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The maximal drawdown of the Brownian meander

by

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Summary. Motivated by evaluating the limiting distribution of randomly biased random walks on trees, we compute the exact value of a negative moment of the maximal drawdown of the standard Brownian meander.

Keywords. Brownian meander, Bessel process, maximal drawdown.

2010 Mathematics Subject Classification. 60J65.

1 Introduction

Let $(X(t), t \in [0, 1])$ be a random process. Its maximal drawdown on $[0, 1]$ is defined by

$$X^\#(1) := \sup_{s \in [0, 1]} [\overline{X}(s) - X(s)],$$

where $\overline{X}(s) := \sup_{u \in [0, s]} X(u)$. There has been some recent research interest on the study of drawdowns from probabilistic point of view ([7], [8]) as well as applications in insurance and finance ([1], [2], [3], [10], [12]).

We are interested in the maximal drawdown $\mathfrak{m}^\#(1)$ of the standard Brownian meander $(\mathfrak{m}(t), t \in [0, 1])$. Our motivation is the presence of the law of $\mathfrak{m}^\#(1)$ in the limiting distribution of randomly biased random walks on supercritical Galton–Watson trees ([4]); in particular, the value of $\mathbb{E}(\frac{1}{\mathfrak{m}^\#(1)})$ is the normalizing constant in the density function of

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this limiting distribution. The sole aim of the present note is to compute $\mathbb{E}(\frac{1}{\mathbf{m}^\#(1)})$, which turns out to have a nice numerical value.

Let us first recall the definition of the Brownian meander. Let $W := (W(t), t \in [0, 1])$ be a standard Brownian motion, and let $\mathbf{g} := \sup\{t \leq 1 : W(t) = 0\}$ be the last passage time at 0 before time 1. Since $\mathbf{g} < 1$ a.s., we can define

$$\mathbf{m}(s) := \frac{|W(\mathbf{g} + s(1 - \mathbf{g}))|}{(1 - \mathbf{g})^{1/2}}, \quad s \in [0, 1].$$

The law of $(\mathbf{m}(s), s \in [0, 1])$ is called the law of the standard Brownian meander. For an account of general properties of the Brownian meander, see Yen and Yor [11].

Theorem 1.1. *Let $(\mathbf{m}(s), s \in [0, 1])$ be a standard Brownian meander. We have*

$$(1.1) \quad \mathbb{E}\left(\frac{1}{\sup_{s \in [0, 1]}[\bar{\mathbf{m}}(s) - \mathbf{m}(s)]}\right) = \left(\frac{\pi}{2}\right)^{1/2},$$

where $\bar{\mathbf{m}}(s) := \sup_{u \in [0, s]} \mathbf{m}(u)$.

The theorem is proved in Section 2.

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N.B. from the first-named coauthors: This note originates from a question we asked our teacher, **Professor Marc Yor (1949–2014)**, who passed away in January 2014, during the preparation of this note. He provided us, in November 2012, with the essential of the material in Section 2.

2 Proof

Let $R := (R(t), t \geq 0)$ be a three-dimensional Bessel process with $R(0) = 0$, i.e., the Euclidean modulus of a standard three-dimensional Brownian motion. The proof of Theorem 1.1 relies on an absolute continuity relation between $(\mathbf{m}(s), s \in [0, 1])$ and $(R(s), s \in [0, 1])$, recalled as follows.

Fact 2.1. (Imhof [5]) *Let $(\mathbf{m}(s), s \in [0, 1])$ be a standard Brownian meander. Let $(R(s), s \in [0, 1])$ be a three-dimensional Bessel process with $R(0) = 0$. For any measurable and non-negative functional F , we have*

$$\mathbb{E}\left[F(\mathbf{m}(s), s \in [0, 1])\right] = \left(\frac{\pi}{2}\right)^{1/2} \mathbb{E}\left[\frac{1}{R(1)} F(R(s), s \in [0, 1])\right].$$

We now proceed to the proof of Theorem 1.1. Let

$$L := \mathbb{E} \left(\frac{1}{\sup_{s \in [0, 1]} [\bar{\mathbf{m}}(s) - \mathbf{m}(s)]} \right).$$

Write $\bar{R}(t) := \sup_{u \in [0, t]} R(u)$ for $t \geq 0$. By Fact 2.1,

$$\begin{aligned} L &= \left(\frac{\pi}{2} \right)^{1/2} \mathbb{E} \left[\frac{1}{R(1)} \frac{1}{\sup_{s \in [0, 1]} [\bar{R}(s) - R(s)]} \right] \\ &= \left(\frac{\pi}{2} \right)^{1/2} \int_0^\infty \mathbb{E} \left[\frac{1}{R(1)} \mathbf{1}_{\{\sup_{s \in [0, 1]} [\bar{R}(s) - R(s)] < \frac{1}{a}\}} \right] da, \end{aligned}$$

the last equality following from the Fubini–Tonelli theorem. By the scaling property, $\mathbb{E} \left[\frac{1}{R(1)} \mathbf{1}_{\{\sup_{s \in [0, 1]} [\bar{R}(s) - R(s)] < \frac{1}{a}\}} \right] = \mathbb{E} \left[\frac{a}{R(a^2)} \mathbf{1}_{\{\sup_{u \in [0, a^2]} [\bar{R}(u) - R(u)] < 1\}} \right]$ for all $a > 0$. So by means of a change of variables $b = a^2$, we obtain:

$$L = \left(\frac{\pi}{8} \right)^{1/2} \int_0^\infty \mathbb{E} \left[\frac{1}{R(b)} \mathbf{1}_{\{\sup_{u \in [0, b]} [\bar{R}(u) - R(u)] < 1\}} \right] db.$$

Define, for any random process X ,

$$\tau_1^X := \inf \{ t \geq 0 : \bar{X}(t) - X(t) \geq 1 \},$$

with $\bar{X}(t) := \sup_{s \in [0, t]} X(s)$. For any $b > 0$, the event $\{\sup_{u \in [0, b]} [\bar{R}(u) - R(u)] < 1\}$ means $\{\tau_1^R > b\}$, so

$$L = \left(\frac{\pi}{8} \right)^{1/2} \int_0^\infty \mathbb{E} \left[\frac{1}{R(b)} \mathbf{1}_{\{\tau_1^R > b\}} \right] db = \left(\frac{\pi}{8} \right)^{1/2} \mathbb{E} \left(\int_0^{\tau_1^R} \frac{1}{R(b)} db \right),$$

the second identity following from the Fubini–Tonelli theorem. According to a relation between Bessel processes of dimensions three and four (Revuz and Yor [9], Proposition XI.1.11, applied to the parameters $p = q = 2$ and $\nu = \frac{1}{2}$),

$$R(t) = U \left(\frac{1}{4} \int_0^t \frac{1}{R(b)} db \right), \quad t \geq 0,$$

where $U := (U(s), s \geq 0)$ is a four-dimensional *squared* Bessel process with $U(0) = 0$; in other words, U is the square of the Euclidean modulus of a standard four-dimensional Brownian motion.

Let us introduce the increasing functional $\sigma(t) := \frac{1}{4} \int_0^t \frac{1}{R(b)} db$, $t \geq 0$. We have $R = U \circ \sigma$, and

$$\begin{aligned} \tau_1^R &= \inf \{ t \geq 0 : \bar{R}(t) - R(t) \geq 1 \} \\ &= \inf \{ t \geq 0 : \bar{U}(\sigma(t)) - U(\sigma(t)) \geq 1 \} \\ &= \inf \{ \sigma^{-1}(s) : s \geq 0 \text{ and } \bar{U}(s) - U(s) \geq 1 \} \end{aligned}$$

which is $\sigma^{-1}(\tau_1^U)$. So $\tau_1^U = \sigma(\tau_1^R)$, i.e.,

$$\int_0^{\tau_1^R} \frac{1}{R(b)} db = 4\tau_1^U,$$

which implies that

$$L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U).$$

The Laplace transform of τ_1^U is determined by Lehoczky [6], from which, however, it does not seem obvious to deduce the value of $\mathbb{E}(\tau_1^U)$. Instead of using Lehoczky's result directly, we rather apply his method to compute $\mathbb{E}(\tau_1^U)$. By Itô's formula, $(U(t) - 4t, t \geq 0)$ is a continuous martingale, with quadratic variation $4 \int_0^t U(s) ds$; so applying the Dambis–Dubins–Schwarz theorem (Revuz and Yor [9], Theorem V.1.6) to $(U(t) - 4t, t \geq 0)$ yields the existence of a standard Brownian motion $B = (B(t), t \geq 0)$ such that

$$U(t) = 2B\left(\int_0^t U(s) ds\right) + 4t, \quad t \geq 0.$$

Taking $t := \tau_1^U$, we get

$$U(\tau_1^U) = 2B\left(\int_0^{\tau_1^U} U(s) ds\right) + 4\tau_1^U.$$

We claim that

$$(2.1) \quad \mathbb{E}\left[B\left(\int_0^{\tau_1^U} U(s) ds\right)\right] = 0.$$

Then $\mathbb{E}(\tau_1^U) = \frac{1}{4} \mathbb{E}[U(\tau_1^U)]$; hence

$$(2.2) \quad L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U) = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)].$$

Let us admit (2.1) for the moment, and prove the theorem by computing $\mathbb{E}[U(\tau_1^U)]$ using Lehoczky [6]'s method; in fact, we determine the law of $U(\tau_1^U)$.

Lemma 2.2. *The law of $U(\tau_1^U)$ is given by*

$$\mathbb{P}\{U(\tau_1^U) > a\} = (a + 1)e^{-a}, \quad \forall a > 0.$$

In particular,

$$\mathbb{E}[U(\tau_1^U)] = \int_0^\infty (a + 1)e^{-a} da = 2.$$

Since $L = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)]$ (see (2.2)), this yields $L = \left(\frac{\pi}{2}\right)^{1/2}$ as stated in Theorem 1.1.

The rest of the note is devoted to the proof of Lemma 2.2 and (2.1).

Proof of Lemma 2.2. Fix $b > 1$. We compute the probability $\mathbb{P}\{\bar{U}(\tau_1^U) > b\}$ which, due to the equality $\bar{U}(\tau_1^U) = U(\tau_1^U) + 1$, coincides with $\mathbb{P}\{U(\tau_1^U) > b - 1\}$. By applying the strong Markov property at time $\sigma_0^U := \inf\{t \geq 0 : U(t) = 1\}$, we see that the value of $\mathbb{P}\{\bar{U}(\tau_1^U) > b\}$ does not change if the squared Bessel process U starts at $U(0) = 1$. Indeed, observing that $\sigma_0^U \leq \tau_1^U$, $U(\sigma_0^U) = 1$ and that $\bar{U}(\tau_1^U) = \sup_{s \in [\sigma_0^U, \tau_1^U]} U(s)$, we have

$$\mathbb{P}\{\bar{U}(\tau_1^U) > b\} = \mathbb{P}\left\{ \sup_{s \in [\sigma_0^U, \tau_1^U]} U(s) > b \right\} = \mathbb{P}_1\{\bar{U}(\tau_1^U) > b\},$$

the subscript 1 in \mathbb{P}_1 indicating the initial value of U . More generally, for $x \geq 0$, we write $\mathbb{P}_x(\bullet) := \mathbb{P}(\bullet | U(0) = x)$; so $\mathbb{P} = \mathbb{P}_0$.

Let $b_0 = 1 < b_1 < \dots < b_n := b$ be a subdivision of $[1, b]$ such that $\max_{1 \leq i \leq n} (b_i - b_{i-1}) \rightarrow 0$, $n \rightarrow \infty$. Consider the event $\{\bar{U}(\tau_1^U) > b\}$: since $U(0) = 1$, this means U hits position b before time τ_1^U ; for all $i \in [1, n-1] \cap \mathbb{Z}$, starting from position b_i , U must hit b_{i+1} before hitting $b_i - 1$ (caution: not to be confused with b_{i-1}). More precisely, let $\sigma_i^U := \inf\{t \geq 0 : U(t) = b_i\}$ and let $U_i(s) := U(s + \sigma_i^U)$, $s \geq 0$; then

$$\{\bar{U}(\tau_1^U) > b\} \subset \bigcap_{i=1}^{n-1} \{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$

By the strong Markov property, the events $\{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}$, $1 \leq i \leq n-1$, are independent (caution : the processes $(U_i(s), s \geq 0)$, $1 \leq i \leq n-1$, are not independent). Hence

$$(2.3) \quad \mathbb{P}_1\{\bar{U}(\tau_1^U) > b\} \leq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$

Conversely, let $\varepsilon > 0$, and if $\max_{1 \leq i \leq n} (b_i - b_{i-1}) < \varepsilon$, then we also have

$$\mathbb{P}_1\{\bar{U}(\tau_{1+\varepsilon}^U) > b\} \geq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\},$$

with $\tau_{1+\varepsilon}^U := \inf\{t \geq 0 : \bar{U}(t) - U(t) \geq 1 + \varepsilon\}$. By scaling, $\bar{U}(\tau_{1+\varepsilon}^U)$ has the same distribution as $(1 + \varepsilon)\bar{U}(\tau_1^U)$. So, as long as $\max_{1 \leq i \leq n} (b_i - b_{i-1}) < \varepsilon$, we have

$$\mathbb{P}_1\{\bar{U}(\tau_1^U) > b\} \leq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} \leq \mathbb{P}_1\{\bar{U}(\tau_1^U) > \frac{b}{1 + \varepsilon}\}.$$

Since $\frac{1}{x}$ is a scale function for U , we have

$$\mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} = \frac{\frac{1}{b_{i-1}} - \frac{1}{b_i}}{\frac{1}{b_{i-1}} - \frac{1}{b_{i+1}}} = 1 - \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{\frac{1}{b_{i-1}} - \frac{1}{b_{i+1}}}.$$

If $\lim_{n \rightarrow \infty} \max_{0 \leq i \leq n-1} (b_{i+1} - b_i) = 0$, then for $n \rightarrow \infty$,

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{\frac{1}{b_{i-1}} - \frac{1}{b_{i+1}}} &= \sum_{i=1}^{n-1} \frac{b_i - 1}{b_i} (b_{i+1} - b_i) + o(1) \\ &\rightarrow \int_1^b \frac{r-1}{r} dr \\ &= b - 1 - \log b. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} = e^{-(b-1-\log b)} = b e^{-(b-1)}.$$

Consequently,

$$\mathbb{P}\{\bar{U}(\tau_1^U) > b\} = b e^{-(b-1)}, \quad \forall b > 1.$$

We have already noted that $U(\tau_1^U) = \bar{U}(\tau_1^U) - 1$. This completes the proof of Lemma 2.2.

□

Proof of (2.1). The Brownian motion B being the Dambis–Dubins–Schwarz Brownian motion associated with the continuous martingale $(U(t) - 4t, t \geq 0)$, it is a $(\mathcal{G}_r)_{r \geq 0}$ -Brownian motion (Revuz and Yor [9], Theorem V.1.6), where, for $r \geq 0$,

$$\mathcal{G}_r := \mathcal{F}_{C(r)}, \quad C(r) := A^{-1}(r), \quad A(t) := \int_0^t U(s) ds,$$

and A^{-1} denotes the inverse of A . [We mention that $\mathcal{F}_{C(r)}$ is well defined because $C(r)$ is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time.] As such,

$$\int_0^{\tau_1^U} U(s) ds = A(\tau_1^U).$$

For all $r \geq 0$, $\{A(\tau_1^U) > r\} = \{\tau_1^U > C(r)\} \in \mathcal{F}_{C(r)} = \mathcal{G}_r$ (observing that τ_1^U is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time), which means that $A(\tau_1^U)$ is a $(\mathcal{G}_r)_{r \geq 0}$ -stopping time. If $A(\tau_1^U) = \int_0^{\tau_1^U} U(s) ds$ has a finite expectation, then we are entitled to apply the (first) Wald identity to see that $\mathbb{E}[B(A(\tau_1^U))] = 0$ as claimed in (2.1).

It remains to prove that $\mathbb{E}[A(\tau_1^U)] < \infty$.

Recall that U is the square of the Euclidean modulus of an \mathbb{R}^4 -valued Brownian motion. By considering only the first coordinate of this Brownian motion, say β , we have

$$\mathbb{P}\left\{\sup_{s \in [0, a]} U(s) < a^{1-\varepsilon}\right\} \leq \mathbb{P}\left\{\sup_{s \in [0, a]} |\beta(s)| < a^{(1-\varepsilon)/2}\right\} = \mathbb{P}\left\{\sup_{s \in [0, 1]} |\beta(s)| < a^{-\varepsilon/2}\right\};$$

so by the small ball probability for Brownian motion, we obtain:

$$\mathbb{P}\left\{\sup_{s \in [0, a]} U(s) < a^{1-\varepsilon}\right\} \leq \exp(-c_1 a^\varepsilon),$$

for all $a \geq 1$ et all $\varepsilon \in (0, 1)$, with some constant $c_1 = c_1(\varepsilon) > 0$. On the event $\{\sup_{s \in [0, a]} U(s) \geq a^{1-\varepsilon}\}$, if $\tau_1^U > a$, then for all $i \in [1, a^{1-\varepsilon} - 1] \cap \mathbb{Z}$, the squared Bessel process U , starting from i , must first hit position $i + 1$ before hitting $i - 1$ (which, for each i , can be realized with probability $\leq 1 - c_2$, where $c_2 \in (0, 1)$ is a constant that does not depend on i , nor on a). Accordingly,¹

$$\mathbb{P}\left\{\sup_{s \in [0, a]} U(s) \geq a^{1-\varepsilon}, \tau_1^U > a\right\} \leq (1 - c_2)^{\lfloor a^{1-\varepsilon} - 1 \rfloor} \leq \exp(-c_3 a^{1-\varepsilon}),$$

with some constant $c_3 > 0$, uniformly in $a \geq 2$. We have thus proved that for all $a \geq 2$ and all $\varepsilon \in (0, 1)$,

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_3 a^{1-\varepsilon}) + \exp(-c_1 a^\varepsilon).$$

Taking $\varepsilon := \frac{1}{2}$, we see that there exists a constant $c_4 > 0$ such that

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_4 a^{1/2}), \quad \forall a \geq 2.$$

On the other hand, U being a squared Bessel process, we have, for all $a > 0$ and all $b \geq a^2$,

$$\mathbb{P}\{A(a) \geq b\} = \mathbb{P}\{A(1) \geq \frac{b}{a^2}\} \leq \mathbb{P}\left\{\sup_{s \in [0, 1]} U(s) \geq \frac{b}{a^2}\right\} \leq e^{-c_5 b/a^2},$$

for some constant $c_5 > 0$. Hence, for $b \geq a^2$ and $a \geq 2$,

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \mathbb{P}\{\tau_1^U > a\} + \mathbb{P}\{A(a) \geq b\} \leq \exp(-c_4 a^{1/2}) + e^{-c_5 b/a^2}.$$

Taking $a := b^{2/5}$ gives that

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \exp(-c_6 b^{1/5}),$$

for some constant $c_6 > 0$ and all $b \geq 4$. In particular, $\mathbb{E}[A(\tau_1^U)] < \infty$ as desired. \square

¹This is the special case $b_i := i$ of the argument we have used to obtain (2.3).

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