



**HAL**  
open science

# Graph manifold $Z$ -homology 3-spheres and taut foliations

Michel Boileau, Steven Boyer

► **To cite this version:**

Michel Boileau, Steven Boyer. Graph manifold  $Z$ -homology 3-spheres and taut foliations. *Journal of topology*, 2015, 8 (2), pp.571-585. 10.1112/jtopol/jtv006 . hal-01302045

**HAL Id: hal-01302045**

**<https://hal.science/hal-01302045>**

Submitted on 5 Jul 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# GRAPH MANIFOLDS $\mathbb{Z}$ -HOMOLOGY 3-SPHERES AND TAUT FOLIATIONS

MICHEL BOILEAU AND STEVEN BOYER

**ABSTRACT.** We show that a graph manifold which is a  $\mathbb{Z}$ -homology 3-sphere not homeomorphic to either  $S^3$  or  $\Sigma(2, 3, 5)$  admits a horizontal foliation. This combines with known results to show that the conditions of *not* being an L-space, of having a left-orderable fundamental group, and of admitting a co-oriented taut foliation, are equivalent for graph manifold  $\mathbb{Z}$ -homology 3-spheres.

October 18, 2018

Throughout this paper we shall often use  $\mathbb{Q}$ -homology 3-sphere to abbreviate *rational homology 3-sphere* and  $\mathbb{Z}$ -homology 3-sphere to abbreviate *integer homology 3-sphere*.

Heegaard Floer theory is a package of 3-manifold homology invariants developed by Ozsváth and Szabó [OS3], [OS2] which provides relatively powerful tools to distinguish between manifolds. For a rational homology 3-sphere  $M$ , the simplest version of these invariants comes in the form of  $\mathbb{Z}/2$ -graded abelian groups  $\widehat{HF}(M)$  whose Euler characteristic satisfies:  $\chi(\widehat{HF}(M)) = |H_1(M)|$ . In particular,  $\text{rank } \widehat{HF}(M) \geq |H_1(M)|$ .

Ozsváth and Szabó defined the family of *L-spaces* as the class of rational homology 3-spheres  $M$  for which the Heegaard Floer homology is as simple as possible. In other words,  $\text{rank } \widehat{HF}(M) = |H_1(M)|$ . Examples of L-spaces include the 3-sphere, lens spaces, and, more generally, manifolds admitting elliptic geometry. By Perelman's proof of the geometrization conjecture, these are the closed 3-manifolds with finite fundamental group. Beyond these examples, Ozsváth and Szabó have shown that the 2-fold branched covering of any non-split alternating link is an L-space, thus providing infinitely many examples of hyperbolic L-spaces. None of these examples are integer homology 3-spheres, except for  $S^3$  and the Poincaré sphere  $\Sigma(2, 3, 5)$ .

The last decade has shown that the conditions of *not* being an L-space, of having a left-orderable fundamental group, and of admitting a  $C^2$  co-oriented taut foliation, are strongly correlated for an irreducible  $\mathbb{Q}$ -homology 3-sphere  $W$ :

- the three conditions are equivalent for non-hyperbolic geometric manifolds (cf. [BRW], [LS], [BGW]).

---

Michel Boileau was partially supported by l'Institut Universitaire de France.

Steven Boyer was partially supported by NSERC grant RGPIN 9446-2008.

2000 MATHEMATICS SUBJECT CLASSIFICATION. PRIMARY 57M25, 57M50, 57M99

- Ozsváth and Szábo have shown that if  $W$  admits a  $C^2$  co-orientable taut foliation then it is not an L-space [OS1, Theorem 1.4].
- Calegari and Dunfield have shown that the existence of a co-orientable taut foliation on an atoroidal  $W$  implies that the commutator subgroup  $[\pi_1(W), \pi_1(W)]$  is a left-orderable group [CD, Corollary 7.6].
- Boyer, Gordon and Watson have conjectured that  $W$  has a left-orderable fundamental group if and only if it is not an L-space and have provided supporting evidence in [BGW].
- Lewallen and Levine have shown that strong L-spaces do not have left-orderable fundamental groups [LL].

Recall that a *graph manifold* is a compact, irreducible, orientable 3-manifold whose Jaco-Shalen-Johannson (JSJ) pieces are Seifert fibred spaces. In this paper we focus on the case that  $W$  is an integer homology 3-sphere, and in particular one which is a graph manifold.

We begin with the statement of the *Heegaard-Floer Poincaré conjecture*, due to Ozsváth and Szábo.

**Conjecture 0.1.** (Ozsváth-Szábo) *An irreducible integer homology 3-sphere is an L-space if and only if it is either  $S^3$  or the Poincaré homology 3-sphere  $\Sigma(2, 3, 5)$ .*

The truth of this striking conjecture would imply that among prime 3-manifolds, the 3-sphere is characterized by its Heegaard-Floer homology together with the vanishing of its Casson invariant (or even its  $\mu$  invariant). It is known to hold in many instance, for example for integer homology 3-spheres obtained by surgery on a knot in  $S^3$  [HW, Proposition 5]. It lends added interest to the questions:

- Which  $\mathbb{Z}$ -homology 3-spheres admit co-oriented taut foliations?
- Which  $\mathbb{Z}$ -homology 3-spheres have left-orderable fundamental groups?

We assume throughout this paper that foliations are  $C^2$ -smooth. The works of Eisenbud-Hirsh-Neumann [EHN], Jankins-Neumann [JN] and Naimi [Na] give necessary and sufficient conditions for a Seifert fibred 3-manifold to carry a horizontal foliation. It follows from their work that a Seifert manifold  $\mathbb{Z}$ -homology 3-sphere is an L-space if and only if it is either  $S^3$  or the Poincaré homology 3-sphere  $\Sigma(2, 3, 5)$  (cf. Proposition 2.2; see also [LS], [CM]). More recently, Clay, Lidman and Watson have shown that the fundamental group of a graph manifold  $\mathbb{Z}$ -homology 3-sphere is left-orderable if and only if it is neither  $S^3$  nor  $\Sigma(2, 3, 5)$  [CLW]. (By convention, the trivial group is *not* left-orderable.) The main result of this paper proves Ozsváth-Szábo conjecture for  $\mathbb{Z}$ -homology 3-spheres which are graph manifolds: we show that a graph manifold  $\mathbb{Z}$ -homology 3-sphere admits a co-oriented taut foliation if and only if it is neither  $S^3$  nor  $\Sigma(2, 3, 5)$ . Before stating the precise version of our result, we need to introduce some definitions.

A *transverse loop* to a codimension one foliation  $\mathcal{F}$  on a 3-manifold  $M$  is a loop in  $M$  which is everywhere transverse to  $\mathcal{F}$ . A codimension one foliation on a 3-manifold  $M$  is *taut* if each of its leaves meets a transverse loop.

A foliation is  $\mathbb{R}$ -*covered* if the leaf space of the pull-back foliation on the universal cover  $\widetilde{M}$  of  $M$  is homeomorphic to the real line.

A foliation on a  $\mathbb{Z}$ -homology 3-sphere is always co-orientable.

We assume that the pieces of a graph manifold are equipped with a fixed Seifert structure. Note that this structure is unique up to isotopy when the graph manifold is a  $\mathbb{Z}$ -homology 3-sphere (cf. Proposition 1.1(2)).

A surface in a graph manifold  $W$  is *horizontal* if it is transverse to the Seifert fibres of each piece of  $W$ . It is *rational* if its intersection with each JSJ torus is a union of simple closed curves. A codimension 1 foliation of  $W$  is *horizontal*, respectively *rational*, if each of its leaves has this property. Horizontal foliations are obviously taut and they are known to be  $\mathbb{R}$ -covered [Br2, Proposition 7]. Rational foliations on graph manifold  $\mathbb{Z}$ -homology 3-spheres are necessarily horizontal (Lemma 2.1). Here is our main result.

**Theorem 0.2.** *Let  $W$  be a graph manifold which is a  $\mathbb{Z}$ -homology 3-sphere and suppose that  $W$  is neither  $S^3$  nor  $\Sigma(2, 3, 5)$ . Then  $W$  admits a rational foliation.*

An action of a group  $G$  on the circle is called *minimal* if each orbit is dense.

A homomorphism  $\rho : G \rightarrow \text{Homeo}_+(S^1)$  is called *minimal* if the associated action on  $S^1$  is minimal.

**Corollary 0.3.** *Let  $W$  be a graph manifold which is a  $\mathbb{Z}$ -homology 3-sphere and suppose that  $W$  is neither  $S^3$  nor  $\Sigma(2, 3, 5)$ . Then*

- (1)  $W$  is not an L-space.
- (2)  $\pi_1(W)$  admits a minimal homomorphism  $\rho$  with values in  $\text{Homeo}_+(S^1)$  whose image contains a nonabelian free group.
- (3) (Clay-Lidman-Watson [CLW])  $\pi_1(W)$  is left-orderable.

*Proof.* Since  $W$  is a  $\mathbb{Z}$ -homology 3-sphere, the taut foliation  $\mathcal{F}$  given by Theorem 0.2 is co-orientable. Thus  $W$  cannot be an L-space [OS1, Theorem 1.4]. Assertion (3) is a consequence of the assertion (2); since  $H^2(W) \cong \{0\}$ , the homomorphism  $\pi_1(W) \rightarrow \text{Homeo}_+(S^1)$  lifts to a homomorphism  $\pi_1(W) \rightarrow \widetilde{\text{Homeo}}_+(S^1) \leq \text{Homeo}_+(\mathbb{R})$  with non-trivial image. Theorem 1.1(1) of [BRW] now implies that  $\pi_1(W)$  is left-orderable. (This also follows from the fact that  $\pi_1(W)$  acts non-trivially on  $\mathbb{R}$  by orientation-preserving homeomorphisms since  $\mathcal{F}$  is co-oriented and  $\mathbb{R}$ -covered [Br2, Proposition 7].) Finally, assertion (2) follows from Lemma 0.4 below.  $\square$

**Lemma 0.4.** *Let  $M$  be a  $\mathbb{Z}$ -homology 3-sphere which admits a taut foliation  $\mathcal{F}$ . Then  $\pi_1(M)$  admits a minimal homomorphism  $\rho : \pi_1(M) \rightarrow \text{Homeo}_+(S^1)$  whose image contains a nonabelian free group.*

*Proof.* A theorem of Margulis [Gh, Corollary 5.15] shows that the image of a minimal representation  $\rho : \pi_1(M) \rightarrow \text{Homeo}_+(S^1)$  is either abelian or contains a nonabelian free group. The former is not possible since  $\pi_1(M)$  is perfect, so to complete the proof we must show that such a representation exists.

Since  $M$  is a  $\mathbb{Z}$ -homology 3-sphere, the co-orientability of  $\mathcal{F}$  implies that it has no compact leaves ([Go, Proposition 2.1]. See also [God, Part II, Lemma 3.8]). Then by Plante's results [Pla, Theorem 6.3, Corollaries 6.4 and 6.5], every leaf of  $\mathcal{F}$  has exponential growth, and thus  $\mathcal{F}$  admits no non-trivial holonomy-invariant transverse measure. Hence Candel's uniformization theorem [CC1, Theorem 12.6.3] applies to show that there is a Riemannian metric on  $M$  such that  $\mathcal{F}$  is leaf-wise hyperbolic. In this setting, Thurston's universal circle construction yields a homomorphism  $\rho_{univ}$  of  $\pi_1(M)$  with values in  $\text{Homeo}_+(S^1)$  [CD].

If  $L$  denotes the leaf space of the pullback  $\tilde{\mathcal{F}}$  of the foliation  $\mathcal{F}$  to the universal cover  $\tilde{M}$  of  $M$ , then either  $L$  is Hausdorff and  $\mathcal{F}$  is  $\mathbb{R}$ -covered or  $L$  has branching points. We treat these cases separately.

First suppose that  $\mathcal{F}$  is  $\mathbb{R}$ -covered. Then Proposition 2.6 of [Fen] implies that after possibly collapsing at most countably many foliated  $I$ -bundles, we can suppose that  $\mathcal{F}$  is a minimal foliation (i.e. each leaf is dense). If  $\mathcal{F}$  is ruffled ([Ca1, Definition 5.2.1]), Lemma 5.2.2 of [Ca1] shows that the associated action of  $\pi_1(M)$  on the universal circle of  $\mathcal{F}$  is minimal, so we take  $\rho = \rho_{univ}$ . If  $\mathcal{F}$  is not ruffled, it is uniform and so by [Ca1, Theorem 2.1.7], after possibly blowing down some pockets of leaves, we can suppose that  $\mathcal{F}$  slithers over the circle ([Ca1, Definition 2.1.6]). Thus if  $\tilde{M}$  denotes the universal cover of  $M$ , there is a locally trivial fibration  $\tilde{M} \rightarrow S^1$  whose fibres are unions of leaves of the pull back of  $\mathcal{F}$  to  $\tilde{M}$ . Further, the deck transformations of the cover  $\tilde{M} \rightarrow M$  act by bundle maps and so determine a homomorphism of  $\pi_1(M)$  with values in  $\text{Homeo}_+(S^1)$ . If this representation has a finite orbit, then a finite index subgroup of  $\pi_1(M)$  acts freely and properly discontinuously on a fibre of the fibration  $\tilde{M} \rightarrow S^1$ . This is impossible as each fibre is a surface and a finite index subgroup of  $\pi_1(M)$  is the fundamental group of a closed 3-manifold. Therefore by [Gh, Propositions 5.6 and 5.8], the associated action on  $S^1$  is semiconjugate to a minimal action  $\rho : \pi_1(M) \rightarrow \text{Homeo}_+(S^1)$ .

In the case that  $L$  branches,  $\rho_{univ} : \pi_1(M) \rightarrow \text{Homeo}_+(S^1)$  is faithful. (See the last line of the first paragraph of [CD, §6.28].) If it branches in both directions, an application of [Ca3, Lemma 5.5.3] to any finite cover of  $M$  implies that  $\rho_{univ}(\pi_1(M))$  has no periodic orbit. The conclusion then follows as above from [Gh, Propositions 5.7 and 5.8]. Thus we are left with the case where  $\mathcal{F}$  has one-sided branching, say in the negative direction (cf. [Ca2]). As in the case of  $\mathbb{R}$ -covered foliations, we can suppose every leaf dense by [Ca2, Theorem 2.2.7]. We need only show that the action associated to the faithful representation  $\rho_{univ} : \pi_1(M) \rightarrow \text{Homeo}_+(S^1)$  has no finite orbits as otherwise [Mat, Theorem 1.2] implies that  $\rho_{univ}$  is semiconjugate to an abelian representation, which is trivial since  $\pi_1(M)$  is perfect. Hence the action of  $\rho_{univ}(\pi_1(M))$  on  $S^1$  has an uncountable compact set  $\Sigma$  of global fixed points. By [Ca2, Theorem 3.2.2] the image of  $\Sigma$  is dense in almost every circle at infinity of the leaves of  $\tilde{\mathcal{F}}$ , and hence in  $S_{univ}^1$  by the construction of the universal circle, see [Ca2, Theorem 3.4.1]. This contradicts the faithfulness of  $\rho_{univ}$ . When  $M$  is hyperbolic, we can also obtain a contradiction to the existence of a finite

orbit from that of topologically pseudo-Anosov elements of  $\rho_{univ}(\pi_1(M))$  which have at most finitely many fixed points in  $S^1_{univ}$ , see [Ca2, Lemma 4.2.5]. This completes the proof of the lemma and therefore that of Corollary 0.3.  $\square$

The conclusion of Lemma 0.4 combines with the two questions above to motivate the following question:

**Question 0.5.** *For which aspherical  $\mathbb{Z}$ -homology 3-spheres  $M$  does  $\pi_1(M)$  admit a minimal representation to  $\text{Homeo}_+(S^1)$ ?*

Our discussion above yields the following corollary.

**Corollary 0.6.** *The following conditions are equivalent for  $W$  a graph manifold  $\mathbb{Z}$ -homology 3-sphere:*

(a)  $\pi_1(W)$  is left-orderable.

(b)  $W$  is not an  $L$ -space.

(c)  $W$  admits a rational foliation.  $\square$

Sections 1 and 2 contain background material on, respectively, the pieces of graph manifold  $\mathbb{Z}$ -homology 3-spheres and strongly detected slopes on the boundaries of Seifert fibered  $\mathbb{Z}$ -homology solid tori. Theorem 0.2 is proven in §3.

### 1. PIECES OF GRAPH MANIFOLD $\mathbb{Z}$ -HOMOLOGY 3-SPHERES

A torus  $T$  in a  $\mathbb{Z}$ -homology 3-sphere  $W$  splits  $W$  into two  $\mathbb{Z}$ -homology solid tori  $X$  and  $Y$ . Let  $\lambda_X$  and  $\lambda_Y$  be primitive classes in  $H_1(T)$  which are trivial in  $H_1(X)$  and  $H_1(Y)$  respectively. The associated slopes on  $T$ , which we also denote by  $\lambda_X$  and  $\lambda_Y$ , are well-defined. We refer to these slopes as the *longitudes* of  $X$  and  $Y$ . A simple homological argument shows that  $X(\lambda_Y)$  and  $Y(\lambda_X)$  are  $\mathbb{Z}$ -homology 3-spheres while  $X(\lambda_X)$  and  $Y(\lambda_Y)$  are  $\mathbb{Z}$ -homology  $S^1 \times S^2$ 's.

Let  $K$  be a knot in a  $\mathbb{Z}$ -homology 3-sphere with exterior  $M_K$ . The *longitude*  $\lambda_K$  of  $K$  is the longitude of  $M_K$ . The *meridian*  $\mu_K$  of  $K$  is the longitude of the tubular neighbourhood  $\overline{W} \setminus \overline{M_K}$  of  $K$ . The pair  $\mu_K, \lambda_K$  forms a basis for  $H_1(\partial M_K)$ .

**Lemma 1.1.** *Suppose that  $T$  is a torus in a  $\mathbb{Z}$ -homology 3-sphere  $W$  and let  $X, Y$  be the components of  $W$  cut open along  $T$ . Suppose that  $Y = P \cup Y_0$  where  $P \cap Y_0 = \partial P \setminus T$  and  $P$  is a Seifert manifold or than  $S^1 \times D^2$  and  $S^1 \times S^1 \times I$ . Then*

(1) *the underlying space  $B$  of the base orbifold of  $P$  is planar, hyperbolic, and the multiplicities of the exceptional fibres in  $P$  are pairwise coprime;*

(2)  *$P$  has a unique Seifert structure;*

(3) *if  $\phi$  is the  $P$ -fibre slope on  $T$  and  $P$  has an exceptional fibre, then  $\phi \notin \{\lambda_X, \lambda_Y\}$ .*

*Proof.* If  $B$  is non-orientable, or is orientable of positive genus, or has two exceptional fibres whose multiplicities are not coprime, then  $W$  admits a degree 1 map to a manifold with non-trivial first homology group, which is impossible. Thus (1) holds. Assertion (2) is a consequence of (1) and the classification of Seifert structures on 3-manifolds (cf. [Ja, §VI.16]). Finally observe that as  $H_1(Y(\lambda_X)) \cong \{0\}$  and  $H_1(Y(\lambda_Y)) \cong \mathbb{Z}$ , neither  $Y(\lambda_X)$  nor  $Y(\lambda_Y)$  has a lens space summand. On the other hand, if  $P$  has an exceptional fibre, then  $Y(\phi)$  does have such a summand. This completes the proof.  $\square$

## 2. HORIZONTAL FOLIATIONS AND STRONGLY DETECTED SLOPES IN SEIFERT FIBRED $\mathbb{Z}$ -HOMOLOGY SOLID TORI

The set  $\mathcal{S}_{rat}(T)$  of (rational) slopes on a torus  $T$  is naturally identified with the subset  $P(H_1(T; \mathbb{Q}))$  of the projective space  $\mathcal{S}(T) = P(H_1(T; \mathbb{R})) \cong S^1$ . We endow  $\mathcal{S}_{rat}(T)$  with the induced topology as a subset of  $\mathcal{S}(T)$ . The projective class of an element  $\alpha \in H_1(T; \mathbb{R})$  will be denoted by  $[\alpha]$ , though we sometimes abuse notation and write  $\alpha \in \mathcal{S}_{rat}(T)$  for a non-zero class  $\alpha$  in  $H_1(T)$ .

For a 3-manifold  $X$  whose boundary is a torus  $T$ , set  $\mathcal{S}_{rat}(X) = \mathcal{S}_{rat}(T)$ . We say that  $[\alpha] \in \mathcal{S}_{rat}(X)$  is *strongly detected* by a taut foliation  $\mathcal{F}$  on  $X$  if  $\mathcal{F}$  restricts on  $T$  to a fibration of slope  $[\alpha]$ . In this case we call  $[\alpha]$  the *slope of  $\mathcal{F}$* .

When  $X$  is Seifert fibred and  $T$  is a boundary component of  $X$ , we say that  $[\alpha] \in \mathcal{S}_{rat}(X)$  is *horizontal* if it is not the fibre slope.

**Lemma 2.1.** *Suppose that  $\mathcal{F}$  is a co-oriented taut foliation on a  $\mathbb{Z}$ -homology 3-sphere  $W$ .*

(1) *If  $\mathcal{F} \cap T$  is a fibration by simple closed curves for some boundary component  $T$  of a piece  $P$  of  $W$ , then the slope of  $T$  represented by these curves is horizontal.*

(2) *If  $\mathcal{F}$  is rational, then it is horizontal.*

*Proof.* Suppose that  $\mathcal{F} \cap T$  is a fibration by simple closed curves of vertical slope  $\phi$  and let  $P'$  be the manifold obtained by the  $(T, \phi)$ -Dehn filling  $P$ . Since  $P$  has base orbifold of the form  $B(a_1, \dots, a_n)$  for a planar surface  $B$  (Lemma 1.1),  $P'$  is homeomorphic to  $(\#_{i=1}^n L_{a_i}) \# (\#_{j=1}^{r-1} S^1 \times D^2)$  where  $r = |\partial P| - 1$ . On the other hand,  $\mathcal{F}$  extends to a co-oriented taut foliation  $\mathcal{F}'$  on  $P'$  and so  $P'$  is either prime or  $S^2 \times I$  (see e.g. [CC2, Corollary 9.1.9]). As the latter case does not arise, we have  $n + (r - 1) \leq 1$ . Thus  $P$  is either a solid torus or  $S^1 \times S^1 \times I$ , which is impossible for a piece of  $W$ . Thus part (1) the lemma holds.

Next suppose that  $\mathcal{F}$  is rational and let  $P$  be a piece of  $W$ . By part (1), for each boundary component  $T$  of  $P$ ,  $\mathcal{F} \cap T$  is a fibration by simple closed horizontal curves. Since the base orbifold of  $P$  is planar (Lemma 1.1), we can now argue as in the proof of [Br1, Proposition 3] to see that if  $\mathcal{F}$  is not horizontal in  $P$ , it contains a vertical, separating leaf homeomorphic to a torus. This is impossible as it contradicts the assumption that  $\mathcal{F}$  is co-oriented and taut ([Go, Proposition 2.1]). Thus part (2) holds.  $\square$

Here is a special case of our main theorem.

**Proposition 2.2.** *Let  $W$  be a Seifert fibred  $\mathbb{Z}$ -homology 3-sphere. Then the following conditions are equivalent:*

- (a)  $\pi_1(W)$  is left-orderable.
- (b)  $W$  is not an L-space.
- (c)  $W$  admits a co-oriented horizontal foliation.

Further,  $W$  satisfies these conditions if and only if it is neither  $S^3$  nor  $\Sigma(2, 3, 5)$ .

*Proof.* Lemma 1.1 implies that the base orbifold  $\mathcal{B}$  of  $W$  has underlying space  $S^2$ . In this case the equivalence of (a) and (c) was established in [BRW], while those of (b) and (c) was established in [LS] (see also [CM]).

Next suppose that  $W$  is either  $S^3$  or  $\Sigma(2, 3, 5)$ . Then the fundamental group of  $W$  is finite so its fundamental group is not left-orderable,  $W$  is an L-space [OS4, Proposition 2.3] and therefore it does not admit a co-oriented horizontal foliation [OS1, Theorem 1.4].

Conversely suppose that  $W \neq S^3, \Sigma(2, 3, 5)$ . Equivalently,  $\chi(\mathcal{B}) \leq 0$ . If  $\chi(\mathcal{B}) = 0$ ,  $\mathcal{B}$  would support a Euclidean structure and would therefore be one of  $S^2(2, 3, 6), S^2(2, 4, 4), S^2(3, 3, 3)$  or  $S^2(2, 2, 2, 2)$ . But then  $H_1(\mathcal{B}) \neq \{0\}$  contrary to the fact that  $H_1(W) = \{0\}$ . Thus  $\chi(\mathcal{B}) < 0$ , so  $\mathcal{B}$  is hyperbolic. It follows that there is a discrete faithful representation  $\pi_1(\mathcal{B}) \rightarrow PSL_2(\mathbb{R})$  and therefore a non-trivial homomorphism  $\pi_1(W) \rightarrow PSL_2(\mathbb{R})$ . As  $H^2(W) = \{0\}$ , this homomorphism factors through  $\widetilde{SL}_2 \leq \widetilde{\text{Homeo}}_+(S^1) \leq \text{Homeo}_+(\mathbb{R})$ . Hence  $\pi_1(W)$  is left-orderable (cf. [BRW, Theorem 1.1(1)]). It follows from the first paragraph of the proof that  $W$  is not an L-space and it admits a co-oriented horizontal foliation.  $\square$

Let  $X$  be a Seifert fibered  $\mathbb{Z}$ -homology solid torus and set

$$\mathcal{D}_{\text{rat}}^{\text{str}}(X) = \{[\alpha] \in \mathcal{S}_{\text{rat}}(X) : [\alpha] \text{ is strongly detected by a rational foliation on } X\}$$

Clearly  $\mathcal{D}_{\text{rat}}^{\text{str}}(X)$  coincides with the set of slopes  $\alpha$  on  $\partial X$  such that  $X(\alpha)$  admits a horizontal foliation (cf. Lemma 2.1). The work of a number of people ([EHN], [JN], [Na]) shows that the latter set is completely determined by the Seifert invariants of  $X(\alpha)$ . In particular, we have the following result.

**Proposition 2.3.** *Let  $X$  be a Seifert manifold which is a  $\mathbb{Z}$ -homology solid torus with incompressible boundary. Then there is a connected open proper subset  $U$  of  $\mathcal{S}(X)$  such that*

- (1)  $\mathcal{D}_{\text{rat}}^{\text{str}}(X) = U \cap \mathcal{S}_{\text{rat}}(X)$ .
- (2) If  $X$  is not contained in  $S^3$  and  $\Sigma(2, 3, 5)$ , then  $U$  contains all the slopes  $\alpha$  on  $\partial X$  such that  $X(\alpha)$  is a  $\mathbb{Z}$ -homology 3-sphere.

*Proof.* The base orbifold of  $X$  is of the form  $D^2(a_1, a_2, \dots, a_n)$  where  $n$  and each  $a_i$  are at least 2. Since  $X$  is a  $\mathbb{Z}$ -homology solid torus, the  $a_i$  are pairwise coprime. We can assume that the



Seifert invariants  $(a_1, b_1), \dots, (a_n, b_n)$  satisfy  $0 < b_i < a_i$  for each  $i$ . Then

$$\pi_1(X) = \langle y_1, y_2, \dots, y_n, h : h \text{ central, } y_1^{a_1} = h^{b_1}, y_2^{a_2} = h^{b_2}, \dots, y_n^{a_n} = h^{b_n} \rangle$$

Further,

$$h^* = y_1 y_2 \dots y_n$$

is a peripheral element of  $\pi_1(X)$  dual to  $h$ . That is,  $H_1(\partial X) = \pi_1(\partial X)$  is generated by  $h$  and  $h^*$ .

Set  $\gamma_i = \frac{b_i}{a_i}$ . If  $\alpha = ah + bh^*$  is a slope on  $\partial X$ , then  $X(\alpha)$  has Seifert invariants  $(0; 0; \gamma_1, \dots, \gamma_n, \frac{a}{b})$  and therefore also  $(0; -\lfloor \frac{a}{b} \rfloor; \gamma_1, \dots, \gamma_n, \{\frac{a}{b}\})$  where  $\{\frac{a}{b}\} = \frac{a}{b} - \lfloor \frac{a}{b} \rfloor$ . According to [EHN], [JN], [Na],  $X(\alpha)$  admits a horizontal foliation if and only if one of the following conditions holds:

- (1)  $1 - n < \frac{a}{b} < -1$ ;
- (2)  $\lfloor \frac{a}{b} \rfloor = -1$  and there are coprime integers  $0 < A < M$  and some permutation  $(\frac{A_1}{M}, \frac{A_2}{M}, \dots, \frac{A_{n+1}}{M})$  of  $(\frac{A}{M}, \frac{M-A}{M}, \frac{1}{M}, \dots, \frac{1}{M})$  such that  $\gamma_i < \frac{A_i}{M}$  for  $1 \leq i \leq n$  and  $\{\frac{a}{b}\} < \frac{A_{n+1}}{M}$ ;
- (3)  $\lfloor \frac{a}{b} \rfloor = 1 - n$  and there are coprime integers  $0 < A < M$  and some permutation  $(\frac{A_1}{M}, \frac{A_2}{M}, \dots, \frac{A_{n+1}}{M})$  of  $(\frac{A}{M}, \frac{M-A}{M}, \frac{M-1}{M}, \dots, \frac{M-1}{M})$  such that  $\gamma_i > \frac{A_i}{M}$  for  $1 \leq i \leq n$  and  $\{\frac{a}{b}\} > \frac{A_{n+1}}{M}$ .

Let  $V \subset \mathbb{R}$  be the convex hull of the set of rationals  $\frac{a}{b}$  determined these three conditions. We leave it to the reader to verify that  $V$  is an open interval if and only if  $n > 2$  or  $n = 2$  and  $\gamma_1 + \gamma_2 \neq 1$  (cf. [BC, Proposition A.4]). On the other hand, our hypothesis that  $X$  is a  $\mathbb{Z}$ -homology solid torus rules out the possibility that  $n = 2$  and  $\gamma_1 + \gamma_2 = 1$ . Thus if  $U$  is the connected proper subset of  $\mathcal{S}(X)$  corresponding to  $V$  under the identification  $\frac{a}{b} \leftrightarrow [ah + bh^*]$ , then  $U$  is open and  $\mathcal{D}_{rat}^{str}(X) = U \cap \mathcal{S}_{rat}(X)$ , which proves (1). Part (2) then follows from Proposition 2.2.  $\square$

The case when  $X$  is contained in  $S^3$  or  $\Sigma(2, 3, 5)$  is dealt with in the following two propositions.

**Proposition 2.4.** *Let  $X$  be a  $(p, q)$  torus knot exterior where  $p, q \geq 2$  and fix a meridian-longitude pair  $\mu, \lambda$  for  $X$  such that the Seifert fibre of  $X$  has slope  $pq\mu + \lambda$ . Identify the non-meridional slopes on  $\partial X$  with  $\mathbb{Q}$  in the usual way:  $m\mu + n\lambda \leftrightarrow \frac{m}{n}$ . Then there is a co-oriented horizontal foliation of slope  $r \in \mathbb{Q}$  in  $X$  if and only if  $r < pq - (p + q)$ . In particular, the result holds for each  $r < 1$ .*

*Proof.* Fix integers  $a, b$  such that  $1 = bp + aq$  and  $0 < a < p$ . Note that  $b < 0$  but  $p(q + b) > aq + pb = 1$ , so  $0 < b_0 = b + q < q$ . There is a Seifert structure on  $X$  with base orbifold  $D^2(p, q)$  where the two exceptional fibres have Seifert invariants  $(p, a)$  and  $(q, b)$ . Hence if  $r = \frac{n}{m} \neq pq$  is a reduced rational fraction where  $m > 0$ , the Dehn filling  $X(r)$  of  $X$  is a Seifert fibred manifold with Seifert invariants  $(0; 0; \frac{a}{p}, \frac{b}{q}, \frac{m}{n-mpq}) = (0; 0; \frac{a}{p}, \frac{b}{q}, \frac{1}{r-pq})$ . Then  $X(r)$  also has a Seifert structure with Seifert invariants  $(0; 1 - \lfloor \frac{1}{pq-r} \rfloor; \frac{a'}{p}, -\frac{b}{q}, \{\frac{1}{pq-r}\})$  where  $a' = p - a$ . Assume that  $\{\frac{1}{pq-r}\} \neq 0$ . Then arguing as in the proof of Proposition 2.3, if  $X(r)$  admits a horizontal foliation, we have  $\lfloor \frac{1}{pq-r} \rfloor \in \{-1, 0\}$ . If  $\lfloor \frac{1}{pq-r} \rfloor = -1$ , then  $X(r)$  has Seifert invariants  $(0; 1; \frac{a}{p}, \frac{b_0}{q}, 1 - \{\frac{1}{pq-r}\})$  and there are positive integers  $A_1, A_2$  coprime with an integer

$M < A_1, A_2$  such that  $\frac{a}{p} < \frac{A_1}{M}, \frac{b_0}{q} < \frac{A_2}{M}$  and  $\frac{A_1+A_2}{M} \leq 1$ . But this is impossible since then  $\frac{A_1+A_2}{M} > \frac{a}{p} + \frac{b_0}{q} = 1 + \frac{1}{pq}$ . Hence  $\lfloor \frac{1}{pq-r} \rfloor = 0$  and therefore  $0 < \frac{1}{pq-r} < 1$  and  $X(r)$  has Seifert invariants  $(0; 1; \frac{a'}{p}, -\frac{b}{q}, \{\frac{1}{pq-r}\})$ . It follows that  $r < pq - 1$ . A straightforward, though tedious, calculation yields the bound stated in the proposition. This calculation can be avoided if we are willing to appeal to results from Heegaard-Floer theory. For instance, the  $(p, q)$  torus knot  $K$  is an L-space knot since  $pq - 1$  surgery on  $K$  yields a lens space. Hence as the genus of  $K$  is  $\frac{1}{2}(p-1)(q-1)$ ,  $K(r)$  is an L-space if and only if  $r \geq pq - (p+q)$  ([OS5, Proposition 9.5]. See also [Hom, Fact 2, page 221]). Hence, according to Proposition 2.2,  $X(r)$  admits a horizontal foliation if and only if  $r < pq - (p+q)$ .  $\square$

**Proposition 2.5.** *Let  $X$  be a Seifert manifold which is the exterior of a knot  $K$  in  $\Sigma(2, 3, 5)$ , the Poincaré homology 3-sphere.*

- (1)  $K$  is a fibre in a Seifert structure on  $\Sigma(2, 3, 5)$ .
- (2)  $X$  has base orbifold  $D^2(2, 3), D^2(2, 5), D^2(3, 5)$ , or  $D^2(2, 3, 5)$ .
- (3) Suppose that  $K$  has multiplicity  $j \geq 1$ . Then there is a choice of meridian  $\mu$  and longitude  $\lambda$  of  $K$  such that  $X$  admits a horizontal foliation detecting the slope  $a\mu + b\lambda$  if and only if

$$\frac{a}{b} > -29 \text{ if } j = 1$$

and

$$\frac{a}{b} < \begin{cases} 7 & \text{if } j = 2 \\ 3 & \text{if } j = 3 \\ 1 & \text{if } j = 5 \end{cases}$$

In particular, there is a sequence of slopes  $\alpha_n$  on  $\partial X$  which converge projectively to the meridian of  $K$  such that  $X$  admits a horizontal foliation of slope  $\alpha_n$  for each  $n$ .

- (4) There is a unique slope on  $\partial X$  such that  $X(\alpha) \cong \Sigma(2, 3, 5)$ .

*Proof.* The boundary of  $X$  is incompressible since the fundamental group of  $\Sigma(2, 3, 5)$  is non-abelian. It follows from Lemma 1.1 that  $X$  has base orbifold of the form  $D^2(a_1, a_2, \dots, a_n)$  where each  $a_i \geq 2$  and  $n \geq 2$ . Since  $\Sigma(2, 3, 5)$  has no lens space summands, the meridian of  $K$  cannot be the fibre slope of  $X$ . Thus the Seifert structure on  $X$  extends to one on  $\Sigma(2, 3, 5)$  in which  $K$  is a fibre. This implies assertions (1) and (2) of the proposition.

Next we deal with (3). Let  $K_j$  be a fibre of multiplicity  $j$  in  $\Sigma(2, 3, 5)$  for  $j = 1, 2, 3, 5$  and let  $X_0$  be the exterior of  $K_1 \cup K_2 \cup K_3 \cup K_5$ . Denote by  $T_j$  the boundary component of  $X_0$  corresponding to  $K_j$  and by  $\mu_j$  the meridional slope of  $K_j$  on  $T_j$ . Let  $\phi_j$  be the fibre slope on  $T_j$ . Note that  $X_0$  is a trivial circle bundle over a 4-punctured sphere  $Q$ . Orient  $Q$ . Since  $\Sigma(2, 3, 5)$  has Seifert invariants  $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ , there is a section of this bundle with image  $\tilde{Q} \subset X_0$  such that if  $\sigma_j$  is the slope of  $\tilde{Q} \cap T_j$  oriented by the induced orientation from  $Q$ . Orient the fibre of  $X_0$  so that for each  $j$ ,  $\sigma_j \cdot \phi_j = 1$ .

There is a horizontal foliation on  $X_j$  detecting the slope  $n\sigma_j + m\phi_j$  if and only if the  $(n\sigma_j + m\phi_j)$ -Dehn filling of  $X_j$  admits a horizontal foliation. The latter problem has been resolved in the

papers [EHN], [JN], and [Na]. First we prove that  $X_j$  has a horizontal foliation if and only if  $\frac{m}{n} \in (-1, 0)$  for  $j = 1$  and  $\frac{m}{n} \in (0, \frac{1}{j})$  for  $j > 1$ .

The exterior  $X_j$  of  $K_j$  is obtained from  $X_0$  by performing the  $(T_k, \mu_k)$ -filling for  $k \neq j$ . It follows that the  $(n\sigma_j + m\phi_j)$ -Dehn filling of  $X_j$  has Seifert invariants

- $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{m}{n})$  if  $j = 1$ ;
- $(0; -1, \frac{1}{3}, \frac{1}{5}, \frac{m}{n})$  if  $j = 2$ ;
- $(0; -1, \frac{1}{2}, \frac{1}{5}, \frac{m}{n})$  if  $j = 3$ ;
- $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{m}{n})$  if  $j = 5$ .

Suppose first that  $j = 1$ . If  $n = 0$ ,  $X_1(n\sigma_1 + m\phi_1) = X_1(\phi_1)$  is a connected sum of lens spaces of orders 2, 3, and 5 so does not admit a taut foliation (see e.g. [CC2, Corollary 9.1.9]). If  $|n| = 1$ , then  $\Delta(n\sigma_1 + m\phi_1, \phi_1) = 1$ , so  $X_1(n\sigma_1 + m\phi_1)$  admits a Seifert structure with base orbifold  $S^2(2, 3, 5)$ . Hence it has a finite fundamental group and so does not admit a horizontal foliation. Assume then that  $|n| > 1$ , and therefore  $0 < \{\frac{m}{n}\} = \frac{m}{n} - \lfloor \frac{m}{n} \rfloor < 1$ . In this case,  $X_1(n\sigma_1 + m\phi_1)$  has Seifert invariants  $(0; \lfloor \frac{m}{n} \rfloor - 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$ . Theorem 2 of [JN] implies that when  $\lfloor \frac{m}{n} \rfloor = -1$  there is a horizontal foliation for all values of  $\{\frac{m}{n}\}$ . In other words, whenever  $\frac{m}{n} \in (-1, 0)$ . It also shows that there is no horizontal foliation when  $\lfloor \frac{m}{n} \rfloor < -2$  or  $\lfloor \frac{m}{n} \rfloor > 0$

If  $\lfloor \frac{m}{n} \rfloor = 0$ , then  $X_1(n\sigma_1 + m\phi_1)$  has Seifert invariants  $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$ . Conjecture 2 of [JN] was verified in [Na] so in this case  $X_1(n\sigma_1 + m\phi_1)$  has a horizontal foliation if and only if we can find coprime integers  $0 < A < M$  such that for some permutation  $\{\frac{a_1}{m_1}, \frac{a_2}{m_2}, \frac{a_3}{m_3}, \frac{a_4}{m_4}\}$  of  $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\}\}$  satisfies  $\frac{a_1}{m_1} < \frac{1}{M}$ ,  $\frac{a_2}{m_2} < \frac{1}{M}$ ,  $\frac{a_3}{m_3} < \frac{1}{M}$  and  $\frac{a_4}{m_4} < \frac{M-A}{M}$ . It is elementary to verify that there is no such pair  $A, M$ .

If  $\lfloor \frac{m}{n} \rfloor = -2$ , then  $X_1(n\sigma_1 + m\phi_1)$  has Seifert invariants  $(0; -3, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$  and therefore also  $(0; -1, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}, 1 - \{\frac{m}{n}\})$ . As in the previous paragraph,  $X_1(n\sigma_1 + m\phi_1)$  never admits a horizontal foliation on this case. We conclude that  $X_1(n\sigma_1 + m\phi_1)$  admits a horizontal foliation if and only if  $\frac{m}{n} \in (-1, 0)$ .

We proceed similarly when  $j = 2$ . As above we can rule out the cases  $n = 0$  and  $|n| = 1$ . When  $|n| > 1$ , so  $0 < \{\frac{m}{n}\} = \frac{m}{n} - \lfloor \frac{m}{n} \rfloor < 1$ ,  $X_2(n\sigma_2 + m\phi_2)$  has Seifert invariants  $(0; \lfloor \frac{m}{n} \rfloor - 1, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$ . By Theorem 2 of [JN], there is no horizontal foliation when  $\lfloor \frac{m}{n} \rfloor < -1$  or  $\lfloor \frac{m}{n} \rfloor > 0$ . If  $\lfloor \frac{m}{n} \rfloor = 0$ ,  $X_2(n\sigma_2 + m\phi_2)$  has Seifert invariants  $(0; -1, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$ . Conjecture 2 of [JN] was verified in [Na] so in this case  $X_2(n\sigma_2 + m\phi_2)$  has a horizontal foliation if and only if we can find coprime integers  $0 < A < M$  such that for some permutation  $\{\frac{a_1}{m_1}, \frac{a_2}{m_2}, \frac{a_3}{m_3}\}$  of  $\{\frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\}\}$  satisfies  $\frac{a_1}{m_1} < \frac{1}{M}$ ,  $\frac{a_2}{m_2} < \frac{1}{M}$  and  $\frac{a_3}{m_3} < \frac{M-A}{M}$ . It is elementary to verify that there is a solution to this problem if and only if  $\frac{m}{n} \in (0, \frac{1}{2})$ . On the other hand, if  $\lfloor \frac{m}{n} \rfloor = -1$ ,  $X_2(n\sigma_2 + m\phi_2)$  has Seifert invariants  $(0; -2, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$  and therefore  $(0; -1, \frac{2}{3}, \frac{4}{5}, 1 - \{\frac{m}{n}\})$ . As above,  $X_2(n\sigma_2 + m\phi_2)$  never admits a horizontal foliation on this case. We conclude that  $X_2(n\sigma_2 + m\phi_2)$  admits a horizontal foliation if and only if  $\frac{m}{n} \in (0, \frac{1}{2})$ .

We leave the cases  $j = 3, 5$  to the reader.

To complete the proof of (3) we must express the conclusions we have just obtained in terms of appropriately chosen meridians and longitudes for the knots  $K_j$ . We proceed as follows. The euler number of  $X_j(n\sigma_j + m\phi_j)$  is given, up to sign, by the sum of its Seifert invariants. Further, since  $H_1(X_j(\lambda_j)) \cong \mathbb{Z}$ , we can solve for the coefficients  $n, m$  of  $\lambda_j$ . For instance for  $j > 1$ , set  $\{j, p, q\} = \{2, 3, 5\}$ . If  $\lambda_j = n\sigma_j + m\phi_j$ , then  $0 = |e(X_j(n\sigma_j + m\phi_j))| = |-1 + \frac{1}{p} + \frac{1}{q} + \frac{m}{n}|$ . Thus  $\frac{m}{n} = \frac{pq - (p+q)}{pq}$ . Since  $\gcd(pq, pq - (p+q)) = 1$ , we have

$$\lambda_j = -pq\sigma_j + (p+q-pq)\phi_j$$

Similarly for  $j = 1$  we have  $\frac{m}{n} = 1 - (\frac{1}{2} + \frac{1}{3} + \frac{1}{5}) = -\frac{1}{30}$ . Hence

$$\lambda_1 = -30\sigma_1 + \phi_1$$

The  $\mu_j$  Dehn filling of  $X_j$  yields  $\Sigma(2, 3, 5)$  and it is known that  $|e(\Sigma(2, 3, 5))| = \frac{1}{30}$ . Combined with the identity  $\Delta(\mu_j, \lambda_j) = 1$  we can solve for the coefficients of  $\mu_j$ :

$$\mu_j = \begin{cases} \sigma_1 & \text{if } j = 1 \\ j\sigma_j + \phi_j & \text{if } j > 1 \end{cases}$$

With these choices, it is easy to verify that the set of detected slopes  $a\mu_1 + b\lambda_1$  corresponds to the interval specified in (3).

To prove (4), let  $\alpha = a\mu_j + b\lambda_j$  be a slope on  $\partial X_j$  such that  $X_j(\alpha) \cong \Sigma(2, 3, 5)$ . Since  $\Sigma(2, 3, 5)$  is a  $\mathbb{Z}$ -homology 3-sphere,  $1 = \Delta(\alpha, \lambda_j) = |a|$ . Without loss of generality we can suppose that  $a = 1$ . On the other hand, the core of the filling torus in  $X_j(\alpha)$  is  $K_j$ , so

$$\begin{aligned} j = \Delta(\alpha, \phi_j) &= \begin{cases} \Delta(\mu_j + b\lambda_j, 30\mu_1 + \lambda_1) & \text{if } j = 1 \\ \Delta(\mu_j + b\lambda_j, pq\mu_j + j\lambda_j) & \text{if } j > 1 \end{cases} \\ &= \begin{cases} |1 - 30b| & \text{if } j = 1 \\ |j - pqb| & \text{if } j > 1 \end{cases} \end{aligned}$$

Hence there is an  $\epsilon \in \{\pm 1\}$  such that  $j\epsilon = \begin{cases} 1 - 30b & \text{if } j = 1 \\ j - pqb & \text{if } j > 1 \end{cases}$ . It follows that  $b = 0$  so that  $\alpha = \mu_j$ . This proves (4).  $\square$

**Corollary 2.6.** *Suppose that  $K$  is a knot in either  $S^3$  or  $\Sigma(2, 3, 5)$  whose exterior  $X$  is Seifert fibered and let  $U$  be the connected open subset of  $\mathcal{S}(X)$  described in Proposition 2.3.*

(1) *If  $X$  is the trefoil exterior, then  $U$  contains all the slopes  $\alpha$  on  $\partial X$  such that  $X(\alpha)$  is a  $\mathbb{Z}$ -homology 3-sphere other than  $S^3$  and  $\Sigma(2, 3, 5)$ . The two slopes yielding the latter two manifolds are the end-points of  $\overline{U}$ .*

(2) *If  $X$  is not the trefoil exterior, then  $U$  contains all the slopes  $\alpha$  on  $\partial X$  such that  $X(\alpha)$  is a  $\mathbb{Z}$ -homology 3-sphere other than the meridian of  $K$ , which is an end-point of  $\overline{U}$ .*  $\square$

3. EXISTENCE OF RATIONAL FOLIATIONS ON ASPHERICAL GRAPH  $\mathbb{Z}$ -HOMOLOGY 3-SPHERES

We prove Theorem 0.2 in this section by induction on the number of its JSJ pieces, the base case being dealt with in Proposition 2.2. We suppose below that  $W$  is a non-Seifert graph manifold  $\mathbb{Z}$ -homology 3-sphere.

**Lemma 3.1.** *Suppose that  $M$  is a graph manifold  $\mathbb{Z}$ -homology solid torus with incompressible boundary. If  $\alpha$  and  $\beta$  are slopes on  $\partial M$  whose associated fillings are  $\mathbb{Z}$ -homology 3-spheres which are either  $S^3$ ,  $\Sigma(2, 3, 5)$  or reducible, then  $\Delta(\alpha, \beta) \leq 1$ .*

*Proof.* If  $M$  is Seifert fibred, it has base orbifold  $D^2(a_1, \dots, a_n)$  where  $n$  and each  $a_i$  are at least 2. Further, the  $a_i$  are pairwise coprime. In this case  $M$  admits no fillings which are simultaneously reducible and  $\mathbb{Z}$ -homology 3-spheres. Thus  $M(\alpha)$  and  $M(\beta)$  are either  $S^3$  or  $\Sigma(2, 3, 5)$ . If  $\alpha$  and  $\beta$  are distinct slopes, then  $M(\alpha)$  and  $M(\beta)$  cannot both be  $S^3$  as torus knots admit unique  $S^3$ -surgery slopes. Similarly Proposition 2.5 implies that  $M(\alpha)$  and  $M(\beta)$  cannot both be  $\Sigma(2, 3, 5)$ . On the other hand, if one of  $M(\alpha)$  and  $M(\beta)$  is  $S^3$  and the other  $\Sigma(2, 3, 5)$ , then  $M$  must be the trefoil knot exterior and  $\Delta(\alpha, \beta) = 1$ .

Next suppose that  $M$  is not Seifert fibred. If  $M(\alpha)$  is reducible, then the main result of [GLu] combines with [BZ2, Theorem 1.2] to show that  $\Delta(\alpha, \beta) \leq 1$ . On the other hand, if  $M(\alpha)$  and  $M(\beta)$  are either  $S^3$  or  $\Sigma(2, 3, 5)$  and  $\Delta(\alpha, \beta) \geq 2$ , then [BZ1, Theorem 1.2(1)] implies that  $M$  has two pieces, one a cable space and the other a Seifert manifold  $M_0$  with base orbifold a 2-disk with two cone points. The proof of [BZ1, Theorem 1.2(1)] (see §8 of [BZ1]) now implies that  $M_0$  admits two Dehn fillings yielding  $S^3$  or  $\Sigma(2, 3, 5)$  whose slopes are of distance at least 8, which is impossible. (See the discussion which follows the statement of [BZ1, Theorem 1.2].) Thus  $\Delta(\alpha, \beta) \leq 1$ .  $\square$

Let  $X$  be a piece of  $W$  whose boundary is a torus. (Thus  $X$  corresponds to a leaf of the JSJ-graph of  $W$ .) If  $Y = \overline{W \setminus X}$  is the exterior of  $X$  in  $W$ , then  $T = X \cap Y$  is an essential torus. Let  $\lambda_X$  and  $\lambda_Y$  be the longitudes of  $X$  and  $Y$ . For slopes  $\alpha$  and  $\beta$  on  $T$  we have

$$|H_1(X(\alpha))| = \Delta(\alpha, \lambda_X) \quad \text{and} \quad |H_1(Y(\beta))| = \Delta(\beta, \lambda_Y)$$

Hence as we noted in §1 that  $\Delta(\lambda_X, \lambda_Y) = 1$ , both  $X(\lambda_Y)$  and  $Y(\lambda_X)$  are  $\mathbb{Z}$ -homology 3-spheres.

Let  $\phi_X$  and  $\phi_Y$  be primitive elements of  $H_1(T)$  representing, respectively, the slopes of the Seifert fibre of  $X$  and that of the piece  $P$  of  $Y$  incident to  $T$ . Since  $X$  has exceptional fibres,  $\pm\phi_X \notin \{\lambda_X, \lambda_Y\}$  (Lemma 1.1(3)). It follows that  $X(\lambda_X)$  and  $X(\lambda_Y)$  are irreducible Seifert manifolds (Lemma 1.1(1)).

*Proof of Theorem 0.2.* For an integer  $n$ , set

$$\alpha_n = \lambda_X + n\lambda_Y$$

and observe that  $\lim_{|n|} [\alpha_n] = [\lambda_Y] \in \mathcal{S}_{rat}(T)$ . Since  $X(\lambda_Y)$  is a  $\mathbb{Z}$ -homology 3-sphere,  $\alpha_n$  is strongly detected by a horizontal foliation in  $X$  for  $n \gg 0$  or for  $n \ll 0$  or for both (Proposition

2.3 and Corollary 2.6). To complete the proof it suffices to find a rational foliation of  $Y$  which strongly detects  $\alpha_n$  for all large  $|n|$ .

Since  $\Delta(\alpha_n, \lambda_Y) = 1$ , the manifolds  $Y(\alpha_n)$  are  $\mathbb{Z}$ -homology 3-spheres, and since  $Y$  is irreducible and  $\Delta(\alpha_n, \alpha_m) = |n - m|$ , there are at most two  $n$  such that  $Y(\alpha_n)$  is either reducible,  $S^3$  or  $\Sigma(2, 3, 5)$ , and if two, they are successive integers (Lemma 3.1). Thus for  $|n|$  large,  $Y(\alpha_n)$  is an irreducible graph manifold  $\mathbb{Z}$ -homology 3-sphere which is neither  $S^3$  nor  $\Sigma(2, 3, 5)$ . Hence our inductive hypothesis implies that  $Y(\alpha_n)$  admits a rational foliation  $\mathcal{F}_n$  for large  $|n|$ . If  $\lambda_Y \neq \phi_Y$ , then as  $\Delta(\alpha_n, \phi_Y) = |\alpha_n \cdot \phi_Y| \geq \frac{|n|}{|\lambda_Y|} |\lambda_Y \cdot \phi_Y| - |\lambda_X \cdot \phi_Y|$ , for large  $|n|$  the JSJ pieces of  $Y(\alpha_n)$  are  $P(\alpha_n)$  and the JSJ pieces of  $\overline{Y \setminus P}$ . Thus  $\mathcal{F}_n$  induces a rational foliation of slope  $\alpha_n$  on  $Y$ , which completes the proof.

Suppose then that  $\lambda_Y = \phi_Y$ . Then Lemma 1.1(3) implies that  $P$  is a product  $F \times S^1$  where  $F$  is a planar surface with  $|\partial P| \geq 3$  boundary components. Since  $\Delta(\alpha_n, \phi_Y) = \Delta(\alpha_n, \lambda_Y) = 1$ , each  $P(\alpha_n)$  is a product  $\bar{F} \times S^1$  where  $\bar{F}$  is a planar surface with  $|\partial P| - 1 \geq 2$  boundary components. If  $|\partial P| \geq 4$ , the JSJ pieces of  $Y(\alpha_n)$  are  $P(\alpha_n)$  and the JSJ pieces of  $\overline{Y \setminus P}$ , so we can proceed as above.

Finally assume that  $|\partial P| = 3$  and let  $Y_1, Y_2$  be the components of  $\overline{Y \setminus P}$ . Denote the JSJ torus  $Y_i \cap P$  by  $T_i$ , so  $\partial P = \partial Y \cup T_1 \cup T_2$ . For each  $n$  we have  $P(\alpha_n) \cong S^1 \times S^1 \times I$ , so  $Y(\alpha_n) \cong Y_1 \cup Y_2 \not\cong S^3, \Sigma(2, 3, 5)$ . By induction, there is a rational foliation  $\mathcal{F}_n$  on  $Y(\alpha_n)$ . Since there is no vertical annulus in  $P$  which is cobounded by the Seifert fibres of the two pieces of  $Y$  incident to  $P$ , the reader will verify that there is at most one value of  $n$  for which there is an annulus in  $P(\alpha_n)$  cobounded by these fibres. Thus for  $|n| \gg 0$ ,  $Y(\alpha_n)$  is a graph manifold  $\mathbb{Z}$ -homology 3-sphere whose pieces are the JSJ pieces of  $\overline{Y \setminus P}$ . Fix such an  $n$  and note that up to isotopy, we can suppose that  $\mathcal{F}_n$  is a product fibration on  $P(\alpha_n) \cong S^1 \times S^1 \times I$  whose fibre is an annulus. It follows that we can choose primitive classes  $\beta_n^1 \in H_1(T_1)$  and  $\beta_n^2 \in H_1(T_2)$  representing the slopes of  $\mathcal{F}_n$  on  $T_1, T_2$  and an integer  $k$  such that  $k\alpha_n + \beta_n^1 + \beta_n^2 = 0$  in  $H_1(P)$ .

Let  $p : P = F \times S^1 \rightarrow F$  be the projection and denote by  $a, b_1, b_2 \in H_1(F)$  the classes associated to the boundary components of  $F$ , where  $a$  corresponds to  $p(T)$ ,  $b_1$  to  $p(T_1)$ , and  $b_2$  to  $p(T_2)$ . We may assume that  $a + b_1 + b_2 = 0$ . Since  $\Delta(\alpha_n, \phi_Y) = 1$ , we can also assume that the projection  $p : P \rightarrow F$  sends  $\alpha_n$  to  $a$ . Fix integers  $k_1, k_2$  so that  $p_*(\beta_n^j) = k_j b_j$ . Clearly  $|k_j| = \Delta(\beta_n^j, \phi_j)$  where  $\phi_j$  is the slope on  $T_j$  determined by the Seifert structure on  $P$ . Then we have

$$0 = p_*(k\alpha_n + \beta_n^1 + \beta_n^2) = ka + k_1 b_1 + k_2 b_2$$

in  $H_1(F)$ . This can only happen if  $k = k_1 = k_2$ . Thus if  $k \neq 0$ , the fibration in  $P(\alpha_n)$  determined by  $\mathcal{F}_n$  is horizontal in  $P$  and of slope  $\alpha_n$  on  $T$ , so we are done.

Suppose then that  $k = 0$ , so  $0 = |k_j| = \Delta(\beta_n^j, \phi_j)$ . Thus  $[\beta_n^1] = [\phi_1]$  and  $[\beta_n^2] = [\phi_2]$  are vertical in  $P$ . By construction,  $Y(\lambda_Y) = Y(\phi_Y) = Y_1(\phi_1) \# Y_2(\phi_2) = Y_1(\beta_n^1) \# Y_2(\beta_n^2)$  and as  $\mathbb{Z} \cong H_1(Y(\lambda_Y)) = H_1(Y_1(\phi_1)) \oplus H_1(Y_2(\phi_2))$ , we can suppose that  $H_1(Y_1(\beta_n^1)) \cong \mathbb{Z}$  and  $H_1(Y_2(\beta_n^2)) \cong \{0\}$ . Thus  $\phi_1 = \beta_n^1 = \lambda_{Y_1}$  and  $\Delta(\phi_2, \lambda_{Y_2}) = \Delta(\beta_n^2, \lambda_{Y_2}) = 1$ .

Fix  $\delta_0 \in H_1(T_1)$  such that  $1 = \Delta(\delta_0, \lambda_{Y_1}) = \Delta(\delta_0, \phi_1)$  and  $p_*(\delta_0) = b_1$ . Then  $p_*(\lambda_X + \delta_0 + \lambda_{Y_2}) = a + b_1 + b_2 = 0 \in H_1(F)$  and therefore  $\lambda_X + \delta_0 + \lambda_{Y_2} = j\phi_Y \in H_1(P)$  for some integer  $j$ . After

replacing  $\delta_0$  by  $\delta_0 - j\phi_1$  we can suppose that

$$\lambda_X + \delta_0 + \lambda_{Y_2} = 0 \in H_1(P)$$

With this choice, set  $\delta_m = \delta_0 + m\phi_1$ .

**Claim 3.2.** *For all but at most finitely many  $m$ ,  $Y_1$  admits a rational foliation of slope  $\delta_m$ .*

*Proof.* Since  $\Delta(\delta_m, \lambda_{Y_1}) = 1$  for all  $m$ ,  $Y_1(\delta_m)$  is a  $\mathbb{Z}$ -homology 3-sphere. Let  $\phi_{Y_1}$  be the primitive element of  $H_1(T_1)$  representing the slope of the Seifert fibre of the piece  $P_1$  of  $Y_1$  incident to  $T_1 = \partial Y_1$ , then  $\Delta(\lambda_{Y_1}, \phi_{Y_1}) \geq 1$ , since  $\lambda_{Y_1} = \phi_Y$  and  $T_1$  is a JSJ-torus of  $Y$ . Therefore our inductive hypothesis combines with Lemma 3.1 to show, as in the first part of the proof, that for all but at most finitely many  $m$ ,  $Y_1$  admits a rational foliation of slope  $\delta_m$ .  $\square$

**Claim 3.3.**  *$Y_2$  admits a rational foliation of slope  $\gamma = p\lambda_{Y_2} + q\phi_2$  where  $p$  and  $q$  are relatively prime and non-zero.*

*Proof.* Let  $\phi_{Y_2}$  be the primitive element of  $H_1(T_2)$  representing the slope of the Seifert fibre of the piece  $P_2$  of  $Y_2$  incident to  $T_2 = \partial Y_2$ . If  $\Delta(\lambda_{Y_2}, \phi_{Y_2}) \geq 1$ , the assertion follows from the proof of Claim 3.2 by taking  $\gamma = p\lambda_{Y_2} + \phi_2$ , for some  $|p|$  sufficiently large.

We consider now the case where  $\lambda_{Y_2} = \phi_{Y_2}$ . Let  $E \subset S^3$  be the trefoil exterior,  $\mu_E \in H_1(\partial E)$  its meridional slope and  $\nu_E \in H_1(\partial E)$  the unique slope such that  $E(\nu_E) \cong \Sigma(2, 3, 5)$ . Then  $\Delta(\mu_E, \nu_E) = 1$ . Further,  $E$  does not admit a horizontal foliation of slope  $\mu_E$  or  $\nu_E$ . We build a  $\mathbb{Z}$ -homology 3-sphere  $W_2 = E \cup Y_2$  by gluing  $E$  and  $Y_2$  along their boundaries in such a way that the slope  $\mu_E$  is identified with the slope  $\lambda_{Y_2}$  and the slope  $\nu_E$  is identified with the slope  $\phi_2$ . Since the fiber slope  $\phi_{Y_2} = \lambda_{Y_2}$  is identified with the meridional slope  $\mu_E$ , the Seifert fibrations on  $E$  and  $P_2$  do not match up, and the torus  $\partial Y_2 = \partial E$  is a JSJ-torus of  $W_2$ . Hence  $W_2$  is a graph  $\mathbb{Z}$ -homology 3-sphere whose JSJ pieces are  $E$  and the JSJ pieces of  $Y_2$ . In particular,  $W_2$  has fewer pieces than  $W$ . By the inductive hypothesis  $W_2$  carries a rational foliation which intersects the JSJ torus  $\partial Y_2 = \partial E$  in a circle fibration of some slope  $\gamma$ . Hence  $Y_2$  admits a rational foliation of slope  $\gamma$ . Moreover  $\Delta(\gamma, \lambda_{Y_2}) \geq 1$  and  $\Delta(\gamma, \phi_2) \geq 1$  since  $E$  cannot admit a horizontal foliation of slope  $\mu_E$  or  $\nu_E$ .  $\square$

Now we complete the proof of Theorem 0.2.

For  $|m|$  sufficiently large, let  $\delta_m = \delta_0 + m\phi_1 \in H_1(T_1)$  be the slope of a rational foliation on  $Y_1$  given by Claim 3.2, and  $\gamma = p\lambda_{Y_2} + q\phi_2 \in H_1(T_2)$  the slope of a rational foliation on  $Y_2$  given by Claim 3.3. Since  $\lambda_Y = \phi_Y = \phi_1 = \phi_2$  and  $\lambda_X + \delta_0 + \lambda_{Y_2} = 0$  in  $H_1(P)$ , the sum  $\zeta_m + p\delta_m + \gamma = 0 \in H_1(P)$  where  $\zeta_m = p\lambda_X - (pm + q)\lambda_Y \in H_1(T)$  is a primitive class. Thus there is a properly embedded, horizontal surface  $F_m$  in  $P$  with boundary curves of slope  $\zeta_m, \delta_m$  and  $\gamma$ . Hence  $P$  fibres over the circle with fibre  $F_m$  and  $Y$  admits a rational foliation of slope  $\zeta_m$  for large  $|m|$ . Now, it is easy to verify that  $\lim_{|m|} [\zeta_m] = [\lambda_Y]$  and that for large  $|m|$ , reversing the sign of  $m$  sends  $[\zeta_m]$  from one side of  $[\lambda_Y]$  to the other. Since  $X(\lambda_Y)$  is a  $\mathbb{Z}$ -homology 3-sphere, Proposition 2.3 and Corollary 2.6 imply that  $X$  admits a horizontal foliation of slope

$\delta_m$  for  $m \gg 0$  or for  $m \ll 0$  or for both. This completes the induction and the proof of Theorem 0.2.  $\square$

## REFERENCES

- [BC] S. Boyer and A. Clay, *On foliations, orders, representations, L-spaces and graph manifolds*, in preparation.
- [BGW] S. Boyer, C. McA. Gordon and L. Watson, *On L-spaces and left-orderable fundamental groups*, preprint 2011, arXiv:1107.5016
- [BRW] S. Boyer, D. Rolfsen and B. Wiest, *Orderable 3-manifold groups*, Ann. Inst. Fourier (Grenoble), **55** (2005), 243–288.
- [BZ1] S. Boyer and X. Zhang, *Finite surgery on knots*, J. Amer. Math. Soc. **9** (1996), 1005–1050.
- [BZ2] ———, *On Culler-Shalen seminorms and Dehn filling*, Ann. Math **148** (1998), 737–801.
- [Br1] M. Brittenham, *Essential laminations in Seifert-fibered spaces*, Topology **32** (1993), 61–85.
- [Br2] ———, *Tautly foliated 3-manifolds with no  $\mathbb{R}$ -covered foliations*, Foliations: geometry and dynamics (Warsaw, 2000), 213–224, World Sci. Publ., River Edge, NJ, 2002.
- [CM] S. Caillat-Gibert and D. Matignon, *Existence of taut foliations on Seifert fibered homology 3-spheres*, preprint 2011, arXiv:1101.3710.
- [Ca1] D. Calegari, *The geometry of  $\mathbb{R}$ -covered foliations*, Geom. & Top. **4** (2000), 457–515.
- [Ca2] ———, *Foliations with one-sided branching*, Geom. Dedicata **96** (2003), 1–53.
- [Ca3] ———, *Promoting essential laminations*, Invent. Math. **166** (2006), 583–643.
- [CD] D. Calegari and N. Dunfield, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. **152** (2003), 149–204.
- [CC1] A. Candel and L. Conlon, *Foliations I*, Graduate Studies in Mathematics **23**, Amer. Math. Soc., 2003.
- [CC2] ———, *Foliations II*, Graduate Studies in Mathematics **60**, Amer. Math. Soc., 2003.
- [CLW] A. Clay, T. Lidman and L. Watson, *Graph manifolds, left-orderability and amalgamation*, preprint 2011, arXiv:1106.0486v1.
- [EHN] D. Eisenbud, U. Hirsch and W. Neumann, *Transverse foliations on Seifert bundles and self-homeomorphisms of the circle*, Comm. Math. Helv. **56** (1981), 638–660.
- [Fen] R. Fenley *Foliations, topology and geometry of 3-manifolds:  $\mathbb{R}$ -covered foliations and transverse pseudo-Anosov flows*, Comment. Math. Helv. **77** (2002), 415–490.
- [Gh] E. Ghys, *Groups acting on the circle*, Enseign. Math. **47** (2001), 329–407.
- [GLu] C. McA. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996), 385–409.
- [God] C. Godbillon, *Feuilletages*, Progress in Mathematics **98**, Birkhauser Verlag, Basel, 1991.
- [Go] S. Goodman, *Closed leaves in foliations of codimension one*, Comm. Math. Helv. **50** (1975), 383–388.
- [HW] M. Hedden and L. Watson, *Does Khovanov homology detect the unknot?*, Amer. J. Math. **132** (2010), 1339–1345.
- [Hom] J. Hom, *A note on cabling and L-space surgeries*, Alg. & Geom. Top. **11** (2011), 219–223.
- [Ja] W. Jaco, *Lectures on three-manifolds topology*, CBMS Regional Conf. Ser. Math. **43** (1980).
- [JN] M. Jankins and W. Neumann, *Rotation numbers and products of circle homomorphisms*, Math. Ann. **271** (1985), 381–400.
- [LL] A. Levine and S. Lewallen, *Strong L-spaces and left orderability*, preprint 2011, arXiv:1110.0563.



- [LS] P. Lisca and A. Stipsicz, *Ozsváth-Szabó invariants and tight contact 3-manifolds. III*, J. Symplectic Geom. **5** (2007), 357–384.
- [Mat] S. Matsumoto, *Numerical invariants for semiconjugacy of homeomorphisms of the circle*, Proc. Amer. Math. Soc. **98** (1986), 163–168.
- [Na] R. Naimi, *Foliations transverse to fibers of Seifert manifolds*, Comm. Math. Helv. **69** (1994), 155–162.
- [OS1] P. Ozsváth and Z. Szábo, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.
- [OS2] ———, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), 58–116.
- [OS3] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **159** (2004), 1027–1158.
- [OS4] ———, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), 1281–1300.
- [OS5] ———, *Knot Floer homology and rational surgeries*, preprint 2005, arXiv:GT/0504404.
- [Pla] J.F. Plante, *Foliations with measure preserving holonomy*, Ann. of Math. **102** (1975), 327–361.

INSTITUT DE MATHÉTIQUES DE TOULOUSE, UMR 5219 ET INSTITUT UNIVERSITAIRE DE FRANCE, UNIVERSITÉ PAUL SABATIER 31062 TOULOUSE CEDEX 9, FRANCE.

*E-mail address:* `boileau@math.univ-toulouse.fr`

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, 201 AVENUE DU PRÉSIDENT-KENNEDY, MONTRÉAL, QC H2X 3Y7.

*E-mail address:* `boyer.steven@uqam.ca`

*URL:* <http://www.cirget.uqam.ca/boyer/boyer.html>