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Numerical specified homogenization of a discrete model with a local perturbation and application to traffic flow

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Abstract

The goal of this work is to present a numerical homogenization of a non-local PDE deriving from a first order discrete model for traffic flow that simulates the presence of a local perturbation. In a previous work, we have shown that the solution of the discrete microscopic model converges to the (unique) solution of a Hamilton-Jacobi equation posed on a network and with a junction condition (it can be seen as a flux limiter that keeps the memory of the local perturbation). The goal of this paper is to provide a numerical scheme able to provide an approximation of this flux-limiter. We will show the convergence of this scheme and we will provide some numerical results.

AMS Classification: 35D40, 90B20, 35B27, 35F20, 45K05, 65M06.

Keywords: numerical specified homogenization, Hamilton-Jacobi equations, viscosity solutions, traffic flow, microscopic models, macroscopic models, convergence of numerical scheme.

1 Introduction

The problem of simulating traffic flow is very important, particularly because it allows us to know how the traffic would react to a change in the infrastructure of the road. Traffic flow can be simulated at different scales: the microscopic scale (which describes the dynamics of all the vehicles), the macroscopic scale (which describes macroscopic quantities such as the vehicle density, the average speed,...) and the mesoscopic scale (between the microscopic and the macroscopic scale). In this paper we will only consider the microscopic and the macroscopic scales.

In this paper, we are interested in the numerical homogenization of a non-local PDE that derives from a microscopic first order model for traffic flow that simulates the presence of a local perturbation that does not depend on time (for instance a school zone, a moderator,...). The PDE was introduced in [12] and derives from a first order traffic flow model of the type "follow-the-leader".

In [12], the homogenization of the PDE was obtained and it turns out that the homogenized system is defined by a function call the effective Hamiltonian and by a constant call the flux limiter. The effective Hamiltonian describes the dynamics of the traffic flow and the flux-limiter will defined how the local perturbation affects the macroscopic (homogenized) model. In fact, the effective Hamiltonian has been explicitly determined, however the flux-limiter constant is only implicitly determined. In this paper, we will provide a numerical scheme for the computation of an approximation of the flux-limiter.

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1.1 General model: first order model with a local perturbation

For the readers convenience, we detail the microscopic model from which derives the PDE we will study later in this paper. We consider the following model where all the vehicles are considered as points placed in the real line, for all $t > 0$,

$$\dot{U}_j(t) = V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)), \quad (1.1)$$

where $U_j : [0, +\infty) \rightarrow \mathbb{R}$ denotes the position of the j -th vehicle and \dot{U}_j is its velocity. The function $\phi : \mathbb{R} \rightarrow [0, 1]$ simulates the presence of a local perturbation around the origin. We denote by r the radius of influence of the perturbation.

The function V is called the optimal velocity function and we make the following assumptions on V and ϕ :

Assumption (A)

- (A1) $V : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on \mathbb{R} .
- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) There exists a real $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing on $[-1/h_0, p_0)$ and increasing on $[p_0, 0)$.
- (A6) The function $\phi : \mathbb{R} \rightarrow [0, 1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$.

Remark 1.1. *Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity functions, we have added assumption (A4) to work with V' with a bounded support. But by modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all the assumptions. For instance, in the case of the Greenshields based models [15](see also [4, 9, 19, 14] for other classical optimal velocity functions):*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^2 \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^2 \right) & \text{for } h > h_{max}. \end{cases}$$

In Figure 1 we give a schematic representation of an optimal velocity function satisfying assumption (A).

1.2 Injecting the system of ODEs into a single PDE

In order to obtain an homogenization result, the authors borrowed the idea from [11] and injected the system of ODE (1.1) into a single PDE. To do this, in [12], it was introduced the following "cumulative distribution function" of vehicles ρ^ε :

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right), \quad (1.2)$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \quad (1.3)$$

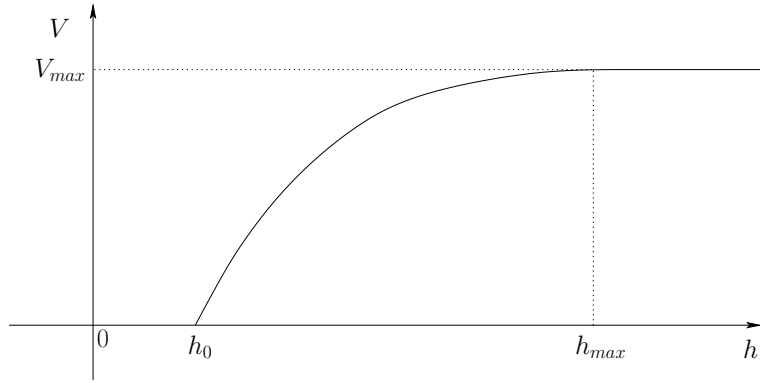


Figure 1: Schematic representation of the optimal velocity function V .

It was proven that under assumption (A) the function ρ^ε satisfies in the viscosity sense the following non-local equation

$$u_t^\varepsilon + M^\varepsilon \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \cdot \phi \left(\frac{x}{\varepsilon} \right) \cdot |u_x^\varepsilon| = 0 \quad \text{on } (0, +\infty) \times \mathbb{R}, \quad (1.4)$$

where M^ε is a non-local operator defined by

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max} \quad (1.5)$$

with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}. \quad (1.6)$$

1.3 Convergence result

We define $k_0 = 1/h_0$ and $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\bar{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V \left(\frac{-1}{p} \right) |p| & \text{for } -k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (1.7)$$

Note that such a \bar{H} is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}(p) = +\infty \right)$ and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \bar{H} \text{ is decreasing on } (-\infty, p_0), \\ \bar{H} \text{ is increasing on } (p_0, +\infty). \end{cases} \quad (1.8)$$

We denote by

$$H_0 = \min_{p \in \mathbb{R}} \bar{H}(p) = \bar{H}(p_0) \quad (1.9)$$

and we refer to Figure 2 for an schematic representation of \bar{H} .

From [12], we have the following homogenization result.

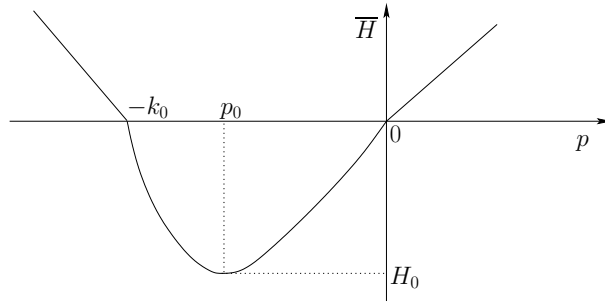


Figure 2: Schematic representation of \bar{H} .

Theorem 1.2 (Junction condition by homogenisation). *Assume (A) and that at the initial time, we have, for all $i \in \mathbb{Z}$,*

$$U_i(0) \leq U_{i+1}(0) - h_0.$$

We define a function u_0 , satisfying $-k_0 \leq (u_0)_x \leq 0$, such that for all $\varepsilon > 0$,

$$\rho^\varepsilon(0, x) = \varepsilon \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor.$$

Then there exists $\bar{A} \in [H_0, 0]$ such that the function ρ^ε defined by (1.2) converges uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0 towards the unique solution u^0 of

$$\begin{cases} u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (-\infty, 0) \\ u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (0, +\infty) \\ u_t^0 + F_{\bar{A}}^-(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.10)$$

where \bar{A} has to be determined and $F_{\bar{A}}^-$ is defined by

$$F_{\bar{A}}^-(p_-, p_+) = \max\left(\bar{A}, \bar{H}^+(p_-), \bar{H}^-(p_+)\right), \quad (1.11)$$

with

$$\bar{H}^-(p) = \begin{cases} \bar{H}(p) & \text{if } p \leq p_0, \\ \bar{H}(p_0) & \text{if } p \geq p_0, \end{cases} \quad \text{and} \quad \bar{H}^+(p) = \begin{cases} \bar{H}(p_0) & \text{if } p \leq p_0, \\ \bar{H}(p) & \text{if } p \geq p_0. \end{cases} \quad (1.12)$$

Moreover, u^0 satisfies $-k_0 \leq u_x^0 \leq 0$.

Remark 1.3. We notice that in the case of traffic flow, (1.10) is equivalent (deriving in space) to a LWR model (see [17, 21]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and u_x^0 corresponds to the density of vehicles.

2 Construction of the flux-limiter

As we can see from Theorem 1.2, in order to have a complete homogenization result we only need to determine the flux limiter \bar{A} . In the rest of the paper, we will focus on obtaining a numerical approximation of the flux limiter. First we will give a few characterizations of \bar{A} that can be founded in [12].

In fact, the flux limiter is the unique constant λ such that there exists a solution w of the following Hamilton-Jacobi equation

$$M[w](x) \cdot \phi(x) \cdot |w_x| = \lambda \quad \text{for } x \in \mathbb{R}, \quad (2.1)$$

with the right slopes at infinity. We refer to [13] and [12] on a more detailed explanation on these slopes.

In order to construct the corrector for the junction w and \bar{A} , in [12] the authors used the idea from [13], [1] and from the lectures of Lions at "College de France" [18], which is to construct the correctors in a truncated domain with good boundary conditions and then to expand the domain.

The effective flux limiter is then obtained as the limit as $l \rightarrow +\infty$, and then $R \rightarrow +\infty$ of $\lambda_{l,R}$ which is the unique constant for which the following truncated cell problem admits a solution (see [12, Proposition 6.4]): for $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$, find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} G_R(x, [w^{l,R}], w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in (-l, l) \\ \bar{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } x = -l \\ \bar{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } x = l, \end{cases} \quad (2.2)$$

with

$$G_R(x, [U], q) = \psi_R(x)\phi(x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \bar{H}(q), \quad (2.3)$$

and $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \rightarrow [0, 1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R - 10, R + 10], \end{cases} \quad \text{and } \psi_R(x) < 1 \ \forall x \notin [-R, R]. \quad (2.4)$$

Remark 2.1. *The operator G_R is used to have a local operator near the boundary and then to well define the boundary conditions.*

To an approximation of $\lambda_{l,R}$ we follow the construction done in [12] and consider the approximated truncated cell problem, for all $\delta > 0$.

$$\begin{cases} \delta v^\delta + \psi_R(x)M[v^\delta](x) \cdot \phi(x) \cdot |v_x^\delta| + (1 - \psi_R(x))\bar{H}(v_x^\delta) = 0 & \text{for } x \in (-l, l) \\ \delta v^\delta + \bar{H}^-(v_x^\delta) = 0 & \text{for } x = -l \\ \delta v^\delta + \bar{H}^+(v_x^\delta) = 0 & \text{for } x = l. \end{cases} \quad (2.5)$$

It is proven in [12] that $-\delta v^\delta(0) \rightarrow \lambda_{l,R}$ as $\delta \rightarrow 0$.

2.1 Organization of the paper

In Section 3 we introduce the definition of a viscosity solution for (2.5), we also present some classical results like the comparison principle and the stability. In the same section, we will present a numerical scheme for (2.5), and we will announce our main result which is a convergence result for the scheme. In Section 4, we will provide the proof of the convergence result. In Section 5, we will prove some properties regarding our numerical scheme that we will use in practice to obtain a numerical approximation of the solution of (2.5). Finally, Section 6 contains some numerical simulations for different types of perturbations and we numerical verify some of the qualitative properties of \bar{A} that were proven in [12].

3 Viscosity solutions for the approximated cell problem

In this section, we study (2.5). In order to simplify the notations, we drop the index δ in v^δ . We also present the definition of viscosity solutions for (2.5). We refer to the user's guide of Crandall, Ishii, Lions [7] and the book of Barles [2] for a good introduction to viscosity solutions.

3.1 Viscosity solution for the continuous approximated cell problem

For $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$, we consider the following problem

$$\begin{cases} \delta v + \psi_R(x)M[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) = 0 & \text{for } x \in (-l, l) \\ \delta v + \overline{H}^-(v_x) = 0 & \text{for } x = -l \\ \delta v + \overline{H}^+(v_x) = 0 & \text{for } x = l, \end{cases} \quad (3.1)$$

with

$$M[U](x) = \int_{h_0}^{h_{max}} J(z)E(U(x+z) - U(x)) dz - \frac{3}{2}V_{max} \quad (3.2)$$

with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}.$$

Similarly, we define

$$\tilde{M}[U](x) = \int_{h_0}^{h_{max}} J(z)\tilde{E}(U(x+z) - U(x)) dz - \frac{3}{2}V_{max} \quad (3.3)$$

with

$$\tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0 \\ 1/2 & \text{if } -1 < z \leq 0 \\ 3/2 & \text{if } z \leq -1, \end{cases} \quad (3.4)$$

In order to simplify the notations, we introduce the following function

$$F(x, [v], v_x) = \begin{cases} \psi_R(x)M[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) & \text{if } x \in (-l, l), \\ \overline{H}^-(v_x) & \text{if } x = -l, \\ \overline{H}^+(v_x) & \text{if } x = l. \end{cases}$$

Similarly, we define \tilde{F} by replacing M by \tilde{M} .

In this paper, we work with viscosity solutions, and the boundary conditions of (3.1) will be interpreted in the classical sense of viscosity solutions with Neumann boundary conditions. That is why we introduce

$$\mathcal{I}(x, [v], v_x) = \begin{cases} \psi_R(x)M[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) & \text{if } x \in (-l, l), \\ \min(\overline{H}(v_x), \overline{H}^-(v_x)) & \text{if } x = -l, \\ \min(\overline{H}(v_x), \overline{H}^+(v_x)) & \text{if } x = l, \end{cases}$$

and

$$\tilde{\mathcal{I}}(x, [v], v_x) = \begin{cases} \psi_R(x)\tilde{M}[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) & \text{if } x \in (-l, l), \\ \max(\overline{H}(v_x), \overline{H}^-(v_x)) & \text{if } x = -l, \\ \max(\overline{H}(v_x), \overline{H}^+(v_x)) & \text{if } x = l. \end{cases}$$

Definition 3.1 (Viscosity solution for the continuous approximated cell problem). *An upper semi-continuous function (resp. lower semi-continuous) $v : [-l, l] \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. a super-solution) of (3.1) in $[-l, l]$, if for all $x \in [-l, l]$ and all $\varphi \in C^1([-l, l])$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point x , we have*

$$\delta v(x) + \mathcal{I}(x, [v], \varphi') \leq 0 \quad (\text{resp. } \delta v(x) + \tilde{\mathcal{I}}(x, [v], \varphi') \geq 0)$$

We say that a function v is a solution of (3.1) if v^* and v_* are respectively a sub and a super-solution of (3.1).

Remark 3.2. Like in [12], we use this type of definition in order to have a stability result for the non-local term. We refer to [8, 22] for such kind of definition.

We now give a slightly stronger result than a stability result for the sub and super-solutions of (3.1) that will be used to prove the consistency of the numerical scheme we present later in this paper.

Proposition 3.3 (Stability for (3.1)). *Let $(u_m)_m$ be a sequence of measurable functions and let \bar{u} denote $\limsup^* u_m$ (resp. $\underline{u} = \liminf_* u_m$). Let $(x_m, p_m) \rightarrow (x_0, p)$ in $[-l, l] \times \mathbb{R}$ be such that $u_m(x_m) \rightarrow \bar{u}(x_0)$ (resp. $u_m(x_m) \rightarrow \underline{u}(x_0)$). Then*

$$\liminf_{m \rightarrow +\infty} F(x_m, [u_m], p_m) \geq \mathcal{I}(x_0, [\bar{u}], p) \quad (3.5)$$

$$\left(\text{resp. } \limsup_{m \rightarrow +\infty} \tilde{F}(x_m, [u_m], p_m) \leq \tilde{\mathcal{I}}(x_0, [\underline{u}], p) \right). \quad (3.6)$$

In order to prove Proposition 3.3, we use the following lemma which proof can be found in [22].

Lemma 3.4. *Let $(f_m)_m$ be a sequence of measurable functions on \mathbb{R} , and consider*

$$\bar{f} = \limsup^* f_m$$

and

$$\underline{f} = \liminf_* f_m. \quad (3.7)$$

Let $(a_m)_m$ be a sequence of \mathbb{R} converging to zero. Then

$$\mathcal{L}(\{f_m \geq a_m\} \setminus \{\bar{f} \geq 0\}) \rightarrow 0 \quad \text{as } m \rightarrow +\infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \setminus \{f_m > a_m\}) \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

where $\mathcal{L}(A)$ denotes the Lebesgue measure of measurable set A .

Proof of Proposition 3.3. We just do the proof for \bar{u} . We distinguish two cases.

Case 1: $x_0 \in (-l, l)$. From the definition of F , we can see that the only problem we have is the non-local operator. We claim that for m big enough we have

$$M[u_m](x_m) \geq M[\bar{u}](x_0) - \varepsilon_m \quad \text{with } \lim_{m \rightarrow +\infty} \varepsilon_m = 0. \quad (3.8)$$

Using (1.6), we have that

$$E(\beta) = \frac{1}{2} \mathbf{1}_{\{\beta \in [-1, 0)\}} + \frac{3}{2} \mathbf{1}_{\{\beta < -1\}} = \frac{1}{2} \mathbf{1}_{\{\beta < 0\}} + \mathbf{1}_{\{\beta < -1\}}.$$

We get that

$$\begin{aligned} & \int_{\mathbb{R}} J(z) E(u_m(x_m + z) - u_m(x_m)) dz - \int_{\mathbb{R}} J(z) E(\bar{u}(x_0 + z) - \bar{u}(x_0)) dz \\ &= \int_{\mathbb{R}} J(z) \left\{ \mathbf{1}_{\{u_m(x_m + z) - u_m(x_m) < -1\}} - \mathbf{1}_{\{\bar{u}(x_0 + z) - \bar{u}(x_0) < -1\}} \right\} dz \\ & \quad + \int_{\mathbb{R}} \frac{1}{2} J(z) \left\{ \mathbf{1}_{\{u_m(x_m + z) - u_m(x_m) < 0\}} - \mathbf{1}_{\{\bar{u}(x_0 + z) - \bar{u}(x_0) < 0\}} \right\} dz \end{aligned} \quad (3.9)$$

Using Lemma 3.4, we have for n big enough,

$$\begin{aligned}
& \int_{\mathbb{R}} J(z) \{1_{\{\bar{u}(x_0+z) - \bar{u}(x_0) \geq -1\}} - 1_{\{u_m(x_m+z) - u_m(x_m) \geq -1\}}\} dz \\
& \geq - \int_{\mathbb{R}} J(z) 1_{\{A_m(z) \setminus A(z)\}} \geq -\frac{\varepsilon_m}{2}, \\
& \int_{\mathbb{R}} \frac{1}{2} J(z) \{1_{\{\bar{u}(x_0+z) - \bar{u}(x_0) \geq 0\}} - 1_{\{u_m(x_m+z) - u_m(x_m) \geq 0\}}\} dz \\
& \geq -\frac{1}{2} \int_{\mathbb{R}} J(z) 1_{\{B_m(z) \setminus B(z)\}} \geq -\frac{\varepsilon_m}{2},
\end{aligned} \tag{3.10}$$

with

$$\begin{cases} A_m(z) = \{u_m(x_m + z) - u_m(x_m) \geq -1\} \cup \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq -1\}, \\ A(z) = \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq -1\} \end{cases}$$

and

$$\begin{cases} B_m(z) = \{u_m(x_m + z) - u_m(x_m) \geq 0\} \cup \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq 0\}, \\ B(z) = \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq 0\}. \end{cases}$$

Using (3.9) and (3.10), we prove (3.8).

Give that $x_0 \in (-l, l)$, for m big enough we have $x_m \in (-l, l)$. Using the definition of F and (3.8), we have

$$\begin{aligned}
& \psi_R(x_m) M[u_m](x_m) \cdot \phi(x_m) \cdot |p_m| + (1 - \psi_R(x_m)) \bar{H}(p_m) \\
& \geq \psi_R(x_m) M[\bar{u}](x_0) \cdot \phi(x_m) \cdot |p_m| + (1 - \psi_R(x_m)) \bar{H}(p_m) - \varepsilon_m \psi_R(x_m) \cdot \phi(x_m) \cdot |p_m|.
\end{aligned}$$

Using the fact that the terms on the right are continuous, we pass to the limit as m goes to infinity to obtain (3.5).

Case 2: $x_0 = -l$ or $x_0 = l$. In this case, using Definition 3.1 and the continuity of \bar{H} , we obtain (3.5). This ends the proof of Proposition 3.3. \square

Theorem 3.5 (Comparison principle). *Let u and v be respectively a sub and a super-solution of (3.1) then we have for all $x \in [-l, l]$,*

$$u(x) \leq v(x).$$

Proof. The proof of this Theorem comes from the comparison principle [11, Theorem 4.4] for the non-local term. The only remaining difficulty is proving this result at the boundaries. However, for x close to l_2 , the function G_R is actually the effective Hamiltonian \bar{H} . Therefore, we can proceed as in the proof of [13, Proposition 4.1] in which the authors consider the boundaries a network composed of a single lane and a junction point then they use the results from [16] and so we skip the rest of the proof. Notice that Definition 3.1 is equivalent at the boundaries to the definition of relaxed viscosity solution [16, Definition 2.1] in the case of a single lane with a junction point. \square

Theorem 3.6 (Existence and uniqueness). *Assume (A). There exists a unique solution v^δ of (3.1). Moreover, this solution is continuous and we have that for all $x \in [-l, l]$*

$$0 \leq v^\delta(x) \leq \frac{C_0}{\delta}.$$

Finally, we have the following result from [12, Proof of Proposition 6.4], which justifies considering (3.1) and looking for a numerical approximation of v^δ .

Proposition 3.7. *We have the following limit as δ goes to 0 (up to a subsequence)*

$$\lim_{\delta \rightarrow 0} -\delta v^\delta(0) = \lambda_{l,R}. \tag{3.11}$$

3.2 Numerical scheme for (3.1)

The numerical scheme we use was inspired by the one from [5, 10] for the non-local operator and by [6] for the local operator. We consider a uniform grid of the interval $[-l, l]$ with $2n + 1$ points, $n \in \mathbb{N} \setminus \{0\}$, and we denote by $\Delta x = l/n$ the discretization step. For all $i \in \{-n, \dots, n\}$, we denote by $x_i = \Delta x \cdot i$ the nodes of the grid. In particular we have that $x_0 = 0$, $x_{-n} = -l$ and $x_n = l$.

For every discrete function $v : \{-n, \dots, n\} \rightarrow \mathbb{R}$, we denote by v_{\sharp} its piecewise constant extension to \mathbb{R} , defined by

$$v_{\sharp}(x) = \sum_{i=1}^n v_i \cdot \chi_{Q_i}(x) \quad \text{with} \quad Q_i = \begin{cases} [-l, -l + \Delta x/2) & \text{if } i = -n, \\ [x_i - \Delta x/2, x_i + \Delta x/2) & \text{if } i \in \{-n+1, \dots, n-1\} \\ [l - \Delta x/2, l] & \text{if } i = n. \end{cases} \quad (3.12)$$

Discretization of the non-local operator For all discrete function $v : \{-n, \dots, n\} \rightarrow \mathbb{R}$, we define the following discrete non-local operators

$$M_i^d[v] := M[v_{\sharp}](x_i) \quad \text{and} \quad \tilde{M}_i^d[v] := \tilde{M}[v_{\sharp}](x_i). \quad (3.13)$$

Discretization of the gradient We consider the standard forward and backward first order differences:

$$D^+v(x_i) = \frac{v(x_{i+1}) - v(x_i)}{\Delta x} \quad \text{and} \quad D^-v(x_i) = \frac{v(x_i) - v(x_{i-1}))}{\Delta x}. \quad (3.14)$$

Finally, we consider $Dv_i = (D^-v(x_i), D^+v(x_i))$.

Similarly to [5], we consider the following Osher, Sethian [20] upwind discretization of the modulus of the gradient. Let $S = (p, q) \in \mathbb{R}^2$, we define the following function, that we will use for the discretization of the gradient

$$G^+(S) = (\max(p, 0)^2 + \min(q, 0)^2)^{1/2}.$$

Discretization of the local operator Concerning the local operator $\overline{H}(\cdot)$, like in [6] we consider the following discretization

$$\overline{H}_d(Dv_i) = \max \left\{ \overline{H}^+(D^-v_i), \overline{H}^-(D^+v_i) \right\}, \quad (3.15)$$

where \overline{H}^- and \overline{H}^+ were defined in (1.12).

Finally, we introduce for any discrete function $v : \{-n, \dots, n\} \rightarrow \mathbb{R}$,

$$R_i[v] := \mathcal{R}_i([v], Dv_i) = \psi_R(x_i) \cdot M_i^d[v] \cdot \phi(x_i) \cdot G^+(Dv_i) + (1 - \psi_R(x_i)) \cdot \overline{H}_d(Dv_i). \quad (3.16)$$

Similarly, we define \tilde{R} and $\tilde{\mathcal{R}}$ by replacing M^d with \tilde{M}^d .

To summarize, in the rest of the paper, for all discrete function $v : \{-n, \dots, n\} \rightarrow \mathbb{R}$, we will consider the following numerical scheme.

$$\begin{cases} \delta v_i + R_i[v] = 0 & \text{for } i \in \{-n+1, \dots, n-1\} \\ \delta v_i + \overline{H}^-(D^+v_i) = 0 & \text{for } i = -n \\ \delta v_i + \overline{H}^+(D^-v_i) = 0 & \text{for } i = n. \end{cases} \quad (3.17)$$

Remark 3.8 (Notation for the discretization of the non-local operator). *Since the function J inside the non-local operator is of bounded support, we introduce the following notations, which are the discrete equivalents of h_0 and h_{max} ,*

$$j_0 = \max \{j \in \{-n, \dots, n\} \text{ s.t. } x_j - \Delta x/2 < h_0\} \quad (3.18)$$

and

$$j_{max} = \min \{j \in \{-n, \dots, n\} \text{ s.t. } x_j + \Delta x/2 > h_{max}\}. \quad (3.19)$$

3.3 Viscosity solution for the numerical scheme for the approximated cell problem

To simplify the notations, we introduce the following notation

$$F_i([v], Dv_i) = \begin{cases} \mathcal{R}_i([v], Dv_i) & \text{if } x_i \in (-l, l), \\ \overline{H}^-(D^+v_i) & \text{if } x_i = -l, \\ \overline{H}^+(D^-v_i) & \text{if } x_i = l. \end{cases} \quad (3.20)$$

Similarly, we define \tilde{F} by replacing \mathcal{R} with $\tilde{\mathcal{R}}$. We introduce the following definition of viscosity solution for (3.17).

Definition 3.9 (Viscosity solution for the approximated cell problem scheme). *Let us consider a function $v : \{-n, \dots, n\} \rightarrow \mathbb{R}$. We say that v is a sub-solution (resp. a super-solution) of (3.17) if for all $i \in \{-n, \dots, n\}$ we have*

$$\delta v_i + F_i([v], Dv_i) \leq 0 \quad (\text{resp. } \delta v_i + \tilde{F}_i([v], Dv_i) \geq 0).$$

Then we say that v is a solution of (3.17) if and only if it is a sub and a super-solution.

Remark 3.10. *The notion of discrete viscosity solutions is necessary here because of the discontinuity inside the non-local operator. In fact, we could not work with a regularise version of E because we do not have a stability result with respect to E . Moreover, in Section 4 and Section 5 we can see the interest in working with such a definition. The proof of existence of solutions for (3.17), will be postponed until Section 5.*

The main result of this paper is the following convergence result which proof is postponed until Section 4.

Theorem 3.11 (Convergence). *Using the same notations as in (3.12). Let $(v_i^{\Delta x})_{i \in \{-n, \dots, n\}}$ be a solution of (3.17), then the function $v_{\frac{\Delta x}{2}}$ (defined as in (3.12)) converges locally uniformly as $\Delta x \rightarrow 0$ to the unique continuous viscosity solution of (3.1).*

Remark 3.12 (Condition on the discretization step). *In the rest of the paper, we consider that the integer n is big enough (Δx is small enough) so that $j_0 > 1$. In fact, given the standard values of $h_0 (> 2$ meters), this is not a very restrictive condition but it helps to simplify the computations (regarding the monotonicity of the scheme) since for any $i \in \{-n, \dots, n\}$ the term v_{i+1} appears only on the gradient in $F_i([v], Dv_i)$.*

Remark 3.13 (Bounds on the non-local operator). *Given the definition of the non-local operators M and \tilde{M} , we have that for any function $U : \mathbb{R} \rightarrow \mathbb{R}$ and any $x \in \mathbb{R}$,*

$$-\frac{3}{2}V_{max} \leq M[U](x) \leq \tilde{M}[U](x) \leq 0.$$

Given the definition of M^d and \tilde{M}^d , this inequality hold for the discrete non-local operators.

Moreover, we have the following properties regarding the numerical scheme (3.17).

Lemma 3.14 (Monotonicity of \tilde{F} and F). *Assume (A). Let v, w be two discrete functions such that*

$$v_j \leq w_j \quad \text{for all } j \in \mathbb{Z}, \quad (3.21)$$

also assume that there exists an index $i \in \mathbb{Z}$ such that $v_i = w_i$, then we have

$$F_i([v], Dv_i) \geq F_i([w], Dw_i) \quad \text{and} \quad \tilde{F}_i([v], Dv_i) \geq \tilde{F}_i([w], Dw_i).$$

Proof. We present the proof for \tilde{F}_i and we skip it for F_i since the proof is similar. Let us begin by proving the monotonicity for the non-local term, first we want to prove that $\tilde{M}_i^d[v] \geq \tilde{M}_i^d[w]$. In fact, using the notations from Remark 3.8, we have

$$\begin{aligned} \tilde{M}^d[v_{\sharp}](x_i) &= \int_{h_0}^{h_{max}} J(z)E^*(v_{\sharp}(x_i+z) - v_{\sharp}(x_i)) dz - \frac{3}{2}V_{max} \\ &= \sum_{j=j_0}^{j_{max}} \int_{x_j-\Delta x/2}^{x_i+\Delta x/2} J(z)E^*(v_{\sharp}(x_i+z) - v_{\sharp}(x_j)) dz - \frac{3}{2}V_{max} \\ &= \sum_{j=j_0}^{j_{max}} \int_{\mathcal{Q}_j} J(z) dz \{E^*(v_{i+j} - v_i)\} - \frac{3}{2}V_{max} \\ &\geq \sum_{j=j_0}^{j_{max}} \int_{\mathcal{Q}_j} J(z) dz \{E^*(w_{i+j} - w_i)\} - \frac{3}{2}V_{max} = \tilde{M}_i^d[w], \end{aligned}$$

where we have used for the last line the fact that \tilde{E} is non-increasing, with (3.21) and the fact that J is non-negative.

Moreover, using (3.21), we have that

$$D^+v_i \leq D^+w_i \quad \text{and} \quad D^-v_i \geq D^-w_i.$$

This implies in particular that

$$\max(D^+v_i, 0)^2 \leq \max(D^+w_i, 0)^2 \quad \text{and} \quad \min(D^-v_i, 0)^2 \leq \min(D^-w_i, 0)^2.$$

Combining the previous inequalities we have that $G^+(Dv_i) \leq G^+(Dw_i)$. We recall that \tilde{M}^d is non-positive (Remark 3.13) and therefore, we have

$$\tilde{M}_i^d[v] \cdot G^+(Dv_i) \geq \tilde{M}_i^d[v] \cdot G^+(Dw_i) \geq \tilde{M}_i^d[w] \cdot G^+(Dw_i). \quad (3.22)$$

Let us now prove the monotonicity for the local term, using the fact that \overline{H}^+ is non-decreasing and that \overline{H}^- is non-increasing, we have that

$$\overline{H}^+(D^-v_i) \geq \overline{H}^+(D^-w_i) \quad \text{and} \quad \overline{H}^-(D^+v_i) \geq \overline{H}^-(D^+w_i),$$

this implies in particular that

$$\overline{H}^d(Dv_i) = \max(\overline{H}^-(D^+v_i), \overline{H}^+(D^-v_i)) \geq \max(\overline{H}^-(D^+w_i), \overline{H}^+(D^-w_i)) = \overline{H}^d(Dw_i). \quad (3.23)$$

Combining (3.22) and (3.23), we get $\tilde{F}_i([v], Dv_i) \geq \tilde{F}_i([w], Dw_i)$. □

4 Convergence of the numerical scheme for the approximated cell problem

This section is devoted to the proof of Theorem 3.11 which is an adaptation of the proof of convergence from [3] to a non-local PDE. Before passing to the proof of Theorem 3.11, we give some preliminary results, concerning the discrete barriers of the solutions of (3.17).

Lemma 4.1 (Existence of discrete barriers for (3.1)). *Assume (A). The discrete functions defined by*

$$v_i^- = 0 \quad \text{and} \quad v_i^+ = \frac{C_0}{\delta} \quad \text{for all } i \in \{-n, \dots, n\}$$

with $C_0 = |H_0| = |\min_{p \in \mathbb{R}} \overline{H}(p)|$ are respectively a sub and a super-solution of (3.1).

Proof. We only prove that v^+ is a super-solution, since the sub-solution case is similar and even simpler. Using the form of (3.1), we have that for all $i \in \{-n, \dots, n\}$

$$\delta v_i^+ + \tilde{F}_i([v^+], (0, 0)) \geq \delta v_i^+ + H_0 = |H_0| + H_0 = 0.$$

Therefore, v^+ is a super-solution of (3.1). □

Proposition 4.2 (Discrete barriers). *Let $u_i^- = 0$ and $v_i^+ = C_0/\delta$ for all $i \in \{-n, \dots, n\}$ with $C_0 = |H_0|$. Then every solution v of (3.17) satisfies*

$$u^- \leq v \leq v^+.$$

Proof. Let us begin by proving that $v - v^+ \leq 0$. We introduce $M = \max_{i \in \{-n, \dots, n\}} \{v_i - v_i^+\}$, we assume the maximum is reached for an index $i_0 \in \{-n, \dots, n\}$. Therefore, we have

$$v_{i_0} - M = v_{i_0}^+ \quad \text{and} \quad \tilde{v}_i := v_i - M \leq v_i^+ \quad \text{for all } i \in \{-n, \dots, n\}. \quad (4.1)$$

We notice that $F_i([v], Dv_i)$ is invariant by addition of constant to v and therefore, $F_i([\tilde{v}], D\tilde{v}_i) = F_i([v], Dv_i)$. Moreover, using Lemme 3.14 and (4.1), we have

$$F_{i_0}([v], Dv_{i_0}) = F_{i_0}([\tilde{v}], D\tilde{v}_{i_0}) \geq F_{i_0}([v^+], Dv_{i_0}^+). \quad (4.2)$$

Using the fact that v is a solution of (3.17) and in particular a sub-solution, we have

$$0 \geq \delta v_{i_0} + F_{i_0}([v], Dv_{i_0}) \geq \delta v_{i_0}^+ + \delta M + F_{i_0}([v^+], Dv_{i_0}^+).$$

In particular, replacing $v_{i_0}^+ = C_0/\delta$ we have

$$\delta M \leq -C_0 - F_{i_0}([v^+], Dv_{i_0}^+) = \begin{cases} 0 & \text{if } i_0 = -n \\ -C_0 & \text{if } i_0 \in \{-n+1, \dots, n\}, \end{cases}$$

where we have used for the equality the definition of F and the fact that $Dv_{i_0}^+ = (0, 0)$ (in this case the only term that is not equal to 0 is $\overline{H}^-(0) = H_0 = -C_0$). Therefore, $M \leq 0$. The proof that $v^- - v \leq 0$ is similar to the previous one and we skip it. □

Proof of Theorem 3.11. We introduce

$$\bar{v}(x) = \limsup_{\substack{y \rightarrow x \\ \Delta x \rightarrow 0}} v_{\#}^{\Delta x}(y) \quad \text{and} \quad \underline{v}(x) = \liminf_{\substack{y \rightarrow x \\ \Delta x \rightarrow 0}} v_{\#}^{\Delta x}(y).$$

Here we have added the superscript Δx in order for the proof to be clearer. Like in (3.12) the function $v_{\#}^{\Delta x}$ is the piecewise extension of a discrete function $v^{\Delta x}$ which is a solution of (3.17) (see Definition 3.9).

We want to prove that \bar{v} and \underline{v} are respectively a sub and a super-solution of (3.1). In fact if that is true, the comparison principle for (3.1) implies that $\bar{v} \leq \underline{v}$ on $[-l, l]$. However by construction we have that $\underline{v} \leq \bar{v}$, which will imply that $\underline{v} = \bar{v} = v^\delta$ the unique continuous solution of (3.1), this implies the local uniform convergence of $v_{\#}^{\Delta x}$.

Let us now prove that \bar{v} is a sub-solution of (3.1). We only do the proof in the sub-solution case, since the super-solution case is very similar and we skip it. We argue by contradiction and assume that there exists a function $\varphi \in C^1([-l, l])$ and a point $\bar{x} \in [-l, l]$ such that $\bar{v} - \varphi$ reaches a strict local maximum at \bar{x} and that we have

$$\delta \bar{v}(\bar{x}) + \mathcal{I}(\bar{x}, [\bar{v}], \varphi_x(\bar{x})) = \theta > 0.$$

Moreover, without any loss of generality, we assume that $\bar{u}(\bar{x}) = \varphi(\bar{x})$ and that $\varphi \geq 2 \sup_{\Delta x} \|v_{\sharp}^{\Delta x}\|_{\infty}$ outside the ball $B(\bar{x}, r)$, where $r > 0$ is such that

$$\bar{u}(x) - \varphi(x) \leq 0 = \bar{u}(\bar{x}) - \varphi(\bar{x}) \quad \text{in } B(\bar{x}, r). \quad (4.3)$$

Then there exists sequences $\Delta x_m \in [0, +\infty)$ and $y_m \in [-l, l]$, such that as $m \rightarrow +\infty$

$$\begin{aligned} \Delta x_m &\rightarrow 0, \quad y_m \rightarrow \bar{x}, \quad v_{\sharp}^{\Delta x_m}(y_m) \rightarrow \bar{v}(\bar{x}), \quad \text{and} \\ y_m &\text{ is a global maximum point of } v_{\sharp}^{\Delta x_m}(\cdot) - \varphi(\cdot). \end{aligned}$$

We denote by $\xi_m = v_{\sharp}^{\Delta x_m}(y_m) - \varphi(y_m)$, and we have that $\xi_m \rightarrow 0$ as $m \rightarrow +\infty$. Moreover, we have for all $x \in [-l, l]$

$$v_{\sharp}^{\Delta x_m}(x) \leq \varphi(x) + \xi_m \quad \text{and} \quad v_{\sharp}^{\Delta x_m}(y_m) = \varphi(y_m) + \xi_m.$$

We denote by x_{i_m} the point in the grid such that $y_m \in \mathcal{Q}_{i_m}$, therefore we have that $|x_{i_m} - y_m| \leq \Delta x_m$ and

$$\begin{aligned} 0 &\geq \delta v_{i_m}^{\Delta x_m} + F_{i_m} \left([v^{\Delta x_m}], Dv_{i_m}^{\Delta x_m} \right) \\ &\geq \delta v_{i_m}^{\Delta x_m} + F_{i_m} \left([v^{\Delta x_m}], D\varphi(y_m) \right) \\ &\geq \delta v_{\sharp}^{\Delta x_m}(y_m) + F \left(x_{i_m}, [v_{\sharp}^{\Delta x_m}], \varphi'(x_0) + o(\Delta x_m) \right), \end{aligned}$$

where we have used the fact that $v^{\Delta x_m}$ is a sub-solution of (3.17) for the first line. For the second line, we have considered

$$D\varphi(y_m) = \left(\frac{\varphi(y_m) - \varphi(y_m - \Delta x_m)}{\Delta x_m}, \frac{\varphi(y_m + \Delta x_m) - \varphi(y_m)}{\Delta x_m} \right)$$

and have used the monotonicity of the discrete operator F_{i_m} (Lemma 3.14). Finally, for the third line, we have used the fact that $\varphi \in C^1$ and the definition of F_{i_m} . We now pass to the limit in the previous inequality

$$\begin{aligned} 0 &\geq \liminf_{m \rightarrow +\infty} \left(\delta v_{\sharp}^{\Delta x_m}(i_m) + F(x_{i_m}, [v_{\sharp}^{\Delta x_m}], \varphi'(x_0) + o(\Delta x_m)) \right) \\ &\geq \delta \bar{v}(\bar{x}) + \mathcal{I}(\bar{x}, [\bar{v}], \varphi'(\bar{x})), \end{aligned}$$

where we have used Proposition 3.3 with $u_m = v_{\sharp}^{\Delta x_m}$, $x_m := x_{i_m}$, $p_m := \varphi'(\bar{x}) + o(\Delta x_m)$ and notice that $v_{\sharp}^{\Delta x_m}(x_{i_m}) = v_{\sharp}^{\Delta x_m}(y_m)$. The last inequality provides us with a contradiction which ends the proof of Theorem 3.11. \square

5 Discrete approximated cell problem

The following results are similar to the ones in [5], for the reader convenience, we recall them and adapt them to our problem. The idea is to obtain some results on the numerical scheme that will help us to construct a solution to (3.17).

5.1 Comparisons for the numerical scheme

In this section, we prove the existence of solutions for the numerical scheme (3.17). We will use the following notations, for every $s \in \mathbb{R}$ and every discrete function v , we define

$$D_{v_i}^+(s) = \frac{v_{i+1} - s}{\Delta x}, \quad D_{v_i}^-(s) = \frac{s - v_{i-1}}{\Delta x}, \quad D_{v_i}(s) = (D_{v_i}^-(s), D_{v_i}^+(s)).$$

For every $i \in \{-n, \dots, n\}$, we set

$$M_i^d[v](s) = \int_{h_0}^{h_{max}} J(z) E(v_{\sharp}(x_i + z) - s) dz - \frac{3}{2} V_{max}$$

$$\tilde{M}_i^d[v](s) = \int_{h_0}^{h_{max}} J(z) \tilde{E}(v_{\sharp}(x_i + z) - s) dz - \frac{3}{2} V_{max}$$

and

$$\mathcal{R}_i[v](s) = \psi_R(x_i) \cdot \phi(x_i) \cdot M_i^d[v](s) \cdot G^+(D_{v_i}(s)) + (1 - \psi_R(x_i)) \cdot \overline{H}^d(D_{v_i}(s)).$$

We define similarly $\tilde{\mathcal{R}}_i$ by replacing M_i^d with \tilde{M}_i^d . Finally, we define for all $i \in \{-n, \dots, n\}$,

$$F_i[v](s) = \begin{cases} \mathcal{R}_i[v](s) & \text{if } i \in \{-n+1, \dots, n-1\} \\ \overline{H}^-(D_{v_i}^+(s)) & \text{if } i = -n \\ \overline{H}^+(D_{v_i}^-(s)) & \text{if } i = n. \end{cases}$$

Similarly we define \tilde{F}_i by replacing \mathcal{R}_i by $\tilde{\mathcal{R}}_i$.

Remark 5.1 (Monotonicity of the numerical scheme). *By the definition of F_i and \tilde{F}_i and by Lemma 3.14, we have for all $i \in \{-n, \dots, n\}$,*

- (i) $F_i[v](s)$ and $\tilde{F}_i[v](s)$ are non-decreasing with respect to s .
- (ii) $F_i[v](s)$ and $\tilde{F}_i[v](s)$ are non-increasing with respect to v .
- (iii) $F_i[v](v_i) = F_i([v], Dv_i)$ and $\tilde{F}_i[v](v_i) = \tilde{F}_i([v], Dv_i)$.
- (iv) $\tilde{F}_i[v](s) \leq F_i[v](t)$ for all $s < t$.

Lemma 5.2 (Comparison). *Let v be a discrete function and let $i \in \{-n, \dots, n\}$. There exists a unique $s_i \in \mathbb{R}$ such that*

$$-\tilde{F}_i[v](s_i) \leq \delta s_i \leq -F_i[v](s_i). \quad (5.1)$$

Moreover, let w be a discrete function. Then the following implications hold:

- (i) $\delta w_i + F_i[w](w_i) \leq 0$ and $w \leq v \Rightarrow w_i \leq s_i$.
- (ii) $\delta w_i + F_i[w](w_i) > 0$ and $w \geq v \Rightarrow w_i > s_i$.
- (iii) $\delta w_i + \tilde{F}_i[w](w_i) \geq 0$ and $w \geq v \Rightarrow w_i \geq s_i$.
- (iv) $\delta w_i + \tilde{F}_i[w](w_i) < 0$ and $w \leq v \Rightarrow w_i < s_i$.

Proof. The existence of a $s_i \in \mathbb{R}$ satisfying (5.1) comes from the fact that $-F_i[v](s)$ and $-\tilde{F}_i[v](s)$ are non-increasing in s and that δs is strictly increasing in s . As for the uniqueness of s_i , let us argue by contradiction and assume there exists s_i^1 and s_i^2 with $s_i^1 < s_i^2$ such that

$$-\tilde{F}_i[v](s_i^1) \leq \delta s_i^1 \leq -F_i[v](s_i^1) \quad \text{and} \quad -\tilde{F}_i[v](s_i^2) \leq \delta s_i^2 \leq -F_i[v](s_i^2).$$

However, using (iv) from Remark 5.1, we have that $-F_i[v](s_i^2) \leq -\tilde{F}_i[v](s_i^1)$, using this and combining the previous inequalities we obtain that

$$F_i[v](s_i^1) - \tilde{F}_i[v](s_i^2) \leq \delta(s_i^2 - s_i^1) \leq \tilde{F}_i[v](s_i^1) - F_i[v](s_i^2) \leq 0. \quad (5.2)$$

This gives us that $s_2 - s_1 \leq 0$ which is a contradiction.

We now prove the implication (i), again we argue by contradiction and assume that $w_i > s_i$. Then using (5.1), we have

$$\delta w_i > \delta s_i \geq -\tilde{F}_i[v](s_i) \geq -\tilde{F}_i[w](s_i) \geq -F_i[w](w_i) \geq \delta w_i,$$

where we have used for the third and fourth inequality respectively (ii) and (iv) from Remark 5.1. The previous inequality gives us the desired contradiction.

We now prove implication (ii). We argue by contradiction and assume that $w_i \leq s_i$, using (5.1), we have

$$\delta w_i \leq \delta s_i \leq -F_i[v](s_i) \leq -F_i[w](s_i) \leq -F_i[w](w_i) < \delta w_i,$$

where we have used for the third and fourth inequality respectively (ii) and (i) from the Remark 5.1. The previous inequality gives us the desired contradiction. The proofs of implications (iii) and (iv) are similar and we skip them. \square

5.2 Construction of minimal and maximal solutions

This section is devoted to the proof of existence and to the construction of minimal and maximal solutions. In order to prove the existence of discrete solution for (3.17), we will provide a constructive method that will also provide us with the minimal and maximal solutions.

Proposition 5.3 (Definition of the map Φ). *There exists a map $\Phi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ satisfying the following properties*

a) Let u^- be a sub-solution of (3.17), i.e.,

$$\delta u_i^- + F_i[u^-](u_i^-) \leq 0 \quad \text{for all } i \in \{-n, \dots, n\}.$$

Then

- i) $u^- \leq \Phi[u^-]$ (with $u^- = \Phi[u^-]$ if and only if u^- is a solution).
- ii) $\Phi[u^-]$ is a sub-solution of (3.17).

b) Let u^+ be a super-solution of (3.17), i.e.,

$$\delta u_i^- + \tilde{F}_i[u^-](u_i^-) \geq 0 \quad \text{for all } i \in \{-n, \dots, n\}.$$

Then

- i) $u^+ \geq \Phi[u^+]$ (with $u^+ = \Phi[u^+]$ if and only if u^+ is a solution).
- ii) $\Phi[u^+]$ is a super-solution of (3.17).

Proof. Let us prove the result for the sub-solutions (a).

Using Lemma 5.2, for every $i \in \{-n, \dots, n\}$ there exists a unique $s_i^- \in \mathbb{R}$ such that

$$-\tilde{F}_i[u^-](s_i^-) \leq \delta s_i^- \leq -F_i[u^-](s_i^-) \quad \text{and} \quad u_i^- \leq s_i^-,$$

the second inequality comes from (i) in Lemma 5.2 considering $w = u^-$. Using the real numbers s_i^- we construct the map Φ :

$$\Phi[u^-]_i = s_i^- \quad \text{for all } i \in \{-n, \dots, n\}.$$

By construction we have $u^- \leq \Phi[u^-]$, with the equality if and only if u^- is a solution of (3.17) (which proves (i) in (a)). Let us now prove that $\Phi[u^-]$ is a sub-solution of (3.17). For all $i \in \{-n, \dots, n\}$, by the definition of s_i^- , we have

$$0 \geq \delta s_i^- + F_i[u^-](s_i^-) \geq \delta s_i^- + F_i[\Phi[u^-]](s_i^-) = \delta \Phi[u^-]_i + F_i[\Phi[u^-]](\Phi[u^-]_i),$$

where we have used for the second inequality the fact that $F_i[\cdot](s)$ is non-increasing and that $u^- \leq \Phi[u^-]$. Therefore, $\Phi[u^-]$ is a sub-solution of (3.17).

In the case of the super-solutions, to prove (b), we define the map Φ in the same way. Using Lemma 5.2, for every $i \in \{-n, \dots, n\}$ there exists a unique $s_i^+ \in \mathbb{R}$ such that

$$-\tilde{F}_i[u^+](s_i^+) \leq \delta s_i^+ \leq -F_i[u^+](s_i^+) \quad \text{and} \quad u_i^+ \geq s_i^+.$$

Like before, the map Φ is constructed using the real numbers s_i^+ :

$$\Phi[u^+]_i = s_i^+ \quad \text{for all } i \in \{-n, \dots, n\}.$$

Proceeding like before we can prove the rest of (b). We skip the rest of the proof and this ends the proof of Proposition 5.3. \square

Proposition 5.4 (Partial comparison principle). *Let u^- and u^+ be respectively a sub and a super-solution of (3.17) such that $u^- \leq u^+$ then $\Phi[u^-] \leq \Phi[u^+]$.*

Proof. We argue by contradiction and assume that there exists a $i_0 \in \{-n, \dots, n\}$ such that

$$\Phi[u^-]_{i_0} = s_{i_0}^- > s_{i_0}^+ = \Phi[u^+]_{i_0}.$$

By definition of $s_{i_0}^+$, we have

$$0 \leq \delta s_{i_0}^+ + \tilde{F}_{i_0}[u^+](s_{i_0}^+) < \delta s_{i_0}^- + \tilde{F}_{i_0}[u^+](s_{i_0}^+) \leq \delta s_{i_0}^- + F_{i_0}[u^+](s_{i_0}^-) \leq \delta s_{i_0}^- + F_{i_0}[u^-](s_{i_0}^-), \quad (5.3)$$

where we have used the fact that $s_{i_0}^- > s_{i_0}^+$ for the second inequality, and (i) and (ii) from Remark 5.1 for the third and fourth inequalities respectively. This inequality gives us the desired contradiction because of the definition of $s_{i_0}^-$. \square

Proposition 5.5 (Construction of solutions). *Let u^- and u^+ be respectively a sub and a super-solution of (3.17) such that $u^- \leq u^+$. We consider for every $k \in \mathbb{N}$,*

$$u^{-,k+1} = \Phi[u^{-,k}], \quad \text{with } u^{-,0} = u^-,$$

and

$$v^{+,k+1} = \Phi[v^{+,k}], \quad \text{with } v^{+,0} = u^+.$$

There exist two discrete functions u and v such that $u^{-,k} \rightarrow u$ and $v^{+,k} \rightarrow v$ as $k \rightarrow +\infty$. Moreover u and v are two solutions of (3.17). We define

$$\Psi[u^-] := u \quad \text{and} \quad \Psi[v^+] := v.$$

Then we have

$$u^- \leq \Psi[u^-] \leq \Psi[v^+] \leq u^+,$$

and $\Psi[u] = u$ if and only if u is a solution of (3.17).

Proof. Using Proposition 5.4 we get the following inequalities

$$u^- \leq \Phi[u^{-,k}] \leq \Phi[u^{-,k+1}] \leq \dots \leq \Phi[v^{+,k+1}] \leq \Phi[v^{+,k}] \leq v^+.$$

Therefore, the sequence $(u^{-,k})_k$ is non-decreasing and bounded from above by v^+ , and the sequence $(v^{+,k})_k$ is non-increasing and bounded from below by u^- . Passing to the limit as k goes to infinity in the previous inequalities, we obtain

$$u^- \leq \lim_{k \rightarrow +\infty} u^{-,k} =: u =: \Psi[u^-] \leq \Psi[v^+] =: v =: \lim_{k \rightarrow +\infty} v^{+,k} \leq v^+.$$

Let us now prove that u is a solution of (3.17) (The proof for v is similar and we skip it). By definition of the sequence $(u^{-,k})_k$, we have for all $k \in \mathbb{N}$ and for all $i \in \{-n, \dots, n\}$,

$$-\tilde{F}_i[u^{-,k}](u_i^{-,k+1}) \leq \delta u_i^{-,k+1} \leq -F_i[u^{-,k}](u_i^{-,k+1}). \quad (5.4)$$

We recall that E and \tilde{E} are respectively lower and upper semi-continuous, which implies that R and \tilde{R} are also lower and upper semi-continuous and in particular that $-R$ and $-\tilde{R}$ are respectively upper and lower semi-continuous. Adding this to the continuity of the discrete gradient and of the functions \bar{H} , \bar{H}^+ and \bar{H}^- , passing to the limit as k goes to $+\infty$ in (5.4) implies

$$-\tilde{F}_i[u](u_i) \leq \delta u_i \leq -F_i[u](u_i),$$

which means that u is a solution of (3.17). Finally, the fact that $\Psi[u] = u$ if and only if u is a solution comes from the properties of Φ (see Proposition 5.3). \square

Proposition 5.6 (Extremal solutions in the interval (u^-, v^+)). *Let u^- and v^+ be respectively a sub and super-solution of (3.17) such that $u^- \leq v^+$. Let $\Psi[u^-]$ and $\Psi[v^+]$ be the two solutions provided by Proposition 5.5. Then every solution v of (3.17) such that $u^- \leq v \leq v^+$ satisfies*

$$u^- \leq \Psi[u^-] \leq v \leq \Psi[v^+] \leq v^+.$$

Proof. Considering v as a super-solution of (3.17), and using Proposition 5.5, we get

$$u^- \leq \Psi[u^-] \leq \Psi[v] = v.$$

Similarly, considering v as a sub-solution of (3.17), using Proposition 5.5, we have

$$v = \Psi[v] \leq \Psi[v^+] \leq v^+.$$

\square

Combining Lemma 4.1, Proposition 4.2, and Proposition 5.6 we obtain the following result.

Corollary 5.7. *Let $u_i^- = 0$ and $v_i^+ = C_0/\delta$ for all $i \in \{-n, \dots, n\}$. Let $\Psi[u^-]$ and $\Psi[v^+]$ be the solutions provided by Proposition 5.5, then every solution v of (3.17) satisfies*

$$\Psi[u^-] \leq v \leq \Psi[v^+].$$

6 Numerical simulations

This section contains the application of the results from the previous sections. First we provide the algorithm used to obtain a numerical approximation of the flux-limiter and then we provide some numerical tests.

6.1 The algorithm

The following algorithm is inspired by the one in [5], and by the results from Section 5. The idea of the algorithm is to build the extremal solutions from Corollary 5.7, to build the biggest sub-solution and the smallest super-solution. Therefore obtaining an interval that contains all the solutions of (3.17) and therefore obtaining an approximation of the solution of (3.1).

We introduce two parameters ε_d and ε_c respectively a tolerance to quit the dichotomy process updating the sub and super-solutions (numerical equivalent of Proposition 5.5) and a tolerance for the convergence of the numerical scheme.

1) Initialization: for $i = -n, \dots, n$,

$$u_i^- = 0 \quad \text{and} \quad v_i^+ = \frac{C_0}{\delta}.$$

2) Initialize dichotomy intervals: for $i = -n, \dots, n$,

$$s_{left,i}^- = u_i^- \quad \text{and} \quad s_{right,i}^- = u_i^- + 0,1k_i^-$$

with k_i^- the first integer such that

$$\delta s_{right,i}^- + F_i[u^-](s_{right,i}^-) > 0,$$

and

$$s_{left,i}^- = v_i^+ - 0,1k_i^+ \quad \text{and} \quad s_{right,i}^+ = v_i^+$$

with k_i^+ the first integer such that

$$\delta s_{left,i}^- + \tilde{F}_i[v^+](s_{left,i}^-) < 0.$$

3) Dichotomy process: for $i = -n, \dots, n$ optimize respectively in $s_i^- \in [s_{left,i}^-, s_{right,i}^-]$ and $s_i^+ \in [s_{left,i}^+, s_{right,i}^+]$, the inequalities

$$\delta s_i^- + F_i[u^-](s_i^-) \leq 0 \quad \text{and} \quad \delta s_i^+ + \tilde{F}_i[v^+](s_i^+) \geq 0$$

until $s_{right,i}^- - s_{left,i}^- < \varepsilon_d$ and $s_{right,i}^+ - s_{left,i}^+ < \varepsilon_d$.

If $\|u^- - s_{left,i}^-\|_\infty \leq \varepsilon_c$ and $\|v^+ - s_{left,i}^+\|_\infty \leq \varepsilon_c$ go to Step 4 else swap $u^- \leftrightarrow s_{left,i}^-$ and $v^+ \leftrightarrow s_{right,i}^+$ and go to Step 2.

4) The interval $[u^-, v^+]$ contains all the solutions of the numerical scheme (3.17) and therefore gives an approximation of the solution of (3.1). In particular, the value of \bar{A} is approximated by the interval $[-\delta v_0^+, -\delta u_0^-]$.

Remark 6.1. Notice that this algorithm can be extremely costly computationally. However, we can easily apply parallel programming to steps 2) and 3) to accelerate the process.

6.2 Setting of the computation

We consider an uniform grid of the interval $[-l, l]$ with $2n + 1$ points, $n \in \mathbb{N} \setminus \{0\}$, and we denote by $\Delta x = l/n$ the discretization step. For all $i \in \{-n, \dots, n\}$, we denote by $x_i = \Delta x \cdot i$ the nodes of the grid.

For all the numerical computations, we will consider for equation (3.1), the following values for the different parameters.

$$l = 200, \quad R = 100, \quad \varepsilon_c = \varepsilon_d = 0.001, \quad \text{and} \quad \delta = 0.001. \quad (6.1)$$

For all the numerical computations, regarding the values for the discretization, we consider

$$n = 400 \quad \text{and} \quad \Delta x = 0.5. \quad (6.2)$$

For the computation of the discrete non-local operator, we recall the following result, using

the notations from Remark 3.8 and from (3.12),

$$\begin{aligned}
M[v_{\sharp}](x_i) &= \int_{j_0}^{j_{max}} J(z) E(v_{\sharp}(x_i + z) - v_{\sharp}(x_i)) dz - \frac{3}{2} V_{max} \\
&= \sum_{j=j_0}^{j_{max}} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} J(z) E(v_{\sharp}(x_i + z) - v_{\sharp}(x_j)) dz - \frac{3}{2} V_{max} \\
&= \sum_{j=j_0}^{j_{max}} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} J(z) E(v_{i+j} - v_i) dz - \frac{3}{2} V_{max} \\
&= \sum_{j=j_0}^{j_{max}} E(v_{i+j} - v_i) \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} J(z) dz - \frac{3}{2} V_{max} \\
&= \sum_{j=j_0}^{j_{max}} E(v_{i+j} - v_i) J_j - \frac{3}{2} V_{max},
\end{aligned}$$

with $J_j = V(x_j + \Delta x/2) - V(x_j - \Delta x/2)$.

6.3 Qualitative properties of \bar{A}

Before passing to the numerical tests, we recall a final result from [12] regarding the qualitative properties of \bar{A} , which we will numerically verify in the next section.

Proposition 6.2 (Qualitative properties of the flux limiter). *Assume (A). We have the following qualitative properties on the flux limiter.*

(i) **(Monotonicity of the flux-limiter).** *Let $\phi_1, \phi_2 : \mathbb{R} \rightarrow [0, 1]$ be two functions satisfying (A6). Let \bar{A}_1 and \bar{A}_2 be their respective flux limiters. If, for all $x \in \mathbb{R}$, we have*

$$\phi_1(x) \leq \phi_2(x),$$

then

$$\bar{A}_1 \geq \bar{A}_2.$$

(ii) **(Flux interruption)** *Let ϕ be a function satisfying (A6). If $\phi = 0$ on an open interval, then we have*

$$\bar{A} = 0.$$

6.4 Numerical tests

Influence of ϕ_0

First, we would like to numerically verify (i) from Proposition 6.2 and to see the influence of $\phi_0 = \min_x \phi(x)$. We consider a Greenshields optimal velocity function,

$$V(h) = \begin{cases} 0 & \text{if } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^2 \right) & \text{if } h > h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^2 \right) & \text{if } h > h_{max}. \end{cases} \quad (6.3)$$

For the perturbation, we consider a function $\phi : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \geq r \\ \phi_0 & \text{if } |x| \leq r/8 \\ -\frac{8x(1-\phi_0)}{7r} + \frac{8\phi_0-1}{7} & \text{if } x \in (-r, -r/8) \\ \frac{8x(1-\phi_0)}{7r} + \frac{8\phi_0-1}{7} & \text{if } x \in (r/8, r). \end{cases} \quad (6.4)$$

For the values of the different parameters for the optimal velocity function, we take

$$\begin{cases} V_{max} = 58 \text{ km/h}, \\ h_0 = 2 \text{ m}, \\ h_{max} = 25 \text{ m}. \end{cases} \quad (6.5)$$

For the local perturbation, we consider the radius of the perturbation $r = 45 \text{ m}$. In Figure 3, we have an example of the local perturbation for two different values of ϕ_0 . Notice that given the

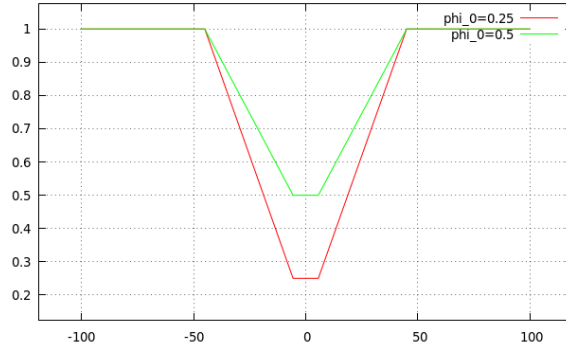


Figure 3: Example of ϕ with $r = 45 \text{ m}$ and $\phi_0 = 0, 25$ (red) and $\phi_0 = 0, 5$ (green).

definition (6.4), we have that if we consider two functions ϕ^1, ϕ^2 with their respective minima $\phi_0^1 < \phi_0^2$ then for all $x \in \mathbb{R}$, we have $\phi^1(x) \leq \phi^2(x)$.

In order to see the influence of $\phi_0 \in [0, 1]$ on \bar{A} , we discretize the interval $[0, 1] \ni \phi_0$ in 21 points (a step of 0.05) and we compute our estimate of \bar{A} for each of those ϕ_0 . For each ϕ_0 , we plot two points, since we have an interval that approximates \bar{A} .

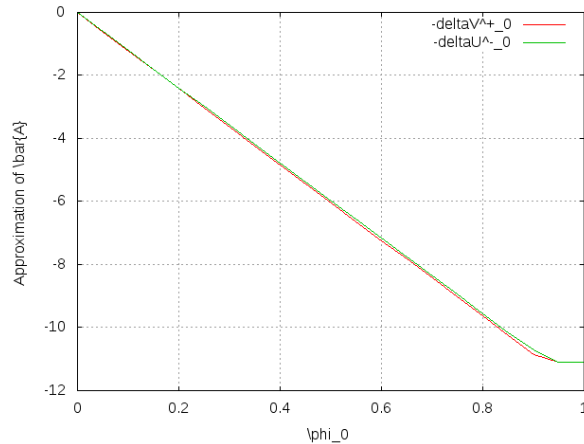


Figure 4: Approximation of \bar{A} plotted versus different values of ϕ_0 .

From Figure 4, we notice that the approximation is decreasing with ϕ_0 , which numerically confirms (i) of Proposition (6.2) and notice that for $\phi_0 = 0$, we have that \bar{A} is close to 0 which numerically confirms (ii) of (6.2).

In the case $\phi_0 = 1$, the model is actually equivalent to a model without a perturbation. Therefore, we should not have a flux-limiting condition. Given the definition (1.11) of $F_{\bar{A}}$ this can only happen if $\bar{A} = \min_p \bar{H}(p)$. In our computational setting we have $H_0 \simeq -11.16$ and we have an approximation of -11.11 which is not very far and which also validates our numerical approach.

Influence of the radius of influence of the perturbation

We consider the same optimal velocity function as before and the same perturbation and we make the radius of influence of the perturbation vary in the interval $[25, 75]$. Figure 5 contains the approximation of \bar{A} for $\phi_0 = 0, 25$ and for different values of $r \in [25, 75]$.

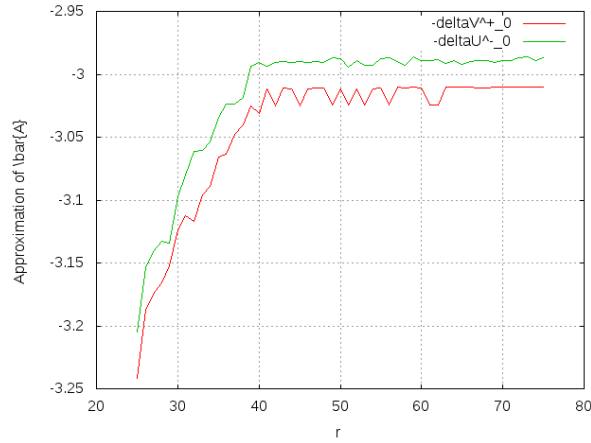


Figure 5: Approximation of \bar{A} plotted versus different values of r and $\phi_0 = 0.25$.

From Figure 5, we can see that in this case the approximation of \bar{A} increases with the radius of the perturbation. However, for $r \geq 40m$ the approximation remains the same which could imply that for r big enough, the radius of the perturbation does not influence the value of \bar{A} . Moreover, for smaller values of r , we can see that \bar{A} is smaller (meaning that the flux is less limited) which is logical, since for a radius $r = 0$, we expect to have $\bar{A} = H_0$ (there is no perturbation).

Two different perturbations

Now we would like to take into account two different perturbations and see how does our approximation of \bar{A} changes with each perturbation. We consider the same perturbation as before and we introduce the following perturbation

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{if } |x| \geq r \\ \frac{(1 - \phi_0)x^2}{r^2} + \phi_0 & \text{if } |x| \leq r. \end{cases} \quad (6.6)$$

We consider the same radius of influence $r = 45 m$. In Figure 6, we plotted the two perturbations ϕ and $\tilde{\phi}$ with $\phi_0 = 0, 25$.

Like before, we make ϕ_0 vary inside the interval $[0, 1]$. In Figure 7, we compare the upper and lower bound of the approximation of \bar{A} for different values of ϕ_0 .

From Figure 7, we first notice that the approximation of \bar{A} for $\tilde{\phi}$ numerically verifies Proposition 6.2. Moreover, we notice that the values for both perturbations are very similar. This actually could imply that the form of the perturbation does not influence the result but it is only ϕ_0 that

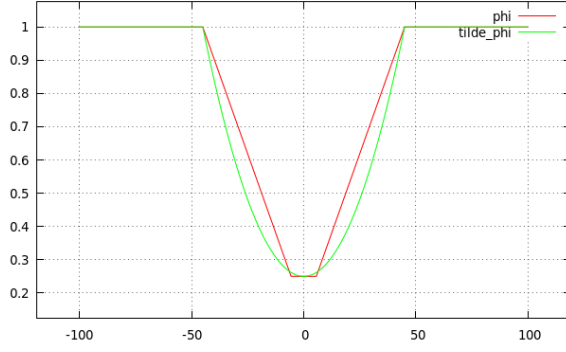


Figure 6: The functions ϕ and $\tilde{\phi}$, with $r = 45 m$ and $\phi_0 = 0, 25$.

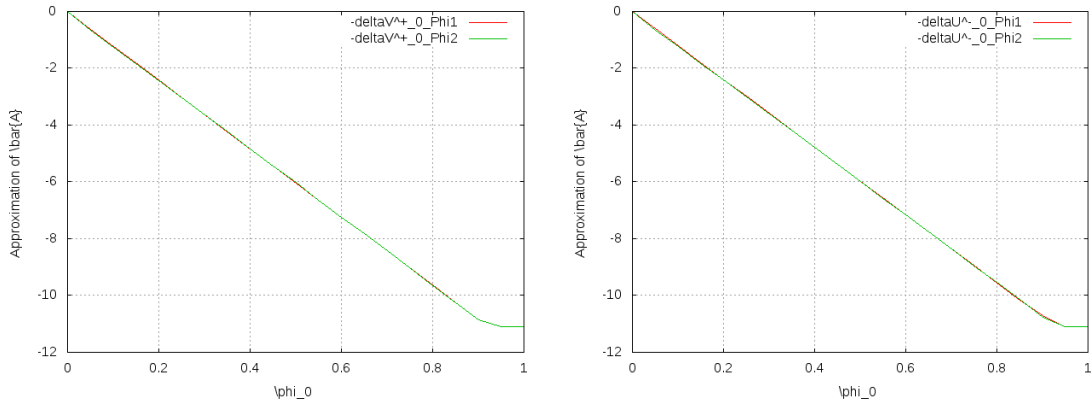


Figure 7: Lower bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (left) upper bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (right).

determines the value of the flux limiter. In Figure 8, we plot the absolute difference between the two approximations and we notice the difference is very small.

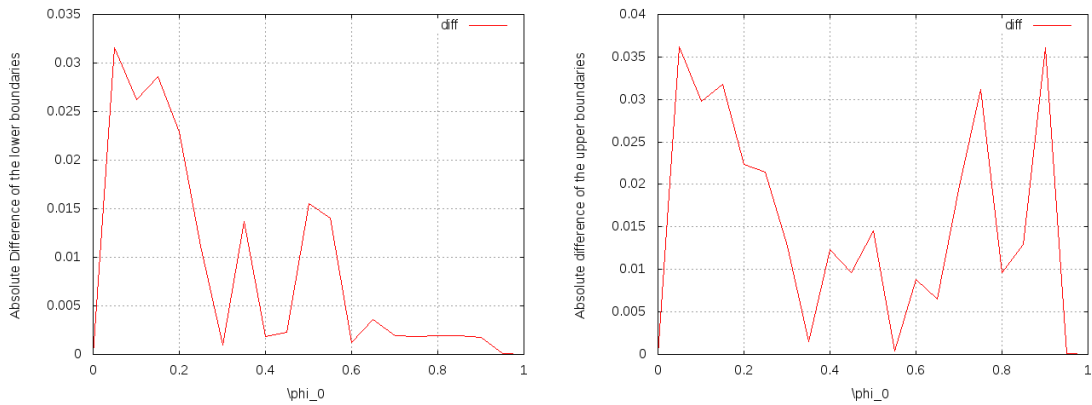


Figure 8: Absolute difference of the lower bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (left). Absolute difference of the upper bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (right).

Influence of δ

We consider the optimal velocity function (6.3) with (6.5) and the perturbation (6.4), with $r = 45 m$ and $\phi_0 = 0, 25$. To see the influence of δ on the approximation of \bar{A} , we fix l and R to the values of (6.1) and we make δ vary in $[0.001, 0.1]$ with a step of 0.001.

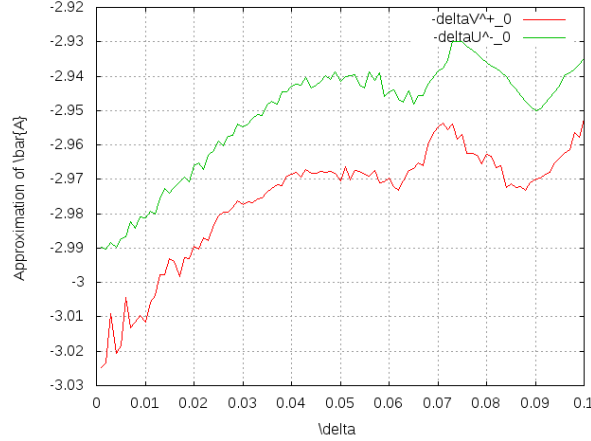


Figure 9: Approximation of \bar{A} versus different values of δ .

From Figure 9, we first we notice that there is a lot of oscillations on the behaviour of the estimates of \bar{A} . However, the upper and lower bound remain close to each other. Given that the difference between the estimates is small, we can assume that considering $\delta = 0.001$ gives a good enough approximation of the flux-limiter.

Influence of R (transition between the non-local and local operators)

We consider the same optimal velocity function and perturbation as before, with the same parameters. To see the influence of R on the approximation of \bar{A} , we fix l and δ to the values of (6.1) and we make R vary in $[80, 150]$ with a step of 1.

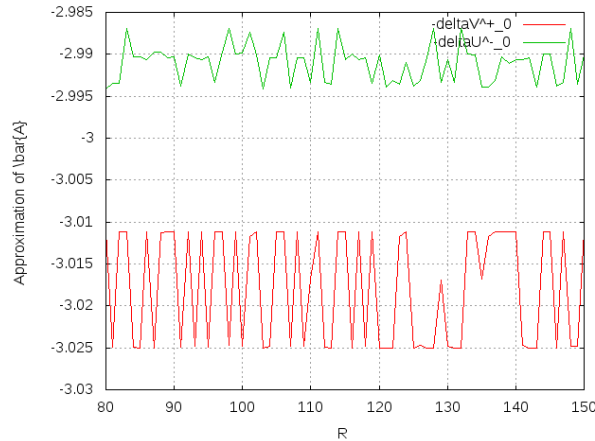


Figure 10: Approximation of \bar{A} versus different values of R .

From Figure 10, we notice a lot a oscillation on the behaviour of the estimates for \bar{A} when we make R vary. However, we notice that for $R > 80$ the difference between the upper and lower estimate is very small (less than 0.4). This would suggest that considering $R > 80$ is enough for a good approximation of \bar{A} .

Influence of l

We consider the same optimal velocity function and perturbation as before, with the same parameters. To see the influence of l on the approximation of \bar{A} , we fix R and δ to the values of (6.1) and we make l vary in $[180, 300]$ with a step of 1.

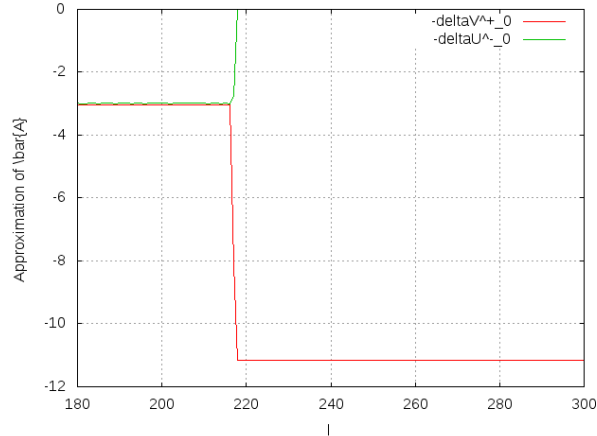


Figure 11: Approximation of \bar{A} versus different values of l .

From Figure 11 we remark that for $l \in [180, 220]$ the approximation of \bar{A} remains almost the same. The behaviour of the approximation for bigger values of l can be explain by the fact that first δ should go to 0 before passing to the limit as l goes to infinity. Therefore, there is a compromise to be made between l and δ . However, for $\delta = 0.001$ taking $l = 200$ seems to give a reasonable approximation.

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