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This paper is dedicated to Professor M. F. Beatty on the occasion of his 70th birthday.

A method of obtaining a full (two-dimensional) nonlinear stability analysis of *inhomogeneous* deformations of arbitrary incompressible hyperelastic materials is presented. The analysis that we develop replaces the second variation condition expressed as an integral involving two arbitrary perturbations, with an equivalent (third-order) system of ordinary differential equations. The positive-definiteness condition is thereby reduced to the simple numerical evaluation of zeros of a well-behaved function. The general theory is illustrated by applying it to the problem of the inflation of axially stretched thick-walled tubes. The bifurcation theory of such deformations is well known and we compare the bifurcation results with the new stability analysis.

1. Introduction

The purpose of this paper is twofold. First, we develop a method to study the stability of inhomogeneous deformations. Second, we study the relationship between stability and bifurcation for a particular problem in nonlinear elasticity.

Stability and bifurcation are two important topics in mathematical physics in general, and in nonlinear elasticity in particular. Both provide stages for theoretical development of mathematical analysis, as well as for various applications of practical interest. In nature, the two theories deal with different phenomena: stability theory concerns the behaviour of deformed bodies under disturbances, while bifurcation theory concerns the non-uniqueness of solutions. Nevertheless, examples have suggested possible connections between stability conditions and bifurcation conditions. For instance, the buckling of an Euler column corresponds to a bifurcation point, and it has been shown that, at this point, the unbuckled deformation becomes unstable. Such examples, however, are too few, especially for inhomogeneous deformations, to give us a complete understanding of the issue.

Some general discussions of the relation between stability and bifurcation can be found in Ogden (1997) and Beatty (1987). One source of difficulty is the lack of a

universally accepted stability criterion. Two approaches are common. One is to use dynamic stability criteria. For example, Fu (1993, 1995, 1998), Fu & Rogerson (1994) and Fu & Ogden (1999) have looked at wave propagation in elastic plates (see Fu (2001) for a review of dynamic stability). They have not only investigated the nonlinear stability of near-neutral modes, but have also looked at the evolution of the near-neutral modes that require a third-order analysis. The features of the underlying deformation that rendered the problem tractable was the assumption of incompressibility coupled with a plane-strain homogeneous deformation of the plate. Another approach uses static stability criteria, usually based on energy considerations. Some early works include those of Bryan (1888*a*, *b*) and Pearson (1956). Under an energy-stability criterion, all finite deformation solutions, both trivial and non-trivial, will satisfy the Euler–Lagrange equations for the minimization of the energy. Stability, in the Hadamard sense, is determined by the sign of the second variation of the energy, unless this is zero, in which case higher-order variations will decide. It is interesting to note that Ericksen & Toupin (1956) and Hill (1957) showed that Hadamard stability implies uniqueness for a class of boundary-value problems. This also states that a bifurcation point is a point of *neutral* stability. Beyond this, we have been unable to find much analysis in the literature concerning the relation between stability and bifurcation. Historically, the actual stability of trivial and post-bifurcation solutions has not been determined for many problems due to the essential nonlinear character of the stability calculation.

While the energy-stability criterion has been used to determine the stability of homogeneous deformations, little has been done for the stability of inhomogeneous deformations. The purpose of this paper is to demonstrate how it is possible to successfully complete a full nonlinear stability analysis of an *inhomogeneous* deformation. We have not found any other such nonlinear analysis in the literature. There are two simplifying features of our analysis. First, we assume incompressibility, and, second, we restrict our analysis to the consideration of two-dimensional perturbations. This allows the introduction of a stream function, which enables us to consider a single dependent variable in the analysis. However, there are a large number of problems that fit these restrictions and which could all be treated in a similar way to that shown in the following. Also, some of the key steps in the analysis may be of use in other similar calculations where the use of a stream function is not possible.

Specifically, we look at the problem of a thick-walled cylinder that is inflated by an internal pressure and subjected to a prescribed axial stretch. For a given material and specified undeformed geometry of the tube, there may be critical values of the pressure and axial stretch at which the tube forms an axisymmetric bulge. The bifurcation analysis of the problem was previously treated for an arbitrary incompressible material by Haughton & Ogden (1979) (see also Ogden 1997).

In § 2, we derive basic equations and inequalities for the stability and bifurcation analysis. A minimization problem is formulated by using the energy-stability criterion. The first variation condition leads to the equations of equilibrium, and the second variation condition to the stability condition. The cylindrical deformations are considered in § 3. It is found that a unique cylindrical solution exists under a certain condition on the strain-energy function. Section 4 contains a bifurcation analysis. The incremental equations of axisymmetric solutions from the cylindrical solutions are presented. At a bifurcation point, these equations admit non-trivial solutions. The stability analysis is given in § 5. The main development is to solve an integral

inequality involving two arbitrary perturbation functions. By using the incompressibility constraint and Fourier analysis, this integral inequality is reduced to one with a single perturbation. The integral, however, has variable coefficients, since the cylindrical deformation is inhomogeneous. The solution of this latter integral inequality is examined via a system of first-order differential equations. It is shown that the integral inequality, and hence the second variation condition, is satisfied if and only if a matrix associated with the solution of the differential equations be positive semi-definite. Interestingly, it turns out that, for this problem, the bifurcation criterion requires the solution of a fourth-order system of linear ordinary differential equations, but the full nonlinear stability criterion requires only the solution of a third-order system of ordinary differential equations. In the concluding § 6, these equations are solved numerically for a class of Ogden materials. The solutions reveal the result that one somewhat expects: the cylindrical deformation indeed changes stability at a bifurcation point. Specifically, as the pressure increases, the cylindrical deformation becomes unstable at the first bifurcation point and may become stable again if the second bifurcation point occurs. The result presented in this work, besides its theoretical value, could be of practical importance as a combined stability and bifurcation analysis gives one significantly more insight into the behaviour of a physical system than can a stability analysis or a bifurcation analysis alone.

2. Basic equations

We consider the extension and inflation of an elastic body that has a cylindrical shape in a reference configuration Ω defined by

$$\Omega = \{(R, \Theta, Z) \in \mathbb{R}^3 : A \leqslant R \leqslant B,\ 0 \leqslant \Theta < 2\pi,\ 0 \leqslant Z \leqslant L\},$$

where R, Θ and Z are the cylindrical material coordinates, L is the length of the cylinder and A and B its inner and outer radii, respectively. A representative material point is denoted by $\boldsymbol{X} \in \Omega$. The boundary $\partial\Omega$ of Ω is divided into four parts:

$$\begin{aligned}
\varphi_1 &= \{\boldsymbol{X} \in \partial\Omega : R = A\},\\
\varphi_2 &= \{\boldsymbol{X} \in \partial\Omega : R = B\},\\
\varphi_3 &= \{\boldsymbol{X} \in \partial\Omega : Z = 0\},\\
\varphi_4 &= \{\boldsymbol{X} \in \partial\Omega : Z = L\}.
\end{aligned}$$

A deformation of the body is expressed by a smooth mapping $\boldsymbol{x} : \Omega \to \mathbb{R}^3$. The deformation is subjected to the incompressibility constraint

$$\det \boldsymbol{F} = 1, \tag{2.1}$$

where

$$\boldsymbol{F} = \nabla \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$$

is the deformation gradient. The cylinder is stretched (or compressed) by two end plates that specify the axial displacements of the ends. Hence the deformation satisfies the following boundary conditions,

$$\boldsymbol{e}_z \cdot \boldsymbol{x}|_{\varphi_3} = 0, \qquad \boldsymbol{e}_z \cdot \boldsymbol{x}|_{\varphi_4} = l, \tag{2.2}$$

where $\boldsymbol{e}_z$ is the unit vector in the axial direction and l is the prescribed deformed length of the cylinder. It is noted that the end plates do not impose restrictions on the end displacements in the radial and azimuthal directions.

The body is comprised of a homogeneous, isotropic and incompressible elastic material in the reference configuration. There exists a strain-energy function $W(\boldsymbol{F})$ whose value gives the strain energy per unit undeformed volume. In addition to being stretched, the cylinder is inflated by a prescribed internal pressure P. The total potential energy associated with the deformed system is then given by

$$E = \int_{\Omega} W(\boldsymbol{F})\,\mathrm{d}V - Pv, \tag{2.3}$$

where v is the volume enclosed by the deformed inner surface of the cylinder and the end plates. It is given by

$$v = \int_{\Omega_0} \det \boldsymbol{F}\,\mathrm{d}V, \tag{2.4}$$

where Ω_0 is the region enclosed by the undeformed inner surface,

$$\Omega_0 = \{(R, \Theta, Z) \in \mathbb{R}^3 : 0 \leqslant R \leqslant A,\ 0 \leqslant \Theta < 2\pi,\ 0 \leqslant Z \leqslant L\}.$$

In (2.4), we have extended $\boldsymbol{x}$ smoothly to $\Omega \cup \Omega_0$ in such a way that the extended function also satisfies the boundary conditions (2.2). The extended function is nevertheless not subjected to the incompressibility constraint (2.1) in Ω_0. It is noted that this method of computing v may not be the standard approach, but does allow a concise derivation.

By the energy-stability criterion, a stable equilibrium deformation $\boldsymbol{x}$ is a relative minimum of the total energy (2.3) among all kinematically admissible deformations. A standard method of solving this constrained minimization problem is to introduce a Lagrange multiplier and minimize the relaxed total potential energy function in the space of deformations not subjected to the incompressibility constraint (2.1). Here, we shall follow the approach of Fosdick & MacSithigh (1986)† and work directly in the set of the kinematically admissible deformations. To this end, we introduce a one-parameter family of deformations $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, \epsilon)$ that satisfy the kinematic constraints‡ (2.1) and (2.2) for each value of ϵ; this renders the potential energy E minimum when evaluated at $\epsilon = 0$. Substituting $\boldsymbol{x}(\boldsymbol{X}, \epsilon)$ into (2.1) and (2.2), and differentiating with respect to ϵ, leads to

$$\boldsymbol{F}^{-\mathrm{T}} \cdot \nabla \dot{\boldsymbol{x}} = 0, \tag{2.5 a}$$

$$\boldsymbol{F}^{-\mathrm{T}} \cdot \nabla \ddot{\boldsymbol{x}} = \mathrm{tr}(\boldsymbol{F}^{-1} \nabla \dot{\boldsymbol{x}})^2, \tag{2.5 b}$$

and

$$\boldsymbol{e}_z \cdot \dot{\boldsymbol{x}}|_{\varphi_3 \cup \varphi_4} = 0, \qquad \boldsymbol{e}_z \cdot \ddot{\boldsymbol{x}}|_{\varphi_3 \cup \varphi_4} = 0, \tag{2.6}$$

where the superscript '$-$T' denotes the inverse transpose and a superimposed dot denotes the partial derivative with respect to ϵ.

† Their work concerns the problems with dead-load boundary traction, while the problem at hand involves pressure loads.

‡ Again, condition (2.1) is not imposed on $\boldsymbol{x}$ $\quad \boldsymbol{x}(\boldsymbol{X}, \epsilon)$ in Ω_0.

Substituting $\boldsymbol{x}(\boldsymbol{X},\epsilon)$ into (2.3), differentiating with respect to ϵ, and using the divergence therorem, we obtain

$$\begin{aligned}\dot{E} &= \int_\Omega W_{\boldsymbol{F}}\cdot\nabla\dot{\boldsymbol{x}}\,\mathrm{d}V - P\int_{\Omega_0}(\det\boldsymbol{F})F^{-\mathrm{T}}\cdot\nabla\dot{\boldsymbol{x}}\,\mathrm{d}V\\ &= \int_{\partial\Omega}\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{F}}\boldsymbol{N}\,\mathrm{d}A - \int_\Omega\dot{\boldsymbol{x}}\cdot\operatorname{div}W_{\boldsymbol{F}}\,\mathrm{d}V - P\int_{\partial\Omega_0}(\det\boldsymbol{F})\dot{\boldsymbol{x}}\cdot\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N}\,\mathrm{d}A\\ &= \int_{\partial\Omega}\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{F}}\boldsymbol{N}\,\mathrm{d}A - \int_\Omega\dot{\boldsymbol{x}}\cdot\operatorname{div}W_{\boldsymbol{F}}\,\mathrm{d}V + P\int_{\varphi_1}\dot{\boldsymbol{x}}\cdot\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N}\,\mathrm{d}A,\end{aligned}\tag{2.7}$$

where the subscripts to W denote the partial derivative, 'div' denotes the divergence operator with respect to $\boldsymbol{X}$, and $\boldsymbol{N}$ is the unit outward normal to the surface in the reference configuration. We note that the outward normals to $\partial\Omega_0$ and to φ_1 are opposite. Since $\boldsymbol{x}(\boldsymbol{X},0)$ is a minimizing function of E, the last expression in (2.7) must vanish at $\epsilon=0$. By an argument used by Fosdick & MacSithigh (1986)†, there exist smooth functions $p:\Omega\to\mathbb{R}$ and $g:\varphi_3\cup\varphi_4\to\mathbb{R}$ such that

$$\operatorname{div}(-p\boldsymbol{F}^{-\mathrm{T}}+W_{\boldsymbol{F}})=0\quad\text{in }\Omega\tag{2.8}$$

and

$$(-p\boldsymbol{F}^{-\mathrm{T}}+W_{\boldsymbol{F}})\boldsymbol{N}=\begin{cases}-P\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N} & \text{on }\varphi_1,\\ 0 & \text{on }\varphi_2,\\ g\boldsymbol{e}_z & \text{on }\varphi_3\cup\varphi_4.\end{cases}\tag{2.9}$$

In (2.8) and (2.9), all terms are now evaluated at the minimizing function $\boldsymbol{x}(X)$.

We now turn our attention to the second variation condition. Substituting $\boldsymbol{x}(\boldsymbol{X},\epsilon)$ into (2.3), differentiating with respect to ϵ twice, using the divergence theorem and (2.2), (2.5), (2.6), (2.8) and (2.9), we find that

$$\begin{aligned}\ddot{E} &= \int_\Omega\{W_{\boldsymbol{F}}\cdot\nabla\ddot{\boldsymbol{x}}+\nabla\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{F}F}[\nabla\dot{\boldsymbol{x}}]\}\,\mathrm{d}V\\ &\quad - P\int_{\Omega_0}(\det\boldsymbol{F})[(\boldsymbol{F}^{-\mathrm{T}}\cdot\nabla\dot{\boldsymbol{x}})^2-\operatorname{tr}(\boldsymbol{F}^{-1}\nabla\dot{\boldsymbol{x}})^2+\boldsymbol{F}^{-\mathrm{T}}\cdot\nabla\ddot{\boldsymbol{x}}]\,\mathrm{d}V\\ &= \int_\Omega\{p\operatorname{tr}(\boldsymbol{F}^{-1}\nabla\dot{\boldsymbol{x}})^2+\nabla\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{FF}}[\nabla\dot{\boldsymbol{x}}]\}\,\mathrm{d}V+\int_{\partial\Omega}\ddot{\boldsymbol{x}}\cdot(-p\boldsymbol{F}^{-\mathrm{T}}+W_{\boldsymbol{F}})\boldsymbol{N}\,\mathrm{d}A\\ &\quad - P\int_{\partial\Omega_0}(\det\boldsymbol{F})[(\boldsymbol{F}^{-\mathrm{T}}\cdot\nabla\dot{\boldsymbol{x}})\boldsymbol{F}^{-1}\dot{\boldsymbol{x}}-\boldsymbol{F}^{-1}\nabla\dot{\boldsymbol{x}}\boldsymbol{F}^{-1}\dot{\boldsymbol{x}}+\boldsymbol{F}^{-1}\ddot{\boldsymbol{x}}]\cdot\boldsymbol{N}\,\mathrm{d}A\\ &= \int_\Omega\{p\operatorname{tr}(\boldsymbol{F}^{-1}\nabla\dot{\boldsymbol{x}})^2+\nabla\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{FF}}[\nabla\dot{\boldsymbol{x}}]\}\,\mathrm{d}V-P\int_{\varphi_1}(\nabla\dot{\boldsymbol{x}}\boldsymbol{F}^{-1}\dot{\boldsymbol{x}})\cdot\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N}\,\mathrm{d}A\\ &= \int_\Omega\{p\operatorname{tr}(\boldsymbol{F}^{-1}\nabla\dot{\boldsymbol{x}})^2+\nabla\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{FF}}[\nabla\dot{\boldsymbol{x}}]\}\,\mathrm{d}V\\ &\quad+\int_{\partial\Omega}(\nabla\dot{\boldsymbol{x}}\boldsymbol{F}^{-1}\dot{\boldsymbol{x}})\cdot(-p\boldsymbol{F}^{-\mathrm{T}}+W_{\boldsymbol{F}})\boldsymbol{N}\,\mathrm{d}A\end{aligned}$$

† See the proof of theorem 3.2 therein.

$$= \int_\Omega \{p\,\mathrm{tr}(\boldsymbol{F}^{-1}\nabla\dot{\boldsymbol{x}})^2 + \nabla\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{FF}}[\nabla\dot{\boldsymbol{x}}] + (-\ \ p\boldsymbol{F}^{-\mathrm{T}} + W_{\boldsymbol{F}})\cdot\nabla(\nabla\dot{\boldsymbol{x}}\boldsymbol{F}^{-1}\dot{\boldsymbol{x}})\}\,\mathrm{d}V$$
$$= \int_\Omega \{\nabla\dot{\boldsymbol{x}}\cdot W_{\boldsymbol{FF}}[\nabla\dot{\boldsymbol{x}}] + W_{\boldsymbol{F}}\cdot\nabla(\nabla\dot{\boldsymbol{x}}\boldsymbol{F}^{-1}\dot{\boldsymbol{x}})\}\,\mathrm{d}V$$
$$\geqslant 0. \tag{2.10}$$

The quadratic integral inequality (2.10) will be used to determine the stability of the deformation $\boldsymbol{x}(X)$. The derivation of (2.10) is for the family of deformations of general form. As a result, the coefficients of the integrand in (2.10) are functions of all material coordinates, and the corresponding eigenvalue problem consists of partial differential equations of variable coefficients. Analytical solutions of such equations are prohibitively difficult. In the remaining sections of this work, we shall consider the stability of cylindrical deformations in the class of axisymmetric deformations, as well as the bifurcation from cylindrical deformations to axisymmetric deformations. It will be shown that for such a problem inequality (2.10) is equivalent to an integral inequality involving one scalar perturbation function, which effectively reduces the associated eigenvalue problem to ordinary differential equations.

3. Cylindrical deformations

A deformation $\boldsymbol{x}$ is cylindrical if it has the following component form,

$$r = r(R), \qquad \theta = \Theta, \qquad z = \lambda_3 Z, \tag{3.1}$$

where r, θ and z are the cylindrical spatial coordinates and λ_3 is the constant axial stretch. In the cylindrical coordinates, the deformation gradient of the cylindrical deformation has the following component form:

$$\boldsymbol{F} = \begin{pmatrix} r' & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Here and henceforth, a prime denotes the derivative with respect to R. The incompressibility constraint (2.1) now reads

$$\frac{\lambda_3 r r'}{R} = 1. \tag{3.2}$$

Equation (3.2) can be integrated to yield

$$r(R) = \sqrt{a^2 + \frac{R^2 - A^2}{\lambda_3}}, \tag{3.3}$$

where a is a constant of integration corresponding to the inner radius of the deformed cylinder.

By the isotropy, the strain-energy function W depends on $\boldsymbol{F}$ through the principal stretches λ_1, λ_2 and λ_3:

$$W(\boldsymbol{F}) = W(\lambda_1, \lambda_2, \lambda_3).$$

Here we have used W to denote different functions. For the cylindrical deformation (3.1), we have

$$\lambda_1 = r', \qquad \lambda_2 = r/R, \tag{3.4}$$

and λ_3 is identical to that in (3.1). For the cylindrical deformation, the tensor $W_{\boldsymbol{F}}$ has the following component form,

$$W_{\boldsymbol{F}} = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & W_3 \end{pmatrix}, \tag{3.5}$$

where a subscript i to W denotes the partial derivative with respect to λ_i. Substituting (3.5) into (2.8), we find that, for the cylindrical deformations, the hydrostatic pressure p is a function of R only. The only non-trivial equation of (2.8) is

$$-\frac{p'}{\lambda_1} + W_1' + \frac{W_1 - W_2}{R} = 0, \tag{3.6}$$

which can be integrated to give p. The resulting constant of integration and the parameter a appearing in (3.3) are to be determined by the first two boundary conditions of (2.9), which can be written as

$$\begin{aligned} -p + \lambda_1 W_1 &= -P \quad \text{for } R = A, \\ -p + \lambda_1 W_1 &= 0 \qquad \text{for } R = B. \end{aligned}$$

By taking the difference of the above equations, and using (3.6), (3.2) and (3.3), we find that

$$\begin{aligned} P &= \int_A^B (-p + \lambda_1 W_1)' \, \mathrm{d}R \\ &= \int_A^B \left[\left(\lambda_1' - \frac{\lambda_1}{R} \right) W_1 + \frac{\lambda_1}{R} W_2 \right] \mathrm{d}R \\ &= \int_A^B \frac{\lambda_1}{\lambda_1 - \lambda_2} (\lambda_1' W_1 + \lambda_2' W_2) \, \mathrm{d}R \\ &= \int_A^B \frac{\lambda_1}{\lambda_1 - \lambda_2} W' \, \mathrm{d}R. \end{aligned} \tag{3.7}$$

For the inflation of a cylinder, at a fixed axial extension, the inflating pressure P can be regarded as a function of the deformed inner radius a defined by (3.7), with (3.3) and (3.4). It is well known that, for this deformation, the internal radius a may not be unique for a given pressure P, or a cylindrical deformation may not even exist. However, it could be argued that certain restrictions on the strain-energy function describing the material should provide the existence and uniqueness of such a deformation for any prescribed pressure P. To investigate this, we first note that (3.7) can be rewritten as

$$P = \int_{\lambda_a}^{\lambda_b} \frac{1}{1 - \lambda_2^2 \lambda_3} \frac{\partial \hat{W}(\lambda_2, \lambda_3)}{\partial \lambda_2} \, \mathrm{d}\lambda_2,$$

where we define the reduced strain-energy function

$$\hat{W}(\lambda_2, \lambda_3) \equiv W\left(\frac{1}{\lambda_2 \lambda_3}, \lambda_2, \lambda_3 \right), \tag{3.8}$$

having used (3.2) and (3.4), and define

$$\lambda_a \equiv \lambda_2(A) = \frac{a}{A}, \qquad \lambda_b \equiv \lambda_2(B) = \frac{1}{B}\sqrt{a^2 + \frac{B^2 - A^2}{\lambda_3}}. \tag{3.9}$$

Now define a continuous function

$$g(a) \equiv \int_{a/A}^{(1/B)\sqrt{a^2+(B^2-A^2)/\lambda_3}} \frac{1}{1-\lambda_2^2\lambda_3} \frac{\partial \hat{W}(\lambda_2, \lambda_3)}{\partial \lambda_2}\, \mathrm{d}\lambda_2. \tag{3.10}$$

By the symmetry of $W(\lambda_1, \lambda_2, \lambda_3)$, we have

$$\hat{W}(\lambda_2, \lambda_3) = \hat{W}\left(\frac{1}{\lambda_2\lambda_3}, \lambda_3\right),$$

and hence

$$\left.\frac{\partial \hat{W}(\lambda_2, \lambda_3)}{\partial \lambda_2}\right|_{\lambda_2^2\lambda_3=1} = 0.$$

This implies, by the smoothness of $\hat{W}$, that the integrand in (3.10) is bounded at $\lambda_2^2\lambda_3 = 1$. Therefore, we have

$$g\left(\frac{A}{\sqrt{\lambda_3}}\right) = \int_{1/\sqrt{\lambda_3}}^{1/\sqrt{\lambda_3}} \frac{1}{1-\lambda_2^2\lambda_3} \frac{\partial \hat{W}(\lambda_2, \lambda_3)}{\partial \lambda_2}\, \mathrm{d}\lambda_2 = 0.$$

Moreover, we note that

$$\lim_{a\to\infty} \frac{\lambda_b}{\lambda_a} = \frac{A}{B}.$$

It is now easy to see that a sufficient condition for the existence of a solution for any positive pressure P is that

$$\hat{W}_2(\lambda_2, \lambda_3) \to \gamma\lambda_2^\beta \quad \text{as } \lambda_2 \to \infty, \tag{3.11}$$

where γ is a positive constant and $\beta > 1$.

To study the uniqueness of solution, we observe, from (3.10) and (3.9), that

$$\begin{aligned} g'(a) &= \frac{a}{B^2\lambda_b(1-\lambda_b^2\lambda_3)}\hat{W}_2(\lambda_b, \lambda_3) - \frac{1}{A(1-\lambda_a^2\lambda_3)}\hat{W}_2(\lambda_a, \lambda_3) \\ &= \frac{a(B^2-A^2)}{A^2B^2\lambda_3(\lambda_b^2-\lambda_a^2)}\left[\frac{\hat{W}_2(\lambda_b, \lambda_3)}{\lambda_b} - \frac{\hat{W}_2(\lambda_a, \lambda_3)}{\lambda_a}\right], \end{aligned}$$

where a subscript to $\hat{W}$ again denotes the derivative. A sufficient condition for the uniqueness of solution (within the set of cylindrical deformations) is then

$$\frac{1}{\lambda_b - \lambda_a}\left[\frac{\hat{W}_2(\lambda_b, \lambda_3)}{\lambda_b} - \frac{\hat{W}_2(\lambda_a, \lambda_3)}{\lambda_a}\right] > 0 \quad \text{for all } \lambda_a, \lambda_b, \tag{3.12}$$

or

$$\lambda_2 \hat{W}_{22} - \hat{W}_2 > 0. \tag{3.13}$$

We note that condition (3.12) requires that $\hat{W}_2(\lambda_2, \lambda_3)/\lambda_2$ be monotone increasing in λ_2. This result was obtained by Haughton & Ogden (1979) as the non-existence of so-called pressure turning-points.

To put this in context, consider the Ogden materials

$$\hat{W}(\lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left(\frac{1}{\lambda_2^{\alpha_i} \lambda_3^{\alpha_i}} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right), \tag{3.14}$$

where N, μ_i and α_i, $i = 1, 2, \ldots, N$, are material parameters. Condition (3.11) requires the existence of at least one pair (α_i, μ_i) satisfying

$$|\alpha_i| > 2, \quad \alpha_i \mu_i > 0. \tag{3.15}$$

Furthermore, condition (3.13) holds if inequalities (3.15) are satisfied for each $i = 1, 2, \ldots, N$.

The existence and uniqueness of a cylindrical deformation does not mean this deformation can actually be observed in experiment. As the internal pressure increases, a cylindrical solution branch may lose stability, and may give rise to non-cylindrical solutions that are stable. In the following sections, we examine the conditions under which the cylindrical solution bifurcates to non-cylindrical axisymmetric solutions, and the conditions under which the cylindrical solution becomes unstable in the class of axisymmetric deformations.

4. Bifurcation analysis

The bifurcation criterion for the problem under consideration is the existence of non-trivial solutions to the linearized equations of (2.8) and (2.9). These equations can be found in Haughton & Ogden (1979, eqns (53)–(55)). For completeness, we rewrite these (Eulerian) equations using Lagrangian coordinates and arrive at an incremental equation,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}R}\Big\{ B_{1313}\lambda_2^3\lambda_3^3 f''' + \lambda_2\lambda_3^2[(5 - 3\lambda_2^2\lambda_3)B_{1313} + \lambda_2^2\lambda_3 R B'_{1313}]\frac{f''}{R} \\
&\qquad + \lambda_3[(1 - 5\lambda_2^2\lambda_3 + 3\lambda_2^4\lambda_3^2)B_{1313} + \lambda_2^2\lambda_3(2 - \lambda_2^2\lambda_3)RB'_{1313}]\frac{f'}{\lambda_2 R^2} \\
&\qquad\qquad - (\lambda_2^3\lambda_3 R B'_{1313} - B_{1313})\frac{f}{\lambda_2^3 R^3}\Big\} \\
&\quad + \frac{\alpha^2}{\lambda_2^2\lambda_3^2}\Big\{ \lambda_2^4\lambda_3^2(2B_{1313} - \lambda_3^2\hat{W}_{33})f'' \\
&\qquad + \lambda_2^2\lambda_3\Big[2(1 - \lambda_2^2\lambda_3)B_{1313} + 2\sigma_1 - \lambda_3^2\hat{W}_{33}(2 - \lambda_2^2\lambda_3) \\
&\qquad\qquad + R\frac{\mathrm{d}}{\mathrm{d}R}(2B_{1313} - \lambda_3^2\hat{W}_{33})\Big]\frac{f'}{R}
\end{aligned}$$

$$+\left[\lambda_2^4\lambda_3^2R^2B''_{1313}+\lambda_2^2\lambda_3RB'_{1313}(2-\lambda_2^2\lambda_3)-2B_{1313}\right.$$
$$+\lambda_2^2\lambda_3(\lambda_2\hat{W}_2+\lambda_2^2\hat{W}_{22})+\lambda_3^2\hat{W}_{33}-2\lambda_2\lambda_3\hat{W}_{23}$$
$$\left.+\lambda_2^2\lambda_3R\frac{\mathrm{d}}{\mathrm{d}R}(\lambda_2\lambda_3\hat{W}_{23}-\lambda_3^2\hat{W}_{33})\right]\frac{f}{R^2}\Bigg\}$$
$$+\alpha^4\lambda_2^2B_{1313}f=0, \tag{4.1}$$

where

$$\alpha=n\pi/L, \tag{4.2}$$

n being an integer,

$$B_{1313}=\frac{\lambda_3\hat{W}_3}{\lambda_2^2\lambda_3^4-1}$$

and

$$\sigma_1=\lambda_1\frac{\partial W}{\partial\lambda_1}.$$

(Note that this is written in terms of the original strain-energy function W and not $\hat{W}$.) The unknown function $f(R)$ in (4.1) corresponds to the radial component of the incremental deformation. Precisely, the radial component of the deformation is given by

$$r=r(R)+f(R)\sin\alpha Z,$$

where $r(R)$ is given by (3.3). The axial component of the incremental deformation can be expressed in terms of f by the incompressibility constraint (2.1). We also have the boundary conditions

$$\lambda_2^4\lambda_3^2R^2f''+\lambda_2^2\lambda_3(2-\lambda_2^2\lambda_3)Rf'+\left(\frac{\lambda_2^2R^2\alpha^2}{\lambda_3^2}-1\right)f=0,\quad R=A,B, \tag{4.3}$$

and

$$\lambda_2^2\lambda_3^3\hat{W}_{33}Rf'+(\lambda_2\hat{W}_2+\lambda_3^3\hat{W}_{33}-\lambda_2\lambda_3\hat{W}_{23})f=0,\quad R=A,B. \tag{4.4}$$

The condition for bifurcation is then the existence of non-trivial solutions to these homogeneous equations. We have conducted some numerical calculations to compare the bifurcation points predicted by (4.1), (4.3), (4.4) and the stability condition to be derived in the next section. These numerical solutions will be presented in § 6. For convenience, we have taken the integer n in (4.2) to be unity and then present the numerical results in terms of the physical parameter L only.

5. Stability analysis

The stability criterion for the problem under consideration is based on the second variation condition (2.10). We wish to find the conditions for the cylindrical deformation to be stable among all axisymmetric deformations. Let $u(R,Z)$ and $v(R,Z)$ be the radial and axial components of an axisymmetric function $\dot{\boldsymbol{x}}$. The first boundary conditions of (2.6) can be written as

$$v(R,0)=v(R,L)=0. \tag{5.1}$$

In the cylindrical coordinate system, $\nabla\dot{\boldsymbol{x}}$ has the following component form,

$$\nabla\dot{\boldsymbol{x}} = \begin{pmatrix} u_R & 0 & u_Z \\ 0 & u/R & 0 \\ v_R & 0 & v_Z \end{pmatrix},$$

where a subscript denotes the partial derivative. Equation (2.5 a) then takes the form

$$\frac{u_R}{\lambda_1} + \frac{u}{R\lambda_2} + \frac{v_Z}{\lambda_3} = 0. \tag{5.2}$$

The non-zero components of $W_{\boldsymbol{FF}}$ are given by (see, for example, Ogden 1997, § 6.2)

$$(W_{\boldsymbol{FF}})_{iijj} = W_{ij}, \quad i, j, \quad \text{no sum},$$

and

$$(W_{\boldsymbol{FF}})_{ijij} = \frac{\lambda_i W_i - \lambda_j W_j}{\lambda_i^2 - \lambda_j^2}, \quad (W_{\boldsymbol{FF}})_{ijji} = \frac{\lambda_j W_i - \lambda_i W_j}{\lambda_i^2 - \lambda_j^2}, \quad i \neq j, \quad \text{no sum}.$$

The first term of the integrand in (2.10) is then

$$\begin{aligned} \nabla\dot{\boldsymbol{x}} \cdot W_{\boldsymbol{F}F}[\nabla\dot{\boldsymbol{x}}] = {} & W_{11}u_R^2 + W_{22}\frac{u^2}{R^2} + W_{33}v_Z^2 + 2W_{12}\frac{uu_R}{R} + 2W_{23}\frac{uv_Z}{R} \\ & + 2W_{13}u_R v_Z + \frac{\lambda_1 W_1 - \lambda_3 W_3}{\lambda_1^2 - \lambda_3^2}(u_Z^2 + v_R^2) + 2\frac{\lambda_3 W_1 - \lambda_1 W_3}{\lambda_1^2 - \lambda_3^2}u_Z v_R. \end{aligned} \tag{5.3}$$

The second term in (2.10) can be written in a form that is more convenient for mathematical treatment. By using (5.2) and (5.1), and the fact that all the coefficients are independent of Z, we find that

$$\begin{aligned} & \int_\Omega W_{\boldsymbol{F}} \cdot \nabla(\nabla\dot{\boldsymbol{x}}\boldsymbol{F}^{-1}\dot{\boldsymbol{x}})\,\mathrm{d}V \\ & \quad = 2\pi \int_0^L \int_A^B \left[W_1 \left(\frac{uu_R}{\lambda_1} + \frac{vu_Z}{\lambda_3} \right)_R + \frac{W_2}{R}\left(\frac{uu_R}{\lambda_1} + \frac{vu_Z}{\lambda_3} \right) \right. \\ & \qquad \left. + W_3 \left(\frac{uv_R}{\lambda_1} + \frac{vv_Z}{\lambda_3} \right)_Z \right] R\,\mathrm{d}R\,\mathrm{d}Z \\ & \quad = 2\pi \int_0^L \int_A^B \left[W_1 \left(-\frac{u^2}{R\lambda_2} - \frac{uv_Z}{\lambda_3} + \frac{vu_Z}{\lambda_3} \right)_R + \frac{W_2}{R}\left(\frac{uu_R}{\lambda_1} + \frac{vu_Z}{\lambda_3} \right) \right] R\,\mathrm{d}R\,\mathrm{d}Z \\ & \quad = 2\pi \int_0^L \int_A^B \left[W_1 \left(\frac{\lambda_1 u^2}{R^2\lambda_2^2} - \frac{2uu_R}{R\lambda_2} - \frac{2u_R v_Z}{\lambda_3} + \frac{2u_Z v_R}{\lambda_3} \right) \right. \\ & \qquad \left. + \frac{W_2}{R}\left(\frac{2uu_R}{\lambda_1} + \frac{u^2}{R\lambda_2} \right) \right] R\,\mathrm{d}R\,\mathrm{d}Z. \end{aligned} \tag{5.4}$$

By (5.3) and (5.4), inequality (2.10) can be written as

$$\int_0^L \int_A^B \left[\boldsymbol{\omega} \cdot M^0 \boldsymbol{\omega} + \frac{\lambda_1 W_1 - \lambda_3 W_3}{\lambda_1^2 - \lambda_3^2} \begin{pmatrix} u_Z \\ v_R \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda_1/\lambda_3 \\ \lambda_1/\lambda_3 & 1 \end{pmatrix} \begin{pmatrix} u_Z \\ v_R \end{pmatrix} \right] R\,\mathrm{d}R\,\mathrm{d}Z \geqslant 0, \tag{5.5}$$

where

$$\boldsymbol{\omega} \equiv \left(u_R, \frac{u}{R}, v_Z\right)^{\mathrm{T}}$$

and

$$M^0 \equiv \begin{pmatrix} W_{11} & W_{12} - W_1/\lambda_2 + W_2/\lambda_1 & W_{31} - W_1/\lambda_3 \\ W_{12} - W_1/\lambda_2 + W_2/\lambda_1 & W_{22} + \lambda_1 W_1/\lambda_2^2 + W_2/\lambda_2 & W_{23} \\ W_{31} - W_1/\lambda_3 & W_{23} & W_{33} \end{pmatrix}.$$

The quadratic inequality (5.5) involves an integral of two perturbation functions u and v. We now reduce it to one of a single perturbation by using the incompressibility constraint (5.2). Suppose that two functions $u(R, Z)$ and $v(R, Z)$ are given that satisfy (5.1) and (5.2). Define a stream function

$$\phi(R, Z) \equiv a\lambda_3^2 \int_0^Z u(A, \xi)\,\mathrm{d}\xi - \int_A^R \rho v(\rho, Z)\,\mathrm{d}\rho.$$

By using (5.1) and (5.2), it is readily verified that

$$\phi_R(R, 0) = \phi_R(R, L) = 0 \tag{5.6}$$

and

$$u = \frac{\phi_Z}{R\lambda_2\lambda_3^2}, \qquad v = -\frac{\phi_R}{R}. \tag{5.7}$$

Substituting (5.7) into (5.5), we find that

$$\int_0^L \int_A^B \left[\begin{pmatrix} \phi_Z \\ R\phi_{RZ} \end{pmatrix} \cdot M^1 \begin{pmatrix} \phi_Z \\ R\phi_{RZ} \end{pmatrix} + \begin{pmatrix} \phi_R \\ R\phi_{RR} \\ R\phi_{ZZ} \end{pmatrix} \cdot M^2 \begin{pmatrix} \phi_R \\ R\phi_{RR} \\ R\phi_{ZZ} \end{pmatrix} \right] \mathrm{d}R\,\mathrm{d}Z \geqslant 0, \tag{5.8}$$

where M^1 is the symmetric matrix with components

$$M^1_{11} = \frac{\lambda_1^4[\lambda_1^2 W_{11} - 2\lambda_1\lambda_2 W_{12} + \lambda_2^2 W_{22} + 3\lambda_1 W_1 - \lambda_2 W_2]}{R^3},$$

$$M^1_{12} = \frac{\lambda_1^3\lambda_2[-\lambda_1^2 W_{11} + \lambda_1\lambda_2 W_{12} - \lambda_2\lambda_3 W_{23} + \lambda_3\lambda_1 W_{13} - 2\lambda_1 W_1 + \lambda_2 W_2]}{R^3},$$

$$M^1_{22} = \frac{\lambda_1^2\lambda_2^2[\lambda_1^2 W_{11} - 2\lambda_3\lambda_1 W_{13} + \lambda_3^2 W_{33} + 2\lambda_1 W_1]}{R^3}$$

and

$$M^2 \equiv \frac{\lambda_1 W_1 - \lambda_3 W_3}{R^3\lambda_3^2(\lambda_1^2 - \lambda_3^2)} \begin{pmatrix} \lambda_3^2 & -\lambda_3^2 & \lambda_1^2 \\ -\lambda_3^2 & \lambda_3^2 & -\lambda_1^2 \\ \lambda_1^2 & -\lambda_1^2 & \lambda_1^2 \end{pmatrix}.$$

We thus conclude that if (5.8) holds for all $\phi(R, Z)$ satisfying (5.6), then (5.5) holds for all $u(R, Z)$ and $v(R, Z)$ satisfying (5.1) and (5.2). Conversely, for any given $\phi(R, Z)$ that satisfies (5.6), the functions $u(R, Z)$ and $v(R, Z)$ defined by (5.7) satisfy (5.1) and (5.2). This means that (5.5) implies (5.8) as well. Therefore, these two inequalities are equivalent.

In terms of the reduced strain-energy function (3.8), the matrices M^1 and M^2 are

$$M^1 = \frac{1}{R^3}\begin{pmatrix} \dfrac{\lambda_2\hat{W}_{22} - \hat{W}_2}{\lambda_2^3\lambda_3^4} & \dfrac{-\lambda_3\hat{W}_{23} + \hat{W}_2}{\lambda_2\lambda_3^3} \\ \dfrac{-\lambda_3\hat{W}_{23} + \hat{W}_2}{\lambda_2\lambda_3^3} & \hat{W}_{33} \end{pmatrix}$$

and

$$M^2 = \frac{\hat{W}_3}{R^3\lambda_3(\lambda_2^2\lambda_3^4 - 1)}\begin{pmatrix} \lambda_2^2\lambda_3^4 & -\lambda_2^2\lambda_3^4 & 1 \\ -\lambda_2^2\lambda_3^4 & \lambda_2^2\lambda_3^4 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

An obvious sufficient condition for (5.8) to hold is that the matrices M^1 and M^2 both be positive-semi-definite. This leads to the following inequalities:

$$\lambda_2\lambda_3^2 - 1 \geqslant 0, \qquad \hat{W}_3 \geqslant 0 \tag{5.9}$$

and

$$\lambda_2\hat{W}_{22} - \hat{W}_2 \geqslant 0, \qquad \hat{W}_{33} \geqslant 0, \qquad \lambda_3^2(\lambda_2\hat{W}_{22} - \hat{W}_2)\hat{W}_{33} - \lambda_2(\lambda_3\hat{W}_{23} - \hat{W}_2)^2 \geqslant 0. \tag{5.10}$$

If we impose strict inequalities in (5.9) and (5.10), then it follows that strict inequality holds in (5.8), and hence the total energy has a local minimum at the cylindrical deformation.

Inequalities (5.9) and (5.10) are by no means necessary for stability, and may be too strong for the purpose of studying the restriction of stability on the strain-energy function. We now derive a necessary and sufficient condition for (5.8), which involves only one integral in R.

To this end, we expand the function $\phi(R, Z)$ into a Fourier sine series in Z,

$$\phi(R, Z) = \sum_{n=1}^{\infty} b_n(R)\sin\frac{n\pi Z}{L}, \tag{5.11}$$

where the coefficients $b_n(R)$ are given by

$$b_n(R) = \frac{2}{L}\int_0^L \phi(R, Z)\sin\frac{n\pi Z}{L}\,\mathrm{d}Z.$$

Also, we have the Fourier cosine series expansion for the derivative $\phi_Z(R, Z)$ of $\phi(R, Z)$:

$$\phi_Z(R, Z) = \tfrac{1}{2}a_0(R) + \sum_{n=1}^{\infty} a_n(R)\cos\frac{n\pi Z}{L}. \tag{5.12}$$

Integrating (5.12) in Z yields

$$\begin{aligned}\phi(R, Z) &= \phi(R, 0) + \tfrac{1}{2}a_0(R)Z + \sum_{n=1}^{\infty}\frac{L}{n\pi}a_n(R)\sin\frac{n\pi Z}{L} \\ &= \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}[1 - (-1)^n]\phi(R, 0) - \frac{L}{n\pi}(-1)^n a_0(R) + \frac{L}{n\pi}a_n(R)\right\}\sin\frac{n\pi Z}{L},\end{aligned} \tag{5.13}$$

or

$$\phi(R,Z)=\sum_{n=1}^{\infty}\Big\{\frac{2}{n\pi}[1-(-1)^n][\phi(R,L)-\tfrac{1}{2}La_0(R)] \\ -\frac{L}{n\pi}(-1)^n a_0(R)+\frac{L}{n\pi}a_n(R)\Big\}\sin\frac{n\pi Z}{L}. \tag{5.14}$$

It follows from (5.11), (5.13) and (5.14) that

$$a_0(R)=\frac{2}{L}[\phi(R,L)-\phi(R,0)], \tag{5.15}$$

$$a_n(R)=\frac{2}{L}[(-1)^n\phi(R,L)-\phi(R,0)]+\frac{n\pi}{L}b_n(R). \tag{5.16}$$

By virtue of (5.6), the coefficient a_0 is a constant.

The Fourier series of other derivatives of $\phi(R,Z)$ can be derived similarly as

$$\phi_R(R,Z)=\sum_{n=1}^{\infty}c_n(R)\sin\frac{n\pi Z}{L},$$

$$\phi_{ZZ}(R,Z)=\sum_{n=1}^{\infty}d_n(R)\sin\frac{n\pi Z}{L},$$

$$\phi_{RZ}(R,Z)=\tfrac{1}{2}e_0(R)+\sum_{n=1}^{\infty}e_n(R)\cos\frac{n\pi Z}{L},$$

$$\phi_{RR}(R,Z)=\sum_{n=1}^{\infty}f_n(R)\sin\frac{n\pi Z}{L},$$

where

$$\left.\begin{aligned} &c_n(R)=b_n'(R)=\frac{L}{n\pi}a_n'(R), \qquad d_n(R)=-\frac{n\pi}{L}a_n(R), \\ &e_0(R)=a_0'(R)=0, \qquad e_n(R)=a_n'(R), \qquad f_n(R)=b_n''(R)=\frac{L}{n\pi}a_n''(R). \end{aligned}\right\} \tag{5.17}$$

Here, use has been made of (5.14) and (5.6).

By Parseval's theorem, along with (5.15), (5.16) and (5.17), inequality (5.8) is equivalent to

$$\int_A^B\Bigg\{\frac{1}{2}\begin{pmatrix}a_0\\ Re_0(R)\end{pmatrix}\cdot M^1\begin{pmatrix}a_0\\ Re_0(R)\end{pmatrix} \\ +\sum_{n=1}^{\infty}\left[\begin{pmatrix}a_n(R)\\ Re_n(R)\end{pmatrix}\cdot M^1\begin{pmatrix}a_n(R)\\ Re_n(R)\end{pmatrix}+\begin{pmatrix}c_n(R)\\ Rf_n(R)\\ Rd_n(R)\end{pmatrix}\cdot M^2\begin{pmatrix}c_n(R)\\ Rf_n(R)\\ Rd_n(R)\end{pmatrix}\right]\Bigg\}\,\mathrm{d}R$$

$$= \int_A^B \left\{ \begin{matrix} \frac{1}{2} & a_0 \\ 0 & \end{matrix} \cdot M^1 \begin{matrix} a_0 \\ 0 \end{matrix} \right.$$

$$+ \sum_{n=1}^{\infty} \left[\begin{pmatrix} a_n(R) \\ Ra_n'(R) \end{pmatrix} \cdot M^1 \begin{pmatrix} a_n(R) \\ Ra_n'(R) \end{pmatrix} \right.$$

$$\left. \left. + \frac{1}{n^2\pi^2 L^2} \begin{pmatrix} L^2 a_n'(R) \\ RL^2 a_n''(R) \\ -n^2\pi^2 R a_n(R) \end{pmatrix} \cdot M^2 \begin{pmatrix} L^2 a_n'(R) \\ RL^2 a_n''(R) \\ -n^2\pi^2 R a_n(R) \end{pmatrix} \right] \right\} \mathrm{d}R$$

$$\geqslant 0. \tag{5.18}$$

Obviously, inequality (5.18), and therefore (5.8), holds if

$$\int_A^B \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot M^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}R \geqslant 0 \tag{5.19}$$

and

$$\int_A^B \left[\begin{pmatrix} a \\ Ra' \end{pmatrix} \cdot M^1 \begin{pmatrix} a \\ Ra' \end{pmatrix} + \frac{1}{n^2\pi^2 L^2} \begin{pmatrix} L^2 a' \\ RL^2 a'' \\ -n^2\pi^2 R a \end{pmatrix} \cdot M^2 \begin{pmatrix} L^2 a' \\ RL^2 a'' \\ -n^2\pi^2 R a \end{pmatrix} \right] \mathrm{d}R \geqslant 0, \tag{5.20}$$

hold for any integer n and $a \in C^2([A, B]; \mathbb{R})$.

Inequalities (5.19) and (5.20) are, in fact, also necessary for (5.8) to hold. Indeed, setting $\phi(R, Z) = Z$ in (5.8) leads to (5.19). Moreover, for a given integer n and $a \in C^2([A, B]; \mathbb{R})$, we choose

$$\phi(R, Z) = \frac{L}{n\pi} a(R) \sin \frac{n\pi Z}{L} = \alpha^{-1} a(R) \sin \alpha Z. \tag{5.21}$$

This function satisfies (5.6). Substituting (5.21) into (5.8) and carrying out the integration in Z yields (5.20).

Inequalities (5.19) and (5.20) are thus necessary and sufficient for the second variation condition (5.5) to hold. This represents considerable reduction, as (5.5) involves two functions $u(R, Z)$ and $v(R, Z)$, and (5.20) involves only one function $a(R)$.

Whether inequality (5.19) holds can be checked directly, and hence requires no comment. Inequality (5.20), on the other hand, is far more difficult to analyse. The difficulty lies not only on the presence of the arbitrary perturbation function $a(R)$, but also on the fact that the coefficient matrices M^1 and M^2 are functions of R. Here, we present a method which gives the solution of this inequality by solving a system of first-order differential equations. This method can be made applicable to quadratic integral inequalities involving any order of derivatives of the perturbation function. This generalization of the present development will be presented elsewhere (Chen 2002).

For convenience, we rewrite inequality (5.20) as

$$\int_A^B \begin{pmatrix} a(R) \\ a'(R) \\ a''(R) \end{pmatrix} \cdot M(R) \begin{pmatrix} a(R) \\ a'(R) \\ a''(R) \end{pmatrix} \mathrm{d}R \geqslant 0, \tag{5.22}$$

where M is a symmetric 3×3 matrix whose components are

$$\frac{1}{R}\begin{pmatrix} \dfrac{\lambda_2\hat{W}_{22}-\hat{W}_2}{R^2\lambda_2^3\lambda_3^4}+\dfrac{\alpha^2\lambda_2^2\lambda_3^3\hat{W}_3}{\mu} & \dfrac{\hat{W}_2-\lambda_3\hat{W}_{23}}{R\lambda_2\lambda_3^3}-\dfrac{\hat{W}_3}{R\lambda_3\mu} & \dfrac{\hat{W}_3}{\lambda_3\mu} \\ & \hat{W}_{33}+\dfrac{\lambda_2^2\lambda_3^3\hat{W}_3}{\alpha^2R^2\mu} & -\dfrac{\lambda_2^2\lambda_3^3\hat{W}_3}{\alpha^2R\mu} \\ & & \dfrac{\lambda_2^2\lambda_3^3\hat{W}_3}{\alpha^2\mu} \end{pmatrix},$$

where we have written

$$\mu = \lambda_2^2\lambda_3^4 - 1.$$

It follows from an elementary argument that (5.22) implies $M_{33}(R) \geqslant 0$ on $[A, B]$. Here and henceforth, M_{ij}, $i, j = 1, 2, 3$, are the ij elements of the matrix M. We shall assume that $M_{33}(R) > 0$ on $[A, B]$, which is a direct consequence of the Baker–Ericksen inequality.

We consider the following initial-value problem for a system of first-order ordinary differential equations,

$$\left.\begin{aligned} y_1' &= M_{11} - \frac{(M_{13}-y_3)^2}{M_{33}}, \\ y_2' &= M_{22} - 2y_3 - \frac{(M_{23}-y_2)^2}{M_{33}}, \\ y_3' &= M_{12} - y_1 - \frac{(M_{13}-y_3)(M_{23}-y_2)}{M_{33}}, \end{aligned}\right\} \tag{5.23}$$

with

$$y_1(A) = y_2(A) = y_3(A) = 0, \tag{5.24}$$

where $y_i(R)$, $i = 1, 2, 3$, are unknown functions.

The theory for the solution of the initial-value problem (5.23) and (5.24) is well developed (see, for example, Ince 1956). In the sequel, we assume that there is a continuous solution on the interval $[A, B]$, which is the case for the numerical examples presented in the next section. In the case where the solution becomes unbounded, further analysis is needed. It is shown in Chen (2002) that (5.22) holds in this latter case.

Inequality (5.22) can be written as

$$a^2(B)y_1(B) + a'^2(B)y_2(B) + 2a(B)a'(B)y_3(B) + \int_A^B \left[a\frac{M_{13}-y_3}{\sqrt{M_{33}}} + a'\frac{M_{23}-y_2}{\sqrt{M_{33}}} + a''\sqrt{M_{33}}\right]^2 \mathrm{d}R \geqslant 0. \tag{5.25}$$

It is obvious that (5.25) holds for all $a(R)$ if the 2×2 matrix

$$Y \equiv \begin{pmatrix} y_1(B) & y_3(B) \\ y_3(B) & y_2(B) \end{pmatrix} \tag{5.26}$$

is positive-semi-definite. It turns out that this condition is also necessary. Indeed, if matrix Y is not positive-semi-definite, then there are real numbers p and q such that

$$p^2y_1(B) + q^2y_2(B) + 2pqy_3(B) < 0.$$

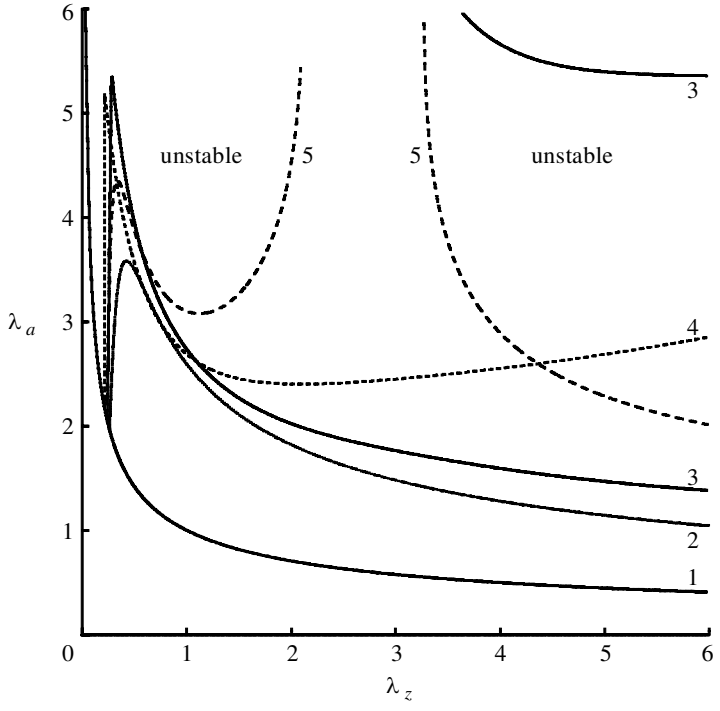

Figure 1. Plot of bifurcation points in the λ_a, λ_z plane for four different materials. Curves 2, 3, 4 and 5 represent, respectively, the Varga material, the three term material, the Neo Hookean material and the Mooney Rivlin material. Curve 1 indicates uniaxial extension. ($A/B = \frac{1}{3}$, $L/B = 10$.)

Subsequently, inequality (5.25) is violated if we choose function $a(R)$ to be the solution of the following linear differential equation

$$a\frac{M_{13} - y_3}{\sqrt{M_{33}}} + a'\frac{M_{23} - y_2}{\sqrt{M_{33}}} + a''\sqrt{M_{33}} = 0, \quad a(B) = p, \quad a'(B) = q.$$

We thus conclude that a necessary and sufficient condition for (5.22) to hold is that matrix Y be positive semi-definite.

6. Comparison of the bifurcation and stability conditions

In the preceding sections, we have derived bifurcation equations (4.1), (4.3), (4.4) and the stability condition that (5.26) be positive-semi-definite. There seems to be no apparent connection between the two sets of conditions. In this section, we present some numerical calculations to compare the bifurcation condition and the stability condition. In particular, we examine whether the stability of the cylindrical deformations changes at a bifurcation point. We have used four commonly used versions of Ogden materials (3.14).

In figure 1 we plot a similar bifurcation diagram to that which can be found in Haughton & Ogden (1979). We plot the critical values of the azimuthal principal stretch λ_2 at the inner surface $\lambda_2(A) = a/A = \lambda_a$ (say) against the axial stretch $\lambda_3 = \lambda_z$. We have taken a thick-walled tube $A/B = \frac{1}{3}$ of reasonable length $L/B = 10$.

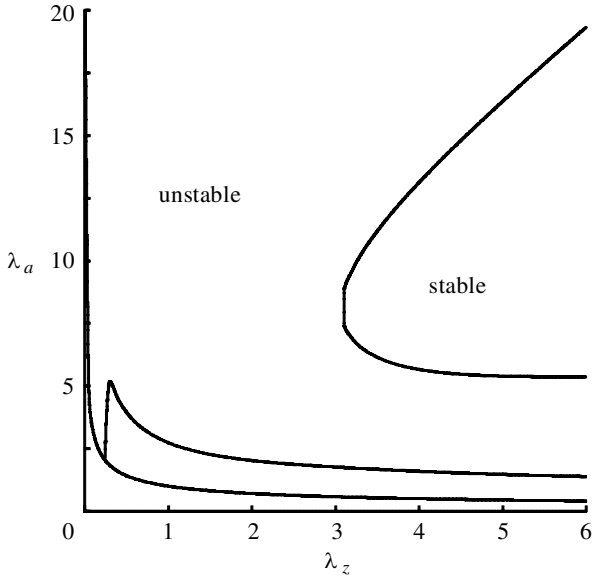

Figure 2. Plot of bifurcation points in the λ_a, λ_z plane for the three term material. ($A/B = \frac{1}{3}$, $L/B = 10$.)

The results can, however, be regarded as being typical. The four materials are used: the Neo–Hookean material ($N = 1$, $\alpha_1 = 2$); the Varga material ($N = 1$, $\alpha_1 = 1$); the Mooney–Rivlin material ($N = 2$, $\alpha_1 = 2$, $\alpha_2 = -2$, with $\mu_1 = 7$, $\mu_2 = -1$); and Ogden's three-term material (see Ogden 1972). In figure 1, the lowest curve corresponds to simple uniaxial extension of the tube $\lambda_a = 1/\sqrt{\lambda_z}$. The only possible deformations lie on or above this curve. If we follow this curve to the left, we find that it intersects the bifurcation curves. These points (one for each different material) correspond to a bulging of the tube due to the axial compression alone. Other bifurcation points rely on a combination of axial compression (or tension) and inflating pressure.

Previously, the stability of the cylindrical deformation may have been inferred from diagrams such as figure 1, but there has been no mathematical justification for any assumptions that may have been made regarding stability. When the stability criterion (5.25) is evaluated, we find that it corresponds exactly (to the accuracy of the numerical calculations) to the bifurcation points. Thus we now have solid mathematical ground for the inclusion of the stable/unstable captions. In figure 1, if we consider a fixed value of the axial stretch λ_z, we then move vertically as the inner radius of the cylinder is increased. When we encounter the first bifurcation curve (depending on the material), the cylindrical deformation becomes unstable. For the single-term materials (Varga and Neo–Hookean), nothing further happens, and the whole of the region above the bifurcation curve is unstable. For the three-term material, there is a second bifurcation point in the upper right-hand corner. Above this second bifurcation point, the cylindrical deformation becomes stable again. (It is tempting to assume that this corresponds to the bulge occupying the whole of the cylinder, but see figure 2.) For the Mooney–Rivlin material, the situation is somewhat different. There is a central region (approximately for $\lambda_z \in (2.1, 3.3)$) for

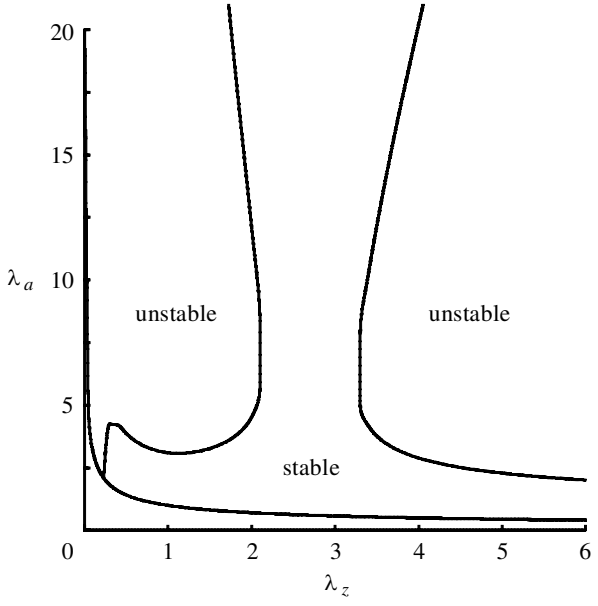

Figure 3. Plot of bifurcation points in the λ_a, λ_z plane for the Mooney Rivlin material. $(A/B = \frac{1}{3},\ L/B = 10.)$

which bifurcation does not occur and so the cylindrical deformation remains stable for all values of pressure.

The range of deformation shown in figure 1 is quite severe and consideration of deformation outside this region is unlikely to represent anything physically meaningful. However, purely for mathematical interest, we include figures 2 and 3, which give a more complete picture of the Bifurcation and stability behaviour.

In figure 2, we show that the three-term material can, in fact, exhibit three bifurcation points for $\lambda_z > 3.1$, approximately, for this combination of initial geometry.

In figure 3, we consider the Mooney–Rivlin material. Here, we see that we can have none, one or two bifurcation points for a fixed λ_z.

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