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# ON THE HOMOGENIZATION OF THE STOKES PROBLEM IN A PERFORATED DOMAIN

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**ABSTRACT.** We consider the Stokes equations on a bounded perforated domain completed with non-zero constant boundary conditions on the holes. We investigate configurations for which the holes are identical spheres and their number  $N$  goes to infinity while their radius  $1/N$  tends to zero. We prove that, under the assumption that there is no concentration in the distribution of holes, the solution is well approximated asymptotically by solving a Stokes-Brinkman problem.

## 1. INTRODUCTION

Let  $\Omega$  a smooth bounded domain in  $\mathbb{R}^3$  and  $N \in \mathbb{N}$ . Given  $h_1^N, \dots, h_N^N$  in  $\Omega$  such that the  $B_i^N = B(h_i^N, 1/N)$  satisfy

$$(A0) \quad B_i^N \Subset \Omega, \quad \overline{B_i^N} \cap \overline{B_j^N} = \emptyset, \quad \text{for } i \neq j \text{ in } \{1, \dots, N\},$$

and a  $N$ -uplet  $(v_i^N)_{i=1, \dots, N} \in (\mathbb{R}^3)^N$ , it is classical that there exists a unique solution to

$$(1) \quad \begin{cases} -\Delta u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \end{cases} \quad \text{on } \mathcal{F}^N := \Omega \setminus \bigcup_{i=1}^N B_i^N,$$

completed with boundary conditions

$$(2) \quad \begin{cases} u &= v_i^N, & \text{on } \partial B_i^N, & \forall i = 1, \dots, N, \\ u &= 0, & \text{on } \partial \Omega. \end{cases}$$

We are interested here in the behavior of this solution when  $N$  goes to infinity and the asymptotics of the data  $(h_i^N, v_i^N)_{i=1, \dots, N}$  are given.

The closely related problem of periodic homogenization of the Stokes equations in a bounded domain perforated by tiny holes is considered in [1]. It is proven therein that there exists a critical value of the ratio between the size of the holes and their minimal distance for which the homogenized problem is a Stokes-Brinkman problem. If the holes are "denser" the homogenized problem is of Darcy type while if the holes are "more dilute" one obtains again a Stokes problem. This former result is an adaptation to the Stokes equations of a previous analysis on the Laplace equation in [3]. We refer the reader to [2, 5] for a review of equivalent results for other fluid models.

In [1], the Stokes equations are completed with vanishing boundary conditions while a volumic source term is added in the bulk. The very problem that we consider herein (1)-(2), with non-zero constant boundary conditions, is introduced in [4] for the modeling of a thin spray in a highly viscous fluid. In this case, the holes represent droplets of another phase called "dispersed phase". This phase can be made of another fluid or small rigid spheres. The Stokes equations should then be completed with evolution equations for this dispersed phase yielding a time-evolution problem with moving holes. With this application in mind, computing the asymptotics of the stationary Stokes problem (1)-(2) is a tool for understanding the instantaneous response of the dispersed phase to the drag forces exerted by the flow on the droplets/spheres. We refer the reader to [4, 11] for more details on the modeling. In [4], the authors adapt the result of [1] on the derivation of the Stokes-Brinkman system. A comparable analysis with another purpose is provided in [10]. We emphasize that there is a significant new difficulty in introducing non-vanishing boundary conditions. Indeed, the boundary conditions on the holes may be highly oscillating (when jumping from one hole to another). Hence, if one were trying to compute the homogenized system for (1)-(2) by lifting the boundary conditions, he would introduce a highly oscillating source term in the Stokes equations that is out of the scope of the analysis in [1].

The result in [4] is obtained under the assumption that the distance between two centers  $h_i^N$  and  $h_j^N$  is larger than  $2/N^{1/3}$ . This assumption is quite restrictive and prevents from extension to a time-dependent problem or a random model (in the spirit of [12]). Furthermore, the proof in [4] relies heavily on explicit formulas for solutions to the Stokes equations in annuli and exterior domains preventing from application to configurations in which the holes have non-spherical shapes or more complex velocities than simple translations. Our main motivation in this paper is to provide another approach that may help to overcome these two difficulties.

In order to consider the limit  $N \rightarrow \infty$ , we make now precise the different assumptions on the data of our Stokes problem (1)-(2). This includes:

- the positions of the centers  $(h_i^N)_{i=1,\dots,N}$ ,
- the velocities prescribed on the holes  $(v_i^N)_{i=1,\dots,N}$ .

First, similarly to [4], we consider finite-energy solutions so that:

$$(A1) \quad \frac{1}{N} \sum_{i=1}^N |v_i^N|^2 \text{ is uniformly bounded.}$$

We also introduce the empiric measure

$$S_N = \frac{1}{N} \sum_{i=1}^N \delta_{h_i^N, v_i^N} \in \mathbb{P}(\mathbb{R}^3 \times \mathbb{R}^3),$$

and we assume:

$$(A2) \quad \int_{\mathbb{R}^3} S_N(dv) \rightharpoonup \rho(x)dx \text{ weakly in the sense of measures on } \mathbb{R}^3,$$

$$(A3) \quad \int_{\mathbb{R}^3} v S_N(dv) \rightharpoonup j(x)dx \text{ weakly in the sense of (vectorial-)measures on } \mathbb{R}^3.$$

We recall that, by assumption (A0), the measure  $S_N$  is supported in  $\Omega \times \mathbb{R}^3$  so that, in the weak limit,  $\rho \geq 0$  and  $\rho$  and  $j$  have support included in  $\Omega$ .

As in [1, 4], we also make precise the dilution regime for the holes that we consider. It is nowadays well documented that the properties of Stokes flows in domains with obstacles change drastically when the distance between obstacles decreases becoming comparable to their diameters (see [8]). We want to avoid this phenomenon in the pairwise as in the global interactions between holes through the flow. To quantify this, we introduce:

$$d_{min}^N = \min_{i=1,\dots,N} \left\{ \text{dist}(h_i^N, \partial\Omega), \min_{j \neq i} |h_i^N - h_j^N| \right\},$$

and, given  $\lambda^N > 0$  to be made precise:

$$M^N = \sup_{x \in \Omega} \left\{ \# \left\{ i \in \{1, \dots, N\} \text{ s.t. } h_i^N \in \overline{B(x, \lambda^N)} \right\} \right\}.$$

We assume below that there exists a sequence  $(\lambda^N)_{N \in \mathbb{N}} \in (0, \infty)^{\mathbb{N}}$  for which the associated  $(M^N)_{N \in \mathbb{N}}$  satisfy:

$$(A4) \quad \lim_{N \rightarrow \infty} \frac{d_{min}^N}{|\lambda^N|^3} = +\infty, \quad \left( \frac{N^{\frac{2}{3}} |\lambda^N|^5}{d_{min}^N} \right)_{N \in \mathbb{N}} \text{ is bounded.}$$

$$(A5) \quad \left( \frac{M^N}{N |\lambda^N|^3} \right)_{N \in \mathbb{N}} \text{ is bounded.}$$

By construction  $d_{min}^N$  is bounded by the diameter of  $\Omega$  while  $M^N$  is bounded from below by 1. Consequently, for any sequence  $(\lambda^N)_{N \in \mathbb{N}}$  satisfying simultaneously the above assumptions, there must exist a nonnegative constant  $c$  such that:

$$(3) \quad \frac{c}{N^{\frac{1}{3}}} < \lambda^N \quad \forall N \in \mathbb{N}, \quad \lim_{N \rightarrow \infty} \lambda^N = 0.$$

Plugging the bound below on  $\lambda^N$  in (A4) we infer also that the above assumptions imply:

$$(4) \quad \lim_{N \rightarrow \infty} N d_{min}^N = +\infty.$$

The second part of condition (A5) restricts the results to the choice of a sequence  $(\lambda_N)_{N \in \mathbb{N}}$  that converges sufficiently fast to 0. However, we note that it allows to choose

$$\lambda_N = \left[ \frac{d_{min}^N}{N^{\frac{2}{3}}} \right]^{\frac{1}{5}} \quad \forall N \in \mathbb{N},$$

which converges to 0 strictly slower than  $1/N^{1/3}$ .

A consequence to the previous computations is that, for  $N$  sufficiently large the  $(B_i^N)_{i=1,\dots,N}$  are disjoint and do not intersect  $\partial\Omega$ . Hence, for  $N$  large enough, assumption (A0) only fixes that the holes are inside  $\Omega$ . There exists then a unique pair  $(u^N, p^N) \in H^1(\mathcal{F}^N) \times L^2(\mathcal{F}^N)$  solution to (1)-(2) (see next section for more details). The pressure is unique up to an additive constant that we may fix by requiring that  $p^N$  has mean 0. It can be seen as the Lagrange multiplier of the divergence-free condition in (1). Hence, we focus on the convergence of the sequence  $(u^N)_{N \in \mathbb{N}}$  and will not go into details on what happens to the pressure (in contrast with [1]). The  $u^N$  are defined on different domains. In order to compute a limit for this sequence of vector-fields, we unify their domain of definition by extending  $u^N$  with the values  $v_i^N$  on  $B_i^N$  for any  $i = 1, \dots, N$ . We still denote  $u^N$  the extension for simplicity. This is now a sequence in  $H_0^1(\Omega)$ .

Our main result reads:

**Theorem 1.** *Let  $(v_i^N, h_i^N)_{i=1,\dots,N}$  be a sequence of data satisfying (A0) for arbitrary  $N \in \mathbb{N}$  and (A1)–(A3) with  $j \in L^2(\Omega)$ ,  $\rho \in L^\infty(\Omega)$ . Assume furthermore that there exists a sequence  $(\lambda^N)_{N \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$  for which (A4)–(A5) hold true. Then, the associated sequence of extended velocity-fields  $(u^N)_{N \in \mathbb{N}}$  converges in  $H_0^1(\Omega)$  –  $w$  to the unique velocity-field  $\bar{u} \in H^1(\Omega)$  such that there exists a pressure  $\bar{p} \in L^2(\Omega)$  for which  $(\bar{u}, \bar{p})$  solves:*

$$(5) \quad \begin{cases} -\Delta \bar{u} + \nabla \bar{p} &= 6\pi(j - \rho \bar{u}), \\ \operatorname{div} \bar{u} &= 0, \end{cases} \quad \text{on } \Omega,$$

*completed with boundary conditions*

$$(6) \quad \bar{u} = 0, \quad \text{on } \partial\Omega.$$

With the assumptions (A1) and (A5), we may extract a subsequence such that the first momentums of  $S_N$  in  $v$  converge to some  $(\rho, j) \in L^\infty(\Omega) \times L^2(\Omega)$ . Hence, assumptions (A2) and (A3) only fix that the whole sequence converges to the same density  $\rho$  and momentum distribution  $j$ . For simplicity, we do not include a source term in (1) even if our result extends in a straightforward way to this case (due to the linearity of the Stokes equations).

The result above is compatible with the previous ones in [1] and [4]. Indeed, our set of assumptions is compatible with the periodic setting of [1] only if the minimal distance between holes behaves like  $1/N^{1/3}$ . This is also the configurations that are considered in [4]. In both references, the authors recover the same Stokes-Brinkman system that we obtain herein. The two other regimes that are considered in [1] are incompatible with our set of assumptions: if the perforation period (modeled by  $d_{min}^N$  in our case) is much larger than  $1/N^{1/3}$ , it is not possible to make  $N$  holes in  $\Omega$ ; if the perforation period is much smaller than  $1/N^{1/3}$ , assumption (A5) is not satisfied. The assumptions of our theorem extend the results in [1] and [4] in a new direction: it provides a broader class of finite-energy configurations with bounded density for which the homogenized system is (5)-(6). To be more precise, if the empiric measures  $S_N$  converge in the sense of (A2) to a bounded density  $\rho \in L^\infty(\Omega)$ , standard measure theory arguments show that (up to the extraction of a subsequence) there exists a sequence  $(\lambda^N)_{N \in \mathbb{N}}$  so that (A5) holds true. Then (A4) might be interpreted as a compatibility condition between the minimal distances  $(d_{min}^N)_{N \in \mathbb{N}}$  and

the sequence  $(\lambda^N)_{N \in \mathbb{N}}$  so that the homogenization process yields the Stokes-Brinkman problem (5)-(6). In a concluding section, we show that assumption (A4) is optimal and discuss the possible homogenized problems when it is not satisfied.

To conclude, one novelty of this paper is that we expect the two assumptions (A4)–(A5) are sufficiently general to tackle the time-evolution problem. Another novelty of the paper stems from the method of proof. We shall apply arguments that are not highly sensitive to the explicit value of solutions to the Stokes problem. The two main ingredients of the proof are the decrease of stokeslets (see (16)) and conservation arguments (see next subsection). In particular, we plan to consider more general shapes of holes and more general boundary conditions on holes in future works.

**1.1. Outline of the proof.** Our proof is based on a classical compactness argument. We first prove that the sequence  $(u^N)_{N \in \mathbb{N}}$  is bounded in  $H_0^1(\Omega)$ . This part is obtained by applying a variational characterization of solutions to Stokes problems and relies only upon (A1) and (4). We may then extract a subsequence (that we do not relabel) converging to some  $\bar{u}$  in  $H_0^1(\Omega)$  (and strongly in any  $L^q(\Omega)$  for  $q \in [1, 6]$ ). In order to identify a system satisfied by  $\bar{u}$  all that remains is devoted to the proof that:

$$I_w := \int_{\Omega} \nabla \bar{u} : \nabla w,$$

satisfies:

$$I_w = 6\pi \int_{\Omega} (j(x) - \rho(x)\bar{u}(x)) \cdot w(x) dx,$$

for arbitrary divergence-free  $w \in C_c^\infty(\Omega)$ . So, we fix a divergence-free  $w \in C_c^\infty(\Omega)$  and we note that, by construction, we have

$$I_w = \lim_{N \rightarrow \infty} I_w^N \quad \text{with} \quad I_w^N = \int_{\Omega} \nabla u^N : \nabla w, \quad \forall N \in \mathbb{N}.$$

We compute then  $I_w^N$  by applying that  $u^N$  is a solution to the Stokes problem (1)-(2). As the support of all the integrals  $I_w^N$  is  $\Omega$  and the support of  $w$  is not adapted to the Stokes problem (1)-(2), this requires special care. So, we introduce a covering  $(T_\kappa^N)_{\kappa \in \mathcal{K}^N}$  of  $\text{Supp}(w)$  with cubes of width  $\lambda^N$  and we split

$$I_w^N = \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w.$$

Given  $N$  and  $\kappa$ , we apply that there are not too many holes in  $T_\kappa^N$  because of assumption (A3). This enables to localize the method of reflections [9, 10] and replace  $w$  with

$$\sum_{i \in \mathcal{I}_\kappa^N} U^N[w(h_i^N)](x - h_i^N),$$

in the integral on  $T_\kappa^N$ . We denote here

- $\mathcal{I}_\kappa^N$  the subset of indices  $i \in \{1, \dots, N\}$  for which  $h_i^N \in T_\kappa^N$ ,

- $(U^N[v](y), P^N[v](y))$  the solution to the Stokes problem outside  $B(0, 1/N)$  with boundary condition  $U[v](y) = v$  on  $\partial B(0, 1/N)$  and vanishing condition at infinity.

We obtain that

$$\int_{T_\kappa^N} \nabla u^N : \nabla w \sim \sum_{i \in \mathcal{I}_\kappa^N} \int_{T_\kappa^N} \nabla u^N : \nabla [U^N[w(h_i^N)]](x - h_i^N).$$

Then, we observe that the pair

$$(U^N[w(h_i^N)](x - h_i^N), P^N[w(h_i^N)](x - h_i^N))$$

is a solution to the Stokes problem outside  $B_i^N$ . Hence, we apply that  $u^N$  is divergence-free, introduce the pressure and integrate by parts to obtain that:

$$\begin{aligned} \int_{T_\kappa^N} \nabla u^N : \nabla w \sim & \sum_{i \in \mathcal{I}_\kappa^N} \int_{\partial B_i^N} (\partial_n U^N[w(h_i^N)] - P^N[w(h_i^N)]n) \cdot v_i^N d\sigma \\ & + \int_{\partial T_\kappa^N} \sum_{i \in \mathcal{I}_\kappa^N} (\partial_n U^N[w(h_i^N)] - P^N[w(h_i^N)]n) \cdot u^N d\sigma. \end{aligned}$$

We skip for conciseness that  $(U^N, P^N)$  depends on  $(x - h_i^N)$  in these last identities. It is classical by the Stokes law that:

$$\int_{\partial B_i^N} (\partial_n U^N[w(h_i^N)] - P^N[w(h_i^N)]n) d\sigma = \frac{6\pi}{N} w(h_i^N),$$

and, by interpreting the Stokes system as the conservation of normal stress, that:

$$\int_{\partial T_\kappa^N} (\partial_n U[w(h_i^N)] - P^N[w(h_i^N)]n) d\sigma = -\frac{6\pi}{N} w(h_i^N).$$

To take advantage of this last identity, we use that the size of  $T_\kappa^N$  decreases to 0 and we replace  $u^N$  by some mean value  $\bar{u}_\kappa^N$  in the integral on  $\partial T_\kappa^N$ . Say for simplicity that:

$$(7) \quad \bar{u}_\kappa^N = \frac{1}{|T_\kappa^N|} \int_{T_\kappa^N} u^N(x) dx,$$

and assume that replacing  $u^N$  by  $\bar{u}_\kappa^N$  induces a small error in the boundary integral. We obtain then that:

$$\int_{T_\kappa^N} \nabla u^N : \nabla w \sim \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \cdot v_i^N - \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \cdot \bar{u}_\kappa^N.$$

Summing over  $\kappa$  yields:

$$I_w^N \sim \sum_{i=1}^N \frac{6\pi}{N} w(h_i^N) \cdot v_i^N - \sum_{\kappa \in \mathcal{K}^N} \left[ \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \right] \cdot \bar{u}_\kappa^N.$$

The first term on the right-hand side converges by assumption (A3) to :

$$6\pi \int_{\Omega} j(x) \cdot w(x) dx.$$

To compute the limit of the second term, we introduce:

$$\sigma^N = \frac{6\pi}{N|\lambda^N|^3} \sum_{\kappa \in \mathcal{K}^N} \left[ \sum_{i \in \mathcal{I}_{\kappa}^N} w(h_i^N) \right] \mathbf{1}_{T_{\kappa}^N},$$

so that:

$$\sum_{\kappa \in \mathcal{K}^N} \left[ \sum_{i \in \mathcal{I}_{\kappa}^N} \frac{6\pi}{N} w(h_i^N) \right] \cdot \bar{u}_{\kappa}^N = \int_{\Omega} \sigma^N \cdot u^N(x) dx.$$

For  $w \in C_c^{\infty}(\Omega)$ , we have that  $\sigma^N$  is bounded in  $L^1(\Omega)$  and, under assumption (A2), it converges to  $\sigma(x) = \rho(x)w(x)$  in  $\mathcal{D}'(\Omega)$ . However, this is not sufficient to compute the limit of this last term. Indeed we have strong convergence of the sequence  $u^N$  in  $L^q(\Omega)$  for  $q < 6$  only. Consequently, we need the supplementary assumption (A5) which entails that  $\sigma^N$  is bounded in  $L^{\infty}(\Omega)$ . Now,  $\sigma^N$  converges in  $L^q(\Omega) - w$  for arbitrary  $q \in (1, \infty)$  (up to the extraction of a subsequence) and combining this fact with the strong convergence of  $u^N$  we obtain that:

$$\lim_{N \rightarrow \infty} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_{\kappa}^N} \frac{6\pi}{N} w(h_i^N) \cdot \bar{u}_{\kappa}^N = \int_{\Omega} \rho(x)w(x) \cdot \bar{u}(x) dx.$$

This would end the proof if we could actually define  $\bar{u}_{\kappa}^N$  as in (7) and prove that it induces a small error by replacing  $u^N$  with the average  $\bar{u}_{\kappa}^N$  in the integral on  $\partial T_{\kappa}^N$ . Unfortunately, for this, we need that the combination of stokeslets to which  $u^N$  is multiplied is a solution to the Stokes equations on the set where the average is taken (in particular we cannot choose  $T_{\kappa}^N$  here contrary to what we have written in (7)). So, we introduce a parameter  $\delta$  (which will be large), we "delete" the holes in a  $\lambda^N/\delta$ -neighborhood of  $\partial T_{\kappa}^N$  and we construct  $\bar{u}_{\kappa}^N$  as the average of  $u^N$  on the  $\lambda^N/(2\delta)$ -neighborhood of  $\partial T_{\kappa}^N$  (inside  $T_{\kappa}^N$ ). By a suitable choice of the covering  $(T_{\kappa}^N)_{\kappa \in \mathcal{K}^N}$  we prove that the cost of this deletion process is  $O(1/\sqrt{\delta})$ . This relies on the two fundamental properties of our choice for the sets on which we average  $u^N$ : they are all obtained from a model annulus by translation and dilation, the non-deleted holes are "far" from this set (with respect to the decay of solutions to Stokes problems in exterior domains). Hence, we obtain that:

$$\left| I_w - 6\pi \int_{\Omega} (j(x) - \rho(x)\bar{u}(x)) \cdot w(x) dx \right| \lesssim \frac{1}{\sqrt{\delta}}$$

for arbitrary large  $\delta$ .



**1.2. Notations.** In the whole paper, for arbitrary  $x \in \mathbb{R}^3$  and  $r > 0$ , we denote  $B_\infty(x, r)$  the open ball with center  $x$  and radius  $r$  for the  $\ell^\infty$  norm. The classical euclidean balls are denoted  $B(x, r)$ . For  $x \in \mathbb{R}^3$  and  $0 < \lambda_1 < \lambda_2$  we also denote:

$$A(x, \lambda_1, \lambda_2) := B_\infty(x, \lambda_2) \setminus \overline{B_\infty(x, \lambda_1)}.$$

The operator distance (between sets) is always computed with the  $\ell^\infty$  norm. We will constantly use a truncation function associated to the parameter  $N$ . This truncation function is constructed in a classical way. We introduce  $\chi \in C_c^\infty(\mathbb{R}^3)$  a truncation function such that  $\chi = 1$  on  $[-1, 1]^3$  and  $\chi = 0$  outside  $[-2, 2]^3$ . We denote  $\chi^N = \chi(N \cdot)$  its rescaled versions. This truncation function satisfies :

- $\chi^N = 1$  on  $B_\infty(0, 1/N)$  and  $\chi^N = 0$  outside  $B_\infty(0, 2/N)$ ,
- $\nabla \chi^N$  has support in  $A(0, 1/N, 2/N)$  and size  $O(N)$ .

When we truncate vector-fields with  $\chi^N$  we shall create *a priori* non divergence-free vector-fields. To lift the divergence of these vector-fields, we use extensively the Bogovskii operator  $\mathfrak{B}_{x, \lambda_1, \lambda_2}$  on the "cubic" annulus  $A(x, \lambda_1, \lambda_2)$  (again  $x \in \mathbb{R}^3$  and  $0 < \lambda_1 < \lambda_2$ ). We recall that  $w = \mathfrak{B}_{x, \lambda_1, \lambda_2}[f]$  is defined for arbitrary  $f \in L^2(A(x, \lambda_1, \lambda_2))$ , whose mean vanishes, and yields an  $H_0^1(A(x, \lambda_1, \lambda_2))$  vector-field such that  $\operatorname{div} w = f$ . As the returned vector-field vanishes on  $\partial A(x, \lambda_1, \lambda_2)$  we extend it by 0 to obtain an  $H^1(\mathbb{R}^3)$  function.

For legibility we also make precise a few conventions. We have the following generic notations:

- $u$  is a velocity-field solution to a Stokes problem, with associated pressure  $p$ ,
- $w$  is a test-function,
- $I$  is an integral while  $\mathcal{I}$  is a set of indices,
- $T$  is a cube, depending on the width we shall use different exponents,
- $n$  denotes the outward normal to the open set under consideration.

We shall also use extensively the symbol  $\lesssim$  to denote that we have an inequality with a non-significant constant. We mean that we denote  $a \lesssim b$  when there exists a constant  $C$ , which is not relevant to our problem, such that  $a \leq Cb$ . In most cases "not relevant" will mean that it does not depend on the parameters  $N$  and/or  $\delta$ . If a more precise statement of this "non-relevance" is required we shall make it precise.

**1.3. Outline of the paper.** As our proof is based on fine properties of the Stokes problem, we recall in next section basics and advanced material on the resolution of this problem in bounded domains, in exterior domains and in a model cell domain. The core of the paper is sections 4 and 5 where a more rigorous statement of our main result is given and the proof is developed. In a concluding section, we provide some remarks and examples on the optimality/limits of our dilution assumptions. Finally, we collect in two appendices technical properties on the Bogovskii operators, Poincaré-Wirtinger inequalities and covering arguments in measure theory.

## 2. ANALYSIS OF THE STOKES PROBLEM

In this section, we recall how can be solved the Stokes problem:

$$(8) \quad \begin{cases} -\Delta u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \end{cases} \quad \text{on } \mathcal{F},$$

completed with boundary conditions

$$(9) \quad u(x) = u_*, \quad \text{on } \partial\mathcal{F},$$

for a lipschitz domain  $\mathcal{F}$  and boundary condition  $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$ . We consider the different cases:  $\mathcal{F}$  is a bounded set, an exterior domain, or a perforated cube. In the second case, we complement the system with a vanishing condition at infinity.

**2.1. Reminders on the Stokes problem in a bounded or an exterior domain.** We first assume that  $\mathcal{F}$  is a bounded domain with a lipschitz boundary  $\partial\mathcal{F}$ . In this setting, a standard way to solve the Stokes problem (8)-(9) is to work with a generalized formulation (see [7, Section 4]). For this, we introduce:

$$D(\mathcal{F}) := \{u \in H^1(\mathcal{F}) \text{ s.t. } \operatorname{div} u = 0\}, \quad D_0(\mathcal{F}) := \{u \in H_0^1(\mathcal{F}) \text{ s.t. } \operatorname{div} u = 0\}.$$

By [7, Theorem III.4.1], we have that  $D_0(\mathcal{F})$  is the closure for the  $H_0^1(\Omega)$ -norm of

$$\mathcal{D}_0(\mathcal{F}) = \{w \in C_c^\infty(\mathcal{F}) \text{ s.t. } \operatorname{div} w = 0\}.$$

We have then the following definition

**Definition 2.** *Given  $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$ , a vector-field  $u \in D(\mathcal{F})$  is called generalized solution to (8)-(9) if*

- $u = u_*$  on  $\partial\mathcal{F}$  in the sense of traces,
- for arbitrary  $w \in D_0(\mathcal{F})$ , there holds:

$$(10) \quad \int_{\mathcal{F}} \nabla u : \nabla w = 0.$$

This generalized formulation is obtained assuming that we have a classical solution, multiplying (8) with arbitrary  $w \in \mathcal{D}_0(\mathcal{F})$  and performing integration by parts. De Rham theory ensures that conversely, if one constructs a generalized solution then it is possible to find a pressure  $p$  such that (8) holds in the sense of distributions. Standard arguments yield:

**Theorem 3.** *Assume that the boundary of the fluid domain  $\partial\mathcal{F}$  splits into  $(N + 1) \in \mathbb{N}$  lipschitz connected components  $\Gamma_0, \Gamma_1, \dots, \Gamma_N$ . Given  $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$  satisfying*

$$(11) \quad \int_{\Gamma_i} u_* \cdot n d\sigma = 0, \quad \forall i \in \{0, \dots, N\},$$

then

- there exists a unique generalized solution  $u$  to (19)-(20);

- *this generalized solution realizes*

$$(12) \quad \inf \left\{ \int_{\mathcal{F}} |\nabla u|^2, u \in D(\mathcal{F}) \text{ s.t. } u|_{\partial\mathcal{F}} = u_* \right\}.$$

*Proof.* Existence and uniqueness of the generalized solution is a consequence of [7, Theorem IV.1.1]. A key argument in the proof of this reference is the property of traces that we state in the following lemma:

**Lemma 4.** *For arbitrary  $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$  satisfying (11) there holds:*

- *there exists  $u_{bdy} \in D(\mathcal{F})$  having trace  $u_*$  on  $\partial\mathcal{F}$ ,*
- *for arbitrary  $u_{bdy} \in D(\mathcal{F})$  having trace  $u_*$  on  $\partial\mathcal{F}$  there holds*

$$\{u \in D(\mathcal{F}) \text{ s.t. } u|_{\partial\mathcal{F}} = u_*\} = u_{bdy} + D_0(\mathcal{F}).$$

Then, given  $u \in D(\mathcal{F})$  the generalized solution to (19)-(20) and  $w \in D_0(\mathcal{F})$ , the fundamental property (10) of  $u$  entails that:

$$\begin{aligned} \int_{\mathcal{F}} |\nabla(u+w)|^2 &= \int_{\mathcal{F}} |\nabla u|^2 + 2 \int_{\mathcal{F}} \nabla u : \nabla w + \int_{\mathcal{F}} |\nabla w|^2, \\ &= \int_{\mathcal{F}} |\nabla u|^2 + \int_{\mathcal{F}} |\nabla w|^2. \end{aligned}$$

Consequently, the norm on the left-hand side is minimal if and only if  $w = 0$ . Combining this remark with the above lemma yields that the generalized solution to (19)-(20) is the unique minimizer of (12) in  $\{v \in D(\mathcal{F}) \text{ s.t. } v|_{\partial\mathcal{F}} = u_*\}$ .  $\square$

As mentioned previously, once it is proven that there exists a unique generalized solution  $u$  to (8)-(9), it is possible to recover a pressure  $p$  so that (8)-(9) holds in the sense of distributions. If the data are smooth (*i.e.*  $\mathcal{F}$  has smooth boundaries and  $u_*$  is smooth) one proves also that  $(u, p) \in C^\infty(\overline{\mathcal{F}})$ .

We turn to the exterior problem as developed in [7, Section 5]. We assume now that  $\mathcal{F} = \mathbb{R}^3 \setminus \overline{B^a}$  where  $B^a = B(0, 1/a)$  and we consider the Stokes problem (8) with boundary condition

$$(13) \quad u = u_* \text{ on } \partial B^a, \quad \lim_{|x| \rightarrow \infty} u(x) = 0,$$

for some  $u_* \in H^{\frac{1}{2}}(\partial B^a)$ . For the exterior problem, we keep the definition of generalized solution up to change a little the function spaces. We denote in this case:

- $\mathcal{D}(\mathcal{F}) = \{w|_{\mathcal{F}}, w \in C_c^\infty(\mathbb{R}^3) \text{ s.t. } \operatorname{div} w = 0\}$ ,
- $D(\mathcal{F})$  is the closure of  $\mathcal{D}(\mathcal{F})$  for the norm:

$$\|w\|_{D(\mathcal{F})} = \left( \int_{\mathcal{F}} |\nabla w|^2 \right)^{\frac{1}{2}}.$$

We keep the definition of  $\mathcal{D}_0(\mathcal{F})$  as in the bounded-domain case and we construct  $D_0(\mathcal{F})$  as the closure of  $\mathcal{D}_0(\mathcal{F})$  with respect to this latter homogeneous  $H^1$ -norm. We note that, in the exterior domain case, we still have that  $D(\mathcal{F}) \subset W_{loc}^{1,2}(\mathcal{F})$  (see [7, Lemma II.6.1]) so that we have a trace operator on  $\partial B^a$  and an equivalent to Lemma 4.

As in the case of bounded domains, the Stokes problem (8)-(13) with boundary conditions  $u_*$  prescribing no flux through  $\partial B^a$  has a unique generalized solution (see [7, Theorem V.2.1], actually the no-flux assumption is not necessary for the exterior problem). Thus, this solution satisfies:

- $\nabla u \in L^2(\mathbb{R}^3 \setminus \overline{B^a})$ ,
- for any  $w \in D_0(\mathbb{R}^3 \setminus \overline{B^a})$  there holds:

$$\int_{\mathbb{R}^3 \setminus \overline{B^a}} \nabla u : \nabla w = 0.$$

Explicit formulas are provided when the boundary condition  $u_* = v$  with  $v \in \mathbb{R}^3$  constant (see [4, Section 6.2] for instance):

$$(14) \quad u(x) = U^a[v](x) := \frac{1}{4a} \left( \frac{3}{|x|} + \frac{1}{a^2|x|^3} \right) v + \frac{3}{4a} \left( \frac{1}{|x|} - \frac{1}{a^2|x|^3} \right) \frac{v \cdot x}{|x|^2} x,$$

$$(15) \quad p(x) = P^a[v](x) := \frac{3}{2a} \frac{v \cdot x}{|x|^3}.$$

We call this classical solution stokeslet in what follows. With these explicit formulas, we remark that:

$$(16) \quad |U^a[v](x)| \lesssim \frac{|v|}{a|x|}, \quad |\nabla U^a[v](x)| + |P^a[v](x)| \lesssim \frac{|v|}{a|x|^2}, \quad \forall x \in \mathbb{R}^3 \setminus \overline{B_a},$$

and we recall that the force exerted by the flow on  $\partial B^a$  reads:

$$(17) \quad \int_{\partial B^a} (\partial_n U^a[v] - P^a[v]n) d\sigma = \frac{6\pi}{a} v.$$

For convenience, the stokeslet  $U^a[v]$  is extended by  $U^a[v] = v$  on  $B^a$  in what follows.

**2.2. Stokes problem in a perforated cube.** In this last subsection, we fix  $(N, M, \lambda) \in (\mathbb{N} \setminus \{0\})^2 \times (0, \infty)$ , and a divergence-free  $w \in C_c^\infty(\mathbb{R}^3)$ . We denote  $T$  an open cube of width  $\lambda$  and  $B_i^N = B(h_i, 1/N) \subset T$  for  $i = 1, \dots, M$ . We assume further that there exists  $d_m$  satisfying

$$(18) \quad \min_{i=1, \dots, M} \left\{ \text{dist}(h_i, \partial T), \min_{j \neq i} (|h_i - h_j|) \right\} \geq d_m > \frac{4}{N}.$$

We consider the Stokes problem:

$$(19) \quad \begin{cases} -\Delta u + \nabla p &= 0, \\ \text{div } u &= 0, \end{cases} \quad \text{on } \mathcal{F} = T \setminus \bigcup_{i=1}^M \overline{B_i^N},$$

completed with boundary conditions

$$(20) \quad \begin{cases} u(x) = w(x), & \text{on } B_i^N, \forall i = 1, \dots, M, \\ u(x) = 0, & \text{on } \partial T. \end{cases}$$

Assumption (18) entails that the  $B_i^N$  do not intersect and do not meet the boundary  $\partial T$ . So, the set  $T \setminus \bigcup_{i=1}^M \overline{B_i^N}$  has a lipschitz boundary that one can decompose in  $M + 1$  connected component corresponding to  $\partial T$  and  $\partial B_i^N$  for  $i = 1, \dots, M$ .

For any  $i = 1, \dots, M$ , direct computations show that:

$$\int_{\partial B_i^N} w \cdot n d\sigma = \int_{B_i^N} \operatorname{div} w = 0.$$

Hence, the problem (19)-(20) is solved by applying Theorem 3 and it admits a unique generalized solution  $u \in H^1(\mathcal{F})$ . We want to compare this solution with:

$$u_s(x) = \sum_{i=1}^M U^N[w(h_i)](x - h_i),$$

where  $U^N$  is the stokeslet as defined in (14). The main result of this subsection is:

**Proposition 5.** *There exists a constant  $K$  independent of  $(N, M, d_m, w, \lambda)$  for which:*

$$\|(u - u_s)\|_{L^6(\mathcal{F})} + \|\nabla(u - u_s)\|_{L^2(\mathcal{F})} \leq K \|w\|_{W^{1,\infty}(\mathbb{R}^3)} \sqrt{\frac{M}{N}} \left[ \frac{1}{N} + \sqrt{\frac{M}{Nd_m}} \right].$$

*Proof.* We split the error term into two pieces. First, we reduce the boundary conditions of the Stokes problem (19)-(20) to constant boundary conditions. Then, we compare the solution to the Stokes problem with constant boundary conditions to the combination of stokeslets  $u_s$ . In the whole proof, the symbol  $\lesssim$  is used when the implicit constant in our inequality does not depend on  $N, M, d_m, w$  and  $\lambda$ .

So, we introduce  $u_c$  the unique generalized solution to the Stokes problem on  $\mathcal{F}$  with boundary conditions:

$$(21) \quad \begin{cases} u_c = w(h_i), & \text{on } B_i^N, \forall i = 1, \dots, M, \\ u_c = 0, & \text{on } \partial T. \end{cases}$$

Again, existence and uniqueness of this velocity-field holds by applying Theorem 3. We split then:

$$\begin{aligned} \|(u - u_s)\|_{L^6(\mathcal{F})} &\leq \|(u - u_c)\|_{L^6(\mathcal{F})} + \|(u_c - u_s)\|_{L^6(\mathcal{F})}, \\ \|\nabla(u - u_s)\|_{L^2(\mathcal{F})} &\leq \|\nabla(u - u_c)\|_{L^2(\mathcal{F})} + \|\nabla(u_c - u_s)\|_{L^2(\mathcal{F})}. \end{aligned}$$

To control the first term on the right-hand sides, we note that  $(u - u_c)$  is the unique generalized solution to the Stokes problem on  $\mathcal{F}$  with boundary conditions:

$$\begin{cases} (u - u_c)(x) = w(x) - w(h_i), & \text{on } B_i^N, \forall i = 1, \dots, M, \\ (u - u_c)(x) = 0, & \text{on } \partial T. \end{cases}$$

Hence, by the variational characterization of Theorem 3,  $\|\nabla(u - u_c)\|_{L^2(\mathcal{F})}$  realizes the minimum of  $\|\nabla\tilde{w}\|_{L^2(\mathcal{F})}$  amongst

$$\left\{ \tilde{w} \in H^1(\mathcal{F}) \text{ s.t. } \operatorname{div} \tilde{w} = 0, \tilde{w}|_{\partial T} = 0, \tilde{w}|_{\partial B_i^N} = w(\cdot) - w(h_i), \forall i = 1, \dots, M \right\}.$$

We construct thus a suitable  $\tilde{w}$  in this space. We set:

$$\tilde{w} = \sum_{i=1}^M \tilde{w}_i$$

with, for  $i = 1, \dots, M$ :

$$\tilde{w}_i = \left( \chi^N(\cdot - h_i)(w(\cdot) - w(h_i)) - \mathfrak{B}_{h_i, 1/N, 2/N} [x \mapsto (w(x) - w(h_i)) \cdot \nabla \chi^N(x - h_i)] \right).$$

We recall that  $\chi^N$  is a chosen function that truncates between  $B_\infty(0, 1/N)$  and  $B_\infty(0, 2/N)$  and that we denote  $\mathfrak{B}_{h_i, 1/N, 2/N}$  the Bogovskii operator on the annulus  $A(h_i, 1/N, 2/N)$ . The properties of this operator are analyzed in Appendix A. The above vector-field  $\tilde{w}_i$  is well-defined as, for  $i = 1, \dots, M$ , there holds:

$$\begin{aligned} \int_{A(h_i, 1/N, 2/N)} (w(x) - w(h_i)) \cdot \nabla \chi^N(x - h_i) dx \\ &= \int_{B_\infty(h_i, 2/N) \setminus B_\infty(h_i, 1/N)} \operatorname{div}(\chi^N(\cdot - h_i)(w(\cdot) - w(h_i))) , \\ &= \int_{\partial B_\infty(h_i, 1/N)} (w(x) - w(h_i)) \cdot n d\sigma , \\ &= \int_{B_\infty(h_i, 1/N)} \operatorname{div}(w) = 0 , \end{aligned}$$

and we can apply the Bogovskii operator to  $x \mapsto (w(\cdot) - w(h_i)) \cdot \nabla \chi^N(\cdot - h_i)$  on the annulus  $A(h_i, 1/N, 2/N)$ . We note that  $\tilde{w}_i$  has support in  $B_\infty(h_i, 2/N)$  so that, as  $d_m > 4/N$ , the  $\tilde{w}_i$  have disjoint supports inside  $T$ . This yields that  $\tilde{w}$  is indeed divergence-free and fits the required boudary conditions. Furthermore, there holds:

$$\|\nabla \tilde{w}\|_{L^2(\mathcal{F})} \leq \left[ \sum_{i=1}^M \|\nabla \tilde{w}_i\|_{L^2(B_\infty(h_i, 2/N))}^2 \right]^{\frac{1}{2}}.$$

For  $i \in \{1, \dots, M\}$  we have by direct computations:

$$\begin{aligned} \|\nabla \chi^N(\cdot - h_i)(w(\cdot) - w(h_i))\|_{L^2(B_\infty(h_i, 2/N))}^2 &\lesssim \frac{\|w\|_{W^{1,\infty}}^2}{N^3}, \\ \|\chi^N(\cdot - h_i) \nabla(w(\cdot) - w(h_i))\|_{L^2(B_\infty(h_i, 2/N))}^2 &\lesssim \frac{\|w\|_{W^{1,\infty}}^2}{N^3}, \end{aligned}$$

and, by applying Lemma 16:

$$\begin{aligned}
& \|\nabla \mathfrak{B}_{h_i, 1/N, 2/N} [x \mapsto (w(x) - w(h_i)) \cdot \nabla \chi^N(x - h_i)]\|_{L^2(B_\infty(h_i, 2/N))}^2 \\
& \lesssim \|x \mapsto (w(x) - w(h_i)) \cdot \nabla \chi^N(x - h_i)\|_{L^2(B_\infty(h_i, 2/N))}^2 \\
& \lesssim \frac{\|w\|_{W^{1,\infty}}^2}{N^3}.
\end{aligned}$$

Gathering all these inequalities in the computation of  $\tilde{w}$  yields finally:

$$\|\nabla \tilde{w}\|_{L^2(\mathcal{F})} \lesssim \sqrt{M} \frac{\|w\|_{W^{1,\infty}}}{N^{\frac{3}{2}}}.$$

The variational characterization of generalized solutions to Stokes problems entails that we have the same bound for  $(u - u_c)$ . At this point, we argue that the straightforward extension of  $u$  and  $u_c$  (by  $w$  and  $w(h_i)$  on the  $B_i^N$  respectively) satisfy  $(u - u_c) \in H_0^1(T) \subset L^6(T)$  so that

$$\begin{aligned}
\|u - u_c\|_{L^6(\mathcal{F})} & \leq \|u - u_c\|_{L^6(T)} \lesssim \|\nabla(u - u_c)\|_{L^2(T)} \\
& \lesssim \left( \|\nabla(u - u_c)\|_{L^2(\mathcal{F})}^2 + M \frac{\|w\|_{W^{1,\infty}}^2}{N^3} \right)^{\frac{1}{2}} \\
& \lesssim \sqrt{M} \frac{\|w\|_{W^{1,\infty}}}{N^{\frac{3}{2}}}.
\end{aligned}$$

We emphasize that, by a scaling argument, the constant deriving from the embedding  $H_0^1(T) \subset L^6(T)$  does not depend on  $\lambda$  so that it is not significant to our problem.

We turn to estimating  $u_c - u_s$ . Due to the linearity of the Stokes equations, we split

$$u_c = \sum_{i=1}^M u_{c,i},$$

where  $u_{c,i}$  is the generalized solution to the Stokes problem on  $\mathcal{F}$  with boundary conditions:

$$\begin{cases} u_{c,i} &= w(h_i), & \text{on } \partial B_i^N, \\ u_{c,i} &= 0, & \text{on } \partial T \cup \bigcup_{j \neq i} \partial B_j^N. \end{cases}$$

We have then

$$(22) \quad \|\nabla(u_c - u_s)\|_{L^2(\mathcal{F})} \leq \sum_{i=1}^M \|\nabla(u_{c,i} - U^N[w(h_i)](\cdot - h_i))\|_{L^2(\mathcal{F})}.$$

Similarly, we expand :

$$u_s = \sum_{i=1}^M U_i, \text{ where } U_i(x) = U^N[w(h_i)](x - h_i), \quad \forall x \in \mathbb{R}^3.$$

For  $i \in \{1, \dots, M\}$  we extend  $u_{c,i}$  by 0 on  $\mathbb{R}^3 \setminus T$  and  $B_j^N$  for  $j \neq i$ . The extension we still denote by  $u_{c,i}$  satisfies  $u_{c,i} \in H^1(\mathbb{R}^3 \setminus \overline{B_i^N})$  and is divergence-free. In particular, we have  $u_{c,i} \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$ . Consequently,  $u_{c,i} - U_i \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$  and:

$$\begin{aligned} \|\nabla(u_{c,i} - U_i(\cdot - h_i))\|_{L^2(\mathcal{F})}^2 &\leq \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla u_{c,i}(x) - \nabla U_i(x)|^2 dx \\ &\leq \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla u_{c,i}|^2 - 2 \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} \nabla u_{c,i} : \nabla U_i + \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla U_i|^2. \end{aligned}$$

To compute the product term, we apply that  $u_{c,i}$  and  $U_i = U^N[w(h_i)](\cdot - h_i)$  have the same trace on  $\partial B_i^N$  and that  $U_i$  is a generalized solution to the Stokes problem on  $\mathbb{R}^3 \setminus \overline{B_i^N}$ . So, integrals of the form  $\int_{\mathbb{R}^3 \setminus \overline{B_i^N}} \nabla U_i : \nabla w$  (for  $w \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$ ) depend only on the trace of  $w$  on  $\partial B_i^N$ . This entails that:

$$\int_{\mathbb{R}^3 \setminus \overline{B_i^N}} \nabla u_{c,i} : \nabla U_i = \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla U_i|^2,$$

and we have:

$$(23) \quad \|\nabla(u_{c,i} - U^N(\cdot - h_i))\|_{L^2(\mathcal{F})}^2 \leq \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla u_{c,i}|^2 - \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla U_i|^2.$$

To conclude, we find a bound from above for

$$\int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\nabla u_{c,i}(x)|^2 dx = \int_{\mathcal{F}} |\nabla u_{c,i}(x)|^2 dx.$$

As  $u_{c,i}$  is a generalized solution to a Stokes problem on  $\mathcal{F}$ , this can be done by constructing a divergence-free  $\bar{w}_i$  satisfying the same boundary condition as  $u_{c,i}$ . We define:

$$\bar{w}_i = \chi_{d_m/4}(\cdot - h_i)U_i - \mathfrak{B}_{h_i, d_m/4, d_m/2} [x \mapsto U_i(x) \cdot \nabla \chi_{d_m/4}(x - h_i)]$$

where  $\chi_{d_m/4} := \chi^{4/d_m}$  (with the family of truncation functions of the introduction). As previously, we have here a divergence-free function which satisfies the right boundary conditions because  $\chi_{d_m/4}(\cdot - h_i) = 1$  on  $B_i^N$  (since  $d_m/4 > 1/N$ ) and vanishes on all the other boundaries of  $\partial \mathcal{F}$  (since the distance between one hole center and the other holes or  $\partial T$  is larger than  $d_m - 1/N > d_m/2$ ). Again, similarly as in the computation of  $\tilde{w}_i$  we apply the properties of the Bogovskii operator  $\mathfrak{B}_{h_i, d_m/4, d_m/2}$  and there exists an absolute constant  $K$  for which:

$$\begin{aligned} \|\nabla \bar{w}_i\|_{L^2(\mathcal{F})}^2 &\leq \int_{\mathbb{R}^3 \setminus \overline{B_i^N}} |\chi_{d_m/4}(\cdot - h_i) \nabla U_i|^2 \\ &\quad + K \left( \int_{A(h_i, d_m/4, d_m/2)} |\nabla U_i(x)|^2 + |\nabla \chi_{d_m/4}(x - h_i) \otimes U_i(x)|^2 dx \right) \end{aligned}$$



As we have the same bound for  $u_{c,i}$ , we plug the right-hand side above in (23) and get:

$$\begin{aligned} \|\nabla(u_{c,i} - U_i)\|_{L^2(\mathcal{F})}^2 &\lesssim \int_{\mathbb{R}^3 \setminus B(h_i, d_m/4)} |\nabla U_i(x)|^2 dx \\ &\quad + \int_{A(h_i, d_m/4, d_m/2)} |\nabla \chi_{d_m/4}(x - h_i) \otimes U_i(x)|^2 dx. \end{aligned}$$

With the explicit decay properties for  $U_i$  (see (16)) and  $\nabla \chi_{d_m/4}$  we derive:

$$\int_{\mathbb{R}^3 \setminus B(h_i, d_m/4)} |\nabla U_i(x)|^2 dx + \int_{A(h_i, d_m/4, d_m/2)} |\nabla \chi_{d_m/4}(x - h_i) \otimes U_i(x)|^2 dx \lesssim \frac{\|w\|_{W^{1,\infty}}^2}{N^2 d_m}.$$

Combining these bounds for  $i = 1, \dots, M$  in (22) we get:

$$\|\nabla(u_c - u_s)\|_{L^2(\mathcal{F})} \leq \frac{M\|w\|_{W^{1,\infty}}}{N\sqrt{d_m}}.$$

By similar arguments, we also have:

$$\|u_c - u_s\|_{L^6(\mathcal{F})} = \|u_c - u_s\|_{L^6(T)} \leq \sum_{i=1}^M \|u_{c,i} - U_i\|_{L^6(\mathbb{R}^3 \setminus \overline{B_i^N})}.$$

As  $u_{c,i}, U_i \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$  and  $u_{c,i}, U_i$  share the same value on  $\partial B_i^N$ , there holds  $u_{c,i} - U_i \in D_0(\mathbb{R}^3 \setminus \overline{B_i^N})$  and we may use the classical inequality (see [7, (II.6.9)]):

$$\|u_{c,i} - U_i\|_{L^6(\mathbb{R}^3 \setminus \overline{B_i^N})} \lesssim \|\nabla u_{c,i} - \nabla U_i\|_{L^2(\mathbb{R}^3 \setminus \overline{B_i^N})}, \quad \forall i = 1, \dots, M,$$

(again the constant arising from this embedding does not depend on  $N$  by a standard scaling argument). This yields again the bound:

$$\|(u_c - u_s)\|_{L^6(\mathcal{F})} \leq \frac{M\|w\|_{W^{1,\infty}}}{N\sqrt{d_m}},$$

and ends the proof of our proposition.  $\square$

### 3. PROOF OF THEOREM 1 – UNIFORM ESTIMATES

From now on, we fix a sequence of data  $(v_i^N, h_i^N)_{i=1,\dots,N}$  associated with  $(B_i^N)_{i=1,\dots,N}$  that satisfy (A0) for arbitrary  $N \in \mathbb{N}$  and such that (A1)–(A3) hold true with

$$j \in L^2(\Omega), \quad \rho \in L^\infty(\Omega).$$

We introduce also  $(\lambda^N)_{N \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$  for which we have (A4)–(A5). Because of assumption (A0), the existence result of the previous section applies so that there exists a unique generalized solution  $u^N \in H^1(\mathcal{F}^N)$  to (1)–(2). In what follows, we extend implicitly  $u^N$  by its boundary values on the  $\partial B_i^N$ :

$$u^N = \begin{cases} v_i^N, & \text{in } B_i^N, \text{ for } i = 1, \dots, N, \\ u^N, & \text{in } \mathcal{F}^N. \end{cases}$$

As the  $B_i^N$  do not overlap and do not meet  $\partial\Omega$ , it is straightforward that these velocity-fields yield a sequence in  $H_0^1(\Omega)$  of divergence-free vector-fields. Moreover, we have the property:

$$\|\nabla u^N\|_{L^2(\mathcal{F}^N)} = \|\nabla u^N\|_{L^2(\Omega)}.$$

Our target result reads:

**Theorem 6.** *The sequence of extended generalized solutions  $(u^N)_{N \in \mathbb{N}}$  converges weakly in  $H_0^1(\Omega)$  to  $\bar{u}$  satisfying*

$$(B1) \quad \bar{u} \in H_0^1(\Omega),$$

$$(B2) \quad \operatorname{div} \bar{u} = 0 \text{ on } \Omega,$$

$$(B3) \quad \text{for any divergence-free } w \in C_c^\infty(\Omega) \text{ we have:}$$

$$(24) \quad \int_{\Omega} \nabla \bar{u} : \nabla w = 6\pi \int_{\Omega} [j - \rho \bar{u}] \cdot w.$$

Theorem 1 is a corollary of this theorem as (B1)-(B2)-(B3) corresponds to the generalized formulation of the Stokes-Brinkman system (5)-(6). The proof of this result is developed in the end of this section and the two next ones.

Let first compute uniform bounds on  $u^N$  by applying the variational characterization of solutions to the Stokes problem (12). Given  $N \in \mathbb{N}$ , we set:

$$v^N(x) = \sum_{i=1}^N \nabla \times \left( \frac{\chi^N(x - h_i^N)}{2} v_i^N \times (x - h_i^N) \right) =: \sum_{i=1}^N v_i(x).$$

Then,  $v^N \in C_c^\infty(\mathbb{R}^3)$  is the curl of a smooth potential vector so that  $\operatorname{div} v^N = 0$ . Because of assumptions (A4)-(A5) (see (4)), there exists a  $N_0 \in \mathbb{N}$  such that:

$$Nd_{min}^N > 4, \quad \forall N > N_0.$$

Let  $N > N_0$  from now on. Because  $\chi^N$  has support in  $B_\infty(0, 2/N)$  we have that  $\operatorname{Supp}(v_i) \subset B_\infty(h_i^N, 2/N)$  and the  $(v_i)_{i=1, \dots, N}$  have disjoint supports. Because  $\chi^N$  is 1 on  $B(0, 1/N) \subset B_\infty(0, 1/N)$  we derive further that, for  $i \in \{1, \dots, N\}$ :

$$\begin{aligned} v_i(x) &= 0, \quad \text{on } \partial\Omega \cup \bigcup_{j \neq i} B_j^N, \\ v_i(x) &= \nabla \times \left( \frac{1}{2} v_i^N \times (x - h_i^N) \right) = v_i^N, \quad \text{on } B_i^N. \end{aligned}$$

By combination, we obtain:

$$\begin{aligned} v^N(x) &= v_i^N, \quad \text{on } B_i^N, \quad \forall i = 1, \dots, N, \\ v^N(x) &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

We have then by Theorem 3 that:

$$(25) \quad \|\nabla u^N\|_{L^2(\mathcal{F}^N)} \leq \|\nabla v^N\|_{L^2(\mathcal{F}^N)} = \left( \sum_{i=1}^N \|\nabla v_i\|_{L^2(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}}.$$

For arbitrary  $N \in \mathbb{N}$  and  $i \in \{1, \dots, N\}$ , there holds:

$$\begin{aligned} |\nabla v_i(x)| &\lesssim |\nabla \chi^N(x - h_i^N)| |v_i^N| + |\nabla^2 \chi^N(x - h_i^N)| |v_i^N| |x - h_i^N| \\ &\lesssim N (|\nabla \chi(N(x - h_i^N))| + |\nabla^2 \chi(N(x - h_i^N))|) |v_i^N|. \end{aligned}$$

Consequently, by a standard scaling argument:

$$\int_{\mathbb{R}^3} |\nabla v_i(x)|^2 dx \lesssim \frac{1}{N} \left( \int_{\mathbb{R}^3} |\nabla \chi(|y|)|^2 + |\nabla^2 \chi(|y|)|^2 dy \right) |v_i^N|^2.$$

Then, for  $N > N_0$ , we combine the previous computation into:

$$\|\nabla v^N\|_{L^2(\mathcal{F}^N)}^2 \lesssim \frac{1}{N} \sum_{i=1}^N |v_i^N|^2.$$

Note that  $\chi$  is fixed a priori so that all constants depending on  $\chi$  may be considered as non-significant. Assumption (A1) then yields that there exists  $\mathcal{E}^\infty < \infty$  so that:

$$(26) \quad \left( \frac{1}{N} \sum_{i=1}^N |v_i^N|^2 \right)^{\frac{1}{2}} \leq \mathcal{E}^\infty, \quad \forall N > N_0.$$

By (25) the norm of  $u^N$  in  $H_0^1(\Omega)$  is also bounded by  $\mathcal{E}^\infty$ . We keep the symbol  $\mathcal{E}^\infty$  to denote the above bound in what follows.

As  $u^N$  is bounded in  $H_0^1(\Omega)$ , it is weakly-compact and we denote by  $\bar{u}$  a cluster-point for the weak topology. It is straightforward that  $\bar{u}$  satisfies  $\operatorname{div} \bar{u} = 0$  on  $\Omega$ . So  $\bar{u}$  satisfies (B1) and (B2) of our theorem. The remainder of the proof consists in showing that it satisfies (B3) also. Indeed, we remark that  $\rho$  is the density of a probability measure. Hence  $\rho \geq 0$  on  $\Omega$ . By a simple energy estimate one may then show that, given  $j \in L^2(\Omega)$ , there exists at most one  $\bar{u} \in H_0^1(\Omega)$  that satisfies simultaneously (B1)-(B2)-(B3). A direct corollary of this remark is that, if we prove that (B3) is satisfied by  $\bar{u}$  we have uniqueness of the possible cluster point to the sequence  $(u^N)_{N \in \mathbb{N}}$  and the whole sequence converges to this  $\bar{u}$  in  $H_0^1(\Omega) - w$ .

#### 4. PROOF OF THEOREM 1 – COMPUTATIONS FOR FINITE $N$

From now on, we assume that  $u^N$  converges weakly to  $\bar{u}$  in  $H_0^1(\Omega)$  (we do not relabel the subsequence for simplicity) and we fix a divergence-free  $w \in C_c^\infty(\Omega)$ . We aim to compute the scalar product:

$$\int_{\Omega} \nabla \bar{u} : \nabla w.$$

By definition, we have:

$$\int_{\Omega} \nabla \bar{u} : \nabla w = \lim_{N \rightarrow \infty} I^N \text{ with } I^N = \int_{\Omega} \nabla u^N : \nabla w, \quad \forall N \in \mathbb{N}.$$

As classical, we want to apply the equation satisfied by  $u^N$  in order to compute  $I^N$  in a way that makes possible to use the assumption on the convergence of the empiric measures  $S_N$ . To do this, we fix an integer  $\delta \geq 4$ , we construct, for fixed  $N$ , a suitable test-function  $w^s$  (depending actually on  $\delta$  and  $N$ ) so that

- we make an error of order  $1/\sqrt{\delta}$  by replacing  $w$  with  $w^s$  in  $I^N$ ,
- replacing  $w$  with  $w^s$  in  $I^N$  we prove that,

$$\int_{\Omega} \nabla u^N : \nabla w^s \rightarrow 6\pi \int_{\Omega} (j - \rho \bar{u}) \cdot w + \text{error},$$

when  $N \rightarrow \infty$ , with an *error* of size  $1/\sqrt{\delta}$ .

As  $\delta$  can be taken arbitrary large, this yields the expected result.

We explain now the construction of  $w^s$ . The integer  $\delta \geq 4$  is fixed in the remainder of this section. For a given  $N \in \mathbb{N}$ , applying the construction in Appendix B, we obtain  $(T_{\kappa}^N)_{\kappa \in \mathbb{Z}^3}$  a covering of  $\mathbb{R}^3$  with cubes of width  $\lambda^N$  such that denoting:

$$\mathcal{Z}_{\delta}^N := \left\{ i \in \{1, \dots, N\} \text{ s.t. } \text{dist} \left( h_i^N, \bigcup_{\kappa \in \mathbb{Z}^3} \partial T_{\kappa}^N \right) < \frac{\lambda^N}{\delta} \right\},$$

there holds:

$$(27) \quad \frac{1}{N} \sum_{i \in \mathcal{Z}_{\delta}^N} (1 + |v_i^N|^2) \leq \frac{12}{\delta} \frac{1}{N} \sum_{i=1}^N (1 + |v_i^N|^2) \leq \frac{12(1 + |\mathcal{E}^{\infty}|^2)}{\delta}.$$

Moreover, for  $N \geq N_w$ , for a  $N_w$  depending only on  $w$  (and the sequence  $(\lambda^N)_{N \in \mathbb{N}}$ ) keeping only the indices  $\mathcal{K}^N$  such that  $T_{\kappa}^N$  intersect  $\text{Supp}(w)$ , we obtain a covering  $(T_{\kappa}^N)_{\kappa \in \mathcal{K}^N}$  of  $\text{Supp}(w)$  such that all the cubes are included in  $\Omega$  (see the appendix for more details). We assume  $N \geq N_w$  from now on. We do not make precise the set of indices  $\mathcal{K}^N$ . The only relevant property to our computations is that

$$(28) \quad \#\mathcal{K}^N \leq |\Omega|/|\lambda^N|^3.$$

This inequality is derived by remarking that the  $T_{\kappa}^N$  are disjoint cubes of volume  $|\lambda^N|^3$  that are all included in  $\Omega$ . Associated to this covering, we introduce the following notations. For arbitrary  $\kappa \in \mathcal{K}^N$ , we set

$$\mathcal{I}_{\kappa}^N := \{i \in \{1, \dots, N\} \text{ s.t. } h_i^N \in T_{\kappa}^N\}, \quad M_{\kappa}^N := \#\mathcal{I}_{\kappa}^N, \quad \mathcal{I}^N := \bigcup_{\kappa \in \mathcal{K}^N} \mathcal{I}_{\kappa}^N.$$

Because of assumption (A5), there exists  $M^{\infty} \in \mathbb{N}$  such that:

$$(29) \quad M_{\kappa}^N \leq M^{\infty} |\lambda^N|^3 N, \quad \forall \kappa \in \mathcal{K}^N, \quad \forall N \in \mathbb{N}.$$

In brief, the set of indices  $\{1, \dots, N\}$  contains the two important subsets:

- the subset  $\mathcal{I}^N$  contains all the indices that are "activated" in our computations,
- the subset  $\mathcal{Z}_\delta^N$  contains the indices that are close to boundaries of the partition.

We emphasize that  $\mathcal{Z}_\delta^N$  contains indices that can be in both  $\mathcal{I}^N$  and its complement.

We construct then  $w^s$  piecewisely on the covering of  $\text{Supp}(w)$ . Given  $\kappa \in \mathcal{K}^N$ , we set:

$$(30) \quad w_\kappa^s(x) = \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} U^N[w(h_i^N)](x - h_i^N), \quad \forall x \in \mathbb{R}^3,$$

and

$$w^s = \sum_{\kappa \in \mathcal{K}^N} w_\kappa^s \mathbf{1}_{T_\kappa^N}$$

We note that  $w^s \notin H_0^1(\mathcal{F}^N)$  because of jumps at interfaces  $\partial T_\kappa^N$ . It will be sufficient for our purpose that  $w^s \in H^1(\overset{\circ}{T}_\kappa^N)$  for arbitrary  $\kappa \in \mathcal{K}^N$ . In a cube  $\overset{\circ}{T}_\kappa^N$  the test function  $w^s$  is thus a combination of stokeslets centered in the  $h_i^N$  that are contained in the cell. We delete from this combination the centers that are too close to  $\partial T_\kappa^N$  (namely  $\lambda^N/\delta$ -close to  $\partial T_\kappa^N$ ). We proceed by proving that we make a small error by replacing  $w$  with  $w^s$  in  $I^N$ :

**Proposition 7.** *There exists  $N_\delta \in \mathbb{N}$  depending only on  $\delta$  and  $w$  for which, given  $N > N_\delta$ , there holds:*

$$(31) \quad \left| \int_\Omega \nabla u^N : \nabla w - \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s \right| \lesssim (1 + |\mathcal{E}^\infty|^2) M^\infty \left( \frac{1}{\delta} + \delta |\lambda^N|^2 + \delta^{\frac{1}{3}} N^{\frac{2}{3}} |\lambda^N|^4 + \frac{N^{\frac{2}{3}} |\lambda^N|^5}{\delta^{\frac{2}{3}} d_{\min}^N} + \frac{|\lambda^N|^3}{d_{\min}^N} \right)^{\frac{1}{2}} \|w\|_{W^{1,\infty}}.$$

*Proof.* We split the proof in several steps by introducing different intermediate test-functions. In this proof, we use symbol  $\lesssim$  to denote inequalities with constants that do not depend on  $N$  and  $\delta$ .

**First step: Construction of auxiliary test-functions.** For arbitrary  $\kappa \in \mathcal{K}^N$ , we consider the Stokes problem on  $\overset{\circ}{T}_\kappa^N \setminus \bigcup_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \overline{B_i^N}$  with boundary conditions:

$$(32) \quad \begin{cases} u(x) = w(x), & \text{on } \partial B_i^N \text{ for } i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N, \\ u(x) = 0, & \text{on } \partial T_\kappa^N. \end{cases}$$

We note that this problem enters the framework of Section 2.2. Indeed, let denote:

$$d_m^\kappa := \min_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \left\{ \text{dist}(h_i^N, \partial T_\kappa^N), \min_{j \neq i} |h_i^N - h_j^N| \right\}$$

Because we deleted the indices of  $\mathcal{Z}_\delta^N$ , we have that:

$$(33) \quad d_m^\kappa \geq \min \left( d_{\min}^N, \frac{\lambda^N}{\delta} \right).$$

In particular, we recall that by combining assumptions (A4)-(A5) we have that  $\lambda^N$  decays at most like  $1/N^{1/3}$  (see (4)). Hence, for  $N$  sufficiently large depending only on  $\delta$  and the sequence  $(\lambda^N)_{N \in \mathbb{N}}$  (say  $N > N_\delta$ ) the  $d_m^\kappa$  satisfy assumption (18) uniformly in  $\kappa \in \mathcal{K}^N$ .

So, for  $N > N_\delta$  the arguments developed in Section 2.2 entail that there exists a unique generalized solution to the Stokes problem on  $\dot{T}_\kappa^N \setminus \bigcup_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \overline{B_i^N}$  with boundary condition (32). We denote this solution by  $\bar{w}_\kappa$ . We keep notation  $\bar{w}_\kappa$  to denote its extension to  $\Omega$  (by  $w$  on the holes and by 0 outside  $\dot{T}_\kappa^N$ ). As  $\dot{T}_\kappa^N \subset \Omega$ , we obtain a divergence-free  $\bar{w}_\kappa \in H_0^1(\Omega)$ . We then add the  $\bar{w}_\kappa$  into:

$$\bar{w} = \sum_{\kappa \in \mathcal{K}^N} \bar{w}_\kappa.$$

This vector-field satisfies:

- $\bar{w} \in H_0^1(\Omega)$ ,
- $\operatorname{div} \bar{w} = 0$  on  $\Omega$ ,
- $\bar{w} = w(x)$  on  $B_i^N$  for all  $i \in \{1, \dots, N\} \setminus \mathcal{Z}_\delta^N$ .

The only statement that needs further explanation is the last one. By construction, we have clearly that  $\bar{w}(x) = w(x)$  on  $B_i^N$  for all  $i \in \mathcal{I}^N \setminus \mathcal{Z}_\delta^N$ . When  $i \in \{1, \dots, N\} \setminus (\mathcal{I}^N \cup \mathcal{Z}_\delta^N)$  we have that  $h_i^N$  is in the  $\lambda^N/\delta$  core of the cube  $T_\kappa^N$  that contains him. As the index  $\kappa$  of this cube is not in  $\mathcal{K}^N$  and  $N > N_\delta$ , we have then that  $B(h_i^N, 1/N) \subset T_\kappa^N$  where there holds that  $\bar{w}(x) = w(x) = 0$ .

We correct now the value of  $\bar{w}$  on the  $B_i^N$  when  $i \in \mathcal{Z}_\delta^N$  in order that it fits the same boundary conditions as  $w$  on  $\mathcal{F}^N$ . We set:

$$\begin{aligned} \tilde{w} &= \sum_{i \in \mathcal{Z}_\delta^N} \left[ \chi^N(\cdot - h_i^N)w - \mathfrak{B}_{h_i^N, 1/N, 2/N}[x \mapsto w(x) \cdot \nabla \chi^N(x - h_i^N)] \right] \\ &\quad + \prod_{i \in \mathcal{Z}_\delta^N} (1 - \chi^N(\cdot - h_i^N))\bar{w} + \sum_{i \in \mathcal{Z}_\delta^N} \mathfrak{B}_{h_i^N, 1/N, 2/N}[x \mapsto \bar{w}(x) \cdot \nabla \chi^N(x - h_i^N)]. \end{aligned}$$

One may interpret the construction of  $\tilde{w}$  as follows. The sum on the first line creates a divergence-free lifting of the boundary conditions prescribed by  $w$  on the  $\partial B_i^N$  for  $i \in \mathcal{Z}_\delta^N$ . On the second line is a divergence-free truncation of  $\bar{w}$  that creates a vector-field vanishing on  $\bigcup_{i \in \mathcal{Z}_\delta^N} B_i^N$ . We remark that this vector-field is well defined because, by similar computations as we did in the proof of Proposition 5, we have:

$$\int_{A(h_i^N, 1/N, 2/N)} \bar{w}(x) \cdot \nabla \chi^N(x - h_i^N) dx = \int_{A(h_i^N, 1/N, 2/N)} w(x) \cdot \nabla \chi^N(x - h_i^N) dx = 0, \quad \forall i \in \mathcal{Z}_\delta^N.$$

Hence, we may apply the Bogovskii operator which lifts the divergence term in the brackets with a vector-field vanishing on the boundaries of  $A(h_i^N, 1/N, 2/N)$  that we extend by 0 on  $\mathbb{R}^3 \setminus A(h_i^N, 1/N, 2/N)$ .

Since  $N > N_0$ , the family of balls  $(B_\infty(h_i^N, 2/N))_{i=1, \dots, N}$  are disjoint and included in  $\Omega$ . Consequently, direct computations show that  $\operatorname{div} \tilde{w} = 0$  on  $\Omega$ , and that the truncation that

we perform in  $\tilde{w}$  do not perturb the value of  $\bar{w}$  neither on the  $B_i^N$  for  $i \in \{1, \dots, N\} \setminus \mathcal{Z}_\delta^N$  nor on  $\partial\Omega$ . This remark entails that

- for  $i \in \{1, \dots, N\} \setminus \mathcal{Z}_\delta^N$ :

$$\tilde{w}(x) = \bar{w}(x) = w(x), \quad \text{on } B_i^N,$$

- for  $i \in \mathcal{Z}_\delta^N$ :

$$\tilde{w}(x) = \chi^N(x - h_i^N)w(x) = w(x), \quad \text{on } B_i^N,$$

- $\tilde{w}(x) = 0$ , on  $\partial\Omega$ .

Consequently, by restriction, there holds that  $w - \tilde{w} \in H_0^1(\mathcal{F}^N)$  is divergence-free. As  $u^N$  is a generalized solution to a Stokes problem on  $\mathcal{F}^N$  we have thus:

$$\int_{\mathcal{F}^N} \nabla u^N : \nabla(w - \tilde{w}) = 0.$$

We rewrite this identity as follows:

$$(34) \quad \int_{\Omega} \nabla u^N : \nabla w = \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s - E_1 - E_2,$$

with :

$$\begin{aligned} E_1 &= \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla(w_\kappa^s - \bar{w}_\kappa), \\ E_2 &= \int_{\Omega} \nabla u^N : \nabla(\bar{w} - \tilde{w}). \end{aligned}$$

**Second step: Control of error term  $E_1$ .** For arbitrary  $\kappa \in \mathcal{K}^N$ , we apply Proposition 5 to  $\bar{w}_\kappa$  and its corresponding combination of stokeslets (namely, the restriction  $w_\kappa^s$  of  $w^s$  to  $T_\kappa^N$ ). By construction,  $d_m^\kappa$  satisfies the requirement  $d_m^\kappa > 4/N$  for  $N > N_\delta$ . We have thus:

$$\|\nabla(w_\kappa^s - \bar{w}_\kappa)\|_{L^2(T_\kappa^N)} \lesssim \sqrt{\frac{M_\kappa^N}{N}} \left( \frac{1}{N} + \sqrt{\frac{M_\kappa^N}{Nd_m^\kappa}} \right) \|w\|_{W^{1,\infty}}.$$

Note here that  $\#(\mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N) \leq \#\mathcal{I}_\kappa^N = M_\kappa^N$ . Consequently, introducing this last bound in the computation of  $E_1$  and applying a standard Cauchy-Schwarz inequality together with (28)-(29) yields:

$$\begin{aligned} |E_1| &\lesssim \sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|_{L^2(T_\kappa^N)} \frac{M_\kappa^N}{N} \left( \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{d_m^\kappa}} \right) \|w\|_{W^{1,\infty}}, \\ &\lesssim \left( \sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|_{L^2(T_\kappa^N)}^2 \right)^{\frac{1}{2}} M^\infty |\lambda^N|^{\frac{3}{2}} \left( \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{d_m^\kappa}} \right) \|w\|_{W^{1,\infty}}. \end{aligned}$$

Here, we note again that, by construction, the  $T_\kappa^N$  are disjoint and included in  $\Omega$  so that

$$\left( \sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|_{L^2(T_\kappa^N)}^2 \right)^{\frac{1}{2}} \leq \|\nabla u^N\|_{L^2(\Omega)}.$$

Applying the uniform bound for  $u^N$  in  $H_0^1(\Omega)$  and introducing (33), we conclude then that:

$$(35) \quad |E_1| \lesssim \mathcal{E}^\infty M^\infty \left( \frac{1}{\sqrt{N}} + \sqrt{\delta} |\lambda^N| + \sqrt{\frac{|\lambda^N|^3}{d_{\min}^N}} \right) \|w\|_{W^{1,\infty}}.$$

**Third step: Control of error term  $E_2$ .** As for the second term, we replace  $\tilde{w}$  by its explicit construction. We remark that because the supports of the  $(\chi^N(\cdot - h_i^N))_{i \in \{1, \dots, N\}}$  are disjoint (as  $d_{\min}^N > 4/N$  for  $N > N_0$ ) we have:

$$1 - \prod_{i \in \mathcal{Z}_\delta^N} (1 - \chi^N(x - h_i^N)) = \sum_{i \in \mathcal{Z}_\delta^N} \chi^N(x - h_i^N), \quad \forall x \in \Omega.$$

Consequently, we split:

$$\begin{aligned} \bar{w} - \tilde{w} &= \sum_{i \in \mathcal{Z}_\delta^N} \left[ \chi^N(\cdot - h_i^N) \bar{w} - \mathfrak{B}_{h_i^N, 1/N, 2/N}[x \mapsto \bar{w}(x) \cdot \nabla \chi^N(x - h_i^N)] \right] \\ &\quad - \sum_{i \in \mathcal{Z}_\delta^N} \left[ \chi^N(\cdot - h_i^N) w - \mathfrak{B}_{h_i^N, 1/N, 2/N}[x \mapsto w(x) \cdot \nabla \chi^N(x - h_i^N)] \right]. \end{aligned}$$

By direct computations and application of Lemma 16 to the Bogovskii operator  $\mathfrak{B}_{h_i^N, 1/N, 2/N}$ , we find  $E_2^i \in L^2(B_\infty(h_i^N, 2/N))$ ,  $i \in \mathcal{Z}_\delta^N$ , for which:

$$\nabla(\bar{w} - \tilde{w}) = \sum_{i \in \mathcal{Z}_\delta^N} E_2^i \mathbf{1}_{B_\infty(h_i^N, 2/N)},$$

and such that:

$$\|E_2^i\|_{L^2(B_\infty(h_i^N, 2/N))}^2 \lesssim \frac{1}{N} \|w\|_{W^{1,\infty}}^2 + N^2 \|\bar{w}\|_{L^2(B_\infty(h_i^N, 2/N))}^2 + \|\nabla \bar{w}\|_{L^2(B_\infty(h_i^N, 2/N))}^2.$$

Introducing these bounds in the computation of  $E_2$ , and reproducing similar computations as for  $E_1$ , we derive:

$$\begin{aligned} |E_2| &\leq \sum_{i \in \mathcal{Z}_\delta^N} \int_{B_\infty(h_i^N, 2/N)} |\nabla u^N| |E_2^i|, \\ &\leq \left( \sum_{i \in \mathcal{Z}_\delta^N} \|\nabla u^N\|_{L^2(B_\infty(h_i^N, 2/N))}^2 \right)^{\frac{1}{2}} \left( \sum_{i \in \mathcal{Z}_\delta^N} \|E_2^i\|_{L^2(B_\infty(h_i^N, 2/N))}^2 \right)^{\frac{1}{2}}, \\ &\lesssim \mathcal{E}^\infty \left( \sum_{i \in \mathcal{Z}_\delta^N} \|E_2^i\|_{L^2(B_\infty(h_i^N, 2/N))}^2 \right)^{\frac{1}{2}}, \end{aligned}$$



where we applied again that the  $(B_\infty(h_i^N, 2/N))_{i \in \mathcal{Z}_\delta^N}$  are disjoint and cover a subset of  $\Omega$ . To complete the proof, it remains to compute:

$$\begin{aligned} \sum_{i \in \mathcal{Z}_\delta^N} \|E_2^i\|_{L^2(B_\infty(h_i^N, 2/N))}^2 \\ \lesssim \sum_{i \in \mathcal{Z}_\delta^N} \frac{1}{N} \|w\|_{W^{1,\infty}}^2 + N^2 \|\bar{w}\|_{L^2(B_\infty(h_i^N, 2/N))}^2 + \|\nabla \bar{w}\|_{L^2(B_\infty(h_i^N, 2/N))}^2. \end{aligned}$$

We rewrite the right-hand side of this inequality:  $E_{2,\infty} + N^2 E_{2,0} + E_{2,1}$ .

First, we recall that, by choice of the covering (see (27)), we have:

$$(36) \quad \#\mathcal{Z}_\delta^N \lesssim \frac{N}{\delta} (1 + |\mathcal{E}^\infty|^2).$$

Consequently, there holds:

$$E_{2,\infty} := \sum_{i \in \mathcal{Z}_\delta^N} \frac{1}{N} \|w\|_{W^{1,\infty}}^2 \lesssim \frac{\|w\|_{W^{1,\infty}}^2}{\delta} (1 + |\mathcal{E}^\infty|^2).$$

Second, as  $(T_\kappa^N)_{\kappa \in \mathcal{K}^N}$  is a covering of  $\text{Supp}(\bar{w})$ , we have:

$$\begin{aligned} E_{2,0} &:= \sum_{i \in \mathcal{Z}_\delta^N} \|\bar{w}\|_{L^2(B_\infty(h_i^N, 2/N))}^2 = \sum_{i \in \mathcal{Z}_\delta^N} \sum_{\kappa \in \mathcal{K}^N} \|\bar{w}\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2, \\ &\lesssim \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{Z}_\delta^N} \|\bar{w}_\kappa - w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2 + \sum_{i \in \mathcal{Z}_\delta^N} \sum_{\kappa \in \mathcal{K}^N} \|w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2. \end{aligned}$$

We compute the terms involving  $w_\kappa^s$  by using the explicit formula (30) and the expansion of stokeslet (16). To this end, we remark that  $B_\infty(h_i^N, 2/N) \cap T_\kappa^N \neq \emptyset$  implies that  $h_i^N$  is in the  $2/N$ -neighborhood of  $T_\kappa^N$ . As the width of a  $T_\kappa^N$  is much larger than  $2/N$  this implies that this property is satisfied by at most 8 cubes. We have thus:

$$\sum_{\kappa \in \mathcal{K}^N} \|w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2 \leq 8 \sup_{\kappa \in \mathcal{K}^N} \|w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N))}^2, \quad \forall i \in \mathcal{Z}_\delta^N.$$

Given  $i \in \mathcal{Z}_\delta^N$  and  $\kappa \in \mathcal{K}^N$ , for  $N > N_0$  and  $j \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N$  the distance between  $h_j^N$  and  $B_\infty(h_i^N, 2/N)$  is larger than  $d_{\min}^N/2$ . Recalling that there are at most  $M_\kappa^N$  indices in  $\mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N$  we derive the bound:

$$|w_\kappa^s(x)| \lesssim \frac{M_\kappa^N}{N d_{\min}^N} \|w\|_{W^{1,\infty}}, \quad \forall x \in B_\infty(h_i^N, 2/N).$$

Consequently, there holds

$$\|w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N))}^2 \lesssim \frac{|M_\kappa^N|^2}{N^5 |d_{\min}^N|^2} \|w\|_{W^{1,\infty}}^2 \lesssim \frac{|M^\infty|^2}{N^3} \|w\|_{W^{1,\infty}}^2.$$

where we applied (29) combined with assumption (A4). This yields, due to our choice of covering:

$$\begin{aligned} \sum_{i \in \mathcal{Z}_\delta^N} \sum_{\kappa \in \mathcal{K}^N} \|w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2 &\lesssim \sum_{i \in \mathcal{Z}_\delta^N} \frac{|M^\infty|^2}{N^3} \|w\|_{W^{1,\infty}}^2 \\ &\lesssim \frac{|M^\infty|^2 \|w\|_{W^{1,\infty}}^2}{\delta N^2} (1 + |\mathcal{E}^\infty|^2). \end{aligned}$$

For the remainder terms, we apply again Proposition 5 combined with (29) and (33) :

$$\begin{aligned} \|(\bar{w}_\kappa - w_\kappa^s)\|_{L^6(T_\kappa^N)}^2 &\lesssim \frac{M_\kappa^N}{N} \left( \frac{1}{N} + \sqrt{\frac{M_\kappa^N}{N d_m^\kappa}} \right)^2 \|w\|_{W^{1,\infty}}^2, \\ &\lesssim |M^\infty|^2 |\lambda^N|^3 \left( \delta |\lambda^N|^2 + \frac{|\lambda^N|^3}{d_{min}^N} \right) \|w\|_{W^{1,\infty}}^2 \end{aligned}$$

For arbitrary  $\kappa \in \mathcal{K}^N$ , we introducing  $\mathcal{Z}_{\delta,\kappa}^N$  the set of indices  $i$  such that  $B_\infty(h_i^N, 2/N) \cap T_\kappa^N \neq \emptyset$ , we infer by a Hölder inequality that:

$$\begin{aligned} \sum_{i \in \mathcal{Z}_{\delta,\kappa}^N} \|(\bar{w}_\kappa - w_\kappa^s)\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2 &\lesssim \frac{|\mathcal{Z}_{\delta,\kappa}^N|^{\frac{2}{3}}}{N^2} \|(\bar{w}_\kappa - w_\kappa^s)\|_{L^6(T_\kappa^N)}^2, \\ &\lesssim |\mathcal{Z}_{\delta,\kappa}^N|^{\frac{2}{3}} \frac{|M^\infty|^2 |\lambda^N|^3}{N^2} \left( \delta |\lambda^N|^2 + \frac{|\lambda^N|^3}{d_{min}^N} \right) \|w\|_{W^{1,\infty}}^2. \end{aligned}$$

Eventually, we obtain, by applying again a Hölder inequality (recall that there are  $O(1/|\lambda^N|^3)$  indices in  $\mathcal{K}^N$ ):

$$\begin{aligned} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{Z}_\delta^N} \|\bar{w}_\kappa - w_\kappa^s\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2 &\lesssim \left[ \sum_{\kappa \in \mathcal{K}^N} |\mathcal{Z}_{\delta,\kappa}^N| \right]^{\frac{2}{3}} \frac{|M^\infty|^2 |\lambda^N|^2}{N^2} \left( \delta |\lambda^N|^2 + \frac{|\lambda^N|^3}{d_{min}^N} \right) \|w\|_{W^{1,\infty}}^2 \end{aligned}$$

At this point, we remark again that one index  $i$  belongs to at most 8 sets  $\mathcal{Z}_{\delta,\kappa}^N$  so that, by our choice of covering and (4):

$$\left[ \sum_{\kappa \in \mathcal{K}^N} |\mathcal{Z}_{\delta,\kappa}^N| \right]^{\frac{2}{3}} \lesssim [|\mathcal{Z}_\delta^N|]^{\frac{2}{3}} \lesssim \left[ \frac{N}{\delta} (1 + |\mathcal{E}^\infty|^2) \right]^{\frac{2}{3}}.$$

This entails:

$$\sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{Z}_\delta^N} \|(\bar{w}_\kappa - w_\kappa^s)\|_{L^2(B_\infty(h_i^N, 2/N) \cap T_\kappa^N)}^2 \leq \frac{(1 + |\mathcal{E}^\infty|^{\frac{4}{3}}) |M^\infty|^2}{N^2} \left( \delta^{\frac{1}{3}} N^{\frac{2}{3}} |\lambda^N|^4 + \frac{N^{\frac{2}{3}} |\lambda^N|^5}{\delta^{\frac{2}{3}} d_{min}^N} \right)$$

Finally:

$$E_{2,0} \lesssim \frac{(1 + |\mathcal{E}^\infty|^2)|M^\infty|^2}{N^2} \left( \frac{1}{\delta} + \delta^{\frac{1}{3}} N^{\frac{2}{3}} |\lambda^N|^4 + \frac{N^{\frac{2}{3}} |\lambda^N|^5}{\delta^{\frac{2}{3}} d_{\min}^N} \right) \|w\|_{W^{1,\infty}}^2.$$

With similar arguments, we prove that:

$$E_{2,1} := \sum_{i \in \mathcal{Z}_\delta^N} \|\nabla \bar{w}\|_{L^2(B_\infty(h_i^N, 2/N))}^2.$$

satisfies a similar bound:

$$E_{2,1} \lesssim (1 + |\mathcal{E}^\infty|^2)|M^\infty|^2 \left( \frac{1}{\delta} + \delta |\lambda^N|^2 + \frac{|\lambda^N|^3}{d_{\min}^N} \right) \|w\|_{W^{1,\infty}}^2.$$

Eventually, we obtain that:

$$(37) \quad E_2 \lesssim (1 + |\mathcal{E}^\infty|^2)|M^\infty| \left( \frac{1}{\delta} + \delta |\lambda^N|^2 + \delta^{\frac{1}{3}} N^{\frac{2}{3}} |\lambda^N|^4 + \frac{N^{\frac{2}{3}} |\lambda^N|^5}{\delta^{\frac{2}{3}} d_{\min}^N} + \frac{|\lambda^N|^3}{d_{\min}^N} \right)^{\frac{1}{2}} \|w\|_{W^{1,\infty}}.$$

Combining (35) and (37) in (34), we obtain the expected result.  $\square$

## 5. PROOF OF THEOREM 1 – ASYMPTOTICS $N \rightarrow \infty$

In this section, we end the proof of Theorem 1 keeping the notations introduced in the previous section. Because of assumption (A4) and remark (3) we have that:

$$N^{\frac{2}{3}} |\lambda^N|^4 \lesssim \frac{d_{\min}^N}{|\lambda^N|} \lesssim |\lambda^N|^2 N d_{\min}^N, \quad \forall N \in \mathbb{N}.$$

A straightforward corollary of Proposition 7 reads then:

**Corollary 8.** *For arbitrary  $\delta \geq 4$ , there holds:*

$$\limsup_{N \rightarrow \infty} \left| \int_{\Omega} \nabla u^N : \nabla w - \sum_{\kappa \in \mathcal{I}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s \right| \lesssim \frac{(1 + |\mathcal{E}^\infty|^2) M^\infty \|w\|_{W^{1,\infty}}}{\delta^{\frac{1}{3}}}.$$

So in this section, we prove the following proposition:

**Proposition 9.** *For arbitrary  $\delta \geq 4$ , there holds:*

$$\limsup_{N \rightarrow \infty} \left| \sum_{\kappa \in \mathcal{I}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s - 6\pi \int_{\Omega} (j - \rho \bar{u}) \cdot w \right| \lesssim \frac{(1 + |\mathcal{E}^\infty|^{\frac{5}{2}}) |M^\infty|^{\frac{1}{4}} \|w\|_{W^{1,\infty}}}{\sqrt{\delta}}.$$

This will end the proof of Theorem 1. Indeed, combining the above corollary and proposition, we obtain that there exists  $K$  which does not depend on  $\delta$  such that, for arbitrary  $\delta \geq 4$ :

$$\limsup_{N \rightarrow \infty} \left| \int_{\Omega} \nabla u^N : \nabla w - 6\pi \int_{\Omega} (j - \rho \bar{u}) \cdot w \right| \leq \frac{K}{\sqrt{\delta}}.$$

As

$$\lim_{N \rightarrow \infty} \int_{\Omega} \nabla u^N : \nabla w = \int_{\Omega} \nabla \bar{u} : \nabla w ,$$

and  $\delta$  can be made arbitrary large, this entails that

$$\int_{\Omega} \nabla \bar{u} : \nabla w = 6\pi \int_{\Omega} (j - \rho \bar{u}) \cdot w ,$$

and we obtain that  $\bar{u}$  satisfies (B3).

We give now a proof of Proposition 9. From now on  $\delta$  is fixed larger than 4 and we assume, with the conventions of the previous section, that:

$$N \geq \max(N_0, N_w, N_\delta) .$$

For such a  $N$ , we denote:

$$\tilde{I}^N = \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s = \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w_\kappa^s .$$

First, let fix  $\kappa \in \mathcal{K}^N$  and simplify

$$\tilde{I}_\kappa^N := \int_{T_\kappa^N} \nabla u^N : \nabla w_\kappa^s .$$

By definition, we have that:

$$w_\kappa^s(x) = \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} U^N[w(h_i^N)](x - h_i^N) , \quad \forall x \in \mathbb{R}^3 ,$$

so that, introducing the associated pressures  $x \mapsto P^N[w(h_i^N)](x - h_i^N)$ , we obtain (recall that  $u^N$  is divergence-free and constant on the  $B_i^N$ ):

$$\begin{aligned} \tilde{I}_\kappa^N &= \int_{T_\kappa^N} \nabla u^N : \nabla w_\kappa^s , \\ &= \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \int_{T_\kappa^N \setminus B_i^N} \nabla u^N(x) : [\nabla U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N) \mathbb{I}_3] dx , \\ &= \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \left( \int_{\partial B_i^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N) n \} \cdot v_i^N d\sigma \right. \\ &\quad \left. + \int_{\partial T_\kappa^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N) n \} \cdot u^N(x) d\sigma \right) , \\ &= \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} I_{i,int}^N + I_{i,ext}^N , \end{aligned}$$

where we denoted:

$$\begin{aligned} I_{i,int}^N &= \int_{\partial B_i^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N)n \} \cdot v_i^N d\sigma, \\ I_{i,ext}^N &:= \int_{\partial T_\kappa^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N)n \} \cdot u^N(x) d\sigma. \end{aligned}$$

Recalling that  $(U^N, P^N)$  is the solution to the Stokes problem in the exterior of a ball of radius  $1/N$ , and that  $v_i^N$  is constant on  $\partial B_i^N$ , we have an explicit value for the interior integral whatever the value of the index  $i$  (see (17)):

$$I_{i,int}^N = \frac{6\pi}{N} w(h_i^N) \cdot v_i^N.$$

For the other term, we apply that the diameter of  $T_\kappa^N$  is small so that we may approximate  $u^N$  on  $\partial T_\kappa^N$  by a constant. Namely, we choose:

$$\bar{u}_\kappa^N = \frac{1}{|[T_\kappa^N]_{2\delta}|} \int_{[T_\kappa^N]_{2\delta}} u^N(x) dx,$$

where  $[T_\kappa^N]_{2\delta}$  is the  $\lambda^N/(2\delta)$ -neighborhood of  $\partial T_\kappa^N$  inside  $\mathring{T}_\kappa^N$ . At this point, we remark that we have actually two notations for the same quantity. Indeed, a simple draw shows that introducing  $x_\kappa^N$  the center of  $T_\kappa^N$ , we have:

$$\mathring{T}_\kappa^N = B_\infty \left( x_\kappa^N, \frac{\lambda^N}{2} \right) \quad \text{while} \quad [T_\kappa^N]_{2\delta} = A \left( x_\kappa^N, \left[ 1 - \frac{1}{\delta} \right] \frac{\lambda^N}{2}, \frac{\lambda^N}{2} \right).$$

So, we replace:

$$\begin{aligned} I_{i,ext}^N &= \int_{\partial T_\kappa^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N)n \} \cdot \bar{u}_\kappa^N d\sigma \\ &\quad + \int_{\partial T_\kappa^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N)n \} \cdot (u^N(x) - \bar{u}_\kappa^N) d\sigma. \end{aligned}$$

For the first term on the right-hand side of this last identity, we apply that the flux through hypersurfaces of the normal stress tensor is conserved by solutions to the Stokes problem so that, applying (17), we have:

$$\begin{aligned} &\int_{\partial T_\kappa^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N)n \} d\sigma \\ &= - \int_{\partial B_i^N} \{ \partial_n U^N[w(h_i^N)](x - h_i^N) - P^N[w(h_i^N)](x - h_i^N)n \} d\sigma \\ &= - \frac{6\pi}{N} w(h_i^N). \end{aligned}$$

Finally, we obtain:

$$(38) \quad \tilde{I}_\kappa^N = \frac{6\pi}{N} \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} (w(h_i^N) \cdot v_i^N - w(h_i^N) \cdot \bar{u}_\kappa^N) + Err_\kappa$$

with:

$$Err_\kappa = \int_{\partial T_\kappa^N} \left\{ \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \partial_n U^N[w(h_i^N)](\cdot - h_i^N) - P^N[w(h_i^N)](\cdot - h_i^N)n \right\} \cdot (u^N - \bar{u}_\kappa^N) d\sigma.$$

We control this error term with the following lemma:

**Lemma 10.** *There exists a constant  $C_\delta$  depending only on  $\delta$  such that,*

$$|Err_\kappa| \leq C_\delta M^\infty \|w\|_{L^\infty} |\lambda^N|^{\frac{5}{2}} \|\nabla u^N\|_{L^2(T_\kappa^N)}, \quad \forall \kappa \in \mathcal{K}^N.$$

*Proof.* For  $N > \max(N_0, N_w, N_\delta)$ , we have that

$$[T_\kappa^N]_{2\delta} \subset T_\kappa^N \setminus \bigcup_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \overline{B_i^N}.$$

Indeed,  $B_i^N = B(h_i^N, 1/N)$  and, for  $i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N$  we have that  $h_i^N$  is  $\lambda^N/\delta$  far from  $\partial T_\kappa^N$ . These centers are thus  $\lambda^N/(2\delta)$  far from  $[T_\kappa^N]_{2\delta}$  which is larger than  $1/N$  since  $N > N_\delta$ . In particular all the stokeslets in  $w_\kappa^s$  satisfy:

$$(39) \quad \begin{cases} \Delta U^N[w(h_i^N)](x - h_i^N) - \nabla P^N[w(h_i^N)](x - h_i^N) &= 0, \\ \operatorname{div} U^N[w(h_i^N)](x - h_i^N) &= 0, \end{cases} \quad \text{on } [T_\kappa^N]_{2\delta}.$$

Consequently, we split

$$\partial[T_\kappa^N]_{2\delta} = \partial T_\kappa^N \cup \partial T_{\kappa,\delta}^N$$

where

$$\partial T_{\kappa,\delta}^N = \{x \in T_\kappa^N \text{ s.t. } \operatorname{dist}(x, \partial T_\kappa^N) = \lambda^N/(2\delta)\}.$$

We remark then that for any divergence-free  $v \in H^1([T_\kappa^N]_{2\delta})$  satisfying

$$\begin{cases} v &= u^N - \bar{u}_\kappa^N, & \text{on } \partial T_\kappa^N, \\ v &= 0, & \text{on } \partial T_{\kappa,\delta}^N, \end{cases}$$

integrating by parts  $Err_\kappa$  and applying (39), we have:

$$Err_\kappa = \int_{[T_\kappa^N]_{2\delta}} \left\{ \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \nabla U^N[w(h_i^N)](\cdot - h_i^N) \right\} : \nabla v,$$

so that:

$$(40) \quad |Err_\kappa| \leq \left\{ \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \|\nabla U^N[w(h_i^N)](\cdot - h_i^N)\|_{L^2([T_\kappa^N]_{2\delta})} \right\} \|\nabla v\|_{L^2([T_\kappa^N]_{2\delta})}.$$

Let choose a suitable  $v$  in order to apply this estimate. We recall that we introduced  $x_\kappa^N$  the center of  $T_\kappa^N$  and that we remarked that

$$T_\kappa^N = B_\infty \left( x_\kappa^N, \frac{\lambda^N}{2} \right), \quad [T_\kappa^N]_{2\delta} = A \left( x_\kappa^N, \left[ 1 - \frac{1}{\delta} \right] \frac{\lambda^N}{2}, \frac{\lambda^N}{2} \right).$$

So, we introduce  $\zeta_\delta \in C^\infty(\mathbb{R}^3)$  such that

$$\zeta_\delta(x) = 0 \text{ in } B_\infty \left( 0, \frac{1-1/\delta}{2} \right) \quad \text{and} \quad \zeta_\delta(x) = 1 \text{ outside } B_\infty \left( 0, \frac{1}{2} \right)$$

and we set

$$\begin{aligned} v(x) = & \zeta_\delta(\lambda^N(x - x_\kappa^N))(u^N(x) - \bar{u}_\kappa^N) \\ & - \mathfrak{B}_{x_\kappa^N, (1-1/\delta)\lambda^N/2, \lambda^N/2}[x \mapsto (u^N(x) - \bar{u}_\kappa^N) \cdot \nabla[\zeta_\delta(\lambda^N(x - x_\kappa^N))]]. \end{aligned}$$

Again  $v$  is well-defined as one shows by direct computations that the argument of the Bogovskii operator has mean zero on  $A(x_\kappa^N, (1-1/\delta)\lambda^N/2, \lambda^N/2)$ . Applying Lemma 16, we have then that there exists a constant  $C_\delta$  depending only on  $\delta$  for which:

$$\|\nabla v\|_{L^2([T_\kappa^N]_{2\delta})} \leq C_\delta [\|\nabla u^N\|_{L^2([T_\kappa^N]_{2\delta})} + \lambda^N \|u^N(x) - \bar{u}_\kappa^N\|_{L^2([T_\kappa^N]_{2\delta})}].$$

Here we note that the  $\bar{u}_\kappa^N$  is exactly the average of  $u^N$  on  $[T_\kappa^N]_{2\delta}$ . Consequently, applying the Poincaré-Wirtinger inequality in the annulus  $[T_\kappa^N]_{2\delta}$  with the remark on the best constant as in Lemma 14 we obtain finally that:

$$(41) \quad \|\nabla v\|_{L^2([T_\kappa^N]_{2\delta})} \leq C_\delta \|\nabla u^N\|_{L^2([T_\kappa^N]_{2\delta})}.$$

As for the stokeslet, we remark again that for any  $i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N$  the minimum distance between  $h_i^N$  and  $[T_\kappa^N]_{2\delta}$  is larger than  $\lambda^N/(2\delta)$ . Hence, applying the expansion (16) of the stokeslet  $U^N[w(h_i^N)]$  we obtain that

$$\begin{aligned} \|\nabla U^N[w(h_i^N)](\cdot - h_i^N)\|_{L^2([T_\kappa^N]_{2\delta})} & \leq \left( \int_{\lambda^N/(2\delta)}^\infty \frac{dr}{N^2 r^2} \right)^{\frac{1}{2}} |w(h_i^N)| \\ & \leq \frac{\sqrt{2\delta}}{N\sqrt{\lambda^N}} |w(h_i^N)|. \end{aligned}$$

Combining these computations for the (at most)  $M_\kappa^N$  indices  $i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N$  entails by (29) that:

$$(42) \quad \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \|\nabla U^N[w(h_i^N)](\cdot - h_i^N)\|_{L^2([T_\kappa^N]_{2\delta})} \leq \sqrt{2\delta} |\lambda^N|^{\frac{5}{2}} M^\infty \|w\|_{L^\infty}.$$

Combining (41) and (42) in (40) yields the expected result.  $\square$

Summing (38) over  $\kappa$ , we obtain that:

$$\begin{aligned}
 \tilde{I}^N &= \frac{6\pi}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} (w(h_i^N) \cdot v_i^N - w(h_i^N) \cdot \bar{u}_\kappa^N) + Err \\
 (43) \quad &= \frac{6\pi}{N} \sum_{i \in \mathcal{I}^N \setminus \mathcal{Z}_\delta^N} w(h_i^N) \cdot v_i^N - \frac{6\pi}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} w(h_i^N) \cdot \bar{u}_\kappa^N + Err.
 \end{aligned}$$

where

$$Err = \sum_{\kappa \in \mathcal{K}^N} Err_\kappa.$$

Hence, applying Lemma 10, a Cauchy-Schwarz inequality and remarking again that the  $(T_\kappa^N)_{\kappa \in \mathcal{K}^N}$  form a partition of a subset of  $\Omega$  with a number of elements satisfying (28), we have:

$$\begin{aligned}
 |Err| &\leq C_\delta M^\infty \|w\|_{L^\infty} \sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|_{L^2(T_\kappa^N)} |\lambda^N|^{\frac{5}{2}} \leq C_\delta M^\infty \|w\|_{L^\infty} \lambda^N \|\nabla u^N\|_{L^2(\Omega)} \\
 (44) \quad &\leq C_\delta \lambda^N M^\infty \mathcal{E}^\infty \|w\|_{L^\infty}.
 \end{aligned}$$

As  $\lambda^N \rightarrow 0$ , the asymptotics of  $\tilde{I}^N$  is given by the two first terms on the right-hand side of (43). We make precise these asymptotics in the two following lemmas:

**Lemma 11.** *For arbitrary  $\delta \geq 4$ , there holds:*

$$\limsup_{N \rightarrow \infty} \left| \frac{6\pi}{N} \sum_{i \in \mathcal{I}^N \setminus \mathcal{Z}_\delta^N} w(h_i^N) \cdot v_i^N - 6\pi \int_\Omega j(x) \cdot w(x) dx \right| \lesssim \frac{(1 + |\mathcal{E}^\infty|^2) \|w\|_{L^\infty}}{\delta}.$$

*Proof.* As  $w \in C_c^\infty(\Omega)$  and  $(T_\kappa^N)_{\kappa \in \mathcal{K}^N}$  is a covering of  $\text{Supp}(w)$  we have by assumption (A3) that:

$$\int_\Omega j(x) \cdot w(x) dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N w(h_i^N) \cdot v_i^N = \lim_{N \rightarrow \infty} \sum_{i \in \mathcal{I}^N} w(h_i^N) \cdot v_i^N.$$

Hence, our proof reduces to find a uniform bound on

$$\sum_{i \in \mathcal{I}^N} w(h_i^N) \cdot v_i^N - \sum_{i \in \mathcal{I}^N \setminus \mathcal{Z}_\delta^N} w(h_i^N) \cdot v_i^N = \sum_{i \in \mathcal{Z}_\delta^N \cap \mathcal{I}^N} w(h_i^N) \cdot v_i^N.$$

However, for large  $N$ , there holds:

$$\left| \frac{1}{N} \sum_{i \in \mathcal{Z}_\delta^N \cap \mathcal{I}^N} w(h_i^N) \cdot v_i^N \right| \leq \left( \frac{1}{N} \sum_{i \in \mathcal{Z}_\delta^N} |v_i^N|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{i \in \mathcal{Z}_\delta^N} |w(h_i^N)|^2 \right)^{\frac{1}{2}}.$$



Here, we apply (27) that has guided our choice for the covering  $(T_\kappa^N)_{\kappa \in \mathcal{K}^N}$  :

$$\begin{aligned} \left( \frac{1}{N} \sum_{i \in \mathcal{Z}_\delta^N} |v_i^N|^2 \right) &\leq \frac{12}{\delta} (1 + |\mathcal{E}^\infty|^2) , \\ \left( \frac{1}{N} \sum_{i \in \mathcal{Z}_\delta^N} |w(h_i^N)|^2 \right) &\leq \frac{12}{\delta} (1 + |\mathcal{E}^\infty|^2) \|w\|_{L^\infty}^2 . \end{aligned}$$

Combining these two estimates, we obtain:

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{i \in \mathcal{Z}_\delta^N \cap \mathcal{I}^N} w(h_i^N) \cdot v_i^N \right| \leq \frac{12}{\delta} (1 + |\mathcal{E}^\infty|^2) \|w\|_{L^\infty} .$$

□

**Lemma 12.** *For  $\delta \geq 4$  there holds:*

$$\limsup_{N \rightarrow \infty} \left| \frac{6\pi}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} w(h_i^N) \cdot \bar{u}_\kappa^N - 6\pi \int_\Omega \rho(x) \bar{u}(x) \cdot w(x) dx \right| \lesssim \frac{|M^\infty|^{\frac{1}{4}} (1 + |\mathcal{E}^\infty|^{\frac{5}{2}})}{\sqrt{\delta}} \|w\|_{L^\infty} .$$

*Proof.* As in the previous proof, let first complete the sum by reintroducing the  $\mathcal{Z}_\delta^N$  indices:

$$(45) \quad \frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} w(h_i^N) \cdot \bar{u}_\kappa^N = \frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \cdot \bar{u}_\kappa^N + \tilde{Err}^N$$

where:

$$\tilde{Err}^N = \frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N \cap \mathcal{Z}_\delta^N} w(h_i^N) \cdot \bar{u}_\kappa^N .$$

For the first term on the right-hand side of (45), we remark that:

$$\frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \cdot \bar{u}_\kappa^N = \left( 1 - \left( 1 - \frac{1}{\delta} \right)^3 \right)^{-1} \frac{1}{|\lambda^N|^3 N} \sum_{\kappa \in \mathcal{K}^N} \int_{[T_\kappa^N]_{2\delta}} \left( \sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \right) \cdot u^N .$$

So, we introduce:

$$\sigma^N = \left( 1 - \left( 1 - \frac{1}{\delta} \right)^3 \right)^{-1} \frac{1}{N |\lambda^N|^3} \sum_{\kappa \in \mathcal{K}^N} \left( \sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \right) \mathbf{1}_{[T_\kappa^N]_{2\delta}} ,$$

for which:

$$\frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \cdot \bar{u}_\kappa^N = \int_\Omega \sigma^N(x) \cdot u^N(x) dx .$$

On the one-hand, we note that:

$$\|\sigma^N\|_{L^1(\Omega)} \leq \frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} M_\kappa^N \|w\|_{L^\infty},$$

where  $\sum_{\kappa \in \mathcal{K}^N} M_\kappa^N \leq N$ , so that:

$$\|\sigma^N\|_{L^1(\Omega)} \leq \|w\|_{L^\infty}.$$

Complementarily, because of assumption (A5), we also have :

$$\begin{aligned} \|\sigma^N\|_{L^\infty(\Omega)} &\leq \left(1 - \left(1 - \frac{1}{\delta}\right)^3\right)^{-1} \sup_{\kappa \in \mathcal{K}^N} \frac{M_\kappa^N}{N|\lambda^N|^3} \|w\|_{L^\infty} \\ &\leq \left(1 - \left(1 - \frac{1}{\delta}\right)^3\right)^{-1} M^\infty \|w\|_{L^\infty}, \end{aligned}$$

and  $\sigma^N$  is bounded in all  $L^q$ -spaces.

On the other hand, for any  $v \in C_c^\infty(\Omega)$  we have

$$\int_{\Omega} \sigma^N(x) \cdot v(x) dx = \frac{1}{N} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \cdot \bar{v}_\kappa^N$$

with

$$\bar{v}_\kappa^N = \frac{1}{|[T_\kappa^N]_{2\delta}|} \int_{[T_\kappa^N]_{2\delta}} v(x) dx.$$

We remark that, for any  $i \in \mathcal{I}_\kappa^N$ ,  $h_i^N$  is inside  $T_\kappa^N$  whose diameter is  $\lambda^N$ . This entails:

$$|\bar{v}_\kappa^N - v(h_i^N)| \lesssim \lambda^N \|\nabla v\|_{L^\infty}.$$

Gathering these identities for all indices  $i$  in all the cubes  $T_\kappa^N$ , we infer :

$$\left| \int_{\Omega} \sigma^N(x) \cdot v(x) dx - \frac{1}{N} \sum_{i \in \mathcal{I}^N} w(h_i^N) \cdot v(h_i^N) \right| \lesssim \lambda^N \|\nabla v\|_{L^\infty} \|w\|_{L^\infty}.$$

Consequently, assumption (A2) implies that:

$$\lim_{N \rightarrow \infty} \int_{\Omega} \sigma^N(x) \cdot v(x) dx = \int_{\Omega} \rho(x) w(x) \cdot v(x) dx,$$

and  $\sigma^N \rightharpoonup \rho w$  weakly in  $L^q(\Omega)$  for arbitrary  $q \in (1, \infty)$ . Combining then the weak convergence of  $\sigma^N$  in  $L^2(\Omega)$  and the strong convergence of  $u^N$  in  $L^2(\Omega)$  (up to the extraction of a subsequence), we have:

$$\lim_{N \rightarrow \infty} \int_{\Omega} \sigma^N \cdot u^N = \int_{\Omega} \rho w \cdot \bar{u}.$$

As for the remainder term, we introduce:

$$\tilde{\sigma}^N = \left(1 - \left(1 - \frac{1}{\delta}\right)^3\right)^{-1} \frac{1}{N|\lambda^N|^3} \sum_{\kappa \in \mathcal{K}^N} \left( \sum_{i \in \mathcal{I}_\kappa^N \cap \mathcal{Z}_\delta^N} |w(h_i^N)| \right) \mathbf{1}_{[T_\kappa^N]_{2\delta}}.$$

so that:

$$|\tilde{E}rr^N| \leq \int_{\Omega} \tilde{\sigma}^N(x) |u^N(x)| dx.$$

With similar arguments as in the previous computations, we have, applying (27):

$$\|\tilde{\sigma}^N\|_{L^1(\Omega)} \leq \frac{1}{N} \#\mathcal{Z}_\delta^N \|w\|_{L^\infty} \leq \frac{1}{\delta} \|w\|_{L^\infty} (1 + |\mathcal{E}^\infty|^2).$$

Furthermore, we have:

$$\|\tilde{\sigma}^N\|_{L^\infty(\Omega)} \lesssim \delta M^\infty \|w\|_{L^\infty}.$$

Consequently, by interpolation, we obtain:

$$\|\tilde{\sigma}^N\|_{L^{\frac{4}{3}}(\Omega)} \lesssim \frac{|M^\infty|^{\frac{1}{4}}}{\sqrt{\delta}} \|w\|_{L^\infty} (1 + |\mathcal{E}^\infty|^2)^{\frac{3}{4}}.$$

As  $u^N$  is bounded in  $L^4(\Omega)$  by sobolev embedding, this yields that:

$$|\tilde{E}rr^N| \lesssim \frac{|M_\infty|^{\frac{1}{4}}}{\sqrt{\delta}} \|w\|_{L^\infty} (1 + |\mathcal{E}^\infty|^2)^{\frac{5}{4}}, \quad \forall N \in \mathbb{N}.$$

This ends the proof.  $\square$

## 6. TWO (COUNTER-)EXAMPLES

In this paper, we derive the Stokes-Brinkman system by homogenizing the Stokes problem in a perforated domain. We recall that we deal only with the dilution regime specified by assumptions (A4)-(A5). Assumption (A5) is motivated by the fact that we want to consider particle distribution functions  $(x, v) \mapsto f(x, v)$  such that the associated density  $x \mapsto \rho(x)$  is bounded. This implies that, for arbitrary  $(x, \lambda) \in \Omega \times (0, \infty)$  the density  $M(x, \lambda)$  of cloud particles in  $B(x, \lambda)$  satisfies

$$M(x, \lambda) \leq \|\rho\|_{L^\infty} \lambda^3.$$

At the discrete level, we require thus that the density of particles in any box of width  $\lambda^N$  satisfies:

$$\frac{M_\kappa^N}{N} \leq M_\infty |\lambda^N|^3, \quad \forall \kappa.$$

One may prove that under the sole assumption that the sequence of discrete density measures  $\rho^N$  converges to  $\rho(x)dx$  with  $\rho \in L^\infty(\Omega)$  implies that there exists a sequence  $(\lambda^N)_{N \in \mathbb{N}}$  converging to 0 for which, up to the extraction of a subsequence, (A5) holds true. So, in this section, we keep this assumption and, on the basis of two examples, we discuss the minimal distances  $(d_{min}^N)_{N \in \mathbb{N}}$  below which we do not expect the homogenized system to correspond to our Stokes-Brinkman problem (5)-(6).

First, as pointed out in the introduction, with assumptions (A4)-(A5), we enforce that  $d_{min}^N$  remains much larger than  $1/N$ . On the opposite, if  $d_{min}^N \sim 1/N$  or  $d_{min}^N << 1/N$ , the distance between holes becomes comparable to their common radius and the influence of the holes on the solution is more intricate. In such a case, we expect that one can pack the holes into sub-groups containing holes between which the distance is smaller or comparable to their common radius. Then, each of these packs of holes has to be considered as one hole with a complicated shape instead of a group of holes. This remark applies in the following example. Let divide the container  $\Omega = [0, 1]^3$  into  $N/2$  cubes  $(T_k^N)_{k=1, \dots, N/2}$  of width  $(2/N)^{1/3}$ . Each of the cubes contains 2 holes so that the centers of these holes are diametrically symmetric on a sphere of radius  $(1+h)/N$  ( $h$  is a positive parameter) centered in the center of the cube (see Figure 1). Broadly, it comes from the proof in the previous sections that the Brinkman term in the limit problem can be computed by zooming in any of the elementary cells (with a scale  $1/N$ ), computing the drag terms involved by the Stokes problem in the cells and summing them after rescaling. In this example, one cell corresponds to a cube  $T_k^N$  which contains two spherical holes. Then, the drag term is computed by considering the Stokes problem in an exterior domain whose shape is the complement of two unit balls. We expect that, after summation, the resulting Brinkman term has a different structure than " $6\pi(j - \rho u)$ ". Especially, it should depend nonlinearly on the parameter  $h$  and anisotropically on  $u$ . To end up the analysis of this example, we mention that it would be natural to choose  $\lambda^N = 1/N^{1/3}$  in order that (A5) holds true. We would then get

$$\lim_{N \rightarrow \infty} \frac{d_{min}^N}{|\lambda^N|^3} = 2h$$

so that (A4) is not satisfied. The value  $h = 0$  is allowed here up to some restriction on the choice of velocities  $(v_i^N)_{i=1, \dots, N}$ .

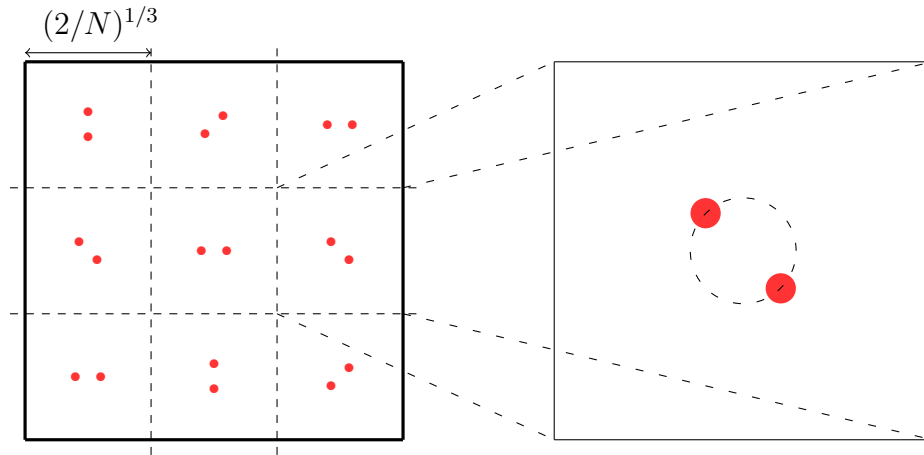


FIGURE 1. First counter-example configuration

The above construction shows that if concentration holds, the microscopic structure of the cells plays a role on the Brinkman term. Now, we provide also an example which shows that the homogenized system might not be of Stokes-Brinkman type. Our example is a variant of the construction in [1]. In particular, we go back to the case of a Stokes problem in a bounded perforated domain with a source term  $f \in L^2(\Omega)$ . We consider vanishing boundary conditions on the holes for simplicity. The holes will be distributed (almost) periodically so that their density converges to a uniform distribution in  $[0, 1]^3$ . In particular, if our main result were extending to this case, the homogenized system should read:

$$\begin{aligned} -\Delta u + \nabla p &= f - 6\pi \mathbf{1}_{[0,1]^3} u & \text{in } \Omega. \\ \operatorname{div} u &= 0 \end{aligned}$$

Nevertheless, let consider  $\Omega$  a smooth bounded domain containing  $[0, 1]^3$  and  $(P^N)_{N \in \mathbb{N}}$  a diverging sequence of integers. We assume that

$$(46) \quad \lim_{N \rightarrow \infty} \frac{P^N}{N} = 0.$$

Given  $N \in \mathbb{N}$  we cover  $\mathbb{R}^3$  with disjoint cubes  $(T_k^N)_{k \in \mathbb{Z}^3}$  of width  $\sigma^N = |P^N/N|^{1/3}$ . For  $k \in \mathbb{Z}^3$ , we denote by  $x_k^N$  the center of  $T_k^N$  so that  $T_k^N = B_\infty(x_k^N, \sigma^N/2)$ . For  $N$  sufficiently large, we extract a list  $\mathcal{K}^N$  containing  $\lfloor N/P^N \rfloor + 1$  indices of cubes  $T_k^N$  that are inside  $\Omega$ . To do this, we first choose all the cubes that are included in  $[0, 1]^3$  and we complement the list by choosing at least one other cube that is included in  $\Omega$ . For the  $\lfloor N/P^N \rfloor$  first cubes of the list  $\mathcal{K}^N$  (including all the ones that are inside  $[0, 1]^3$ ), we perform  $P^N$  holes in  $T_k^N$ . The holes are distributed concentrically around the center  $x_k^N$  of  $T_k^N$  on an orthogonal grid of step  $2d_m^N > 0$ . In particular, we center the grid so that all the perforated sites are inside  $B_\infty(x_k^N, \lfloor (|P^N|^{1/3} + 1) \rfloor d_m^N)$ . To ensure that all the perforated sites remain in  $T_k^N$  and the holes are sufficiently spaced, we assume below that:

$$(47) \quad \lim_{N \rightarrow \infty} N^{\frac{1}{3}} d_m^N = 0, \quad \lim_{N \rightarrow \infty} N d_m^N = +\infty.$$

In the last cube, we perform  $N - \lfloor N/P^N \rfloor P^N$  holes in the same way so that we have eventually  $N$  holes of radius  $1/N$  in  $\Omega$  that we label  $(B(h_i^N, 1/N))_{i=1, \dots, N}$ . See Figure 2 for an illustration.

With these conventions, we introduce  $f \in L^2(\Omega)$  and are interested now in the asymptotic behavior of the unique  $u^N \in H_0^1(\Omega)$  such that there exists a pressure  $p^N$  for which there holds:

$$(48) \quad \begin{cases} -\Delta u^N + \nabla p^N &= f, \\ \operatorname{div} u^N &= 0, \end{cases} \quad \text{on } \mathcal{F}^N := \Omega \setminus \bigcup_{i=1}^N B(h_i^N, 1/N),$$

completed with boundary conditions

$$(49) \quad \begin{cases} u^N &= 0, & \text{on } \partial B(h_i^N, 1/N), \\ u^N &= 0, & \text{on } \partial \Omega. \end{cases}$$

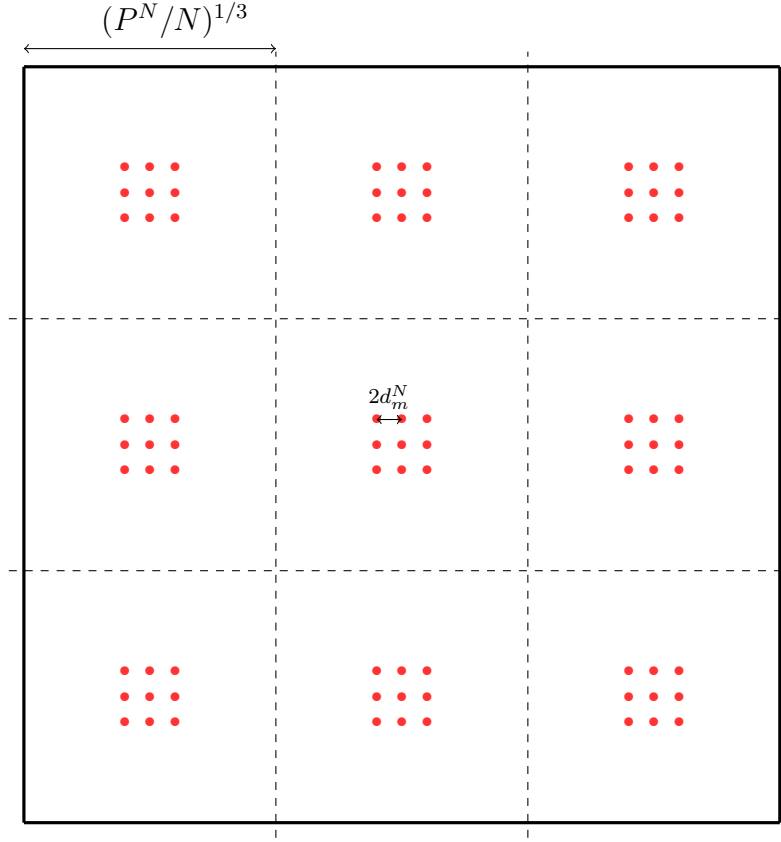


FIGURE 2. Second counter-example configuration.

We observe that the Stokes regime computed in [1] extends to this example:

**Proposition 13.** *Assume that (46)-(47) are in force together with:*

$$(50) \quad \lim_{N \rightarrow \infty} \frac{Nd_m^N}{|P^N|^{\frac{2}{3}}} = 0.$$

*Then  $u^N$  converges in  $H_0^1(\Omega) - w$  to the unique  $\bar{u} \in H_0^1(\Omega)$  such that there exists  $\bar{p} \in L^2(\Omega)$  for which:*

$$\begin{aligned} -\Delta \bar{u} + \nabla \bar{p} &= f \\ \operatorname{div} \bar{u} &= 0 \end{aligned} \quad \text{in } \Omega.$$

*Proof.* First, as  $d_m^N \gg 1/N$ , we may reproduce the arguments in Section 3 to obtain that  $u^N$  is bounded in  $H_0^1(\Omega)$ . We have thus a weak-cluster point in the weak-topology. We then show that any cluster point of the sequence  $u^N$  for the weak topology of  $H_0^1(\Omega)$  is the above  $\bar{u}$ .

To prove this latter property, we introduce  $r^N := (|P^N|^{\frac{1}{3}} + 1)d_m^N$ . Then, given a divergence-free  $w \in C_c^\infty(\Omega)$  we set :

$$\bar{w}^N = w - \left[ \sum_{k \in \mathcal{K}^N} \chi \left( \frac{(x - x_k^N)}{r^N} \right) w - \mathfrak{B}_{x_k^N, r^N, 2r^N} \left[ x \mapsto \nabla \chi \left( \frac{(x - x_k^N)}{r^N} \right) \cdot w(x) \right] \right]$$

As all the holes are contained in the  $B_\infty(x_k^N, r^N)$  for  $k \in \mathcal{K}^N$ , we have that  $\bar{w}^N \in H_0^1(\mathcal{F}^N)$  and is divergence-free. Because  $u^N$  is a solution to the Stokes system in  $\mathcal{F}^N$  we obtain then that:

$$\int_{\Omega} \nabla u^N : \nabla \bar{w}^N = \int_{\Omega} f \cdot \bar{w}^N.$$

Let denote

$$\delta_w^N = \left[ \sum_{k \in \mathcal{K}^N} \chi \left( \frac{(x - x_k^N)}{r^N} \right) w - \mathfrak{B}_{x_k^N, r^N, 2r^N} \left[ x \mapsto \nabla \chi \left( \frac{(x - x_k^N)}{r^N} \right) \cdot w(x) \right] \right]$$

Remarking that the vector-fields in the sum have disjoint supports (see (47)) and applying the properties of the Bogovskii operator of the appendix together with the fact that  $\#\mathcal{K}^N \lesssim N/P^N$ , we obtain:

$$\|\delta_w^N\|_{H_0^1(\Omega)}^2 \lesssim \#\mathcal{K}^N r^N \|w\|_{W^{1,\infty}}^2 \lesssim \frac{Nd_m^N}{|P^N|^{\frac{2}{3}}} \|w\|_{W^{1,\infty}}^2.$$

Consequently, with assumption (50) we have that  $\delta_w^N$  converges strongly to 0 in  $H_0^1(\Omega)$  so that:

$$\lim_{N \rightarrow \infty} \int_{\Omega} \nabla u^N : \nabla \bar{w}^N = \int_{\Omega} \nabla \bar{u} : \nabla \bar{w} \quad \lim_{N \rightarrow \infty} \int_{\Omega} f \cdot \bar{w}^N = \int_{\Omega} f \cdot w.$$

This ends the proof.  $\square$

This last example does not contradict our main result. Indeed, in this example, the role of  $d_{min}^N$  is played by  $d_m^N$ . We wonder then, if we may construct a sequence  $(\lambda^N)_{N \in \mathbb{N}}$  so that (A4) and (A5) hold simultaneously, *i.e.*:

$$\frac{M^N}{N} \lesssim |\lambda^N|^3 \ll d_m^N.$$

However, if we were having such a sequence, given  $N \in \mathbb{N}$  we would have two cases:

- either  $\lambda^N < |P^N|^{\frac{1}{3}} d_m^N$ , so that  $M^N \sim |\lambda^N|^3 / |d_m^N|^3$  and

$$\frac{M^N}{N|\lambda^N|^3} \sim \frac{1}{N|d_m^N|^3} \gg 1 \text{ (because of assumption (47))};$$

- either  $\lambda^N > |P^N|^{\frac{1}{3}} d_m^N$ , so that  $P^N \lesssim M^N$  and, applying that we require  $|\lambda^N|^3 \ll d_m^N$ , we should have:

$$\frac{M^N}{N|\lambda^N|^3} \gg \frac{P^N}{Nd_m^N} \gg 1 \text{ (because of assumption (50))}.$$

Hence, we contradict (A5). Finally, given the way we constructed this second example, it would be natural to choose  $\lambda^N = (P^N/N)^{1/3}$  so that (A5) holds true. We would then have

$$\lim_{N \rightarrow \infty} \frac{d_{min}^N}{|\lambda^N|^3} = \lim_{N \rightarrow \infty} \frac{d_m^N N}{P^N} = 0.$$

Hence (A4) is not satisfied.

To conclude, we stress that the analysis of the two above examples imply that our Theorem 1 seems optimal in the frame of finite-energy bounded-density configurations. Indeed, we already remarked that if the density  $\rho$  of holes is bounded then we may construct a sequence  $(\lambda^N)_{N \in \mathbb{N}}$  so that (A5) holds true. Then if (A4) is not satisfied we should be able to extract a subsequence so that  $d_{min}^N/|\lambda^N|^3$  has a bounded limit. If this limit is strictly positive our first example shows that we need *a priori* more information on the microscopic structure of the cells to compute the Brinkman term. While, if the ratio  $d_{min}^N/|\lambda^N|^3$  converges to 0, a comparison of the two examples shows that it is even not clear which type of homogenized system is to be expected.

#### APPENDIX A. AUXILIARY TECHNICAL LEMMAS

We recall here several standard lemmas that help in the above proofs.

First, we recall the Poincaré-Wirtinger inequality [7, Theorem II.5.4] which states that for arbitrary lipschitz domain  $\mathcal{F}$ , there holds:

$$\left\| u - \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} u(x) dx \right\|_{L^2(\mathcal{F})} \leq C_{PW} \|\nabla u\|_{L^2(\mathcal{F})}.$$

We extensively use this inequality when  $\mathcal{F}$  is an annulus. In this case, a standard scaling argument entails the following remark on the constant  $C_{PW}$ :

**Lemma 14.** *Given  $(x, \lambda, a) \in \mathbb{R}^3 \times (0, \infty) \times (0, 1)$  there exists a constant  $C_a$  depending only on  $a$  (and expecially not on  $(x, \lambda)$ ) for which :*

$$\left\| u - \frac{1}{|A(x, a\lambda, \lambda)|} \int_{A(x, a\lambda, \lambda)} u(x) dx \right\|_{L^2(A(x, a\lambda, \lambda))} \leq C_a \lambda \|\nabla u\|_{L^2(A(x, a\lambda, \lambda))}.$$

Second, we focus on the properties of the Bogovskii operators  $\mathfrak{B}$ . This means we are interested in solving the divergence problem:

$$(51) \quad \operatorname{div} v = f, \quad \text{on } \mathcal{F},$$

whose data is  $f$  and unknown is  $v$ . We recall the result due to M.E. Bogovskii (see [7, Theorem III.3.1]):

**Lemma 15.** *Let  $\mathcal{F}$  be a lipschitz bounded domain in  $\mathbb{R}^3$ . Given  $f \in L^2(\mathcal{F})$  such that*

$$\int_{\mathcal{F}} f(x) dx = 0,$$



there exists a solution  $v := \mathfrak{B}_{\mathcal{F}}[f] \in H_0^1(\mathcal{F})$  to (51) such that

$$\|\nabla v\|_{L^2(\mathcal{F})} \leq C \|f\|_{L^2(\mathcal{F})}$$

with a constant  $C$  depending only on  $\mathcal{F}$ .

In the case of annuli, the above result yields the following lemma by a standard scaling argument:

**Lemma 16.** *Let  $(x, \lambda, a) \in \mathbb{R}^3 \times (0, \infty) \times (0, 1)$ . Given  $f \in L^2(A(x, a\lambda, \lambda))$  such that*

$$\int_{\mathcal{F}} f(x) dx = 0,$$

*there exists a solution  $v := \mathfrak{B}_{x, a\lambda, \lambda}[f] \in H_0^1(A(x, a\lambda, \lambda))$  to (51) such that*

$$\|\nabla v\|_{L^2(A(x, a\lambda, \lambda))} \leq C_a \|f\|_{L^2(A(x, a\lambda, \lambda))},$$

*with a constant  $C_a$  depending only on  $a$  (and especially neither on  $f$  nor on  $(x, \lambda)$ ) .*

## APPENDIX B. PROOF OF A COVERING LEMMA

This appendix is devoted to the construction of coverings that are adapted to the empiric measures  $S_N$ . We prove the following general lemma:

**Lemma 17.** *Let  $(d, \lambda) \in \mathbb{N}^* \times (0, \infty)$ ,  $d \geq 2$ , and  $\mu \in \mathcal{M}_+(\mathbb{R}^3)$  a positive bounded measure. There exists  $(T_\kappa)_{\kappa \in \mathbb{Z}^3}$  a covering of  $\mathbb{R}^3$  with disjoint cubes of width  $\lambda$  such that denoting*

$$\mathcal{C}_d^\lambda := \left\{ x \in \mathbb{R}^3 \text{ s.t. } \text{dist} \left( x, \bigcup_{\kappa \in \mathbb{Z}^3} \partial T_\kappa \right) < \frac{\lambda}{(d+1)} \right\}$$

*there holds*

$$(52) \quad \mu(\mathcal{C}_d^\lambda) \leq \frac{6}{d} \mu(\mathbb{R}^3).$$

In Section 4, we apply the previous lemma for arbitrary  $N \in \mathbb{N}^*$ , with  $\lambda = \lambda^N$ ,  $d = \delta - 1$  and

$$\mu := \frac{1}{N} \sum_{i=1}^N (1 + |v_i^N|^2) \delta_{h_i^N}.$$

We obtain a covering  $(T_\kappa^N)_{\kappa \in \mathbb{Z}^3}$  satisfying (27). Assuming then  $\lambda^N \leq [\text{dist}(\text{Supp}(w), \mathbb{R}^3 \setminus \Omega)/4]$  (this is possible as  $\lambda^N \rightarrow 0$ ) we obtain that the subcovering  $(T_\kappa^N)_{\kappa \in \mathcal{K}^N}$  containing only the cubes that intersect  $\text{Supp}(w)$  is made of cubes  $T_\kappa^N$  that are included in the  $\lambda^N$ -neighborhood of  $\text{Supp}(w)$ . By direct computations, we obtain then that, for  $\kappa \in \mathcal{K}^N$ , the distance between  $T_\kappa^N$  and  $\mathbb{R}^3 \setminus \Omega$  is strictly positive so that  $T_\kappa^N \subset \Omega$ .

*Proof.* By a standard scaling argument, it suffices to prove the result for  $\lambda = 1$ . Let  $d \geq 2$ . First, for arbitrary  $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$  we set:

$$\tilde{T}_k = \left[ \frac{k_1}{d}, \frac{k_1+1}{d} \right] \times \left[ \frac{k_2}{d}, \frac{k_2+1}{d} \right] \times \left[ \frac{k_3}{d}, \frac{k_3+1}{d} \right]$$

These cubes with tildas and index  $k$  are cubes of width  $1/d$ . We call them "small cubes." It is straightforward that  $(\tilde{T}_k)_{k \in \mathbb{Z}^3}$  forms a partition of  $\mathbb{R}^3$ . For arbitrary

$$\kappa = (k_1, k_2, k_3) + \{0, \dots, d-1\}^3,$$

we set then:

$$T_\kappa = \bigcup_{k \in \kappa} \tilde{T}_k = \left[ \frac{k_1}{d}, \frac{k_1}{d} + 1 \right] \times \left[ \frac{k_2}{d}, \frac{k_2}{d} + 1 \right] \times \left[ \frac{k_3}{d}, \frac{k_3}{d} + 1 \right].$$

These cubes without tildas and with index  $\kappa$  are cubes of width 1. We call them "large cubes". We introduce then the  $1/d$ -neighborhood of the boundary of this large cube:

$$[T_\kappa]_d := \bigcup_{k \in \partial\kappa} \tilde{T}_k.$$

where

$$\begin{aligned} \partial\kappa &= \{k \in \{k_1, k_1 + d - 1\} \times \{k_2, \dots, k_2 + d - 1\} \times \{k_3, \dots, k_3 + d - 1\}\} \\ &\cup \{k \in \{k_1, \dots, k_1 + d - 1\} \times \{k_2, k_2 + d - 1\} \times \{k_3, \dots, k_3 + d - 1\}\} \\ &\cup \{k \in \{k_1, \dots, k_1 + d - 1\} \times \{k_2, \dots, k_2 + d - 1\} \times \{k_3, k_3 + d - 1\}\} \end{aligned}$$

(which means taking the small cubes whose indices are in the boundary of  $\kappa$ ). We remark that we may split  $[T_\kappa]_d$  into 6 subsets corresponding to the top, bottom, left, right, front and back faces of the cube  $T_\kappa$ . For instance, the bottom face of  $[T_\kappa]_d$  reads:

$$\bigcup_{k \in \{k_1, \dots, k_1 + d - 1\} \times \{k_2, \dots, k_2 + d - 1\} \times \{k_3\}} \tilde{T}_k.$$

For arbitrary  $k^\ell = \ell(1, 1, 1)$ , with  $\ell \in \{0, \dots, d-1\}$  we also denote

$$\mathcal{K}_\ell = \left\{ \kappa = (k^\ell + \pi + \{0, \dots, d-1\}^3), \quad \pi \in d\mathbb{Z}^3 \right\}$$

We emphasize that  $\mathcal{K}_\ell$  is a set made of sets (corresponding to large cubes). Any set  $\mathcal{K}_\ell$  corresponds to a partition of  $\mathbb{Z}^3$  and then to a covering of  $\mathbb{R}^3$  with disjoint large cubes.

Given  $\ell \in \{0, \dots, d-1\}$  we consider now

$$\mathcal{C}_d^\ell = \left\{ x \in \mathbb{R}^3 \text{ s.t. } \text{dist} \left( x, \bigcup_{\kappa \in \mathcal{K}_\ell} \partial T_\kappa \right) < \frac{1}{(d+1)} \right\}.$$

We remark that, for fixed  $\ell$  there holds:

$$\mathcal{C}_d^\ell \subset \bigcup_{\kappa \in \mathcal{K}_\ell} [T_\kappa]_d.$$

We denote  $\partial\mathcal{K}_l$  the set of indices  $k$  such that  $\tilde{T}_k$  contributes to this  $1/d$ -neighborhood. Setting  $\partial\mathcal{K}_l = \bigcup \{\partial\kappa, \kappa \in \mathcal{K}_l\}$ , we have thus:

$$\mathcal{C}_d^\ell \subset \bigcup_{k \in \partial\mathcal{K}_l} \tilde{T}_k.$$

We can decompose this union of small cubes by regrouping together the cubes that belong to left / right / top / bottom / front / back faces of large cubes. For instance, the indices  $k$  of small cubes belonging to bottom faces of large cubes satisfy

$$k \in \mathbb{Z}^2 \times \{\ell + d\mathbb{Z}\}.$$

For two different  $\ell$  and  $\ell'$  in  $\{0, \dots, d-1\}$  the same index  $k$  cannot belong to the bottom faces of two different cubes in the coverings  $\mathcal{K}_\ell$  and  $\mathcal{K}_{\ell'}$  of  $\mathbb{R}^3$ . We have the same properties for top / right / left / front / back faces. Consequently, in the family of coverings  $(\mathcal{K}_\ell)_{\ell \in \{0, \dots, d-1\}}$  one small cube  $\tilde{T}_k$  belongs at most once to a top / bottom / right / left / front / back face of a large cube so that:

$$(53) \quad \text{any } k \in \mathbb{Z}^3 \text{ belongs to at most 6 different } \partial\mathcal{K}_l.$$

Let now introduce the measure  $\mu$ . For any  $k \in \mathbb{Z}^3$ , we denote:

$$\tilde{\mu}_k = \mu(\tilde{T}_k),$$

and we consider the sum:

$$Rem := \sum_{\ell \in \{0, \dots, d-1\}} \mu(\mathcal{C}_d^\ell).$$

With the previous definitions, we have:

$$Rem \leq \sum_{\ell \in \{0, \dots, d-1\}} \sum_{k \in \partial\mathcal{K}_\ell} \tilde{\mu}_k.$$

Because of (53), we have then that any  $k \in \mathbb{Z}^3$  appears at most 6 times in this sum. Consequently:

$$Rem \leq 6 \sum_{k \in \mathbb{Z}^3} \tilde{\mu}_k \leq 6\mu(\mathbb{R}^3).$$

The measure  $\mu$  being positive and finite, this implies that one of the terms in the sum defining  $Rem$  is less than  $Rem/d$ . In other words, there exists at least one  $\ell^0 \in \{0, \dots, d-1\}$  such that:

$$\mu(\mathcal{C}_d^{\ell^0}) \leq \frac{6}{d}\mu(\mathbb{R}^3).$$

The covering  $(T_\kappa)_{\kappa \in \mathcal{K}_{\ell^0}}$  is then made of disjoint cubes of width 1 satisfying (52). We have obtained the required covering of  $\mathbb{R}^3$ .  $\square$

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