  $B$  an lmc algebra, and  $(\text{Im}\Phi)^-$  is a  $Q$ -algebra, then the continuity of  $\Phi^{-1}/\text{Im}\Phi$  is equivalent to  $\text{Im}\Phi \cap R((\text{Im}\Phi)^-) = \{0\}$ . We also show that if  $A$  is weakly  $\sigma^*$ -compact-regular,  $B$  has continuous product, and  $(\text{Im}\Phi)^-$  is a  $Q$ -algebra, then the continuity of  $\Phi^{-1}/\text{Im}\Phi$  is equivalent to  $\text{Im}\Phi \cap \text{SR}((\text{Im}\Phi)^-) = \{0\}$ .

2. PRELIMINARIES. All algebras considered are over the field  $\mathbb{C}$ , commutative, and having a unit element. A topological algebra is an algebra which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra (lc algebra) is a topological algebra whose topology is locally convex. A locally multiplicatively convex algebra (lmc algebra) is a topological algebra whose topology is determined by a family of submultiplicative seminorms. A uniform seminorm on an algebra  $A$  is a seminorm  $p$  such that  $p(x^2) = p(x)^2$  for all  $x \in A$ . Such a seminorm is submultiplicative [4]. A uniform topological algebra (uT-algebra) is a topological algebra whose topology is determined by a family of uniform seminorms. A uniform normed algebra is a normed algebra  $(A, \|\cdot\|)$  such that  $\|x^2\| = \|x\|^2$  for all  $x \in A$ . Let  $A$  be an algebra and  $x \in A$ , we denote by  $\text{sp}_A(x)$  the spectrum of  $x$  and  $r_A(x)$  the spectral radius of  $x$ . For an algebra  $A$ ,  $M^*(A)$  denotes the set of all nonzero multiplicative linear functionals on  $A$ . For a topological algebra  $A$ ,  $M(A)$  denotes the set of all nonzero continuous multiplicative linear functionals on  $A$ . A topological algebra  $A$  is functionally continuous if  $M^*(A) = M(A)$ . A topological algebra is a  $Q$ -algebra [7] if the set of invertible elements is open. A topological algebra is weakly regular [2] if given a closed subset  $F$  of  $M(A)$ ,  $F \neq M(A)$ , there exists a nonzero  $x \in A$  such that  $f(x) = 0$  for all  $f \in F$ . A topological algebra  $A$  is weakly  $\sigma^*$ -compact-regular [2] if given a compact subset  $K$  of  $M^*(A)$ ,  $K \neq M^*(A)$ , there exists a nonzero  $x \in A$  such that  $f(x) = 0$  for all  $f \in K$ . We use  $R(A)$  to denote the radical of an algebra  $A$ . If  $R(A) = \{0\}$ , we say that  $A$  is semisimple. Let  $A$  be a topological algebra with  $M(A) \neq \emptyset$ , the set  $\{x \in A, f(x) = 0 \text{ for all } f \in M(A)\}$  is called the strong radical of  $A$  and denoted by  $\text{SR}(A)$ . If  $\text{SR}(A) = \{0\}$ , we say that  $A$  is strongly semisimple. Let  $A$  be an lmc algebra, if  $A$  is complete or a  $Q$ -algebra, then  $R(A) = \text{SR}(A)$ .

### 3. RESULTS

Theorem 3.1. Let  $A$  be a weakly regular, functionally continuous, uT-algebra. Let  $B$  be an lmc algebra, and let  $\Phi: A \rightarrow B$  be a one-to-one homomorphism such that  $(\text{Im}\Phi)^-$  is a  $Q$ -algebra.

Then the following are equivalent:

- (1)  $\Phi^{-1}/\text{Im}\Phi$  is continuous.
- (2)  $\text{Im}\Phi$  is functionally continuous.
- (3)  $\Phi^*: M((\text{Im}\Phi)^-) \rightarrow M(A)$ ,  $\Phi^*(f) = f \circ \Phi$ , is surjective.
- (4)  $\text{Im}\Phi \cap R((\text{Im}\Phi)^-) = \{0\}$ .

Proof: (1)  $\Rightarrow$  (2): Let  $F \in M^*(\text{Im}\Phi)$ ,  $F = F \circ \Phi \circ (\Phi^{-1}/\text{Im}\Phi)$  is continuous since  $F \circ \Phi$  and  $\Phi^{-1}/\text{Im}\Phi$  are continuous.

(2)  $\Rightarrow$  (3): Let  $f \in M(A)$  and  $F = fo(\Phi^{-1}/\text{Im}\Phi)$ ,  $F \in M^*(\text{Im}\Phi) = M(\text{Im}\Phi)$  and  $f = Fo\Phi$ .  $F$  can be extended to an  $F^- \in M((\text{Im}\Phi)^-)$ . We have  $f = F^- \circ \Phi$ . This shows that  $\Phi^*$  is surjective.  
(3)  $\Rightarrow$  (1): By [5, Theorem 2.1], the topology of  $A$  is defined by a family  $\{p_s, s \in S\}$  of submultiplicative seminorms such that (i) for all  $x \in A$  and  $s \in S$  with  $p_s(x) = 1$ , there exists  $f \in M(A)$  such that  $|f(x)| = 1$ . Let  $s \in S$  and  $y \in \text{Im}\Phi$  with  $p_s(\Phi^{-1}(y)) \neq 0$ . By (i), there exists  $f \in M(A)$  such that  $|f(\Phi^{-1}(y))| = p_s(\Phi^{-1}(y))$ . Since  $\Phi^*$  is surjective, there exists  $F \in M((\text{Im}\Phi)^-)$  such that  $f = Fo\Phi$ . We have  $p_s(\Phi^{-1}(y)) = |f(\Phi^{-1}(y))| = |F(y)| \leq r_C(y)$ , where  $C = (\text{Im}\Phi)^-$ . Since  $C$  is a Q-algebra,  $r_C$  is continuous at 0 [7, Proposition 13.5]. Then  $\Phi^{-1}/\text{Im}\Phi$  is continuous.  
(3)  $\Rightarrow$  (4): Let  $y \in \text{Im}\Phi \cap R((\text{Im}\Phi)^-)$ , there exists  $x \in A$  such that  $y = \Phi(x)$  and  $F(\Phi(x)) = 0$  for all  $F \in M((\text{Im}\Phi)^-)$ . Then  $f(x) = 0$  for all  $f \in M(A)$  since  $\Phi^*$  is surjective. Hence  $x = 0$  and so  $y = \Phi(x) = 0$  since  $A$  is a uT-algebra.  
(4)  $\Rightarrow$  (3):  $\Phi^*$  is well defined and continuous. Since  $(\text{Im}\Phi)^-$  is a Q-algebra,  $M((\text{Im}\Phi)^-)$  is compact [6, p.187], thus  $\Phi^*(M((\text{Im}\Phi)^-))$  is compact. Suppose that  $\Phi^*$  is not surjective. By the weak regularity of  $A$ , there exists a nonzero  $x \in A$  such that  $f(\Phi(x)) = 0$  for all  $f \in M((\text{Im}\Phi)^-)$ . Since  $(\text{Im}\Phi)^-$  is a Q-algebra, it follows that  $\Phi(x) \in \text{Im}\Phi \cap R((\text{Im}\Phi)^-) = \{0\}$ , and then  $x = 0$ , a contradiction.

**Theorem 3.2.** Let  $A$  be a weakly  $\sigma^*$ -compact-regular, uT-algebra. Let  $B$  be an lc algebra with continuous product, and  $\Phi: A \rightarrow B$  be a one-to-one homomorphism such that  $(\text{Im}\Phi)^-$  is a Q-algebra. The following are equivalent:

- (1)  $\Phi^{-1}/\text{Im}\Phi$  is continuous.
- (2)  $\text{Im}\Phi \cap SR((\text{Im}\Phi)^-) = \{0\}$ .
- (3)  $\Phi^{**}: M((\text{Im}\Phi)^-) \rightarrow M^*(A)$ ,  $\Phi^{**}(f) = fo\Phi$ , is surjective.

**Proof:** (1)  $\Rightarrow$  (2): The topology of  $A$  is determined by a family  $\{p_u, u \in U\}$  of uniform seminorms. For each  $u \in U$ , let  $N_u = \{x \in A, p_u(x) = 0\}$  and  $A_u$  be the Banach algebra obtained by completing  $A/N_u$  in the norm  $\|x_u\|_u = p_u(x)$ ,  $x_u = x + N_u$ . It is clear that  $A_u$  is a uniform Banach algebra. For each  $u \in U$ , let  $M_u(A) = \{f \in M(A), |f(x)| \leq p_u(x) \text{ for all } x \in A\}$ . Let  $u \in U$  and  $x \in A$ ,  $p_u(x) = \|x_u\|_u = r_u(x_u) = \sup\{|g(x_u)|, g \in M(A_u)\} = \sup\{|f(x)|, f \in M_u(A)\}$  by [7, Proposition 7.5] ( $r_u$  is the spectral radius on  $A_u$ ). Let  $u \in U$  and  $y \in \text{Im}\Phi$ ,  $p_u(\Phi^{-1}(y)) = \sup\{|fo\Phi^{-1}(y)|, f \in M_u(A)\}$ . Let  $f \in M_u(A)$ ,  $fo\Phi^{-1} \in M(\text{Im}\Phi) = M((\text{Im}\Phi)^-)$  since  $\Phi^{-1}/\text{Im}\Phi$  is continuous and  $B$  has continuous product. Then  $p_u(\Phi^{-1}(y)) \leq \sup\{|F(y)|, F \in M((\text{Im}\Phi)^-)\}$  for all  $u \in U$  and  $y \in \text{Im}\Phi$ . Let  $y \in \text{Im}\Phi \cap SR((\text{Im}\Phi)^-)$ , we have  $p_u(\Phi^{-1}(y)) = 0$  for all  $u \in U$ , then  $\Phi^{-1}(y) = 0$  and so  $y = 0$ .  
(2)  $\Rightarrow$  (3):  $\Phi^{**}$  is continuous. Since  $(\text{Im}\Phi)^-$  is a Q-algebra,  $M((\text{Im}\Phi)^-)$  is compact [6, p.187], thus  $\Phi^{**}(M((\text{Im}\Phi)^-))$  is compact. Suppose that  $\Phi^{**}$  is not surjective. Since  $A$  is  $\sigma^*$ -compact-regular, there exists a nonzero  $x \in A$  such that  $f(\Phi(x)) = 0$  for all  $f \in M((\text{Im}\Phi)^-)$ . This gives  $\Phi(x) \in \text{Im}\Phi \cap SR((\text{Im}\Phi)^-) = \{0\}$ , and then  $x = 0$ , a contradiction.  
(3)  $\Rightarrow$  (1): Similar to the proof of (3)  $\Rightarrow$  (1) in Theorem 3.1.

Here is an example such that  $A$  is weakly regular, uniform Banach algebra,  $B$  is a Banach algebra,  $\Phi: A \rightarrow B$  is a one-to-one homomorphism, and  $\text{Im}\Phi \cap R((\text{Im}\Phi)^-) = \{0\}$  but  $R((\text{Im}\Phi)^-) \neq \{0\}$ .

Example. Let  $A = C[0,1]$  be the algebra of all complex continuous functions on the closed unit interval  $[0,1]$ .  $A$  is a uniform Banach algebra under the supnorm  $\|\cdot\|$ ,  $A$  is also weakly regular. By [3], there exists a norm  $|\cdot|$  on  $C[0,1]$  such that  $C[0,1]$  is an incomplete normed algebra. It is well known that  $\|\cdot\| \leq |\cdot|$ . Let  $B$  be the completion of  $C[0,1]$  under the norm  $|\cdot|$ . Consider  $\Phi: A \rightarrow B$ ,  $\Phi(f) = f$ , we have  $(\text{Im}\Phi)^- = B$ . If  $B$  is semisimple, then  $\Phi$  is continuous, and consequently the norms  $\|\cdot\|$  and  $|\cdot|$  are equivalent, a contradiction. Since  $\|\cdot\| \leq |\cdot|$ ,  $\Phi^{-1}/\text{Im}\Phi$  is continuous and so  $\text{Im}\Phi \cap R((\text{Im}\Phi)^-) = \{0\}$  by Theorem 3.1.

Remark. The algebra  $A$  considered in the above example is also  $\sigma^*$ -compact-regular and  $\text{Im}\Phi \cap \text{SR}((\text{Im}\Phi)^-) = \{0\}$  but  $\text{SR}((\text{Im}\Phi)^-) \neq \{0\}$ .

The following result is an application of Theorem 3.1.

Theorem 3.3. Let  $A$  be a functionally continuous normed algebra. Then the following assertions are equivalent:

- (1)  $A$  is a uniform normed algebra.
- (2)  $A$  has a largest closed, idempotent, absolutely convex, bounded subset.

Proof. (1)  $\Rightarrow$  (2): Let  $\|\cdot\|$  be a uniform norm defining the topology of  $A$ . Let  $B = \{x \in A, \|x\| \leq 1\}$ ,  $B$  is a closed, idempotent, absolutely convex, bounded subset of  $A$ . Let  $C$  be an idempotent bounded subset of  $A$ . There exists  $M > 0$  such that  $\|x\| \leq M$  for all  $x \in C$ . Let  $x \in C$ ,  $\|x\| = \|x^{2^n}\|^{2^{-n}} \leq M^{2^{-n}}$  for all  $n \geq 1$ , then  $\|x\| \leq 1$  i.e.  $x \in B$ .

(2)  $\Rightarrow$  (1): Let  $B$  be a largest closed, idempotent, absolutely convex, bounded subset of  $A$ . By [1, Proposition 2.15], we have  $A = A(B) = \{tx, t \in \mathbb{C} \text{ and } x \in B\}$ . Let  $\|\cdot\|_B$  be the Minkowski functional of  $B$ ,  $(A, \|\cdot\|_B)$  is a normed algebra. By [1, Proposition 2.15],  $\beta = \|\cdot\|_B$  where  $\beta$  is the radius of boundedness, then  $(A, \|\cdot\|_B)$  is a uniform algebra since  $\beta(x^2) = \beta(x)^2$  for all  $x \in A$ . Let  $A_1$  be the completion of  $A$  under the original norm. It is clear that  $\Phi: (A, \|\cdot\|_B) \rightarrow A_1$ ,  $\Phi(x) = x$ , is continuous, and consequently  $(A, \|\cdot\|_B)$  is functionally continuous. We now remark that we have proved the equivalence of (1), (2) and (3) in Theorem 3.1 without the condition that  $A$  is weakly regular. Using this remark,  $\Phi^{-1}/\text{Im}\Phi$  is continuous, then  $\Phi$  is a homeomorphism (into), so  $A$  is a uniform normed algebra.

## REFERENCES

- [1] G.R.Allan, A spectral theory for locally convex algebras, Proc.London.Math.Soc. (3) 15 (1965), 399-421.
- [2] A.Bedaa, S.J.Bhatt and M.Oudadess, On automatic continuity of homomorphisms, Proc.Amer.Math.Soc.Vol.128, No.4 (1999), 1039-1045.
- [3] H.G.Dales, A discontinuous homomorphism from  $C(X)$ , Amer.J.Math.101 (1979), 647-734.

- [4] H.V.Dedania, A seminorm with square property is automatically submultiplicative, Proc. Indian.Acad.Sci (Math.Sci), 108 (1998), 51-53.
- [5] M.El Azhari, On some automatic continuity theorems, Bull.Belg.Math.Soc. 10 (2003), 627-634.
- [6] A.Mallios, Topological algebras, Selected topics, North-Holland, 1986.
- [7] E.A.Michael, Locally multiplicatively convex topological algebras, Mem.Amer.Math.Soc., No.11 (1952).

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