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INDIVIDUAL VACCINATION EQUILIBRIUM FOR IMPERFECT VACCINE EFFICACY AND LIMITED PERSISTENCE

FRANCESCO SALVARANI AND GABRIEL TURINICI

ABSTRACT. We analyze a model of vaccinal campaign with imperfect vaccine efficacy and limited persistence. We prove the existence of an equilibrium by Kakutani's fixed point theorem in the context of non-persistent immunity. Subsequently, we propose and test a novel numerical method to find the equilibrium. Various issues of the model are then discussed, among which the dependence of the optimal policy with respect to the imperfections of the vaccine.

1. INTRODUCTION

Vaccination is a widely used epidemic control tool and its impact on the epidemic spread has been analyzed from several perspectives.

The first approach which has been historically considered is the *benevolent planner* framework (see [27, 1, 37, 42, 17, 2, 31]). It supposes that a health authority can decide of a vaccination plan, which is then implemented. The plan optimizes the vaccination strategy as a function of the severity of the epidemic, its medical risks and the (economic and medical) costs associated with the vaccine. This model is suitable for compulsory vaccination or when the individuals fully adhere to the recommendations of the authority. On the contrary, when the vaccination is a choice – on a voluntary base – or when there exist debates on the risks or costs of the vaccine, the situation is better described by models which take into account the individual decision level. In such models, the agents decide for themselves whether the vaccination is suitable or not, but they cannot individually influence the epidemic propagation, which is given by the collective choice of all other persons.

Such questions received a firm mathematical ground since the introduction of the *Mean Field Games* (MFG) theory in the pioneering works by Lasry and Lions [34, 33, 35] and by Huang, Malhamé and Caines [29, 28].

One important question of MFG is the existence of an equilibrium, namely a stable collection of individual strategies such that nobody has any incentive to change his own strategy.

This approach is formulated in terms of a Nash equilibrium for the whole population. Early works in this direction include [22, 9, 26] which study the question of disease eradication, market equilibrium and externalities in relationship with vaccination. More recent contributions (see [5, 4, 43, 23, 41, 32]) treat the question of Nash equilibria for a large number of individuals in relation with an epidemic model. They investigated many aspects, such as the impact of the subjective perceptions and individual behaviors on the equilibrium (see, for example, [15, 14, 40]), the presence of several groups having distinct epidemic characteristics (see [25, 16, 12]), particular vaccination strategies or

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specific models about the available information at the individual level (see [10, 6, 21, 19, 20, 24, 46, 8, 18]).

The rich diversity of models mentioned above leads to natural questions about the numerical computation of the equilibrium and the study of the sensitivity to vaccination specificities. For this reason in this work we will consider the realistic situation of an imperfect vaccine (see [48] for a related work). Two attributes of the vaccine are considered:

- the persistence, which is the duration of the immunity conferred by the vaccine; this can span from several months up to several years – see [13, 3] and the literature within;
- the vaccine efficacy (noted VE, an input in our model), which is the theoretical success rate (to be distinguished from the vaccine efficiency, which is the practical observed success and is the output of the model – see [47] for a presentation of the differences between the two). The VE can range from several percents to almost perfect efficacy – see the meta-analysis in [39] and also [38]; other references include [36] and [45].

As such, a first contribution of our model is to study imperfect vaccines.

On the other hand, the individual vaccine model that we propose does not have, up to our knowledge, an explicit solution. Far from being a disadvantage, this situation prompted us into proposing a general numerical method to find the equilibrium; this is a second contribution of this work (see also [44] for some alternatives coming from the physics community for general Mean Field Games). The numerical method is adapted from general works in game theory (see Section 3) and is expected to give accurate results in any situation when an individual chooses the right timing to perform some action (here vaccination) with strategy-dependent costs. This procedure has been extensively tested on our model and performs very well.

The structure of the paper is the following: the model is presented in Section 2 and the theoretical result guaranteeing the existence of an equilibrium in Section 2.3. The numerical algorithm for finding the equilibrium is presented in Section 3 and the numerical results in Section 4. Finally, Section 5 collects some considerations on the pertinence and validity of our approach.

2. THE MODEL

The model studies the dynamics of an epidemic in a population. Each disease has its own features. In what follows we will suppose that

- the infection does not cause the death of the patient and it gives permanent immunity to those who have been infected;
- the incubation period is short when compared to the time scale of the model;
- the individuals can be vaccinated. If the vaccine is successful, the protection of the vaccine is maximal (but possibly not total) after a time delay, it remains high during some period and then it decreases. At the end, the individual is not at all protected by the vaccine.
- the evolution of the epidemic can be influenced by seasonality effects.

We suppose that the time horizon T is finite, and that it can be discretized in $(N + 1)$ ($N \in \mathbb{N}$) time instants $t_0 = 0$, $t_1 = \Delta T$, $t_2 = 2\Delta T, \dots, t_n = n\Delta T, \dots, t_N = T$. The population is composed of

- *susceptible individuals*: S_n is the proportion of individuals in this class at time t_n ;
- *infected individuals*: I_n^ω is the proportion of individuals in this class at time t_n , which have been infected at time $t_{n-\omega}$; moreover we denote by I_n the sum of all I_n^ω ;
- *recovered individuals* upon exiting the class of infected;
- *vaccinated individuals*: V_n^θ is the proportion of individuals that vaccinated at time $t_{n-\theta}$ and have not been infected since;
- *failed vaccinated individuals*: F_n is the proportion of individuals that vaccinated at some time $t \leq t_n$, whose vaccination failed and have not yet been infected since.

The quantities ω and θ are counters. The first one measures the time lapse between the infection instant and the current instant, $\omega = 0, 1, \dots, \Omega \in \mathbb{N}$, whereas the second one measures the time lapse between the vaccination instant and the current instant, $\theta = 0, 1, \dots, \Theta - 1, \Theta \in \mathbb{N}$. The upper bound Θ indicates the maximal duration of the immunity given by the vaccine. Correspondingly, V^Θ is the class of vaccinated individuals that lost the immunity given by the vaccine: since they do not vaccinate twice, we need a specific class for describing them. Similarly, Θ is the maximum time before recovery, and it depends on the properties of the illness itself.

The equations of the model, which conserves the total number of individuals, have the following form:

$$S_{n+1} = (1 - \beta_{\Delta T}^n I_n) (S_n - U_n) \quad (2.1)$$

$$I_{n+1}^0 = \beta_{\Delta T}^n \left[F_n + S_n + \sum_{\theta=0}^{N-1} \alpha_\theta V_n^\theta \right] I_n \quad (2.2)$$

$$I_{n+1}^{\omega+1} = (1 - \gamma_{\Delta T}^\omega) I_n^\omega \quad \omega = 0, \dots, \Omega - 1 \quad (2.3)$$

$$V_{n+1}^0 = (1 - f) \cdot (1 - \beta_{\Delta T}^n I_n) U_n \quad (2.4)$$

$$V_{n+1}^{\theta+1} = (1 - \beta_{\Delta T}^n \alpha_\theta I_n) V_n^\theta, \quad \theta = 0, \dots, \Theta - 2 \quad (2.5)$$

$$V_{n+1}^\Theta = (1 - \beta_{\Delta T}^n \alpha_{\Theta-1} I_n) V_n^{\Theta-1} + (1 - \beta_{\Delta T}^n I_n) V_n^\Theta \quad (2.6)$$

$$F_{n+1} = f \cdot (1 - \beta_{\Delta T}^n I_n) U_n + F_n (1 - \beta_{\Delta T}^n I_n) \quad (2.7)$$

with initial conditions

$$S_0 = S_{0-}, \quad I_0^\omega = I_{0-}^\omega, \quad V_0^\theta = 0, \quad \forall \theta \geq 0, \quad (2.8)$$

where

- U describes the vaccination: U_n is the proportion of people vaccinated after time t_n and before t_{n+1} ; a fraction f of them will never develop any immunity and will enter the *failed vaccination* class;
- the vector $\gamma = (\gamma^0, \dots, \gamma^\Omega) \in (\mathbb{R}_+)^{\Omega+1}$ describes how fast an infected individual recovers and depends on the duration of the illness itself. In particular, when $\gamma^\omega = 0$, there is no hope to recover at the next time instant; on the contrary, when $\gamma^\omega = 1$, the individual will recover with certainty in the next time slot;
- the function $\beta(t)$ quantifies how infectious is a contact between an infected individual and a susceptible one at time t . To take into account the length of the

time interval ΔT , we work with $\beta_{\Delta T}^n := \beta(t_n) \cdot \Delta T$ and $\gamma_{\Delta T}^\omega := \gamma^\omega \cdot \Delta T$. In order to take into account the possible seasonality, $\beta(\cdot)$ is taken time-dependent, see Section 4 for an exemple.

- The vector α_θ are the time instants of a function $A(\cdot)$ ($\alpha_\theta = A(\theta\Delta T)$) with values in $[0, 1]$. This vector quantifies the protection given by the vaccine in terms of the probability of infection if vaccinated. It is known that this protection is not instantaneous, the immunity conferred by the vaccine being maximal after a latency period. Moreover, often (as in the case of influenza vaccine) the protection is not complete, and the effects of the vaccine decrease with time. In what follows, we suppose that there exists an upper bound $\Theta > 0$ to the persistence of the vaccine. Some possible candidates for the function A are shown in Figure 1.

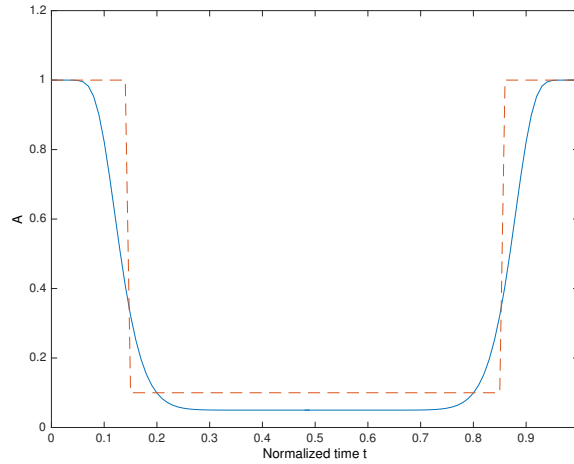


FIGURE 1. Two possible forms for the function A .

2.1. The societal cost and individual cost. Let r_I and r_V be the individual cost for the illness and the vaccination respectively. We work under the meaningful assumption that $r_I > r_V$ (although the alternative $r_I \leq r_V$ may also give non-trivial problems in particular situations, see [31]).

The total societal cost associated to the vaccination strategy U is:

$$J(S_0, I_0, U) = r_I \sum_{n=0}^N I_n + r_V \sum_{n=0}^{N-1} U_n, \quad (2.9)$$

which has to be minimized (see [27, 1, 37, 42, 17, 31]) within the set of all admissible vaccination strategies U .

However this is not what the individuals do. They rather optimize an individual cost function. In order to define it, we have to consider the individual dynamics (see Figure 2 for an illustration). It takes the form of a controlled Markov chain with several states, *susceptible* (S), *failed vaccination* (F), *recovered* (R), *infected* (indexed by the time counter ω : I^0, \dots, I^Ω), and, finally, *vaccinated* states (indexed by the time counter θ : V^0, \dots, V^Θ).

The Markov chain of the individual, denoted M_n , is described in terms of passage probabilities:

$$\begin{aligned}
\mathbb{P}(M_{n+1} = S | M_n = S) &= (1 - \lambda_n)(1 - \beta_{\Delta T}^n I_n) \\
\mathbb{P}(M_{n+1} = I^0 | M_n = S) &= \beta_{\Delta T}^n I_n \\
\mathbb{P}(M_{n+1} = V^0 | M_n = S) &= (1 - f) \cdot \lambda_n (1 - \beta_{\Delta T}^n I_n) \\
\mathbb{P}(M_{n+1} = F | M_n = S) &= f \cdot \lambda_n (1 - \beta_{\Delta T}^n I_n) \\
\mathbb{P}(M_{n+1} = R | M_n = I^\Omega) &= 1 \\
\mathbb{P}(M_{n+1} = R | M_n = I^\omega) &= \gamma_{\Delta T}^\omega, & \omega = 0, \dots, \Omega - 1 \\
\mathbb{P}(M_{n+1} = I^{\omega+1} | M_n = I^\omega) &= 1 - \gamma_{\Delta T}^\omega, & \omega = 0, \dots, \Omega - 1 \\
\mathbb{P}(M_{n+1} = I^0 | M_n = V^\theta) &= \alpha_\theta \beta_{\Delta T}^n I_n, & \theta = 0, \dots, \Theta - 1 \\
\mathbb{P}(M_{n+1} = V^{\theta+1} | M_n = V^\theta) &= 1 - \alpha_\theta \beta_{\Delta T}^n I_n, & \theta = 0, \dots, \Theta - 1 \\
\mathbb{P}(M_{n+1} = I^0 | M_n = V^\Theta) &= \beta_{\Delta T}^n I_n, \\
\mathbb{P}(M_{n+1} = I^0 | M_n = F) &= \beta_{\Delta T}^n I_n.
\end{aligned} \tag{2.10}$$

The conditions $\beta_{\Delta T}^n \leq 1$, $\gamma_{\Delta T}^\omega \leq 1$, $\lambda_n \geq 0$, $\lambda_n \leq 1$ ensure the well-posedness of this definition.

The conditional rates λ_n are derived from a probability density ξ defined on $\{t_0, \dots, t_{N-1}\} \cup \{\infty\}$. In practice, the individual chooses the probability distribution ξ before the dynamics starts. Then, he selects a random number n distributed with the aforementioned probability ξ , which means that before the beginning of the epidemic he knows the time t_n at which he will vaccinate (unless he is already infected by that time).

There is a mapping between $\lambda = (\lambda_n)_{n=0}^{N-1}$ and ξ defined by:

$$\xi_\infty = \prod_{n=0}^{N-1} (1 - \lambda_n), \quad \xi_n = \lambda_n \prod_{k=0}^{n-1} (1 - \lambda_k), \quad n \leq N - 1 \tag{2.11}$$

$$\forall n \leq N - 1: \lambda_n = \begin{cases} \frac{\xi_n}{\xi_n + \dots + \xi_\infty}, & \text{if } \xi_n + \dots + \xi_\infty > 0 \\ 0, & \text{otherwise} \end{cases}. \tag{2.12}$$

The cost of a vaccination strategy depends on ξ (see [32] for a similar situation). The individual pays the price r_I upon arriving in class I^0 and the price r_V upon arriving in class V^0 . The cost for an individual will have three components:

- the cost r_I of being infected before vaccination;
- the cost r_V of vaccination plus a possible cost of being infected while immunity is still building or after the persistence period;
- the cost r_V of failed vaccination plus a possible cost of being infected.

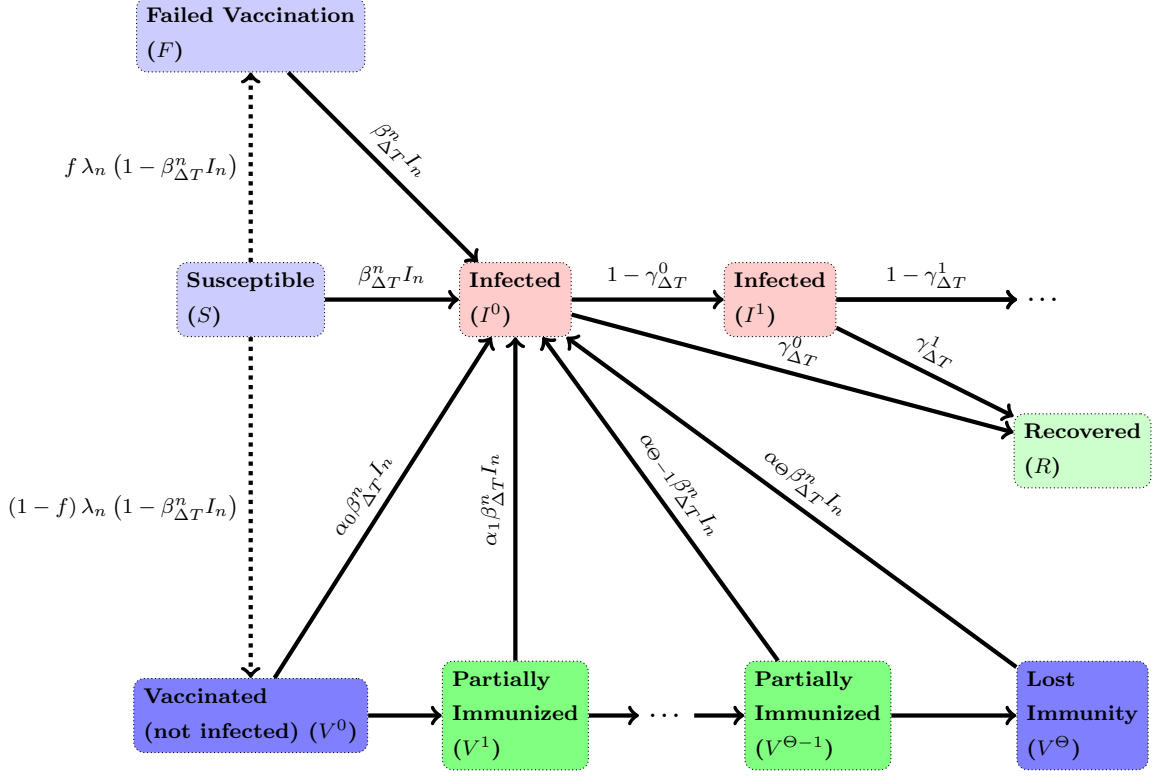


FIGURE 2. Individual model.

Note that an individual may pay both prices if he vaccinates and, moreover, if he is infected. For an individual starting at $M_0 = S$, the total price is:

$$\begin{aligned}
J_{indi}(\xi; U) &= r_V \mathbb{P} \left(\cup_{n < N} \{M_{n+1} = V^0, M_n \neq V^0\} \middle| M_0 = S \right) \\
&+ r_V \mathbb{P} \left(\cup_{n < N} \{M_{n+1} = F, M_n \neq F\} \middle| M_0 = S \right) \\
&+ r_I \mathbb{P} \left(\cup_{n < N} \{M_{n+1} = I^0, M_n \neq I^0\} \middle| M_0 = S \right). \tag{2.13}
\end{aligned}$$

This form for $J_{indi}(\xi; U)$ is impractical and has to be made more explicit. One possibility is to sum over the first passages from S to I^0 , V^0 or F . The following quantities are useful for general n :

- the probability $\varphi_n^{V,I}$ of infection (at time t_{n+1} or later) of an individual that vaccinated in the interval $[t_n, t_{n+1}]$, given by the formula:

$$\varphi_n^{V,I} = 1 - \prod_{k=n}^{\Theta} \left(1 - \beta_{\Delta T}^k \alpha_{k-n-1} I_k \right), \tag{2.14}$$

where we introduce the coefficient $\alpha_{-1} = 1$;

- the conditional probability of being infected (strictly) before t_{n+1} (of a person that did not vaccinate)

$$\varphi_n^I = \mathbb{P} \left[\cup_{k=0}^n \{M_k = I\} \middle| M_0 = S, M_k \neq V^0, M_k \neq F, k \leq n \right],$$

given by the formula:

$$\varphi_n^I = 1 - \prod_{k=0}^n \left(1 - \beta_{\Delta T}^k I_k\right) \quad n = 0, \dots, N-1. \quad (2.15)$$

Note that the probability of being infected after the time n is

$$1 - \frac{1 - \varphi_\infty^I}{1 - \varphi_n^I} = \frac{\varphi_\infty^I - \varphi_n^I}{1 - \varphi_n^I},$$

where

$$\varphi_\infty^I = 1 - \prod_{k=0}^N \left(1 - \beta_{\Delta T}^k I_k\right).$$

Then, after elementary computations:

$$J_{indi}(\xi; U) = r_I \varphi_\infty^I \xi_\infty + \sum_{n=0}^{N-1} \left[r_I \varphi_n^I + (1 - \varphi_n^I)(r_V + (1 - f)r_I \varphi_n^{V,I}) + r_I f(\varphi_\infty^I - \varphi_n^I) \right] \xi_n. \quad (2.16)$$

The individual cannot change U_n neither S_n , I_n^ω nor V_n^θ . He can only choose his vaccination strategy ξ . Denote

$$g_n^U = \begin{cases} r_I \varphi_n^I + (1 - \varphi_n^I)(r_V + (1 - f)r_I \varphi_n^{V,I}) + r_I f(\varphi_\infty^I - \varphi_n^I), & \text{for } n \leq N-1 \\ r_I \varphi_\infty^I & \text{for } n = N. \end{cases} \quad (2.17)$$

Then $J_{indi}(\xi; U) = \langle \xi, g^U \rangle$, where g^U and ξ are seen as vectors in \mathbb{R}^{N+1} , and it is to be minimized under the constraint $\xi_0 + \dots + \xi_{N-1} + \xi_\infty = 1$, $\xi_k \geq 0$. Then any probability distribution ξ with support in $\{n \mid g_n^U \leq g_k^U, k = 0, \dots, N\}$ attains the minimum.

Now, for a given individual policy ξ one can ask whether the equations (2.1)-(2.8) are obtained when *all* individuals follow this vaccination policy and in this case what is the compatibility relationship between ξ and U . Supposing identical initial conditions S_{0-} and I_{0-} , the compatibility relation between the two dynamics is:

$$U_n = \lambda_n S_n. \quad (2.18)$$

2.2. Failed vaccination. A simplified model can be proposed to tackle the possibility of vaccination failure. Note that, for $n \leq N-1$,

$$g_n^U = r_I f \varphi_\infty^I + (1 - f)[r_I \varphi_n^I + (1 - \varphi_n^I)(r_V/(1 - f) + r_I \varphi_n^{V,I})].$$

Therefore, since the term $r_I f \varphi_\infty^I$ does not depend on n and $(1 - f)$ is an overall factor, the cost has exactly the same minimum as that of a model without the class F when we replace r_V by $r_V/(1 - f)$. Therefore, when the efficacy $1 - f$ of the vaccine is not 100%, this can be treated by considering that the cost of the vaccine is multiplied by $(1 - f)^{-1}$. See Section 4 for some numerical illustrations.

Note however that this is a first order approximation as, in practice, the quantities $\varphi_n^{V,I}$ depend on the precise values of I_n^ω and a different model with different classes will change those values.

2.3. Equilibrium. Consider now the following mapping: for any given probability law η on $\{t_0, \dots, t_{N-1}\} \cup \{\infty\}$ define λ by (2.12) (using η_n instead of ξ_n), U_n, S_n, I_n recursively by the relations (2.1)-(2.5) and (2.18). Denote $\mathcal{C}_\eta = g^U$.

Let $\mathcal{J}(\eta)$ be the ensemble containing all optimal individual strategies ξ that minimize the cost $\langle \xi, \mathcal{C}_\eta \rangle$.

The goal of this subsection is to deduce the existence of an equilibrium of the system, i.e. a common strategy which is a Nash equilibrium when it is used by all agents of the population. The following result holds.

Theorem 2.1. *There exists at least one law η such that $\eta \in \mathcal{J}(\eta)$ (i.e., an equilibrium). Moreover, the mapping $\eta \mapsto \mathcal{C}_\eta$ is continuous.*

Proof. Note that any set $\mathcal{J}(\eta)$ is non void, convex and compact. We use Kakutani's fixed point theorem for the function $\mathcal{J}(\cdot)$ defined on the simplex

$$\Sigma_{N+1} = \{(x_0, \dots, x_N) \in \mathbb{R}^{N+1} | x_k \geq 0, x_0 + \dots + x_N = 1\}. \quad (2.19)$$

The only hypothesis left to check is the closed graph property of $\mathcal{J}(\cdot)$. Let η^ℓ be a sequence of probability laws on $\{t_0, \dots, t_{N-1}\} \cup \{\infty\}$ converging to η and $\xi^\ell \in \mathcal{J}(\eta^\ell)$ converging to ξ . We have to prove that $\xi \in \mathcal{J}(\eta)$.

We denote by $S^\ell, U^\ell, I^\ell, g^{U^\ell}$ (respectively S, U, I, g^U) the dynamics corresponding to η^ℓ (respectively η).

Let Z be the first index such that $\eta_Z + \dots + \eta_\infty = 0$. We consider the non-trivial case when $Z > 1$.

Consider λ^ℓ (respectively λ) the rates associated to η^ℓ (respectively η) by the formula (2.12). In particular $\eta_{Z-1} > 0$ and $\lambda_{Z-1}^\ell = 1$.

Although $\eta^\ell \rightarrow \eta$ as $\ell \rightarrow \infty$ we do not have that $\lambda^\ell \rightarrow \lambda$, but we have instead that $\lambda_n^\ell \rightarrow \lambda_n$ for all $n < Z$. In particular $\lambda_n^\ell S_n^\ell \rightarrow \lambda_n S_n$ for all $n < Z$. On the other hand, since $\lambda_{Z-1} = 1$ we have $S_Z = S_{Z-1}(1 - \lambda_{Z-1})(1 - \beta_{\Delta T}^n I_{Z-1}) = 0$ and $\lambda_{Z-1}^\ell \rightarrow 1$ implies $S_Z^\ell \rightarrow 0$; furthermore, S^ℓ being monotonically decreasing we also have $S_n^\ell \rightarrow 0$ for any $n \geq Z$. Since all rates λ^ℓ are bounded by 1 we obtain thus that $\lambda_n^\ell S_n^\ell \rightarrow 0 = \lambda_n S_n$ for all $n \geq Z$ and thus ultimately $\lambda_n^\ell S_n^\ell \rightarrow \lambda_n S_n$ for all $n \leq N$. This, combined with the formulas (2.1)-(2.8) and (2.18) show that $U^\ell \rightarrow U, S^\ell \rightarrow S, I^\ell \rightarrow I$ as $\ell \rightarrow \infty$. Thus we also have $\mathcal{C}_{\eta^\ell} \rightarrow \mathcal{C}_\eta$; therefore the limit of any converging sequence of minimas of \mathcal{C}_{η^ℓ} is also a minima of \mathcal{C}_η which, given its definition, proves the closed graph property of $\mathcal{J}(\cdot)$. \square

Remark 2.2. *The theorem reduces the existence of the equilibrium to the study of the mapping $\eta \mapsto \mathcal{C}_\eta$. This mapping has a well-defined meaning for a large class of vaccination games because the variable ξ is nothing else than the (mixed) individual strategy and the vector \mathcal{C}_ξ collects the price of pure strategies of the individual (given the overall epidemic propagation dynamics). We expect that this methodology can be generalized to other situations.*

3. FINDING THE EQUILIBRIUM

The result of the Section 2.3 guarantees the existence of at least one equilibrium. But, it does not prescribe a constructive method to find it.

For arbitrary strategy ξ , introduce the quantity $E(\xi)$ defined as follows: consider a situation when all individuals use the strategy ξ . Then $E(\xi)$ is the maximum gain

obtained by an individual if he changes unilaterally its strategy (and everybody else remaining with the strategy ξ). In mathematical terms:

$$E(\xi) = \langle \xi, \mathcal{C}_\xi \rangle - \min_{\eta \in \Sigma_{N+1}} \langle \eta, \mathcal{C}_\xi \rangle, \quad (3.1)$$

where Σ_{N+1} is the space of all possible strategies.

Note that $E(\cdot) \geq 0$ and an equilibrium corresponds to a ξ such that $\langle \xi, \mathcal{C}_\xi \rangle \leq \langle \eta, \mathcal{C}_\xi \rangle$ for any other strategy $\eta \in \Sigma_{N+1}$, which means $E(\xi) = 0$. The equilibrium can be rephrased as finding a strategy ξ such that the mapping $\xi \mapsto E(\xi)$ is minimized.

The minimization of $E(\cdot)$ would require to compute, for instance, some gradient of \mathcal{C}_ξ with respect to ξ which involves the introduction of several adjoint states (Lagrange multipliers) and render the computation complex.

Another idea is simpler and intuitively more appealing: consider a candidate ξ_k obtained at the iteration k of the procedure. Construct the cost \mathcal{C}_{ξ_k} obtained if everybody uses the strategy ξ_k . An individual in this population will test whether ξ_k is optimal i.e. if it is a minimum of $E(\cdot)$. If this is the case then the solution is ξ_k ; otherwise the individual will replace its strategy ξ_k with a strategy ξ_{k+1} , which is an adjustment of ξ_k (meaning that is not too far from ξ_k) and which goes towards minimizing $\eta \mapsto \langle \eta, \mathcal{C}_{\xi_k} \rangle$. In practice one can choose (with ideas close to the general framework of gradient flows, see [30] for an entry point to this literature):

$$\xi_{k+1} \in \operatorname{argmin}_{\eta \in \Sigma_{N+1}} \frac{\operatorname{dist}(\eta, \xi_k)^2}{2\tau} + \langle \eta, \mathcal{C}_{\xi_k} \rangle, \quad (3.2)$$

where $\operatorname{dist}(\cdot, \cdot)$ is some suitable distance. Then the procedure is iterated till convergence. This idea is similar to the paradigms of ‘‘Best Reply’’ (see [7]) and ‘‘fictitious play’’ for which some proofs of convergence exist under specific hypotheses (see [11]).

In order to keep the presentation as simple as possible, we used as distance in (3.2) the standard euclidian distance on \mathbb{R}^{N+1} although in principle other distances (such as the 2-Wasserstein distance) may perform better.

In practice, the algorithm applied is the following:

- Step 1** Choose a step $\tau > 0$ and a starting distribution ξ_1 .
Set iteration count $k = 1$.
- Step 2** Compute ξ_{k+1} as in formula (3.2).
- Step 3** If $E(\xi_{k+1})$ is smaller than a given tolerance then stop and exit,
otherwise set $k \rightarrow k + 1$ and go back to Step 1.

In practice **Step 2** is computed with a quadratic programming routine (`quadprog` in Matlab/Gnu Octave) that can accommodate linear constraints.

Remark 3.1. *The procedure proposed above can be extended in a straightforward manner to any rational individual vaccination model, by replacing the vector \mathcal{C}_ξ by a time-dependent function $c(\cdot)$, where $c(t)$ is the cost of the pure strategy consisting in vaccinating at the time t under the assumption that everybody follows the strategy ξ .*

4. NUMERICAL RESULTS

In order to test the model, we simulated the situation of an epidemic with several sets of parameters, as indicated below.

4.1. Preliminary tests. We first tested the procedure for a situation when the analytic result is known (see [23, 32]): we used the parameters in [32, Figure 5] and obtained that the optimum individual strategy is a mixed strategy with $\xi_0 = 33\%$ probability to vaccinate at $t = 0$ and $\xi_\infty = 67\%$ probability to not vaccinate at all; its cost is 0.5067; this compares favorably with the analytic result which is a mixed strategy with $\xi_0 = 34\%$, $\xi_\infty = 66\%$ and a cost 0.5.

4.2. Short persistence, large efficacy. The numerical values used in this simulation are the following: total simulation time $T = 1$ (one year), number of time instants: $N = 365 \times 3$ (three times a day); recovery rate $\gamma^\omega = \gamma = 365/3.2$ (mean recovery time 3.2 days, $\Omega = 20$), reproduction number $R_0 = 1.35$, thus $\beta = \gamma R_0$; initial proportion of susceptibles $S_0 = 0.94$ and infected $I_0 = 2.0 \times 10^{-6}$; relative costs $r_I = 1$, $r_V = 0.005$. To take into account the seasonality of $\beta(t)$ we set $\beta_{min} = \gamma/S_0$ and $\beta(t) = \beta$ for $t \leq t_2^\beta := 1/2$ (6 months) and then $\beta(t) = \beta_{min}$ for $t > t_2^\beta = 1/2$; these parameters model an epidemic having a duration of 6 months.

To define the persistence of the vaccine we set $t_1 = 5/365$, $t_2 = 1/12$ (one month, $\Theta = 93$) and $\alpha_\theta = 1 - \mathbf{1}_{[t_1, t_2]}$. The vaccine efficacy is set to 100%, i.e., we suppose a failure rate $f = 0$. The step is $\tau = 0.1$ and we performed 1000 iterations.

The results are displayed in Figures 3, 4 and 5. A good quality equilibrium is found: the incentive to change the strategy, as measured by the function $E(\xi)$, is more than two orders of magnitude lower at the solution than at the initial guess. The cost of the solution $\langle \xi^{MFG}, \mathcal{C}_{\xi^{MFG}} \rangle$ is 0.0237.

The solution is a strategy ξ^{MFG} supported at several time instants between 0.25 and 0.43 and also having 68% of the mass at the non-vaccinating time $t = \infty$. Note that the cost is adapted accordingly, reaching its minimum at all points in the support of the solution ξ^{MFG} . Generally the vaccination occurs when I_n has large values, except at the end of the epidemic (time 0.5) when people expect the epidemic to end and estimate that their infection probability is low; the individuals have a strategic behavior, in coherence with the model. This is coherent with the model in [6] where the vaccination rate is taken by hypothesis increasing with the number of people infected. The two models agree in a majority of time instants except the end of the epidemic. This behavior has been observed in our simulations across a wide range of vaccination persistence times and initial conditions (the results are not shown here).

It should be mentioned that the solution ξ^{MFG} , with cost 0.0237, is *not* the solution that minimizes the average cost across individuals (see also equation (2.9)) which is $M(\xi) = \langle \xi, \mathcal{C}_\xi \rangle$: for instance the strategy ξ^{min} that vaccinates with certainty at time $t = 0.0$ (unless infected by that time) has $M(\xi^{min}) = 0.005$. This result is not surprising and often appears in such contexts (see [32]). When $M(\xi^{MFG}) > M(\xi^{min})$ the game is said to have a positive *cost of anarchy*. It can be intuitively explained as follows: suppose that everybody uses the strategy ξ^{min} . The cost of an individual with strategy η will be $\langle \eta, \mathcal{C}_{\xi^{min}} \rangle$ and it turns out that there exists some η_1 with $\langle \eta_1, \mathcal{C}_{\xi^{min}} \rangle < \langle \xi^{min}, \mathcal{C}_{\xi^{min}} \rangle$. For instance here η_1 can be a pure non-vaccinator strategy whose cost is very low 8.0×10^{-6} . Therefore any individual with current strategy ξ^{min} has an incentive to change his strategy (and use η_1) by hoping that everybody else remains with the strategy ξ^{min} . This does not happen and everybody slides towards η_1 and so on until the Nash equilibrium ξ^{MFG} is found. In the process the cost of *everybody* will increase and this is the price to pay for equilibrium.

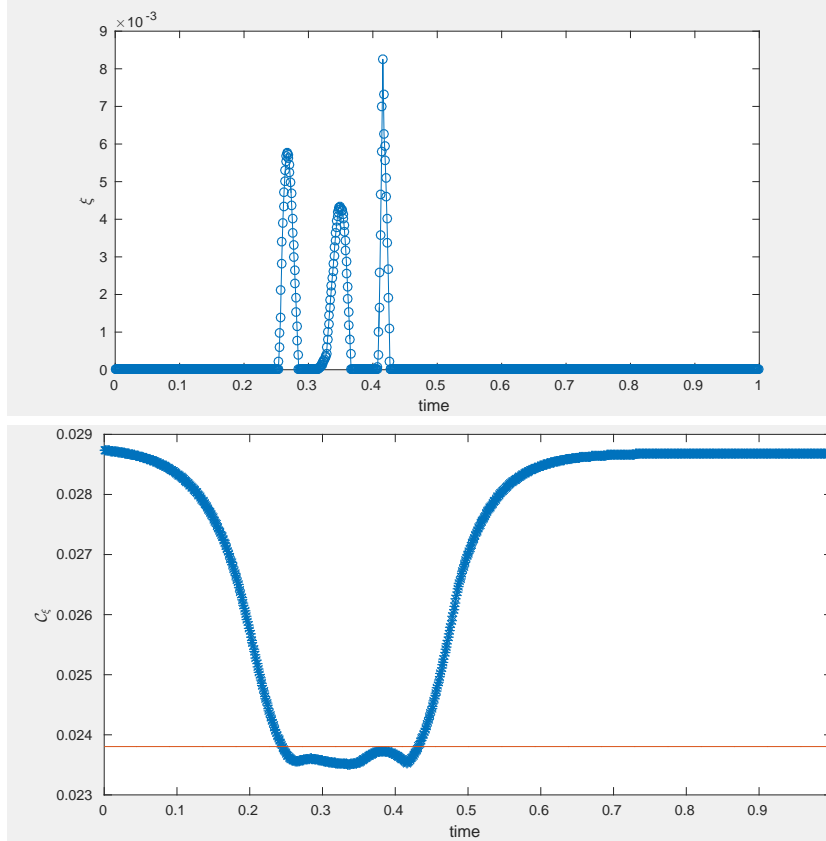


FIGURE 3. Results for Subsection 4.2. Top: the optimal converged strategy ξ^{MFG} at times $\{t_0, \dots, t_{N-1}\}$. The weight of the non-vaccinating pure strategy (i.e., corresponding to time $t = \infty$) is 68%. Bottom: the corresponding cost $\mathcal{C}_{\xi^{MFG}}$. The horizontal thin line corresponds to the cost of the non-vaccinating pure strategy $(\mathcal{C}_{\xi^{MFG}})_{N+1}$.

4.3. Long persistence, 100% efficacy. The parameters are identical as in Subsection 4.2, except the vaccine persistence time t_2 which is set now to 6 months: $t_2 = 1/2$ ($\Theta = 549$). The convergence is quickly attained (100 iterations) and the results are displayed in Figure 6. Although fewer people vaccinate (only 9% here, to compare with 32% in Subsection 4.2), the fact that vaccine persistence is higher improves the outcome. The equilibrium cost becomes 0.005, almost one order of magnitude lower than in the previous test.

4.4. Long persistence, smaller efficacy. In this Subsection, we test a situation when the vaccine efficacy is only 50%. All other inputs are as in Subsection 4.3. The result, not shown here because very similar to those described in the previous tests, has however several differences:

- the probability of the non-vaccinating strategy is now 86% (i.e. 14% of people vaccinate);
- the cost of the optimal strategy is 0.0101.

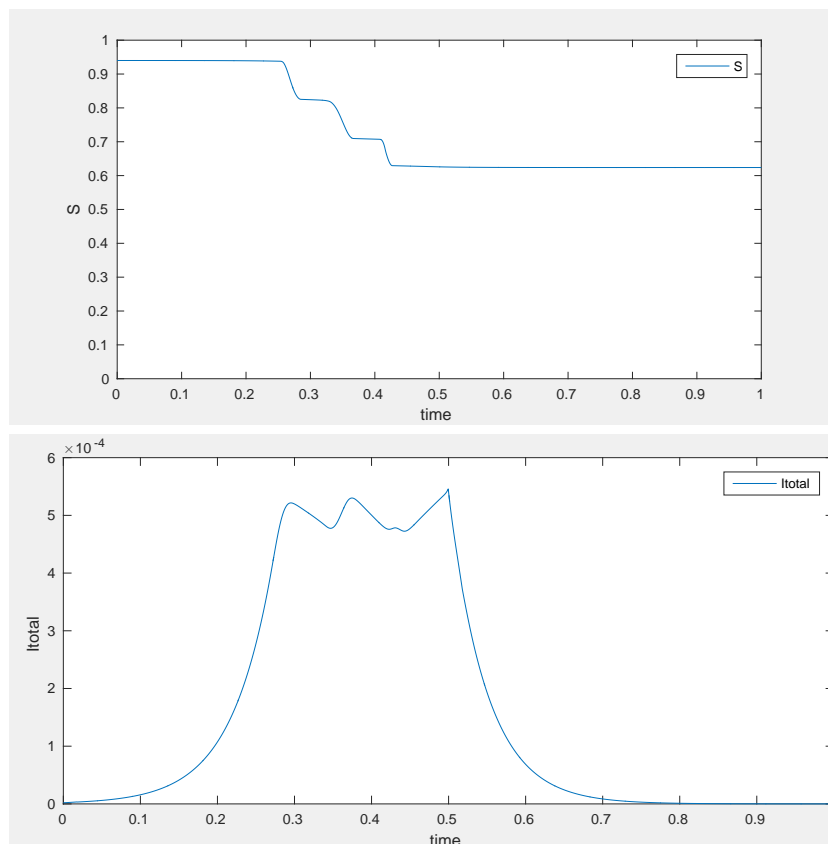


FIGURE 4. Results of Subsection 4.2. Top: the evolution of the susceptible class S_n ; bottom: the (total) infected class I_n .

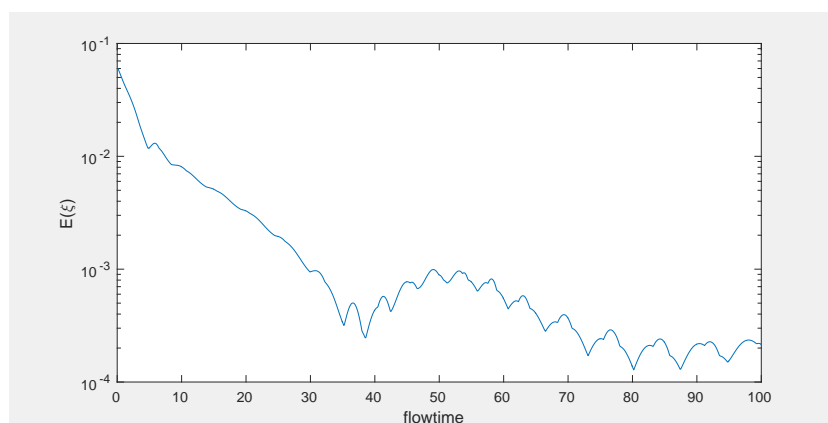


FIGURE 5. Result of Subsection 4.2. The decrease of the incentive to change strategy $E(\xi_k)$. Note that $E(\xi_k)$ does not decrease monotonically. In fact, there is no reason to expect such a behavior, since we are not minimizing $E(\cdot)$ in a monotonic fashion.

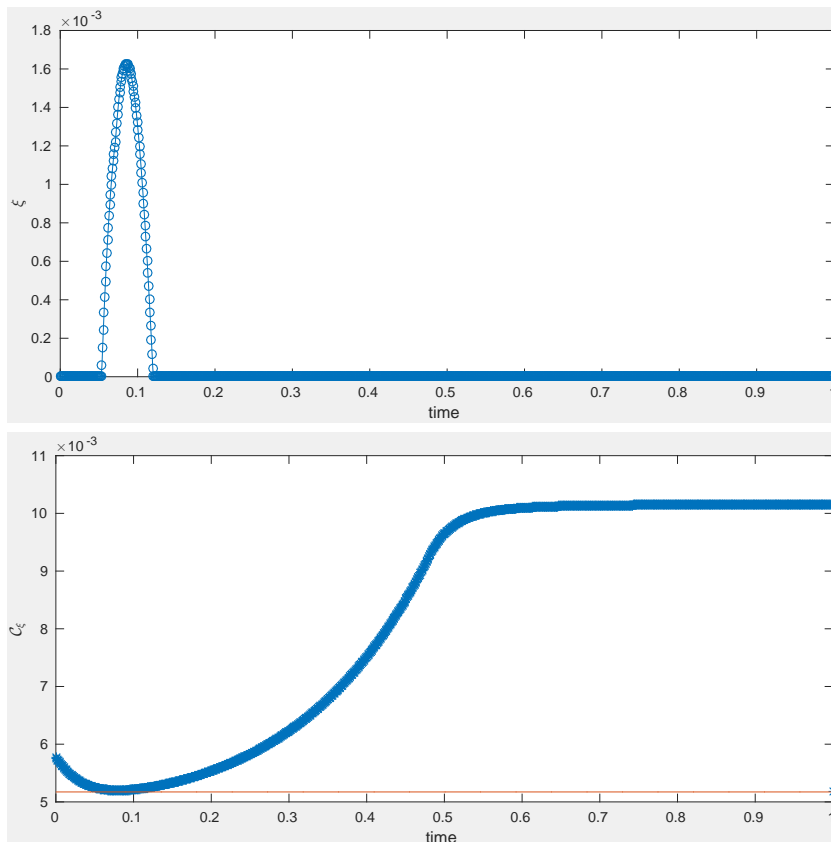


FIGURE 6. Results of Subsection 4.3. Top: the optimal converged strategy ξ^{MFG} . The weight of the non-vaccinating pure strategy (i.e., corresponding to time $t = \infty$) is 91%. Bottom: the corresponding cost $\mathcal{C}_{\xi^{MFG}}$. The thin horizontal line corresponds to the cost of the non-vaccinating pure strategy $(\mathcal{C}_{\xi^{MFG}})_{N+1}$.

Therefore the equilibrium shifts towards a bigger fraction of the population that vaccinate (in order to compensate lower vaccine efficacy). However, the overall number of protected people is lower (50% of 14% being smaller than 100% of 9%) which results in a larger overall equilibrium cost (about twice larger). We tested other settings and these conclusions were consistently obtained: the introduction of imperfect vaccines (here lower efficacy) generates overall lower coverage rates and larger costs.

We also compared this result with result obtained by setting the cost $r_V \rightarrow r_V/(1-f) = 0.01$, $f \rightarrow 0$. In this case the cost is 0.0103 (with 6.6% vaccinators) which is very similar and confirms the qualitative analysis of Subsection 2.2.

4.5. Effects of the failed vaccination rate on the vaccination strategy. We analyze in this subsection the effects of the failed vaccination rate on the overall vaccination policy. The numerical value of the vaccination cost is $r_V = 0.025$, whereas the other parameters are the same as in Subsection 4.4. The results are presented in Table 1.

When the failure rate f is small, the vaccination rate $(1 - \xi_\infty)$ is increasing with f . However, when the failure rate f is larger than a given value (in our numerical simulations, when $f \geq 0.60$), the vaccination rate decreases as f increases.

Failed vaccination rate f	Vaccination rate $1 - \xi_\infty$
0.00	5.04%
0.25	5.94%
0.50	7.02%
0.55	7.20%
0.60	7.29%
0.65	7.23%
0.75	5.74%
0.80	2.93%
0.85	0.00%

TABLE 1. Results for the Subsection 4.5. Individual vaccination policy with respect to the failed vaccination rate of the vaccine.

The first part of this result agrees with the behavior already observed in Subsection 4.4. When the failure rate is small, the individuals tend to vaccinate more to compensate the decrease in efficacy and therefore to contribute to and profit from the group protection. However, after a given threshold, the construction of a group protection is too expensive, and therefore the individuals are oriented to avoid vaccination (if $f = 0.85$, the vaccination rate $(1 - \xi_\infty)$ is zero; in this case, the probability of being infected is 14.38%).

5. DISCUSSION

We analyzed in this work the vaccination equilibrium in a context of rational individual vaccination choices; the situation is modeled as a Nash equilibrium with an infinity of players. A specific focus of our work is the presence of imperfect vaccine. We presented both a theoretical approach (existence of an equilibrium via the Kakutani fixed point theorem) and a numerical algorithm (similar to a gradient flow). Both approaches have the advantage to use rather weak assumptions on the structure of the model and as such we hope that they will be useful in other situations.

The model shows that the imperfections of the vaccine increase the overall cost. But the obtained equilibrium is such that the increased vaccination rate does not compensate the lower efficacy (or persistence) of the vaccine.

When the failure rate is below a given threshold, the cost for building a group protection is advantageous with respect to the infection cost. In this case, a higher vaccination rate can be optimal to compensate an increase in the failure rate. However, this individual policy is far from the societal level optimal strategy, which would consist in a global optimization of the vaccination policy. When the failure rate is above the threshold, the vaccination rate decreases.

Several assumptions in this work may limit the applicability of the results and motivate further studies:

- the individuals are supposed perfectly aware of the past, present and future epidemic dynamics: a model with limited information may be more realistic;
- the individuals are identical. In particular the cost of the illness is exactly the same, irrespective of age: considering several age groups may prove interesting especially if their strategies are different;

- the absence of vital dynamics: the introduction of a birth/death rate into the model can make it applicable to childhood diseases;
- the geographical heterogeneity in the propagation of the epidemic is neglected: travels and intra/inter-community contacts may be important for the epidemic propagation.

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