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Homogenization of unbounded integrals with quasiconvex growth

Omar Anza Hafsa · Jean-Philippe Mandallena · Hamdi Zorgati

Abstract We study homogenization by Γ -convergence of periodic nonconvex integrals when the integrand has quasiconvex growth with fixed convex effective domain.

Keywords Homogenization · Quasiconvex growth · Unbounded integrals · Ru-usc

Mathematics Subject Classification 49J45 · 35B27 · 74Q05

1 Introduction and main result

Let $m, d \geq 1$ be two integers and $p \in [1, \infty[$. Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open set with Lipschitz boundary. We consider the periodic homogenization problem of integral functionals by Γ -convergence. More precisely, for each $\varepsilon > 0$, we define $I_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ by

$$I_\varepsilon(u) := \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

where the integrand $W : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ is Borel measurable and 1-periodic with respect to the first variable. The homogenization of $\{I_\varepsilon\}_{\varepsilon > 0}$ consists to show the Γ -convergence with respect to the strong topology of L^p as $\varepsilon \rightarrow 0$ to an homogenized functional integral and to give a representation of the homogenized integrand.

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Nonconvex homogenization by Γ -convergence of the family $\{I_\varepsilon\}_{\varepsilon>0}$ was mainly studied in the framework of p -polynomial growth conditions on W . Unfortunately, this framework is not compatible with two basic conditions of hyperelasticity: the non-interpenetration of the matter, i.e., $W(x, \xi) = \infty$ if and only if $\det(I + \xi) \leq 0$, and the necessity of an infinite amount of energy to compress a finite volume into zero volume, i.e., for every $x \in \mathbb{R}^d$, $W(x, \xi) \rightarrow \infty$ as $\det(I + \xi) \rightarrow 0$. At present, it seems difficult to take these conditions into account in homogenization problems. Generally, the attempts to go beyond the p -polynomial growth are not easy due to the lack of available techniques. However, in the scalar case, we refer to the book [15] where relaxation and homogenization of unbounded functionals were studied (see also [12–14, 16]). In the vectorial case, i.e., when $\min\{d, m\} > 1$, the homogenization in $W^{1,\infty}$ without growth conditions but with W having fixed bounded convex domain is studied in [5], and in [4] the homogenization with convex growth (independent of x) on W is carried out (for the relaxation problems see [3, 6, 24]).

To go beyond the p -polynomial growth we consider G -growth and p -coercivity conditions on W as follows:

(H₁) G -growth conditions, i.e., there exist $\alpha, \beta > 0$ such that for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{M}^{m \times d}$

$$\alpha G(\xi) \leq W(x, \xi) \leq \beta(1 + G(\xi));$$

(H₂) W is p -coercive, i.e., there exists $c > 0$ such that for every $(x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m \times d}$

$$c|\xi|^p \leq W(x, \xi),$$

where $G : \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ is a Borel measurable function. Denote by \mathbb{G} the effective domain of G , i.e., $\mathbb{G} = \{\xi \in \mathbb{M}^{m \times d} : G(\xi) < \infty\}$. We can remark that (H₁) implies that the effective domain of W is independent of x and $\text{dom} W(x, \cdot) = \mathbb{G}$ for all $x \in \mathbb{R}^d$. In [4] the case G convex with $0 \in \text{int}(\mathbb{G})$ and $p > d$ was studied.

In this paper we consider the following conditions on G :

(C₁) $0 \in \text{int}(\mathbb{G})$;

(C₂) there exists $C > 0$ such that for every $\xi, \zeta \in \mathbb{M}^{m \times d}$ and every $t \in]0, 1[$

$$G(t\xi + (1 - t)\zeta) \leq C(1 + G(\xi) + G(\zeta));$$

(C₃) G is $W^{1,p}$ -quasiconvex, i.e., for every $\xi \in \mathbb{M}^{m \times d}$

$$G(\xi) = \inf \left\{ \int_Y G(\xi + \nabla \varphi(x)) dx : \varphi \in W_0^{1,p}(Y; \mathbb{R}^m) \right\}$$

where $Y =]0, 1[^d$.

Note that (C₂) implies that \mathbb{G} is convex, but G is not necessarily convex (see Sect. 9 for an example). The condition (C₂) prevents the possible “strong bumps” of G .

We say that W is *periodically radially uniformly upper semicontinuous (periodically rusc)* if there exists $a \in L_{\text{loc}}^1(\mathbb{R}^d;]0, \infty[)$ 1-periodic such that

$$\overline{\lim}_{t \rightarrow 1^-} \Delta_W^a(t) \leq 0$$

with

$$\Delta_W^a(t) := \sup_{x \in U} \sup_{\xi \in \text{dom} W(x, \cdot)} \frac{W(x, t\xi) - W(x, \xi)}{a(x) + W(x, \xi)}$$

where $\text{dom} W(x, \cdot)$ is the effective domain of $W(x, \cdot)$ (see Sect. 4.2).

Here is the main result of our paper.

Theorem 1.1 *Assume that $p > d$. Assume that (C_1) , (C_2) , (C_3) , (H_1) and (H_2) hold. If W is periodically ru-usc then $\{I_\varepsilon\}_{\varepsilon>0}$ Γ -converges with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ to $I_0 : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ given by*

$$I_0(u) = \int_{\Omega} \widehat{\mathcal{H}W}(\nabla u(x)) dx$$

with

$$\widehat{\mathcal{H}W}(\xi) = \begin{cases} \liminf_{t \rightarrow 1^-} \inf_{k \in \mathbb{N}^*} \left\{ \int_{kY} W(x, t\xi + \nabla \varphi(x)) dx : \varphi \in W_0^{1,p}(kY; \mathbb{R}^m) \right\} & \text{if } \xi \in \overline{\mathbb{G}} \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 1.1 is an extension of the homogenization result in [4], to the case where W has *quasiconvex growth conditions*.

The assumption that W is periodically ru-usc (already in [4]), allows us to consider a suitable extension (in a radial way) of the homogenized integrand to the boundary $\partial\mathbb{G}$ of \mathbb{G} . The reason is that the weak limits of the sequences of gradients can be located at $\partial\mathbb{G}$ during the homogenization process by Γ -convergence. In fact, we will see that the homogenized integrand $\widehat{\mathcal{H}W}$ is nothing but the lower semicontinuous envelope of the Braides–Müller homogenization formula $\mathcal{H}W$ (see formula (2.1) and Remark 2.2).

The assumption that $p > d$ (already in [4]) allows, by using the p -coercivity condition (H_2) and Sobolev compact imbedding, to work with the convergence in $L^\infty(\Omega; \mathbb{R}^m)$ instead of $L^p(\Omega; \mathbb{R}^m)$. Moreover, the functions of $W^{1,p}(\Omega; \mathbb{R}^m)$ are almost everywhere differentiable in Ω since Sobolev imbedding. When it is combined with the fact that G has a local upper bound property (see Lemma 4.1) this allows, in both proofs of the lower and upper bound of the Γ -limit (see Step 2 of Sect. 6 and Step 3 of Sect. 7), to obtain suitable bounds in cut-off techniques.

The main difficulty of proving Theorem 1.1 comes from the proof of the upper bound of the Γ -limit. Indeed, in the setting of convex growth conditions on W we can use mollifier techniques to construct approximations of Sobolev functions by smooth ones. However, we need to develop other techniques when we deal with quasiconvex growth. We will consider a set function which is a pointwise limit of local Dirichlet minimization problems associated to the family $\{I_\varepsilon\}_{\varepsilon>0}$ and localization arguments introduced by [10] which reduce the proof of the upper bound to cut-off techniques, avoiding then any approximation arguments.

Outline of the paper

In Sect. 2 we present definitions and notations needed in this paper. The proof of Theorem 1.1 is based on two propositions which are stated in this section. Proposition 2.1 is concerned with, first the lower bound of the Γ -limit, and second, with the upper bound of the Γ -limit in the restrictive case where the gradients belong to the interior of the effective domain. Next, we need to extend the homogenized integrand to the boundary of the effective domain, this is the purpose of Proposition 2.2.

In Sect. 3 we show how to recover the classical homogenization theorem with p -polynomial growth in the case $p > d$ from Theorem 1.1.

In Sect. 4 we present some preliminary results needed in the proof of the main result. We first give an analogue property of convex functions for nonconvex integrands satisfying (C_1) and (C_2) . Then, we give the definition and some properties of radially uniformly upper semicontinuous integrands. In Sect. 4.3, we recall some basic facts about subadditive invariant set functions which allow easily to characterize the homogenized formula. Section 4.4 is

devoted to the introduction of the pointwise limit of local Dirichlet minimization problems associated to a family of variational functionals.

In Sect. 5 we prove Proposition 2.2.

In Sect. 6 we prove the lower bound for the Γ -limit by the method of localization and cut-off techniques.

In Sect. 7 we prove the upper bound for the Γ -limit for gradients in the interior of the effective domain in three steps. The first step consists in proving that the Γ -limsup is lower than a suitable envelope (similar to a Carathéodory type envelope in measure theory) of a set function given by the pointwise limit of local Dirichlet minimization problems associated to the family $\{I_\varepsilon\}_{\varepsilon>0}$. This envelope turns out to be a nonnegative finite Radon measure by a domination condition coming from the G -growth conditions. Then, the second step is devoted to prove the local equivalence of the envelope with the set function through Radon–Nikodym derivative. In the last step, we use cut-off functions techniques allowing to substitute the Sobolev functions with their affine tangents maps and we conclude by a subadditive argument which gives the homogenized formula.

In Sect. 8 we prove Theorem 1.1 using Propositions 2.1 and 2.2.

In Sect. 9 we give an example, when $d = m = 2$, of W and G satisfying the requirements of the homogenization result. Moreover, we show that the two basic conditions of hyperelasticity can be considered.

2 Preliminaries

If $L : \Omega \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ is a Borel measurable integrand which is 1-periodic with respect to the first variable, i.e., for every $x \in \Omega$, $z \in \mathbb{Z}^d$ and $\xi \in \mathbb{M}^{m \times d}$ we have

$$L(x + z, \xi) = L(x, \xi),$$

then $\mathcal{H}L : \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ defined by

$$\mathcal{H}L(\xi) := \inf_{k \in \mathbb{N}^*} \inf_{kY} \left\{ \int_{kY} L(x, \xi + \nabla \varphi(x)) dx : \varphi \in W_0^{1,p}(kY; \mathbb{R}^m) \right\} \quad (2.1)$$

is usually called the *Braides–Müller homogenization formula* (see [11,23]). When L has p -polynomial growth, the corresponding homogenized functional by Γ -convergence with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ is given by

$$\int_{\Omega} \mathcal{H}L(\nabla u(x)) dx$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ (see Sect. 3).

Remark 2.1 Some comments concerning the effective domain of $\mathcal{H}W$ and G are in order.

- (i) If (H₁) and (C₃) hold then $\text{dom} \mathcal{H}W = \mathbb{G}$. Indeed, by a change of variables and periodicity arguments we see that

$$\inf_{\varphi \in W_0^{1,p}(nY; \mathbb{R}^m)} \int_{nY} G(\xi + \nabla \varphi(x)) dx = G(\xi)$$

for any $\xi \in \mathbb{M}^{m \times d}$ and $n \in \mathbb{N}$. So, using the G -growth conditions (H₁) we obtain

$$\alpha G(\xi) \leq \mathcal{H}W(\xi) \leq \beta(1 + G(\xi))$$

which implies $\text{dom} \mathcal{H}W = \mathbb{G}$.

(ii) Note that the effective domain \mathbb{G} is *convex* when (C_2) holds. Moreover, if both (C_1) and (C_2) hold, then we have the well-known property

$$t\xi \in \text{int}(\mathbb{G})$$

for all $t \in [0, 1[$ and all $\xi \in \overline{\mathbb{G}}$, where $\overline{\mathbb{G}}$ denotes the closure of \mathbb{G} .

The homogenization of the family $\{I_\varepsilon\}_{\varepsilon>0}$ is achieved by Γ -convergence. We refer to the book of G. Dal Maso [19] for a good introduction to the Γ -convergence theory. We give a brief description and specify some notation. We denote by $\mathcal{O}(\Omega)$ the set of all open subsets of Ω . Define $I_-, I_+ : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ by

$$I_-(u; O) := \inf \left\{ \varliminf_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon; O) : u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\};$$

$$I_+(u; O) := \inf \left\{ \varlimsup_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon; O) : u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}.$$

For any $O \in \mathcal{O}(\Omega)$, the functional $I_-(\cdot; O)$ (resp. $I_+(\cdot; O)$) is called the Γ -liminf (resp. the Γ -limsup) with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ of the family $\{I_\varepsilon(\cdot; O)\}_{\varepsilon>0}$. Note that we always have $I_+(\cdot; O) \geq I_-(\cdot; O)$. When $I_+(\cdot; O) = I_-(\cdot; O)$ we say that the family $\{I_\varepsilon(\cdot; O)\}_{\varepsilon>0}$ Γ -converges with the Γ -limit given by the common value and we write

$$I_0(\cdot; O) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon(\cdot; O).$$

When $O = \Omega$ we simply write

$$I_0(\cdot) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon(\cdot).$$

Note that the functionals $I_-(\cdot; O)$ and $I_+(\cdot; O)$ are lower semicontinuous with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$.

We define the *radial extension* of $\mathcal{H}W$ by

$$\widehat{\mathcal{H}W}(\xi) := \lim_{t \rightarrow 1^-} \mathcal{H}W(t\xi)$$

for all $\xi \in \mathbb{M}^{m \times d}$.

Define $\mathcal{G}, \mathcal{H}I, \widehat{\mathcal{H}I} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ by

$$\diamond \mathcal{G}(u; O) := \int_O G(\nabla u(x)) dx;$$

$$\diamond \mathcal{H}I(u; O) := \int_O \mathcal{H}W(\nabla u(x)) dx;$$

$$\diamond \widehat{\mathcal{H}I}(u; O) := \int_O \widehat{\mathcal{H}W}(\nabla u(x)) dx.$$

The proof of Theorem 1.1 is based on the following result.

Proposition 2.1 *Assume that $p > d$. Assume that (C_1) , (C_2) , (H_1) and (H_2) hold. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $O \in \mathcal{O}(\Omega)$.*

(i) *If W is periodically ru-usc then*

$$I_-(u; O) \geq \widehat{\mathcal{H}I}(u; O).$$

(ii) Let $t \in]0, 1[$. If $\mathcal{G}(tu; O) < \infty$ and if there exists $t_* \in]t, 1[$ such that $\mathcal{G}(t_*u; O) < \infty$ then

$$I_+(tu; O) \leq \mathcal{H}I(tu; O).$$

The following proposition gives some properties of $\mathcal{H}W$ and $\widehat{\mathcal{H}W}$ when W is periodically ru-usc.

Proposition 2.2 *Assume that (C_1) , (C_2) , (C_3) and (H_1) hold. If W is periodically ru-usc then $\mathcal{H}W$ and $\widehat{\mathcal{H}W}$ are ru-usc. Moreover, we have*

$$\widehat{\mathcal{H}W}(\xi) = \lim_{t \rightarrow 1^-} \mathcal{H}W(t\xi)$$

for all $\xi \in \mathbb{M}^{m \times d}$.

Remark 2.2 Under the assumptions of Theorem 1.1 it holds that

$$\widehat{\mathcal{H}W} = \overline{\mathcal{H}W}$$

where the bar denotes the lower semicontinuous envelope of $\mathcal{H}W$. Indeed, first it is easy to see that $\widehat{\mathcal{H}W} \geq \overline{\mathcal{H}W}$. On the other hand, $\widehat{\mathcal{H}W}$ is ru-usc by Proposition 2.2. By Remark 4.1 we have $\widehat{\mathcal{H}W} \leq \mathcal{H}W$ on $\text{dom}\mathcal{H}W$ and $\text{dom}\mathcal{H}W = \mathbb{G}$ by Remark 2.1 (i). So it follows that $\widehat{\mathcal{H}W} \leq \overline{\mathcal{H}W}$. But Theorem 1.1 implies that $\widehat{\mathcal{H}W}$ is lower semicontinuous, i.e., $\widehat{\mathcal{H}W} = \overline{\widehat{\mathcal{H}W}}$, and then $\widehat{\mathcal{H}W} \leq \overline{\mathcal{H}W}$.

3 Application to homogenization with p -polynomial growth

We want to show how to recover, from Theorem 1.1, the classical homogenization result with p -polynomial growth on the integrand for $p > d$ (see [11, 23]).

Let $L : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ be a Borel measurable, 1-periodic with respect the first variable, and satisfying p -polynomial growth: there exist $\alpha, \beta > 0$ such that for every $\xi \in \mathbb{M}^{m \times d}$

$$\alpha|\xi|^p \leq L(x, \xi) \leq \beta(1 + |\xi|^p). \quad (3.1)$$

For each $\varepsilon > 0$ we define $J_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ by

$$J_\varepsilon(u) := \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

Theorem 3.1 *Let $p > d$. The family $\{J_\varepsilon\}_{\varepsilon > 0}$ Γ -converges with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ to $J_0 : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ given by*

$$J_0(u) := \int_{\Omega} \mathcal{H}L(\nabla u(x)) dx. \quad (3.2)$$

Proof Since (3.1) we consider the *quasiconvexification* of L , denoted by W , given by the Dacorogna formula

$$W(x, \xi) = QL(x, \xi) := \inf \left\{ \int_Y L(x, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}$$

for all $(x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m \times d}$. We set $G(\cdot) := |\cdot|^p$.

It is easy to see that G satisfies (C_1) , (C_2) , (C_3) , and W satisfies (H_1) and (H_2) . We show now that $W(=QL)$ is periodically ru-usc. Fix any $t \in [0, 1]$, any $x \in \mathbb{R}^d$ and any $\xi \in \mathbb{G}$. As W is quasiconvex and satisfies (3.1), there exists $K > 0$ such that

$$|W(x, \zeta) - W(x, \zeta')| \leq K |\zeta - \zeta'| (1 + |\zeta|^{p-1} + |\zeta'|^{p-1}) \quad (3.3)$$

for all $x \in \mathbb{R}^d$ and all $\zeta, \zeta' \in \mathbb{M}^{d \times d}$. Using (3.3) with $\zeta = t\xi$ and $\zeta' = \xi$ and taking the left inequality in (3.1) into account, we obtain

$$W(x, t\xi) - W(x, \xi) \leq K'(1-t)(1+W(x, \xi)) \quad (3.4)$$

with $K' := 3K \max\{1, \frac{1}{\alpha}\}$. Dividing by $1 + W(x, \xi)$ and passing to the supremum in $x \in \mathbb{R}^d$ and $\xi \in \mathbb{M}^{m \times d}$, we have

$$\Delta_W^1(t) \leq K'(1-t).$$

Passing to the limit $t \rightarrow 1$, we have that W is periodically ru-usc.

Applying Theorem 1.1 to the family $\{I_\varepsilon\}_{\varepsilon>0}$ defined by

$$W^{1,p}(\Omega; \mathbb{R}^m) \ni u \mapsto I_\varepsilon(u) := \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

We obtain that for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$(\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon)(u) = I_0(u) = \int_{\Omega} \widehat{\mathcal{H}}(\overline{QL})(\nabla u(x)) dx. \quad (3.5)$$

By Remark 2.2, we have $\widehat{\mathcal{H}}(\overline{QL}) = \overline{\mathcal{H}(QL)}$. Using Dacorogna–Acerbi–Fusco relaxation result [1, 17] (see “Dacorogna–Acerbi–Fusco” of Appendix section) we have $\mathcal{H}(QL) = \mathcal{H}L$ since the p -growth (3.1). Assume for the moment that $\mathcal{H}(QL)$ is lower semicontinuous, i.e.,

$$\mathcal{H}(QL) = \overline{\mathcal{H}(QL)} \quad (3.6)$$

then we have

$$\mathcal{H}L = \widehat{\mathcal{H}(\overline{QL})}. \quad (3.7)$$

For each $\varepsilon > 0$ we apply the Dacorogna–Acerbi–Fusco relaxation theorem to J_ε and we have

$$\overline{J}_\varepsilon(u) = \int_{\Omega} QL\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx = I_\varepsilon(u)$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, where the bar over J_ε denotes the lower semicontinuous envelope with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$. On the other hand, it is well known that

$$J_0(u) := (\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} J_\varepsilon)(u) = (\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \overline{J}_\varepsilon)(u) = I_0(u) \quad (3.8)$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Collecting (3.5), (3.7) and (3.8), we finally obtain (3.2).

The only thing which remains to prove is (3.6). Let $\{\xi_\varepsilon\}_{\varepsilon>0}$, $\xi_0 \in \mathbb{M}^{m \times d}$ be such that $\xi_\varepsilon \rightarrow \xi_0$ as $\varepsilon \rightarrow 0$. Without loss of generality, we can assume that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{H}(QL)(\xi_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}(QL)(\xi_\varepsilon) < \infty, \quad \text{and } M := \sup_{\varepsilon>0} \mathcal{H}(QL)(\xi_\varepsilon) < \infty.$$

Fix $\varepsilon \in]0, 1[$. We choose $k_\varepsilon \in \mathbb{N}$ and $\varphi_\varepsilon \in W^{1,p}(k_\varepsilon Y; \mathbb{R}^m)$ such that

$$\begin{aligned} M + 1 &\geq \varepsilon + \mathcal{H}(QL)(\xi_\varepsilon) \geq \int_{k_\varepsilon Y} QL(x, \xi_\varepsilon + \nabla\varphi_\varepsilon(x))dx \\ &= \int_Y QL(k_\varepsilon y, \xi_\varepsilon + \nabla\varphi_\varepsilon(k_\varepsilon y))dy, \end{aligned}$$

where a change of variable is used. Set $\phi_\varepsilon(\cdot) := \frac{1}{k_\varepsilon}\varphi(k_\varepsilon\cdot)$. It is easy to see that QL satisfies (3.1) with the same constants. So, it follows that

$$\sup_{\varepsilon > 0} \int_Y |\xi_\varepsilon + \nabla\phi_\varepsilon(x)|^p dx \leq \frac{M+1}{c}. \quad (3.9)$$

By Hölder inequality and (3.9) we have for every $\varepsilon > 0$

$$\|\xi_\varepsilon + \nabla\phi_\varepsilon\|_{L^{p-1}(Y; \mathbb{R}^m)} \leq \|\xi_\varepsilon + \nabla\phi_\varepsilon\|_{L^p(Y; \mathbb{R}^m)} \leq \left(\frac{M+1}{c}\right)^{\frac{1}{p}}, \quad (3.10)$$

and also by norm inequality

$$\begin{aligned} \|\xi_0 + \nabla\phi_\varepsilon\|_{L^{p-1}(Y; \mathbb{R}^m)} &\leq |\xi_0 - \xi_\varepsilon| + \|\xi_\varepsilon + \nabla\phi_\varepsilon\|_{L^{p-1}(Y; \mathbb{R}^m)} \\ &\leq |\xi_0 - \xi_\varepsilon| + \left(\frac{M+1}{c}\right)^{\frac{1}{p}}. \end{aligned} \quad (3.11)$$

By the definition of $\mathcal{H}(QL)(\xi_0)$, (3.3), (3.10) and (3.11), we have for every $\varepsilon > 0$

$$\begin{aligned} &\varepsilon + \mathcal{H}(QL)(\xi_\varepsilon) \\ &\geq \int_Y QL(k_\varepsilon y, \xi_\varepsilon + \nabla\phi_\varepsilon(y)) - QL(k_\varepsilon y, \xi_0 + \nabla\phi_\varepsilon(y))dy + \mathcal{H}(QL)(\xi_0) \\ &\geq -K|\xi_\varepsilon - \xi_0| \left(1 + \int_Y |\xi_\varepsilon + \nabla\phi_\varepsilon(x)|^{p-1} + |\xi_0 + \nabla\phi_\varepsilon(x)|^{p-1} dx\right) + \mathcal{H}(QL)(\xi_0) \\ &\geq -K|\xi_\varepsilon - \xi_0| \left(1 + \left(\frac{M+1}{c}\right)^{\frac{p-1}{p}} + \left(|\xi_0 - \xi_\varepsilon| + \left(\frac{M+1}{c}\right)^{\frac{1}{p}}\right)^{p-1}\right) + \mathcal{H}(QL)(\xi_0), \end{aligned}$$

letting $\varepsilon \rightarrow 0$ we obtain the lower semicontinuity of $\mathcal{H}(QL)$. \square

4 Auxiliary results

4.1 Consequence of assumptions (C₁) and (C₂)

The following lemma is an extension, for nonconvex functions satisfying (C₁) and (C₂), of the classical local upper bound property for convex functions.

Lemma 4.1 *Let $L : \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ be a Borel measurable integrand. If L satisfies (C₁) and (C₂) then there exists $\rho_0 > 0$ such that*

$$r_0 := \sup_{\zeta \in \overline{B}_{\rho_0}(0)} L(\zeta) < \infty.$$

Proof From (C₁) there exists $\rho_0 > 0$ such that $L(\xi) < \infty$ for all $\xi \in \overline{B}_{\rho_0}(0)$. Each matrix $\xi \in \overline{B}_{\rho_0}(0)$ is identified to the vector

$$\xi = (\xi_{11}, \dots, \xi_{1d}, \dots, \xi_{i1}, \dots, \xi_{id}, \dots, \xi_{m1}, \dots, \xi_{md}).$$

Consider the finite subset

$$\mathcal{S} := \left\{ (\xi_{11}, \dots, \xi_{md}) \in \mathbb{M}^{m \times d} : \xi_{ij} \in \{-\rho_0, 0, \rho_0\} \right\} \subset \overline{B}_{\rho_0}(0)$$

and we define $L^* := \max_{\xi \in \mathcal{S}} L(\xi) < \infty$.

Let $\zeta = (\zeta_{11}, \dots, \zeta_{1d}, \dots, \zeta_{i1}, \dots, \zeta_{id}, \dots, \zeta_{m1}, \dots, \zeta_{md}) \in \mathcal{S}$ and $\xi \in \overline{B}_{\rho_0}(0)$ with $\xi_{ij} = \zeta_{ij}$ for all $i \neq 1$ and $j \neq 1$. If $\xi_{11} \neq 0$ then by (C₂) we have

$$\begin{aligned} L(\xi) &= L \left(\frac{|\xi_{11}|}{\rho_0} \operatorname{sgn}(\xi_{11}) \rho_0 + \left(1 - \frac{|\xi_{11}|}{\rho_0} \right) 0, \dots, \xi_{1d}, \dots, \xi_{m1}, \dots, \xi_{md} \right) \quad (4.1) \\ &\leq C (1 + L(\rho_0, \dots, \xi_{md}) + L(0, \dots, \xi_{md})) \\ &\leq 2C (1 + L^*) \end{aligned}$$

where $\operatorname{sgn}(\xi_{ij})$ denotes the sign of ξ_{ij} . The same upper bound in (4.1) holds for $L(\xi)$ when $\xi_{11} = 0$.

Assume now that $\xi_{ij} = \zeta_{ij}$ for all $i \neq 1$ and $j \notin \{1, 2\}$. Then using (4.1) and (C₂), we have

$$\begin{aligned} L(\xi) &= L \left(\xi_{11}, \frac{|\xi_{12}|}{\rho_0} \operatorname{sgn}(\xi_{12}) \rho_0 + \left(1 - \frac{|\xi_{12}|}{\rho_0} \right) 0, \dots, \xi_{1d}, \dots, \xi_{m1}, \dots, \xi_{md} \right) \\ &\leq C (1 + 2C(1 + L^*) + L^*) \\ &\leq C(1 + 2C) (1 + L^*). \end{aligned}$$

Recursively, we obtain $C^* > 0$ which depends on C only, such that

$$L(\xi) \leq C^*(1 + L^*)$$

for all $\xi \in \overline{B}_{\rho_0}(0)$. □

4.2 Ru-usc functions

Let $U \subset \mathbb{R}^d$ be an open set and let $L : U \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ be a Borel measurable function. For each $x \in U$, we denote the effective domain of $L(x, \cdot)$ by \mathbb{L}_x and, for each $a \in L^1_{\text{loc}}(U;]0, \infty[)$, we define $\Delta_L^a : [0, 1] \rightarrow]-\infty, \infty]$ by

$$\Delta_L^a(t) := \sup_{x \in U} \sup_{\xi \in \mathbb{L}_x} \frac{L(x, t\xi) - L(x, \xi)}{a(x) + L(x, \xi)}.$$

Definition 4.1 We say that L is *radially uniformly upper semicontinuous (ru-usc)* if there exists $a \in L^1_{\text{loc}}(U;]0, \infty[)$ such that

$$\overline{\lim}_{t \rightarrow 1} \Delta_L^a(t) \leq 0.$$

If moreover a is 1-periodic then we say that L is *periodically ru-usc*.

For a detailed study of ru-usc functions see [7].

Remark 4.1 If L is ru-usc then

$$\overline{\lim}_{t \rightarrow 1^-} L(x, t\xi) \leq L(x, \xi) \quad (4.2)$$

for all $x \in U$ and all $\xi \in \mathbb{L}_x$. Indeed, given $x \in U$ and $\xi \in \mathbb{L}_x$, we have

$$L(x, t\xi) \leq \Delta_L^a(t) (a(x) + L(x, \xi)) + L(x, \xi) \quad \text{for all } t \in [0, 1],$$

which gives (4.2) since $a(x) + L(x, \xi) > 0$ and $\overline{\lim}_{t \rightarrow 1} \Delta_L^a(t) \leq 0$.

Define $\widehat{L} : U \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ by

$$\widehat{L}(x, \xi) := \underline{\lim}_{t \rightarrow 1^-} L(x, t\xi).$$

The following lemma gives some properties of \widehat{L} when L is ru-usc (for the proof see also [4, Lemma 3.4 and Theorem 3.5 (ii)]).

Lemma 4.2 *If L is ru-usc and if for every $x \in U$,*

$$t\overline{\mathbb{L}}_x \subset \mathbb{L}_x \quad \text{for all } t \in]0, 1[\quad (4.3)$$

then

(i) *for every $\xi \in \overline{\mathbb{L}}_x$ it holds $\widehat{L}(x, \xi) = \lim_{t \rightarrow 1^-} L(x, t\xi)$ for all $x \in U$;*

(ii) *\widehat{L} is ru-usc.*

Proof First we prove (i). Fix $x \in U$. We have to prove that for every $\xi \in \overline{\mathbb{L}}_x$

$$\underline{\lim}_{t \rightarrow 1^-} L(x, t\xi) = \overline{\lim}_{t \rightarrow 1^-} L(x, t\xi).$$

Fix $\xi \in \overline{\mathbb{L}}_x$. It suffices to prove that

$$\overline{\lim}_{t \rightarrow 1} \Psi(t) \leq \underline{\lim}_{t \rightarrow 1} \Psi(t). \quad (4.4)$$

where $\Psi(t) := L(x, t\xi)$ for all $t \in [0, 1]$. Without loss of generality we can assume that $\underline{\lim}_{t \rightarrow 1} \Psi(t) < \infty$. Choose two sequences $\{t_n\}_n, \{s_n\}_n \subset]0, 1[$ such that $t_n \rightarrow 1, s_n \rightarrow 1, \frac{t_n}{s_n} < 1$ for all $n \in \mathbb{N}$, and

$$\begin{aligned} \overline{\lim}_{t \rightarrow 1} \Psi(t) &= \lim_{n \rightarrow \infty} \Psi(t_n); \\ \underline{\lim}_{t \rightarrow 1} \Psi(t) &= \lim_{n \rightarrow \infty} \Psi(s_n). \end{aligned}$$

It is possible because, once the sequences $\{t_n\}_n, \{s_n\}_n \subset]0, 1[$ satisfying $t_n \rightarrow 1, s_n \rightarrow 1$ choosen, we can extract a subsequence $\{s_{\sigma(n)}\}_n$ such that $\frac{t_n}{s_{\sigma(n)}} < 1$ for all $n \in \mathbb{N}$. Indeed, it suffices to consider the increasing map $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ defined by $\sigma(0) := \min\{v \in \mathbb{N} : s_v > t_0\}$ and $\sigma(n+1) := \min\{v \in \mathbb{N} : v > \sigma(n) \text{ and } s_v > t_{n+1}\}$ for all $n \in \mathbb{N}$.

Since (4.3) we have $t_n\xi \in \mathbb{L}_x$ for all $n \in \mathbb{N}$, so we can assert that for every $n \in \mathbb{N}$

$$\begin{aligned} \Psi(t_n) &= \Psi\left(\frac{t_n}{s_n} s_n \xi\right) = L\left(x, \frac{t_n}{s_n} (s_n \xi)\right) \\ &\leq \Delta_L^a\left(\frac{t_n}{s_n}\right) (a(x) + \Psi(s_n)) + \Psi(s_n). \end{aligned} \quad (4.5)$$

Letting $n \rightarrow \infty$ we deduce (4.4) from (4.5) since L is ru-usc.

It remains to prove (ii), i.e., \widehat{L} is ru-usc. Fix $t \in [0, 1[$ and $\xi \in \overline{\mathbb{L}}_x$. By (i) we can assert that

$$\begin{aligned}\widehat{L}(x, \xi) &= \lim_{s \rightarrow 1} L(x, s\xi) \\ \widehat{L}(x, t\xi) &= \lim_{s \rightarrow 1} L(x, s(t\xi)).\end{aligned}$$

So, we have

$$\frac{\widehat{L}(x, t\xi) - \widehat{L}(x, \xi)}{a(x) + \widehat{L}(x, \xi)} = \lim_{s \rightarrow 1} \frac{L(x, t(s\xi)) - L(x, s\xi)}{a(x) + L(x, s\xi)} \leq \Delta_L^a(t).$$

It follows that $\Delta_L^a(t) \leq \Delta_L^a(t)$. Letting $t \rightarrow 1$, we finish the proof. \square

Assume that $U = \mathbb{R}^d$ and consider $\mathcal{H}L$ given by (2.1). The following result shows that the ru-usc property is stable by homogenization.

Proposition 4.1 *If L is periodically ru-usc then $\mathcal{H}L$ is ru-usc.*

Proof Fix any $t \in [0, 1]$ and any $\xi \in \mathcal{H}\mathbb{L}$, where $\mathcal{H}\mathbb{L}$ denotes the effective domain of $\mathcal{H}L$. By definition, there exists $\{k_n; \phi_n\}_n$ such that

- $\diamond \phi_n \in W_0^{1,p}(k_n Y; \mathbb{R}^m)$ for all $n \geq 1$;
- $\diamond \mathcal{H}L(\xi) = \lim_{n \rightarrow \infty} \int_{k_n Y} L(x, \xi + \nabla \phi_n(x)) dx$;
- $\diamond \xi + \nabla \phi_n(x) \in \mathbb{L}_x$ for all $n \geq 1$ and a.a. $x \in k_n Y$.

Moreover, for every $n \geq 1$,

$$\mathcal{H}L(t\xi) \leq \int_{k_n Y} L(x, t(\xi + \nabla \phi_n(x))) dx$$

since $t\phi_n \in W_0^{1,p}(k_n Y; \mathbb{R}^m)$, and so

$$\mathcal{H}L(t\xi) - \mathcal{H}L(\xi) \leq \lim_{n \rightarrow \infty} \int_{k_n Y} (L(x, t(\xi + \nabla \phi_n(x))) - L(x, \xi + \nabla \phi_n(x))) dx.$$

As L is periodically ru-usc it follows that

$$\mathcal{H}L(t\xi) - \mathcal{H}L(\xi) \leq \Delta_L^a(t) (\langle a \rangle + \mathcal{H}L(\xi))$$

with $\langle a \rangle := \int_Y a(y) dy$, which implies that $\Delta_{\mathcal{H}L}^{\langle a \rangle}(t) \leq \Delta_L^a(t)$ for all $t \in [0, 1]$, and the proof is complete. \square

4.3 Subadditive theorem

Let $\mathcal{O}_b(\mathbb{R}^d)$ be the class of all bounded open subsets of \mathbb{R}^d .

Definition 4.2 Let $\mathcal{S} : \mathcal{O}_b(\mathbb{R}^d) \rightarrow [0, \infty]$ be a set function.

(i) We say that \mathcal{S} is subadditive if

$$\mathcal{S}(A) \leq \mathcal{S}(B) + \mathcal{S}(C)$$

for all $A, B, C \in \mathcal{O}_b(\mathbb{R}^d)$ with $B, C \subset A$, $B \cap C = \emptyset$ and $|A \setminus B \cup C| = 0$.

(ii) We say that S is \mathbb{Z}^d -invariant if

$$S(A + z) = S(A)$$

for all $A \in \mathcal{O}_b(\mathbb{R}^d)$ and all $z \in \mathbb{Z}^d$.

Let $\text{Cub}(\mathbb{R}^d)$ be the class of all open cubes in \mathbb{R}^d and let $Y :=]0, 1[^d$. The following theorem is due to Akcoglu and Krengel (see [8], see also [22], and [4, Theorem 3.11]).

Theorem 4.1 *Let $S : \mathcal{O}_b(\mathbb{R}^d) \rightarrow [0, \infty]$ be a subadditive and \mathbb{Z}^d -invariant set function for which there exists $c > 0$ such that*

$$S(A) \leq c|A| \quad (4.6)$$

for all $A \in \mathcal{O}_b(\mathbb{R}^d)$. Then, for every $Q \in \text{Cub}(\mathbb{R}^d)$,

$$\lim_{\varepsilon \rightarrow 0} \frac{S\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \inf_{k \geq 1} \frac{S(kY)}{k^d}.$$

Given a Borel measurable function $W : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$, we define for each $\xi \in \mathbb{M}^{m \times d}$, $S_\xi : \mathcal{O}_b(\mathbb{R}^d) \rightarrow [0, \infty]$ by

$$S_\xi(A) := \inf \left\{ \int_A W(x, \xi + \nabla \phi(x)) dx : \phi \in W_0^{1,p}(A; \mathbb{R}^m) \right\}. \quad (4.7)$$

It is easy to see that the set function S_ξ is subadditive. If we assume that W is 1-periodic with respect to the first variable, then S_ξ is \mathbb{Z}^d -invariant. Moreover, if W is such that there exist a Borel measurable function $G : \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ and $\beta > 0$ satisfying

$$W(x, \xi) \leq \beta(1 + G(\xi)) \quad (4.8)$$

for all $\xi \in \mathbb{M}^{m \times d}$, then

$$S_\xi(A) \leq \beta(1 + G(\xi))|A|$$

for all $A \in \mathcal{O}_b(\mathbb{R}^d)$.

From the above, we see that the following result is a direct consequence of Theorem 4.1.

Corollary 4.1 *Assume that W is 1-periodic with respect to the first variable and satisfies (4.8). Then, for every $\xi \in \mathbb{G}$*

$$\lim_{\varepsilon \rightarrow 0} \frac{S_\xi\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \inf_{k \geq 1} \frac{S_\xi(kY)}{k^d}.$$

4.4 Local Dirichlet problems associated to a family of functionals

For any family of (variational) functionals $\{H_\delta\}_{\delta > 0}$, $H_\delta : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ we set

$$\begin{aligned} \diamond \quad m_\delta(u; O) &:= \inf \left\{ H_\delta(v; O) : v \in u + W_0^{1,p}(O; \mathbb{R}^m) \right\}; \\ \diamond \quad \mathcal{M}(u; O) &:= \overline{\lim}_{\delta \rightarrow 0} m_\delta(u; O), \end{aligned}$$

where $v \in u + W_0^{1,p}(O; \mathbb{R}^m)$ means that $v \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $v - u = 0$ in $\Omega \setminus O$ (this definition is equivalent to the classical definition of $u + W_0^{1,p}(O; \mathbb{R}^m)$, see for

instance [AH96, Chap. 9, p. 233]). It is easy to see that we may also write $m_\delta(u; O) = \inf \left\{ H_\delta(u + \varphi; O) : \varphi \in W_0^{1,p}(O; \mathbb{R}^m) \right\}$ for all $\delta > 0$ and all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$.

For each $\varepsilon > 0$ and each $O \in \mathcal{O}(\Omega)$, denote by $\mathcal{V}_\varepsilon(O)$ the class of all countable family $\{\overline{Q}_i := \overline{Q}_{\rho_i}(x_i)\}_{i \in I}$ of disjoint (pairwise disjoint) closed balls of O with $x_i \in O$ and $\rho_i \in]0, \varepsilon[$ such that $|O \setminus \cup_{i \in I} Q_i| = 0$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $\varepsilon > 0$. Consider $\mathcal{M}^\varepsilon(u; \cdot) : \mathcal{O}(\Omega) \rightarrow [0, \infty]$ defined by

$$\mathcal{M}^\varepsilon(u; O) := \inf \left\{ \sum_{i \in I} \mathcal{M}(u; Q_i) : \{\overline{Q}_i\}_{i \in I} \in \mathcal{V}_\varepsilon(O) \right\},$$

and define $\mathcal{M}^*(u; \cdot) : \mathcal{O}(\Omega) \rightarrow [0, \infty]$ by

$$\mathcal{M}^*(u; O) := \sup_{\varepsilon > 0} \mathcal{M}^\varepsilon(u; O) = \lim_{\varepsilon \rightarrow 0} \mathcal{M}^\varepsilon(u; O).$$

Lemma 4.3 *Assume that for each $\delta > 0$ and each $v \in W^{1,p}(\Omega; \mathbb{R}^m)$ the set function $H_\delta(v; \cdot)$ is countably subadditive. Let $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$ satisfying*

$$\sum_{i \in I} \sup_{\delta > 0} H_\delta(u; Q_i) < \infty \quad (4.9)$$

for all countable disjointed closed balls $\{\overline{Q}_i\}_{i \in I}$ of O satisfying $|O \setminus \cup_{i \in I} Q_i| = 0$. Then we have

$$\mathcal{M}(u; O) \leq \mathcal{M}^*(u; O). \quad (4.10)$$

Proof Fix $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$ satisfying (4.9). Fix $\varepsilon > 0$. Choose $\{\overline{Q}_i\}_{i \geq 1} \in \mathcal{V}_\varepsilon(O)$ such that

$$\sum_{i \geq 1} \mathcal{M}(u; Q_i) \leq \mathcal{M}^\varepsilon(u; O) + \frac{\varepsilon}{2} \leq \mathcal{M}^*(u; O) + \frac{\varepsilon}{2}. \quad (4.11)$$

Fix $\delta > 0$. For each $i \geq 1$ there exists $\varphi_i \in W_0^{1,p}(Q_i; \mathbb{R}^m)$ such that

$$H_\delta(u + \varphi_i; Q_i) \leq \frac{\delta}{2i+1} + m_\delta(u; Q_i). \quad (4.12)$$

Set $\varphi_{\delta,\varepsilon} := \sum_{i \geq 1} \varphi_i \mathbb{I}_{Q_i} \in W_0^{1,p}(O; \mathbb{R}^m)$. By the countable subadditivity of $H_\delta(u + \varphi_{\delta,\varepsilon}; \cdot)$ and (4.12) we have

$$\mathcal{M}(u; O) \leq \overline{\lim}_{\delta \rightarrow 0} H_\delta(u + \varphi_{\delta,\varepsilon}; O) \leq \overline{\lim}_{\delta \rightarrow 0} \sum_{i \geq 1} H_\delta(u + \varphi_i; Q_i) \leq \overline{\lim}_{\delta \rightarrow 0} \sum_{i \geq 1} m_\delta(u; Q_i). \quad (4.13)$$

But, for every $\delta > 0$ and every $i \geq 1$ it holds

$$\sup_{\eta \in]0, \delta[} m_\eta(u; Q_i) \leq \sup_{\eta > 0} H_\eta(u; Q_i). \quad (4.14)$$

Applying the dominated convergence theorem and using (4.9) together with (4.14), we have

$$\overline{\lim}_{\delta \rightarrow 0} \sum_{i \geq 1} m_\delta(u; Q_i) \leq \sum_{i \geq 1} \overline{\lim}_{\delta \rightarrow 0} m_\delta(u; Q_i) = \sum_{i \geq 1} \mathcal{M}(u; Q_i). \quad (4.15)$$

Collecting (4.11), (4.15) and (4.13) and letting $\varepsilon \rightarrow 0$ we obtain (4.10). \square

Remark 4.2 Let $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$ and let $\mathcal{G} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ be such that $\mathcal{G}(u, O) < \infty$ and $\mathcal{G}(v, \cdot)$ is a measure for all $v \in W^{1,p}(\Omega; \mathbb{R}^m)$. If there exists $\beta > 0$ such that for every $\delta > 0$ it holds

$$H_\delta(u; U) \leq \beta (|U| + \mathcal{G}(u; U))$$

for all $U \in \mathcal{O}(O)$, then (4.9) is satisfied. Indeed, we have

$$\sum_{i \in I} \sup_{\delta > 0} H_\delta(u; Q_i) \leq \sum_{i \in I} \beta (|Q_i| + \mathcal{G}(u; Q_i)) = \beta (|O| + \mathcal{G}(u; O)) < \infty.$$

The following result is needed for the proof of Lemma 7.1.

Lemma 4.4 ([10] and [9, Prop. 2.1.]) *Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. If there exists a finite Radon measure μ_u on Ω such that for every cube $Q \in \mathcal{O}(\Omega)$*

$$\mathcal{M}(u; Q) \leq \mu_u(Q)$$

then $\mathcal{M}^(u; \cdot)$ can be extended to a Radon measure λ_u on Ω satisfying $0 \leq \lambda_u \leq \mu_u$.*

5 Proof of Proposition 2.2

The function $\mathcal{H}W$ is ru-usc since Proposition 4.1, and so $\widehat{\mathcal{H}W}$ is ru-usc by Lemma 4.2 (ii). Since Remark 2.1 (i) we have $\text{dom} \mathcal{H}W = \mathbb{G}$. It is easy to deduce that $\text{dom} \widehat{\mathcal{H}W} = \overline{\mathbb{G}}$. From Lemma 4.2 (i) it holds that

$$\widehat{\mathcal{H}W}(\xi) = \lim_{t \rightarrow 1^-} \mathcal{H}W(t\xi) \text{ for all } \xi \in \overline{\mathbb{G}}.$$

The proof is complete. □

6 Proof of Proposition 2.1 (i)

Let $O \in \mathcal{O}(\Omega)$ and let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ be such that $I_-(u; O) < \infty$. It follows that $\nabla u(\cdot) \in \overline{\mathbb{G}}$ a.e. in O since \mathbb{G} is convex and the coercivity condition (H₂).

We have to prove that

$$I_-(u; O) \geq \int_O \widehat{\mathcal{H}W}(\nabla u(x)) dx. \quad (6.1)$$

Consider $\{u_\varepsilon\}_{\varepsilon > 0} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ satisfying $\|u_\varepsilon - u\|_{L^p(\Omega; \mathbb{R}^m)} \rightarrow 0$. Without loss of generality we can assume that

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon; O) = \lim_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon; O) < \infty, \text{ and so } \sup_{\varepsilon > 0} I_\varepsilon(u_\varepsilon; O) < \infty. \quad (6.2)$$

Then for every $t \in [0, 1[$ it holds

$$t \nabla u_\varepsilon(x) \in \mathbb{G} \text{ for all } \varepsilon > 0 \text{ and for a.a. } x \in O \quad (6.3)$$

and, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^m) \quad (6.4)$$

since W is p -coercive (H₂) and (6.2).

As $p > d$, by the Sobolev compact imbedding and (6.4), we have, for a subsequence, that

$$\|u_\varepsilon - u\|_{L^\infty(\Omega; \mathbb{R}^m)} \rightarrow 0. \quad (6.5)$$

Step 1: Localization

For each $\varepsilon > 0$, we define the nonnegative Radon measure μ_ε on O by

$$\mu_\varepsilon := W\left(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon(\cdot)\right) dx \llcorner O.$$

From (6.2) we see that $\sup_\varepsilon \mu_\varepsilon(O) < \infty$, and so there exists a Radon measure μ on O such that (up to a subsequence) $\mu_\varepsilon \xrightarrow{*} \mu$. By Lebesgue's decomposition theorem, we have $\mu = \mu_a + \mu_s$ where μ_a and μ_s are nonnegative Radon measures such that $\mu_a \ll dx \llcorner O$ and $\mu_s \perp dx \llcorner O$, and from Radon-Nikodym's theorem we deduce that there exists $f \in L^1(O; [0, \infty[)$, given by

$$f(x) = \lim_{\rho \rightarrow 0} \frac{\mu_a(Q_\rho(x))}{\rho^d} = \lim_{\rho \rightarrow 0} \frac{\mu(Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O \quad (6.6)$$

with $Q_\rho(x) := x + \rho Y$, such that

$$\mu_a(A) = \int_A f(x) dx \quad \text{for all measurable sets } A \subset O.$$

Remark 6.1 The support of μ_s , $\text{spt}(\mu_s)$, is the smallest closed subset F of O such that $\mu_s(O \setminus F) = 0$. Hence, $O \setminus \text{spt}(\mu_s)$ is an open set, and so, given any $x \in O \setminus \text{spt}(\mu_s)$, there exists $\hat{\rho} > 0$ such that $\overline{Q_{\hat{\rho}}(x)} \subset O \setminus \text{spt}(\mu_s)$ with $\overline{Q_{\hat{\rho}}(x)} := x + \hat{\rho} \overline{Y}$. Thus, for a.a. $x \in \Omega$, $\mu(Q_\rho(x)) = \mu_a(Q_\rho(x))$ for all $\rho > 0$ sufficiently small.

To prove (6.1) it suffices to show that

$$f(x) \geq \widehat{\mathcal{H}W}(\nabla u(x)) \quad \text{a.e. in } O. \quad (6.7)$$

Indeed, by Alexandrov theorem (see ‘‘Alexandrov theorem’’ of Appendix section) we see that

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon; O) = \lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(O) \geq \mu(O) = \mu_a(O) + \mu_s(O) \geq \mu_a(O) = \int_O f(x) dx.$$

But, by (6.7), we have

$$\int_O f(x) dx \geq \int_O \widehat{\mathcal{H}W}(\nabla u(x)) dx,$$

and (6.1) follows.

Fix $t \in]0, 1[$. Let $t_* \in]t, 1[$ be such that

$$\Delta_W^a(t_*) < \infty. \quad (6.8)$$

Fix $x_0 \in O \setminus N$, where $N \subset O$ is a suitable set such that $|N| = 0$, and such that

$$\nabla u(x_0) \in \overline{\mathbb{G}}; \quad (6.9)$$

$$f(x_0) < \infty; \quad (6.10)$$

$$G(t_* \nabla u(x_0)) < \infty; \quad (6.11)$$

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0); \mathbb{R}^m)} = 0. \quad (6.12)$$

Note that $G(t_* \nabla u(\cdot)) < \infty$ a.e. in O since Remark 2.1 (ii) and $\nabla u(\cdot) \in \overline{\mathbb{G}}$ a.e. in O . Note also that u is almost everywhere differentiable, i.e., $\lim_{\rho \rightarrow 0} \frac{1}{\rho} \|u - u_x\|_{L^\infty(Q_\rho(x); \mathbb{R}^m)} = 0$ a.e. in O since $p > d$ (where $u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x)$ is the affine tangent map of u at $x \in O$).

We have to prove that $f(x_0) \geq \widehat{\mathcal{H}W}(\nabla u(x_0))$.

As $\mu(O) < \infty$ we have $\mu(\partial Q_\rho(x_0)) = 0$ for all $\rho \in]0, 1] \setminus D$ where D is a countable set. From (6.6) and Alexandrov theorem (see ‘‘Alexandrov theorem’’ of Appendix section) we deduce that

$$f(x_0) = \lim_{\rho \rightarrow 0} \frac{\mu(Q_\rho(x_0))}{\rho^d} = \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d},$$

and so we are reduced to show that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx \geq \widehat{\mathcal{H}W}(\nabla u(x_0)). \quad (6.13)$$

Using ru-usc property of W we can see that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx \geq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon(x)\right) dx.$$

So to prove (6.13), it is enough to show that

$$\overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon(x)\right) dx \geq \widehat{\mathcal{H}W}(\nabla u(x_0)). \quad (6.14)$$

Step 2: Cut-off technique to substitute tu_ε with $tv_\varepsilon \in tu_{x_0} + W_0^{1,p}(Q_\rho(x_0); \mathbb{R}^m)$

Fix any $\varepsilon > 0$ and any $s \in]0, 1[$. Let $\phi \in W_0^{1,\infty}(Q_\rho(x_0); [0, 1])$ be a cut-off function between $Q_{s\rho}(x_0)$ and $Q_\rho(x_0)$ such that $\|\nabla \phi\|_{L^\infty(Q_\rho(x_0))} \leq \frac{4}{\rho(1-s)}$. Setting

$$v_\varepsilon := \phi u_\varepsilon + (1 - \phi)u_{x_0}$$

where $u_{x_0}(\cdot) := u(x_0) + \nabla u(x_0)(\cdot - x_0)$, we have

$$tv_\varepsilon \in tu_{x_0} + W_0^{1,p}(Q_\rho(x_0); \mathbb{R}^m), \quad (6.15)$$

and if $\tau := \frac{t}{t_*} \in]0, 1[$ then

$$t \nabla v_\varepsilon := \begin{cases} t \nabla u_\varepsilon & \text{on } Q_{s\rho}(x_0) \\ \tau (\phi t_* \nabla u_\varepsilon + (1 - \phi) t_* \nabla u(x_0)) + (1 - \tau) \Phi_{\varepsilon,\rho} & \text{on } S_\rho \end{cases} \quad (6.16)$$

with $S_\rho := Q_\rho(x_0) \setminus \overline{Q_{s\rho}(x_0)}$ and $\Phi_{\varepsilon,\rho} := \frac{t}{1-\tau} \nabla \phi \otimes (u_\varepsilon - u_{x_0})$.

Using the G -growth conditions (H₁) we have

$$\begin{aligned} & \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla v_\varepsilon\right) dx \\ & \leq \int_{Q_{s\rho}(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon\right) dx + \frac{1}{\rho^d} \int_{S_\rho} W\left(\frac{x}{\varepsilon}, t \nabla v_\varepsilon\right) dx \\ & \leq \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon\right) dx + \beta(1 - s^d) + \frac{\beta}{\rho^d} \int_{S_\rho} G(t \nabla v_\varepsilon) dx. \end{aligned}$$

On the other hand, taking (6.16) into account and using (C₂), we have

$$\begin{aligned} G(t \nabla v_\varepsilon) & \leq 2C_1 \left(1 + G(t_* \nabla u_\varepsilon) + G(t_* \nabla u(x_0)) + G(\Phi_{\varepsilon,\rho})\right) \\ & \leq 2C_1 \left(1 + \frac{1}{\alpha} W\left(\frac{x}{\varepsilon}, t_* \nabla u_\varepsilon\right) + G(t_* \nabla u(x_0)) + G(\Phi_{\varepsilon,\rho})\right) \quad \text{a.e. in } S_\rho \end{aligned}$$

with $C_1 := C^2 + C$. Moreover, it is easy to see that

$$\begin{aligned} & \|\Phi_{\varepsilon, \rho}\|_{L^\infty(Q_\rho(x_0); \mathbb{M}^{m \times d})} \\ & \leq \frac{4t}{(1-\tau)(1-s)} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0); \mathbb{R}^m)} + \frac{4t}{\rho(1-\tau)(1-s)} \|u_\varepsilon - u\|_{L^\infty(\Omega; \mathbb{R}^m)} \end{aligned}$$

where

$$\lim_{\rho \rightarrow 0} \frac{4t}{(1-t)(1-s)} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0); \mathbb{R}^m)} = 0 \quad (6.17)$$

since (6.12), i.e., $\lim_{\rho \rightarrow 0} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0); \mathbb{R}^m)} = 0$, and

$$\lim_{\varepsilon \rightarrow 0} \frac{4t}{\rho(1-t)(1-s)} \|u_\varepsilon - u\|_{L^\infty(\Omega; \mathbb{R}^m)} = 0 \quad \text{for all } \rho > 0 \quad (6.18)$$

since (6.5), i.e., $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{L^\infty(\Omega; \mathbb{R}^m)} = 0$. By Lemma 4.1 we have for some $\rho_0 > 0$

$$r_0 := \sup_{\xi \in \bar{B}_{\rho_0}(0)} G(\xi) < \infty.$$

By (6.17) there exists $\bar{\rho} > 0$ such that $\frac{4t}{(1-t)(1-s)} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(Q_{\bar{\rho}}(x_0); \mathbb{R}^m)} < \frac{\rho_0}{2}$ for all $\rho \in]0, \bar{\rho}[$.

Fix any $\rho \in]0, \bar{\rho}[$. Taking (6.18) into account we can assert that there exists $\varepsilon_\rho > 0$ such that for every $\varepsilon \in]0, \varepsilon_\rho[$

$$G(\Phi_{\varepsilon, \rho}) \leq r_0 \quad \text{a.e. in } Q_\rho(x_0).$$

Thus, for every $\varepsilon \in]0, \varepsilon_\rho[$, we have

$$\begin{aligned} & \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla v_\varepsilon\right) dx \\ & \leq \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon\right) dx + (1-s^d) \left(\beta + \bar{C}(1+r_0+G(t_* \nabla u(x_0)))\right) \\ & \quad + \frac{\bar{C}}{\alpha} \frac{1}{\rho^d} \int_{S_\rho} W\left(\frac{x}{\varepsilon}, t_* \nabla u_\varepsilon\right) dx \end{aligned} \quad (6.19)$$

where $2\beta C_1 := \bar{C}$. Since W is periodically ru-usc, for every $\varepsilon \in]0, \varepsilon_\rho[$ we have the estimate for the last term of (6.19) shown as follows

$$\frac{1}{\rho^d} \int_{S_\rho} W\left(\frac{x}{\varepsilon}, t_* \nabla u_\varepsilon\right) dx \leq \Delta_W^a(t_*) \frac{1}{\rho^d} \int_{S_\rho} a\left(\frac{x}{\varepsilon}\right) dx + \left(1 + \Delta_W^a(t_*)\right) \frac{1}{\rho^d} \mu_\varepsilon(S_\rho). \quad (6.20)$$

Step 3: End of the proof

Taking (6.15) into account we see that for every $\varepsilon \in]0, \varepsilon_\rho[$

$$\int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla v_\varepsilon\right) dx \geq \frac{1}{|\frac{1}{\varepsilon} Q_\rho(x_0)|} \mathcal{S}_{t \nabla u(x_0)} \left(\frac{1}{\varepsilon} Q_\rho(x_0)\right),$$

where $\mathcal{S}_\xi(A)$ is given by (4.7) for all $\xi \in \mathbb{M}^{m \times d}$ and all open set $A \subset \mathbb{R}^d$. By (6.9) we have $\nabla u(x_0) \in \bar{\mathbb{G}}$, and so $t \nabla u(x_0) \in \mathbb{G}$ because \mathbb{G} is convex and $0 \in \text{int}(\mathbb{G})$ since (C₁) and (C₂).

From Corollary 4.1 we deduce that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla v_\varepsilon\right) dx \geq \mathcal{H}W(t \nabla u(x_0)) \quad (6.21)$$

for all $\rho \in]0, \bar{\rho}[$. On the other hand, as $\mu_\varepsilon(S_\rho) \leq \mu_\varepsilon(\bar{S}_\rho)$ for all $\varepsilon \in]0, \varepsilon_\rho[$, \bar{S}_ρ is compact and $\mu_\varepsilon \xrightarrow{*} \mu$, we have $\overline{\lim}_{\varepsilon \rightarrow 0} \mu_\varepsilon(S_\rho) \leq \mu(\bar{S}_\rho)$ by Alexandrov theorem. But $\mu(\bar{S}_\rho) = \mu_a(\bar{S}_\rho)$ since $\bar{S}_\rho \subset \bar{Q}_\rho(x_0) \subset \Omega \setminus \text{spt}(\mu_s)$ (see Remark 6.1). Hence, for every $\rho \in]0, \bar{\rho}[$,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\rho^d} \mu_\varepsilon(S_\rho) \leq \frac{1}{\rho^d} \int_{S_\rho} f(x) dx = \int_{Q_\rho(x_0)} f(x) dx - s^d \int_{Q_{s\rho}(x_0)} f(x) dx,$$

and consequently

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\rho^d} \mu_\varepsilon(S_\rho) \leq (1 - s^d) f(x_0). \quad (6.22)$$

Taking (6.19) and (6.20) into account, from (6.21) and (6.22) we deduce that

$$\begin{aligned} & \mathcal{H}W(t \nabla u(x_0)) \\ & \leq \overline{\lim}_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon\right) dx \\ & + \left(1 - s^d\right) \left(\beta + \bar{C} \left(1 + r_0 + G(t_* \nabla u(x_0)) + \frac{1}{\alpha} (\Delta_W^a(t_*) \langle a \rangle + f(x_0))\right)\right). \end{aligned}$$

Taking (6.10), (6.11) and (6.8) into account and passing to the limits $\rho \rightarrow 0$ and $s \rightarrow 1$, we obtain

$$\mathcal{H}W(t \nabla u(x_0)) \leq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, t \nabla u_\varepsilon\right) dx,$$

and (6.14) follows when $t \rightarrow 1$. \square

7 Proof of Proposition 2.1 (ii)

For each $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$ we recall that

$$m_\varepsilon(u; O) := \inf \left\{ I_\varepsilon(v; O) : v \in W_0^{1,p}(O; \mathbb{R}^m) \right\} \quad \text{and} \quad \mathcal{M}(u; O) := \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u; O).$$

We give a sketch of the proof which is divided into three steps.

The first step consists in proving that $I_+(u; O) \leq \mathcal{M}^*(u; O)$ for all $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$. When we assume that $\mathcal{G}(u; O) < \infty$, Lemma 4.4 and the G -growth conditions imply that $\mathcal{M}^*(u; \cdot)$ is a Radon measure which is absolutely continuous with respect to the Lebesgue measure on O . Thus, we can write

$$I_+(u; O) \leq \mathcal{M}^*(u; O) = \int_O \lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(u; Q_\rho(x))}{\rho^d} dx. \quad (7.1)$$

The second step consists in showing that $\mathcal{M}^*(u; \cdot)$ is locally equivalent to $\mathcal{M}(u; \cdot)$, i.e., for a.a. $x \in O$

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(u; Q_\rho(x))}{\rho^d} = \lim_{\rho \rightarrow 0} \frac{\mathcal{M}(u; Q_\rho(x))}{\rho^d}. \quad (7.2)$$

This is carrying out by measure theoretic arguments (see Step 2).

In the third and last step we replace u by tu with $t \in]0, 1[$ and we show, using cut-off techniques, that for a.a. $x \in O$

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}(tu; Q_\rho(x))}{\rho^d} \leq \lim_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(tu_x; Q_{s\rho}(x))}{(s\rho)^d} \quad (7.3)$$

where $u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x)$. The right hand term of (7.3) is equal to $\mathcal{H}W(t\nabla u(x))$. Indeed, it is easy to see that for any $\varepsilon, \rho > 0$ and any $x \in O$ we can write

$$\begin{aligned} & \frac{m_\varepsilon(tu_x; Q_{s\rho}(x))}{(s\rho)^d} \\ &= \frac{1}{|\frac{1}{\varepsilon}Q_{s\rho}(x)|} \inf \left\{ \int_{\frac{1}{\varepsilon}Q_{s\rho}(x)} W(y, \nabla v(\varepsilon y)) dy : v \in tu_x + W_0^{1,p}(Q_{s\rho}(x); \mathbb{R}^m) \right\} \\ &= \frac{1}{|\frac{1}{\varepsilon}Q_{s\rho}(x)|} \inf \left\{ \int_{\frac{1}{\varepsilon}Q_{s\rho}(x)} W(y, t\nabla u(x_0) + \nabla \phi(\varepsilon y)) dy : \phi \in W_0^{1,p}(Q_{s\rho}(x); \mathbb{R}^m) \right\} \\ &= \frac{1}{|\frac{1}{\varepsilon}Q_{s\rho}(x)|} \mathcal{S}_{t\nabla u(x)} \left(\frac{1}{\varepsilon}Q_{s\rho}(x) \right) \end{aligned}$$

which give

$$\frac{\mathcal{M}(tu_x; Q_{s\rho}(x))}{(s\rho)^d} = \mathcal{H}W(t\nabla u(x))$$

since a subadditive argument (see Corollary 4.1). The proof is achieved by taking (7.2) and (7.3) into account in the inequality (7.1)

$$\begin{aligned} I_+(tu; O) &\leq \int_O \lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(tu; Q_\rho(x))}{\rho^d} dx = \int_O \lim_{\rho \rightarrow 0} \frac{\mathcal{M}(tu; Q_\rho(x))}{\rho^d} dx \\ &\leq \int_O \lim_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(tu_x; Q_{s\rho}(x))}{(s\rho)^d} dx \\ &= \int_O \mathcal{H}W(t\nabla u(x)) dx. \end{aligned}$$

Step 1: Prove that $I_+(u; O) \leq \mathcal{M}^(u; O)$ when $\mathcal{G}(u; O) < \infty$*

Fix $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$ such that $\mathcal{G}(u; O) < \infty$. Without loss of generality we assume that

$$\mathcal{M}^*(u; O) < \infty. \quad (7.4)$$

Fix $\varepsilon \in]0, 1[$. Choose $\{\overline{Q}_i\}_{i \in I} \in \mathcal{V}_\varepsilon(O)$ such that

$$\sum_{i \in I} \mathcal{M}(u; Q_i) \leq \mathcal{M}^\varepsilon(u; O) + \frac{\varepsilon}{2} \leq \mathcal{M}^*(u; O) + \frac{\varepsilon}{2}. \quad (7.5)$$

Fix $\delta \in]0, 1[$. Given any $i \in I$ there exists $v_i \in u + W_0^{1,p}(Q_i; \mathbb{R}^m)$ such that

$$I_\delta(v_i; Q_i) \leq m_\delta(u; Q_i) + \frac{\delta}{2} \frac{|Q_i|}{|O|} \quad (7.6)$$

by definition of $m_\delta(u; Q_i)$. Define $u_{\delta,\varepsilon} \in u + W_0^{1,p}(O; \mathbb{R}^m)$ by

$$u_{\delta,\varepsilon} := \sum_{i \in I} v_i \mathbb{I}_{Q_i} + u \mathbb{I}_{\Omega \setminus \cup_{i \in I} Q_i}.$$

From (7.6) we have that

$$I_\delta(u_{\delta,\varepsilon}; O) = \sum_{i \in I} I_\delta(v_i; Q_i) \leq \sum_{i \in I} m_\delta(u; Q_i) + \frac{\delta}{2}.$$

Letting $\delta \rightarrow 0$ we obtain

$$\overline{\lim}_{\delta \rightarrow 0} I_\delta(u_{\delta,\varepsilon}; O) \leq \overline{\lim}_{\delta \rightarrow 0} \sum_{i \in I} m_\delta(u; Q_i). \quad (7.7)$$

By the G -growth conditions (H₁) we have $\sup_{\eta \in]0, \delta[} m_\eta(u; Q_i) \leq \beta(|Q_i| + \mathcal{G}(u; Q_i))$ for all $\delta > 0$ and all $i \in I$ with

$$\sum_{i \in I} \beta(|Q_i| + \mathcal{G}(u; Q_i)) = \beta(|O| + \mathcal{G}(u; O)) < \infty,$$

then applying the dominated convergence theorem we have

$$\overline{\lim}_{\delta \rightarrow 0} \sum_{i \in I} m_\delta(u; Q_i) \leq \sum_{i \in I} \mathcal{M}(u; Q_i). \quad (7.8)$$

Therefore collecting (7.5), (7.7), (7.8) and passing to the limit $\varepsilon \rightarrow 0$, we have

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} I_\delta(u_{\delta,\varepsilon}; O) \leq \mathcal{M}^*(u; O). \quad (7.9)$$

From the p -coercivity of W (H₂), (7.9) and (7.4), we deduce

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \int_O |\nabla u_{\delta,\varepsilon}|^p dx < \infty. \quad (7.10)$$

By Poincaré inequality there exists $K > 0$ depending only on p and d such that for each $v_i \in u + W_0^{1,p}(Q_i; \mathbb{R}^m)$

$$\int_{Q_i} |v_i - u|^p dx \leq K \varepsilon^p \int_{Q_i} |\nabla v_i - \nabla u|^p dx$$

since $\text{diam}(Q_i) < \varepsilon$. Summing on $i \in I$ we obtain

$$\int_O |u_{\delta,\varepsilon} - u|^p dx \leq 2^{p-1} K \varepsilon^p \left(\int_O |\nabla u_{\delta,\varepsilon}|^p dx + \int_O |\nabla u|^p dx \right)$$

which shows that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \int_\Omega |u_{\delta,\varepsilon} - u|^p dx = 0 \quad (7.11)$$

since (7.10). A simultaneous diagonalization of (7.9) and (7.11) gives a sequence $\{u_\delta := u_{\delta,\varepsilon(\delta)}\}_{\delta > 0} \subset u + W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_\delta \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ and

$$I_+(u; O) \leq \overline{\lim}_{\delta \rightarrow 0} I_\delta(u_\delta; O) \leq \mathcal{M}^*(u; O)$$

since the definition of $I_+(u; O)$. The proof is complete. \square

Step 2: Prove that $\mathcal{M}^*(u; \cdot)$ is locally equivalent to $\mathcal{M}(u; \cdot)$

In this step we use the following result from [9, 10]. For a sake of completeness we give a proof.

Lemma 7.1 *If $\mathcal{G}(u; O) < \infty$ then we have*

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(u; \mathcal{Q}_\rho(x_0))}{\rho^d} = \lim_{\rho \rightarrow 0} \frac{\mathcal{M}(u; \mathcal{Q}_\rho(x_0))}{\rho^d} \quad x_0\text{-a.e. in } O.$$

Proof Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ be such that $\mathcal{G}(u; O) < \infty$. Then for each $U \in \mathcal{O}(O)$

$$\mathcal{M}(u; U) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \int_U W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx \leq \beta \left(|O| + \int_O G(\nabla u) dx \right) < \infty,$$

so, using Lemma 4.4 with $\mu_u := \beta(|\cdot| + G(\nabla u(\cdot))dx) \llcorner_O$, we have $\mathcal{M}(u; \cdot)$ is the trace of a Radon measure λ_u on O satisfying $0 \leq \lambda_u \leq \mu_u$. Since μ_u is absolutely continuous with respect to $dx \llcorner_O$ the Lebesgue measure on O , the limit $\lim_{\rho \rightarrow 0} \frac{\lambda_u(\mathcal{Q}_\rho(x_0))}{\rho^d}$ exists for a.a. $x_0 \in O$ as the Radon-Nikodym derivative of λ_u with respect to $dx \llcorner_O$. Moreover, using Lemma 4.3, the G -growth conditions together with Remark 4.2 we have

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(u; \mathcal{Q}_\rho(x_0))}{\rho^d} \geq \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(u; \mathcal{Q}_\rho(x_0))}{\rho^d} \quad x_0\text{-a.e. in } O.$$

It remains to prove that

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(u; \mathcal{Q}_\rho(x_0))}{\rho^d} \leq \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(u; \mathcal{Q}_\rho(x_0))}{\rho^d} \quad x_0\text{-a.e. in } O. \quad (7.12)$$

Fix any $\theta > 0$. Consider the following sets

$$\begin{aligned} \mathcal{G}_\theta &:= \left\{ \mathcal{Q}_\rho(x) : x \in O, \rho > 0 \text{ and } \mathcal{M}^*(u; \mathcal{Q}_\rho(x)) > \mathcal{M}(u; \mathcal{Q}_\rho(x)) + \theta |\mathcal{Q}_\rho(x)| \right\}, \\ \mathcal{N}_\theta &:= \left\{ x \in O : \forall \delta > 0 \exists \rho \in]0, \delta[\mathcal{Q}_\rho(x) \in \mathcal{G}_\theta \right\}. \end{aligned}$$

It is sufficient to prove that \mathcal{N}_θ is a negligible set for the Lebesgue measure on O . Indeed, given $x_0 \in O \setminus \mathcal{N}_\theta$ there exists $\delta_0 > 0$ such that

$$\mathcal{M}^*(u; \mathcal{Q}_\rho(x_0)) \leq \mathcal{M}(u; \mathcal{Q}_\rho(x_0)) + \theta |\mathcal{Q}_\rho(x_0)|$$

for all $\rho \in]0, \delta_0[$. Hence

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}^*(u; \mathcal{Q}_\rho(x_0))}{|\mathcal{Q}_\rho(x_0)|} \leq \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(u; \mathcal{Q}_\rho(x_0))}{|\mathcal{Q}_\rho(x_0)|} + \theta,$$

then we obtain (7.12) when $\theta \rightarrow 0$.

Fix $\delta > 0$. Consider the set

$$\mathcal{F}_\delta := \left\{ \overline{\mathcal{Q}_\rho(x)} : x \in \mathcal{N}_\theta, \rho \in]0, \delta[\text{ and } \mathcal{Q}_\rho(x) \in \mathcal{G}_\theta \right\}.$$

Using the definition of \mathcal{N}_θ we can see that $\inf_{Q \in \mathcal{F}_\delta} \text{diam}(Q) = 0$. By the Vitali covering theorem there exists a (pairwise) disjointed countable subfamily $\{\overline{Q}_i\}_{i \geq 1}$ of \mathcal{F}_δ such that

$$\left| \mathcal{N}_\theta \setminus \bigcup_{i \geq 1} \overline{Q}_i \right| = 0. \quad (7.13)$$

We have $\mathcal{N}_\theta \subset \cup_{i \geq 1} Q_i \cup \mathcal{N}_\theta \setminus \cup_{i \geq 1} Q_i$. To prove that \mathcal{N}_θ is a negligible set is equivalent to prove that $|V_j| = 0$ for all $j \geq 1$ where

$$V_j := \bigcup_{i=1}^j Q_i.$$

Fix $j \geq 1$. Let $\{Q'_i\}_{i \geq 1} \in \mathcal{V}_\delta(O \setminus \cup_{i=1}^j \overline{Q}_i)$ satisfying

$$\sum_{i \geq 1} \mathcal{M}(u; Q'_i) \leq \mathcal{M}^*(u; O \setminus \bigcup_{i=1}^j \overline{Q}_i) + \delta. \quad (7.14)$$

Recalling that $\mathcal{M}^*(u; \cdot)$ is the trace on $\mathcal{O}(O)$ of a nonnegative finite Radon measure, we see that

$$\begin{aligned} \mathcal{M}^*(u; O) &\geq \mathcal{M}^*(u; O \setminus \bigcup_{i=1}^j \overline{Q}_i) + \mathcal{M}^*(u; V_j) \\ &= \mathcal{M}^*(u; O \setminus \bigcup_{i=1}^j \overline{Q}_i) + \sum_{1 \leq i \leq j} \mathcal{M}^*(u; Q_i). \end{aligned}$$

Since each $Q_i \in \mathcal{G}_\theta$, by (7.14) we have

$$\mathcal{M}^*(u; O) \geq \sum_{i \geq 1} \mathcal{M}(u; Q'_i) - \delta + \sum_{i=1}^j \mathcal{M}(u; Q_i) + \theta |V_j|.$$

It is easy to see that the countable family $\{Q'_i : i \geq 1\} \cup \{Q_i : 1 \leq i \leq j\}$ belongs to $\mathcal{V}_\delta(O)$, thus

$$\mathcal{M}^*(u; O) \geq \mathcal{M}^\delta(u; O) + \theta |V_j| - \delta.$$

Letting $\delta \rightarrow 0$, we have $\mathcal{M}^\delta(u; O) \rightarrow \mathcal{M}^*(u; O)$, and so $|V_j| = 0$ since $\theta > 0$. \square

Step 3: Cut-off technique to locally substitute tu with tu_{x_0} in $\frac{\mathcal{M}(\cdot; Q_\rho(x_0))}{\rho^d}$

Fix $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$. Let $t \in]0, 1[$ such that

$$\mathcal{G}(tu; O) < \infty. \quad (7.15)$$

Our goal here is to prove

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}(tu; Q_\rho(x_0))}{\rho^d} \leq \mathcal{H}W(t \nabla u(x_0)) \quad x_0\text{-a.e. in } O. \quad (7.16)$$

We claim that it is enough to prove that

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}(tu; Q_\rho(x_0))}{\rho^d} \leq \lim_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(tu_{x_0}; Q_{s\rho}(x_0))}{(s\rho)^d} \quad x_0\text{-a.e. in } O, \quad (7.17)$$

where $u_{x_0}(\cdot) := u(x_0) + \nabla u(x_0)(\cdot - x_0)$. Indeed, by Corollary 4.1 we have

$$\begin{aligned} &\lim_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(tu_{x_0}; Q_{s\rho}(x_0))}{(s\rho)^d} \\ &= \lim_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|\frac{1}{\varepsilon} Q_{s\rho}(x_0)|} \mathcal{S}_t \nabla u(x_0) \left(\frac{1}{\varepsilon} Q_{s\rho}(x_0) \right) = \mathcal{H}W(t \nabla u(x_0)) \quad x_0\text{-a.e. in } O. \end{aligned}$$

since $t\nabla u(\cdot) \in \mathbb{G}$ a.e. in O by (7.15).

We are reduced to prove (7.17). Consider $x_0 \in O$ satisfying

$$\lim_{\rho \rightarrow 0} \int_{Q_\rho(x_0)} G(t\nabla u(x)) dx = G(t\nabla u(x_0)) < \infty; \quad (7.18)$$

$$\lim_{\rho \rightarrow 0} \int_{Q_\rho(x_0)} G(t_*\nabla u(x)) dx = G(t_*\nabla u(x_0)) < \infty. \quad (7.19)$$

Fix $s \in]0, 1[$, $\rho > 0$ and $\varepsilon > 0$.

Choose $v \in u_{x_0} + W_0^{1,p}(Q_{s\rho}(x_0); \mathbb{R}^m)$ satisfying

$$I_\varepsilon(tv; Q_{s\rho}(x_0)) \leq m_\varepsilon(tu_{x_0}; Q_{s\rho}(x_0)) + \rho^{d+1}.$$

Consider a cut-off function $\phi \in W_0^{1,\infty}(Q_\rho(x_0); [0, 1])$ between $Q_{s\rho}(x_0)$ and $Q_\rho(x_0)$ such that $\|\nabla\phi\|_{L^\infty(Q_\rho(x_0))} \leq \frac{4}{(1-s)\rho}$.

Define $w := \phi v + (1 - \phi)u$ which belongs to $u + W_0^{1,p}(Q_\rho(x_0); \mathbb{R}^m)$. We have

$$\begin{aligned} m_\varepsilon(tu; Q_\rho(x_0)) &\leq I_\varepsilon(tv; Q_{s\rho}(x_0)) + I_\varepsilon(tw; Q_\rho(x_0) \setminus Q_{s\rho}(x_0)) \\ &\leq m_\varepsilon(tu_{x_0}; Q_{s\rho}(x_0)) + \rho^{d+1} + I_\varepsilon(tw; Q_\rho(x_0) \setminus Q_{s\rho}(x_0)). \end{aligned} \quad (7.20)$$

Let us estimate the last term of (7.20) divided by ρ^d .

Set $\tau := tt_*^{-1}$, $\Phi_\rho := \frac{t}{1-\tau}\nabla\phi \otimes (u_{x_0} - u)$ and $S_\rho := Q_\rho(x_0) \setminus \overline{Q_{s\rho}(x_0)}$. Using the G -growth conditions, (C₂) and (7.19) we have

$$\begin{aligned} &\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(tw; S_\rho)}{\rho^d} \\ &\leq \beta \overline{\lim}_{\rho \rightarrow 0} \left((1 - s^d) + \frac{1}{\rho^d} \int_{S_\rho} G(\tau(\phi t_*\nabla u(x_0) + (1 - \phi)t_*\nabla u) + (1 - \tau)\Phi_\rho) dx \right) \\ &\leq C_1 \overline{\lim}_{\rho \rightarrow 0} \left((1 - s^d) + \frac{1}{\rho^d} \int_{S_\rho} G(\phi t_*\nabla u(x_0) + (1 - \phi)t_*\nabla u) dx + \frac{1}{\rho^d} \int_{S_\rho} G(\Phi_\rho) dx \right) \\ &\leq C_2 \overline{\lim}_{\rho \rightarrow 0} \left((1 - s^d)(1 + G(t_*\nabla u(x_0))) + \frac{1}{\rho^d} \int_{S_\rho} G(t_*\nabla u) dx + \frac{1}{\rho^d} \int_{S_\rho} G(\Phi_\rho) dx \right) \\ &\leq 2C_2 \left((1 - s^d)(1 + G(t_*\nabla u(x_0))) + \overline{\lim}_{\rho \rightarrow 0} \frac{1}{\rho^d} \int_{S_\rho} G(\Phi_\rho) dx \right). \end{aligned} \quad (7.21)$$

where $C_1 := \beta(1 + C)$ and $C_2 := C_1(1 + C)$.

Since (6.12), we choose $\bar{\rho} > 0$ such that for every $\rho \in]0, \bar{\rho}[$ it holds

$$\frac{4t}{(1 - \tau)(1 - s)} \frac{1}{\rho} \|u_{x_0} - u\|_{L^\infty(Q_\rho(x_0); \mathbb{R}^m)} \leq \frac{\rho_0}{2}.$$

It follows that

$$\|\Phi_\rho\|_{L^\infty(Q_\rho(x_0); \mathbb{M}^{m \times d})} \leq \rho_0.$$

Using Lemma 4.1 we deduce that

$$G(\Phi_\rho(\cdot)) \leq r_0 \quad \text{a.e. in } Q_\rho(x_0) \quad (7.22)$$

for all $\rho \in]0, \bar{\rho}[$. From (7.21) and (7.22) we obtain

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(tw; S_\rho)}{\rho^d} \leq 2C_2(1 - s^d)(1 + G(t_* \nabla u(x_0) + r_0)). \quad (7.23)$$

Taking (7.23) into account in the inequality (7.20) we obtain by passing to the limit $\varepsilon \rightarrow 0$ and $\rho \rightarrow 0$

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{M}(tu; Q_\rho(x_0))}{\rho^d} \leq s^d \overline{\lim}_{\rho \rightarrow 0} \frac{\mathcal{M}(tu_{x_0}; Q_{s\rho}(x_0))}{(s\rho)^d} + 2C_2(1 - s^d)(1 + G(t_* \nabla u(x_0) + r_0)).$$

Letting $s \rightarrow 1$, we finally find (7.17). \square

8 Proof of Theorem 1.1

To shorten notation we set

$$\mathcal{G}(\cdot; \Omega) := \mathcal{G}(\cdot), \quad \mathcal{H}I(\cdot; \Omega) := \mathcal{H}I(\cdot), \quad \text{and} \quad \widehat{\mathcal{H}I}(\cdot; \Omega) := \widehat{\mathcal{H}I}(\cdot).$$

Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. The proof of the lower bound is already done in Proposition 2.1 (i), so it remains to prove the upper bound $I_+(\cdot) \leq \widehat{\mathcal{H}I}(\cdot)$.

Assume without loss of generality that $\widehat{\mathcal{H}I}(u) < \infty$.

First, we assume that $\nabla u(\cdot) \in \text{int}(\mathbb{G})$ a.e. in Ω . In this case, from Proposition 2.2, coercivity conditions (H₁) and (C₂) we have

$$\widehat{\mathcal{H}I}(u) = \int_{\Omega} \lim_{t \rightarrow 1^-} \mathcal{H}W(t \nabla u(x)) dx \geq \alpha \int_{\Omega} \underline{\lim}_{t \rightarrow 1^-} G(t \nabla u(x)) dx.$$

But G is $W^{1,p}$ -quasiconvex, so it is continuous in $\text{int}(\mathbb{G})$ (see for instance [20] or [18]). It follows that

$$\underline{\lim}_{t \rightarrow 1^-} G(t \nabla u(x)) = G(\nabla u(x)) \quad \text{a.e. in } \Omega$$

since we assumed that $\nabla u(\cdot) \in \text{int}(\mathbb{G})$ a.e. in Ω . Therefore

$$\alpha \mathcal{G}(u) \leq \widehat{\mathcal{H}I}(u) < \infty.$$

So, using (C₂) and (C₁) we have $\mathcal{G}(tu) \leq C(|\Omega|(1 + G(0)) + \mathcal{G}(u)) < \infty$ for all $t \in]0, 1[$. Applying Proposition 2.1 (ii) we have for every $t \in]0, 1[$

$$I_+(tu) \leq \mathcal{H}I(tu).$$

Since (H₁), we have for any $t \in]0, 1[$ that $\mathcal{H}W(t \nabla u(\cdot)) \leq \beta(1 + G(t \nabla u(\cdot)))$, so applying Lebesgue dominated convergence theorem and Proposition 2.2 we obtain

$$\underline{\lim}_{t \rightarrow 1^-} \mathcal{H}I(tu) \leq \int_{\Omega} \lim_{t \rightarrow 1^-} \mathcal{H}W(t \nabla u(x)) dx = \widehat{\mathcal{H}I}(u).$$

Using the lower semicontinuity of I_+ with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$, we then have

$$I_+(u) \leq \underline{\lim}_{t \rightarrow 1^-} I_+(tu) \leq \underline{\lim}_{t \rightarrow 1^-} \mathcal{H}I(tu) \leq \widehat{\mathcal{H}I}(u).$$

Now, we assume the general case $\nabla u(\cdot) \in \overline{\mathbb{G}}$ a.e. in Ω . We have for every $t \in]0, 1[$ that $t\nabla u(\cdot) \in \text{int}(\mathbb{G})$ a.e. in Ω since \mathbb{G} is convex with $0 \in \text{int}(\mathbb{G})$ (see Remark 2.1 (ii)). We can apply the first part of the proof to get

$$I_+(tu) \leq \widehat{\mathcal{H}I}(tu)$$

for all $t \in]0, 1[$. But $\widehat{\mathcal{H}W}$ is ru-usc since Proposition 2.2, so, for every $t \in]0, 1[$

$$\widehat{\mathcal{H}I}(tu) \leq \Delta_{\widehat{\mathcal{H}W}}^{(a)}(t) (\langle a \rangle |\Omega| + \widehat{\mathcal{H}I}(u)) + \widehat{\mathcal{H}I}(u).$$

Letting $t \rightarrow 1$ and using the lower semicontinuity of I_+ , we obtain the desired result. \square

9 A two-dimensional example

We show, when $m = d = 2$, how to construct an example of W with G -growth conditions satisfying the assumptions of Theorem 1.1. We want to point out that the following discussed example is not covered by [4], indeed the growth considered here are not convex see Remark 9.3.

Consider a set $\mathbb{G} \subset \mathbb{M}^{2 \times 2}$ with the following properties:

- (A₁) $0 \in \text{int}(\mathbb{G})$;
- (A₂) \mathbb{G} is convex;
- (A₃) $\det(I + \xi) > 0$ for all $\xi \in \mathbb{G}$;
- (A₄) $\text{tr}(\text{cof}(I + \xi)^\top(I + \zeta)) > 0$ for all $\xi, \zeta \in \mathbb{G}$,

where I is the identity matrix and $\text{cof}(F)$ is the matrix of cofactors of $F \in \mathbb{M}^{2 \times 2}$.

Remark 9.1 The set $I + \mathbb{G}$ can be interpreted as *internal constraints* of an elastic material. However, the properties of $I + \mathbb{G}$ do not fit with the requirements of the theory of internal constraints as developed by [21]. Indeed, due to the frame indifference principle, we should have

$$\text{SO}(2)(I + \mathbb{G}) \subset I + \mathbb{G}, \quad (9.1)$$

but this is not true. Assume that a such $I + \mathbb{G}$ satisfying (9.1) exists then $\text{SO}(2) \subset I + \mathbb{G}$ since (A₁). Choose any rotation matrix $I + \zeta$ with angle $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $I + \xi = I$, i.e., $\xi = 0$, then

$$\text{tr}(\text{cof}(I + \xi)^\top(I + \zeta)) = \text{tr}(I + \zeta) \leq 0,$$

so (A₄) cannot be satisfied.

Let $g : \mathbb{M}^{2 \times 2} \rightarrow [0, \infty]$ be defined by

$$g(\xi) := \begin{cases} h(\det(I + \xi)) & \text{if } \xi \in \mathbb{G} \\ \infty & \text{otherwise} \end{cases}$$

where $h :]0, \infty[\rightarrow [0, \infty]$ is a nonincreasing convex function satisfying for every $\lambda \in]0, 1[$ and every $x \in]0, \infty[$

$$h(\lambda x) \leq \frac{1}{\lambda^r} h(x) \quad (9.2)$$

where $r \leq 1$. Note that the function h can be chosen to satisfy $\lim_{x \rightarrow 0} h(x) = \infty$.

Proposition 9.1 *We have*

- (i) \tilde{g} is polyconvex where $\tilde{g}(\cdot) := g(\cdot - I)$;
(ii) for every $\xi, \zeta \in \mathbb{G}$ and every $\lambda \in]0, 1[$ it holds

$$g(\lambda\xi + (1 - \lambda)\zeta) \leq g(\xi) + g(\zeta);$$

- (iii) g is ru-usc.

Proof We have (i) because we can write $\tilde{g}(F) = \varphi(F, \det(F))$ with $\varphi : \mathbb{M}^{2 \times 2} \times \mathbb{R} \rightarrow [0, \infty]$ is the convex function defined by

$$\varphi(F, s) := \begin{cases} h(s) & \text{if } F \in I + \mathbb{G} \\ \infty & \text{otherwise.} \end{cases}$$

Now, we show (ii). Fix $\xi, \zeta \in \mathbb{G}$ and $\lambda \in]0, 1[$. Using (A₃), (A₄) and properties of h , we have

$$\begin{aligned} & g(\lambda\xi + (1 - \lambda)\zeta) \\ &= h(\det(\lambda(I + \xi) + (1 - \lambda)(I + \zeta))) \\ &= h(\lambda^2 \det(I + \xi) + (1 - \lambda)^2 \det(I + \zeta) + \lambda(1 - \lambda)\text{tr}(\text{cof}(I + \xi)^T(I + \zeta))) \\ &\leq h(\lambda^2 \det(I + \xi) + (1 - \lambda)^2 \det(I + \zeta)) \\ &\leq \lambda h(\lambda \det(I + \xi)) + (1 - \lambda)h((1 - \lambda) \det(I + \zeta)) \\ &\leq \lambda^{1-r} h(\det(I + \xi)) + (1 - \lambda)^{1-r} h(\det(I + \zeta)) \\ &\leq g(\xi) + g(\zeta). \end{aligned} \tag{9.3}$$

From (9.3) and properties of h we have for every $\xi \in \mathbb{G}$ and every $t \in]0, 1[$

$$\begin{aligned} g(t\xi) &= h(t^2 \det(I + \xi) + (1 - t)^2 + t(1 - t)\text{tr}(I + \xi)) \\ &\leq h(t^2 \det(I + \xi)) \\ &\leq \frac{1}{t^{2r}} h(\det(I + \xi)) \\ &= \frac{1}{t^{2r}} h(\det(I + \xi)) - h(\det(I + \xi)) + g(\xi) \\ &\leq \frac{1 - t^{2r}}{t^{2r}} (1 + g(\xi)) + g(\xi) \end{aligned}$$

which implies $\Delta_g^1(t) \leq \frac{1-t^{2r}}{t^{2r}}$. Letting $t \rightarrow 1$ we obtain (iii). \square

Remark 9.2 We may think that the function \tilde{g} could be convex but it is not the case in general, see Remark 9.3.

We define the function $G : \mathbb{M}^{2 \times 2} \rightarrow [0, \infty]$ by

$$G(\xi) := \begin{cases} |\xi|^p + g(\xi) & \text{if } \xi \in \mathbb{G} \\ \infty & \text{otherwise.} \end{cases} \tag{9.4}$$

Using (A₁), Proposition 9.1 (i) and (ii) it is easy to see

Lemma 9.1 *The function G defined in (9.4) satisfies (C₁), (C₂) and (C₃).*

Let $W : \mathbb{R}^2 \times \mathbb{M}^{2 \times 2} \rightarrow [0, \infty]$ be defined by

$$W(x, \xi) := \begin{cases} \Phi(x, \xi) + g(\xi) & \text{if } \xi \in \mathbb{G} \\ \infty & \text{otherwise,} \end{cases} \quad (9.5)$$

where $\Phi : \mathbb{R}^2 \times \mathbb{M}^{2 \times 2} \rightarrow [0, \infty]$ is a quasiconvex function, 1-periodic with respect to its first variable and of p -polynomial growth, i.e., there exist $c, C > 0$ such that

$$c|\xi|^p \leq \Phi(x, \xi) \leq C(1 + |\xi|^p) \quad (9.6)$$

for all $(x, \xi) \in \mathbb{R}^2 \times \mathbb{M}^{2 \times 2}$.

The following proposition shows that such a W is consistent with the assumptions of Theorem 1.1 as well as with the two basic conditions of hyperelasticity, i.e., the non-interpenetration of the matter and the necessity of an infinite amount of energy to compress a finite volume of matter into zero volume.

Proposition 9.2 *We have*

- (i) W is 1-periodic with respect to the first variable;
- (ii) W satisfies (H_1) and (H_2) with G given by (9.4);
- (iii) for every $(x, \xi) \in \mathbb{R}^d \times \mathbb{G}$, $W(x, \xi) < \infty$ if and only if $\det(I + \xi) > 0$;
- (iv) if $\lim_{x \rightarrow 0} h(x) = \infty$ then for every $x \in \mathbb{R}^d$, $W(x, \xi) \rightarrow \infty$ as $\det(I + \xi) \rightarrow 0$;
- (v) W is periodically ru-usc.

Proof The only not direct property is (v). Fix any $t \in [0, 1]$, any $x \in \mathbb{R}^d$ and any $\xi \in \mathbb{G}$. As Φ is quasiconvex and satisfies (9.6), then repeating the same arguments as in the proof of Theorem 3.1 (Sect. 3) we see that Φ is periodically ru-usc with $a \equiv 1$.

On the other hand, as g is ru-usc by Proposition 9.1 (iii) we have

$$g(t\xi) - g(\xi) \leq \Delta_g^1(t)(1 + g(\xi)). \quad (9.7)$$

From (3.4) and (9.7) we deduce that

$$W(x, t\xi) - W(x, \xi) \leq \max \left\{ \Delta_\Phi^2(t), \Delta_g^1(t) \right\} (2 + W(x, \xi)).$$

Passing to the supremum on x and ξ , we obtain

$$\sup_{x \in \mathbb{R}^d} \sup_{\xi \in \mathbb{G}} \frac{W(x, t\xi) - W(x, \xi)}{2 + W(x, \xi)} \leq \max \left\{ \Delta_\Phi^2(t), \Delta_g^1(t) \right\},$$

and (v) follows when $t \rightarrow 1$. □

Concrete example

Every $\xi \in \mathbb{M}^{2 \times 2}$ is denoted by

$$\xi := \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$$

Define the set

$$\mathbb{G} := \left\{ \xi \in \mathbb{M}^{2 \times 2} : \min \{1 + \xi_{11}, 1 + \xi_{22}\} > \max \{|\xi_{12}|, |\xi_{21}|\} \right\},$$

Property (A_1) is evident. The subset \mathbb{G} is open and convex as intersection of open convex sets, so (A_2) holds. The assertion (A_3) is satisfied because for every $\xi \in \mathbb{G}$ we have

$$\det(I + \xi) = (1 + \xi_{11})(1 + \xi_{22}) - \xi_{12}\xi_{21} > |\xi_{12}\xi_{21}| - \xi_{12}\xi_{21} \geq 0.$$

To verify (A₄), we note that for every $\xi, \zeta \in \mathbb{G}$ it holds

$$\begin{aligned} & \operatorname{tr}(\operatorname{cof}(I + \xi)^\top(I + \zeta)) \\ &= (1 + \xi_{11})(1 + \zeta_{11}) + (1 + \xi_{22})(1 + \zeta_{22}) - \xi_{12}\zeta_{21} - \xi_{21}\zeta_{12} \\ &> |\xi_{12}\zeta_{21}| + |\xi_{21}\zeta_{12}| - \xi_{12}\zeta_{21} - \xi_{21}\zeta_{12} \geq 0. \end{aligned}$$

We can take $g : \mathbb{M}^{2 \times 2} \rightarrow [0, \infty]$ defined by

$$g(\xi) := \begin{cases} h(\det(I + \xi)) & \text{if } \xi \in \mathbb{G} \\ \infty & \text{otherwise} \end{cases}$$

where h is the convex and nonincreasing function defined by $h(x) := \frac{1}{x}$ for all $x > 0$ satisfying (9.2) with $r = 1$.

Remark 9.3 It is easy to see that $\tilde{g}(\cdot) := g(\cdot - I)$ is polyconvex but not convex. Indeed, consider $F \in I + \mathbb{G}$ defined by

$$F := \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$\text{then } \tilde{g}\left(\frac{1}{2}F + \frac{1}{2}F^\top\right) = 1 > \frac{4}{5} = \frac{1}{2}(\tilde{g}(F) + \tilde{g}(F^\top)).$$

Remark 9.4 A necessary condition for \tilde{g} to be frame indifferent is that $P(I + \mathbb{G}) \subseteq I + \mathbb{G}$ for all $P \in \operatorname{SO}(2)$, which in particular means that $\operatorname{SO}(2) \subseteq I + \mathbb{G}$ since (A₁). But, this is not true because the rotation of angle $\frac{\pi}{2}$ does not belong to $I + \mathbb{G}$.

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Appendix

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary.

A) Dacorogna-Acerbi-Fusco relaxation theorem

Let $p \in [1, \infty[$. Let $f : \Omega \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ be a Borel measurable integrand with p -polynomial growth, i.e., there exist $c, C > 0$ such that for every $(x, \xi) \in \Omega \times \mathbb{M}^{m \times d}$ it holds

$$c|\xi|^p \leq f(x, \xi) \leq C(1 + |\xi|^p).$$

Let $J : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ be defined by

$$J(u) := \int_{\Omega} f(x, \nabla u(x)) dx.$$

Then for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$\bar{J}(u) := \inf \left\{ \liminf_{\varepsilon \rightarrow 0} J(u_\varepsilon) : u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\} = \int_{\Omega} Qf(x, \nabla u(x)) dx, \quad (9.8)$$

where $Qf : \Omega \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ is the *quasiconvexification* of f given by the Dacorogna formula

$$Qf(x, \xi) = \inf \left\{ \int_Y f(x, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}.$$

As a consequence we have

$$\inf \left\{ \int_{\Omega} f(x, \nabla u) dx : u \in W^{1,p}(\Omega; \mathbb{R}^m) \right\} = \inf \left\{ \int_{\Omega} Qf(x, \nabla u) dx : u \in W^{1,p}(\Omega; \mathbb{R}^m) \right\}. \quad (9.9)$$

The same equalities (9.8) and (9.9) hold when replacing $W^{1,p}(\Omega; \mathbb{R}^m)$ by $W_0^{1,p}(\Omega; \mathbb{R}^m)$.

B) Alexandrov theorem

If the sequence $\{\mu_\varepsilon\}_{\varepsilon>0}$ of nonnegative finite Radon measures on Ω weakly converges to the Radon measure μ , i.e.,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi d\mu_\varepsilon = \int_{\Omega} \phi d\mu \text{ for all } \phi \in C_c(\Omega),$$

then

- (a) $\varliminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(U) \geq \mu(U)$ for all open sets $U \subset \Omega$;
- (b) $\varlimsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(K) \leq \mu(K)$ for all compact sets $K \subset \Omega$;
- (c) $\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(B) = \mu(B)$ for all Borel sets $B \subset \Omega$ with $\mu(\partial B) = 0$.

References

1. Acerbi, E., Fusco, N.: Semicontinuity problems in the calculus of variations. Arch. Ration. Mech. Anal. **86**(2), 125–145 (1984)
2. Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory, Volume 314 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (1996)
3. Anza Hafsa, O.: On the integral representation of relaxed functionals with convex bounded constraints. ESAIM Control Optim. Calc. Var. **16**(1), 37–57 (2010)
4. Anza Hafsa, O., Mandallena, J.-P.: Homogenization of nonconvex integrals with convex growth. J. Math. Pures Appl. **96**(2), 167–189 (2011)
5. Anza Hafsa, O., Mandallena, J.-P.: Homogenization of unbounded singular integrals in $W^{1,\infty}$. Ric. Mat. **61**(2), 185–217 (2012)
6. Anza Hafsa, O., Mandallena, J.-P.: On the relaxation of unbounded multiple integrals. ArXiv e-prints, July (2012)
7. Anza Hafsa, O., Mandallena, J.-P.: Radial representation of lower semicontinuous envelope. Bollettino dell'Unione Matematica Italiana **7**(1), 1–18 (2014)
8. Akcoglu, M.A., Krengel, U.: Ergodic theorems for superadditive processes. J. Reine Angew. Math. **323**, 53–67 (1981)
9. Bouchitté, G., Bellieud, M.: Regularization of a set function: application to integral representation. Ricerche Mat. **49**, 79–93 (2000)
10. Bouchitté, G., Fonseca, I., Mascarenhas, L.: A global method for relaxation. Arch. Ration. Mech. Anal. **145**(1), 51–98 (1998)
11. Braides, A.: Homogenization of some almost periodic coercive functional. Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **9**(1), 313–321 (1985)
12. Carbone, L., Cioranescu, D., De Arcangelis, R., Gaudiello, A.: Homogenization of unbounded functionals and nonlinear elastomers. The general case. Asymptot. Anal. **29**(3–4), 221–272 (2002)

13. Carbone, L., Cioranescu, D., De Arcangelis, R., Gaudiello, A.: Homogenization of unbounded functionals and nonlinear elastomers. The case of the fixed constraints set. *ESAIM Control Optim. Calc. Var.* **10**(1), 53–83 (2004). (electronic)
14. Cardone, G., Esposito, A.C., Paderni, G.: Homogenization of Dirichlet and Neumann problems with gradient constraints. *Adv. Math. Sci. Appl.* **16**(2), 447–465 (2006)
15. Carbone, L., De Arcangelis, R.: Unbounded functionals in the calculus of variations, volume 125 of Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. Chapman & Hall/CRC, Boca Raton, FL, (2002), Representation, relaxation, and homogenization
16. De Arcangelis, R.: On the relaxation of some classes of pointwise gradient constrained energies. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **24**(1), 113–137 (2007)
17. Dacorogna, B.: Quasiconvexity and relaxation of nonconvex problems in the calculus of variations. *J. Funct. Anal.* **46**(1), 102–118 (1982)
18. Dacorogna, B.: *Direct Methods in the Calculus of Variations*, Volume 78 of Applied Mathematical Sciences. Springer, New York (2008)
19. Dal Maso, G.: An introduction to Γ -Convergence. *Progress in Nonlinear Differential Equations and their Applications*, 8. Birkhäuser Boston Inc., Boston (1993)
20. Fonseca, I.: The lower quasiconvex envelope of the stored energy function for an elastic crystal. *J. Math. Pures Appl.* (9) **67**(2), 175–195 (1988)
21. Gurtin, M.E., Podio-Guidugli, P.: The thermodynamics of constrained materials. *Arch. Ration. Mech. Anal.* **51**, 192–208 (1973)
22. Licht, Ch., Michaille, G.: Global-local subadditive ergodic theorems and application to homogenization in elasticity. *Ann. Math. Blaise Pascal* **9**(1), 21–62 (2002)
23. Müller, S.: Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Ration. Mech. Anal.* **99**(3), 189–212 (1987)
24. Sychev, M.A.: First general lower semicontinuity and relaxation results for strong materials. *J. Convex Anal.* **17**(1), 183–202 (2010)