



The maximal drawdown of the Brownian meander

Yueyun Hu, Zhan Shi, Marc Yor

► To cite this version:

Yueyun Hu, Zhan Shi, Marc Yor. The maximal drawdown of the Brownian meander. Electronic Communications in Probability, 2015, 20 (39), 10.1214/ECP.v20-3945 . hal-01301802

HAL Id: hal-01301802

<https://hal.science/hal-01301802>

Submitted on 13 Apr 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The maximal drawdown of the Brownian meander

by

Yueyun Hu, Zhan Shi and Marc Yor[†]

Université Paris XIII, Université Paris VI & Université Paris VI

Summary. Motivated by evaluating the limiting distribution of randomly biased random walks on trees, we compute the exact value of a negative moment of the maximal drawdown of the standard Brownian meander.

Keywords. Brownian meander, Bessel process, maximal drawdown.

2010 Mathematics Subject Classification. 60J65.

1 Introduction

Let $(X(t), t \in [0, 1])$ be a random process. Its maximal drawdown on $[0, 1]$ is defined by

$$X^\#(1) := \sup_{s \in [0, 1]} [\overline{X}(s) - X(s)],$$

where $\overline{X}(s) := \sup_{u \in [0, s]} X(u)$. There has been some recent research interest on the study of drawdowns from probabilistic point of view ([7], [8]) as well as applications in insurance and finance ([1], [2], [3], [10], [12]).

We are interested in the maximal drawdown $\mathfrak{m}^\#(1)$ of the standard Brownian meander $(\mathfrak{m}(t), t \in [0, 1])$. Our motivation is the presence of the law of $\mathfrak{m}^\#(1)$ in the limiting distribution of randomly biased random walks on supercritical Galton–Watson trees ([4]); in particular, the value of $\mathbb{E}(\frac{1}{\mathfrak{m}^\#(1)})$ is the normalizing constant in the density function of

Partly supported by ANR project MEMEMO2 (2010-BLAN-0125).

this limiting distribution. The sole aim of the present note is to compute $\mathbb{E}(\frac{1}{\mathfrak{m}^\#(1)})$, which turns out to have a nice numerical value.

Let us first recall the definition of the Brownian meander. Let $W := (W(t), t \in [0, 1])$ be a standard Brownian motion, and let $\mathfrak{g} := \sup\{t \leq 1 : W(t) = 0\}$ be the last passage time at 0 before time 1. Since $\mathfrak{g} < 1$ a.s., we can define

$$\mathfrak{m}(s) := \frac{|W(\mathfrak{g} + s(1 - \mathfrak{g}))|}{(1 - \mathfrak{g})^{1/2}}, \quad s \in [0, 1].$$

The law of $(\mathfrak{m}(s), s \in [0, 1])$ is called the law of the standard Brownian meander. For an account of general properties of the Brownian meander, see Yen and Yor [11].

Theorem 1.1. *Let $(\mathfrak{m}(s), s \in [0, 1])$ be a standard Brownian meander. We have*

$$(1.1) \quad \mathbb{E}\left(\frac{1}{\sup_{s \in [0, 1]} [\bar{\mathfrak{m}}(s) - \mathfrak{m}(s)]}\right) = \left(\frac{\pi}{2}\right)^{1/2},$$

where $\bar{\mathfrak{m}}(s) := \sup_{u \in [0, s]} \mathfrak{m}(u)$.

The theorem is proved in Section 2.

We are grateful to an anonymous referee for a careful reading of the manuscript and for many suggestions for improvements.

N.B. from the first-named coauthors: This note originates from a question we asked our teacher, **Professor Marc Yor (1949–2014)**, who passed away in January 2014, during the preparation of this note. He provided us, in November 2012, with the essential of the material in Section 2.

2 Proof

Let $R := (R(t), t \geq 0)$ be a three-dimensional Bessel process with $R(0) = 0$, i.e., the Euclidean modulus of a standard three-dimensional Brownian motion. The proof of Theorem 1.1 relies on an absolute continuity relation between $(\mathfrak{m}(s), s \in [0, 1])$ and $(R(s), s \in [0, 1])$, recalled as follows.

Fact 2.1. (Imhof [5]) *Let $(\mathfrak{m}(s), s \in [0, 1])$ be a standard Brownian meander. Let $(R(s), s \in [0, 1])$ be a three-dimensional Bessel process with $R(0) = 0$. For any measurable and non-negative functional F , we have*

$$\mathbb{E}\left[F(\mathfrak{m}(s), s \in [0, 1])\right] = \left(\frac{\pi}{2}\right)^{1/2} \mathbb{E}\left[\frac{1}{R(1)} F(R(s), s \in [0, 1])\right].$$

We now proceed to the proof of Theorem 1.1. Let

$$L := \mathbb{E} \left(\frac{1}{\sup_{s \in [0, 1]} [\overline{\mathbf{m}}(s) - \mathbf{m}(s)]} \right).$$

Write $\overline{R}(t) := \sup_{u \in [0, t]} R(u)$ for $t \geq 0$. By Fact 2.1,

$$\begin{aligned} L &= \left(\frac{\pi}{2} \right)^{1/2} \mathbb{E} \left[\frac{1}{R(1)} \frac{1}{\sup_{s \in [0, 1]} [\overline{R}(s) - R(s)]} \right] \\ &= \left(\frac{\pi}{2} \right)^{1/2} \int_0^\infty \mathbb{E} \left[\frac{1}{R(1)} \mathbf{1}_{\{\sup_{s \in [0, 1]} [\overline{R}(s) - R(s)] < \frac{1}{a}\}} \right] da, \end{aligned}$$

the last equality following from the Fubini–Tonelli theorem. By the scaling property, $\mathbb{E} \left[\frac{1}{R(1)} \mathbf{1}_{\{\sup_{s \in [0, 1]} [\overline{R}(s) - R(s)] < \frac{1}{a}\}} \right] = \mathbb{E} \left[\frac{a}{R(a^2)} \mathbf{1}_{\{\sup_{u \in [0, a^2]} [\overline{R}(u) - R(u)] < 1\}} \right]$ for all $a > 0$. So by means of a change of variables $b = a^2$, we obtain:

$$L = \left(\frac{\pi}{8} \right)^{1/2} \int_0^\infty \mathbb{E} \left[\frac{1}{R(b)} \mathbf{1}_{\{\sup_{u \in [0, b]} [\overline{R}(u) - R(u)] < 1\}} \right] db.$$

Define, for any random process X ,

$$\tau_1^X := \inf \{ t \geq 0 : \overline{X}(t) - X(t) \geq 1 \},$$

with $\overline{X}(t) := \sup_{s \in [0, t]} X(s)$. For any $b > 0$, the event $\{\sup_{u \in [0, b]} [\overline{R}(u) - R(u)] < 1\}$ means $\{\tau_1^R > b\}$, so

$$L = \left(\frac{\pi}{8} \right)^{1/2} \int_0^\infty \mathbb{E} \left[\frac{1}{R(b)} \mathbf{1}_{\{\tau_1^R > b\}} \right] db = \left(\frac{\pi}{8} \right)^{1/2} \mathbb{E} \left(\int_0^{\tau_1^R} \frac{1}{R(b)} db \right),$$

the second identity following from the Fubini–Tonelli theorem. According to a relation between Bessel processes of dimensions three and four (Revuz and Yor [9], Proposition XI.1.11, applied to the parameters $p = q = 2$ and $\nu = \frac{1}{2}$),

$$R(t) = U \left(\frac{1}{4} \int_0^t \frac{1}{R(b)} db \right), \quad t \geq 0,$$

where $U := (U(s), s \geq 0)$ is a four-dimensional *squared* Bessel process with $U(0) = 0$; in other words, U is the square of the Euclidean modulus of a standard four-dimensional Brownian motion.

Let us introduce the increasing functional $\sigma(t) := \frac{1}{4} \int_0^t \frac{1}{R(b)} db$, $t \geq 0$. We have $R = U \circ \sigma$, and

$$\begin{aligned} \tau_1^R &= \inf \{ t \geq 0 : \overline{R}(t) - R(t) \geq 1 \} \\ &= \inf \{ t \geq 0 : \overline{U}(\sigma(t)) - U(\sigma(t)) \geq 1 \} \\ &= \inf \{ \sigma^{-1}(s) : s \geq 0 \text{ and } \overline{U}(s) - U(s) \geq 1 \} \end{aligned}$$

which is $\sigma^{-1}(\tau_1^U)$. So $\tau_1^U = \sigma(\tau_1^R)$, i.e.,

$$\int_0^{\tau_1^R} \frac{1}{R(b)} db = 4\tau_1^U,$$

which implies that

$$L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U).$$

The Laplace transform of τ_1^U is determined by Lehoczký [6], from which, however, it does not seem obvious to deduce the value of $\mathbb{E}(\tau_1^U)$. Instead of using Lehoczký's result directly, we rather apply his method to compute $\mathbb{E}(\tau_1^U)$. By Itô's formula, $(U(t) - 4t, t \geq 0)$ is a continuous martingale, with quadratic variation $4 \int_0^t U(s) ds$; so applying the Dambis–Dubins–Schwarz theorem (Revuz and Yor [9], Theorem V.1.6) to $(U(t) - 4t, t \geq 0)$ yields the existence of a standard Brownian motion $B = (B(t), t \geq 0)$ such that

$$U(t) = 2B\left(\int_0^t U(s) ds\right) + 4t, \quad t \geq 0.$$

Taking $t := \tau_1^U$, we get

$$U(\tau_1^U) = 2B\left(\int_0^{\tau_1^U} U(s) ds\right) + 4\tau_1^U.$$

We claim that

$$(2.1) \quad \mathbb{E}\left[B\left(\int_0^{\tau_1^U} U(s) ds\right)\right] = 0.$$

Then $\mathbb{E}(\tau_1^U) = \frac{1}{4} \mathbb{E}[U(\tau_1^U)]$; hence

$$(2.2) \quad L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U) = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)].$$

Let us admit (2.1) for the moment, and prove the theorem by computing $\mathbb{E}[U(\tau_1^U)]$ using Lehoczký [6]'s method; in fact, we determine the law of $U(\tau_1^U)$.

Lemma 2.2. *The law of $U(\tau_1^U)$ is given by*

$$\mathbb{P}\{U(\tau_1^U) > a\} = (a+1)e^{-a}, \quad \forall a > 0.$$

In particular,

$$\mathbb{E}[U(\tau_1^U)] = \int_0^\infty (a+1)e^{-a} da = 2.$$

Since $L = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)]$ (see (2.2)), this yields $L = \left(\frac{\pi}{2}\right)^{1/2}$ as stated in Theorem 1.1.

The rest of the note is devoted to the proof of Lemma 2.2 and (2.1).

Proof of Lemma 2.2. Fix $b > 1$. We compute the probability $\mathbb{P}\{\overline{U}(\tau_1^U) > b\}$ which, due to the equality $\overline{U}(\tau_1^U) = U(\tau_1^U) + 1$, coincides with $\mathbb{P}\{U(\tau_1^U) > b - 1\}$. By applying the strong Markov property at time $\sigma_0^U := \inf\{t \geq 0 : U(t) = 1\}$, we see that the value of $\mathbb{P}\{\overline{U}(\tau_1^U) > b\}$ does not change if the squared Bessel process U starts at $U(0) = 1$. Indeed, observing that $\sigma_0^U \leq \tau_1^U$, $U(\sigma_0^U) = 1$ and that $\overline{U}(\tau_1^U) = \sup_{s \in [\sigma_0^U, \tau_1^U]} U(s)$, we have

$$\mathbb{P}\{\overline{U}(\tau_1^U) > b\} = \mathbb{P}\left\{\sup_{s \in [\sigma_0^U, \tau_1^U]} U(s) > b\right\} = \mathbb{P}_1\{\overline{U}(\tau_1^U) > b\},$$

the subscript 1 in \mathbb{P}_1 indicating the initial value of U . More generally, for $x \geq 0$, we write $\mathbb{P}_x(\bullet) := \mathbb{P}(\bullet | U(0) = x)$; so $\mathbb{P} = \mathbb{P}_0$.

Let $b_0 = 1 < b_1 < \dots < b_n := b$ be a subdivision of $[1, b]$ such that $\max_{1 \leq i \leq n} (b_i - b_{i-1}) \rightarrow 0$, $n \rightarrow \infty$. Consider the event $\{\overline{U}(\tau_1^U) > b\}$: since $U(0) = 1$, this means U hits position b before time τ_1^U ; for all $i \in [1, n-1] \cap \mathbb{Z}$, starting from position b_i , U must hit b_{i+1} before hitting $b_i - 1$ (caution: not to be confused with b_{i-1}). More precisely, let $\sigma_i^U := \inf\{t \geq 0 : U(t) = b_i\}$ and let $U_i(s) := U(s + \sigma_i^U)$, $s \geq 0$; then

$$\{\overline{U}(\tau_1^U) > b\} \subset \bigcap_{i=1}^{n-1} \{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$

By the strong Markov property, the events $\{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}$, $1 \leq i \leq n-1$, are independent (caution : the processes $(U_i(s), s \geq 0)$, $1 \leq i \leq n-1$, are not independent). Hence

$$(2.3) \quad \mathbb{P}_1\{\overline{U}(\tau_1^U) > b\} \leq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$

Conversely, let $\varepsilon > 0$, and if $\max_{1 \leq i \leq n} (b_i - b_{i-1}) < \varepsilon$, then we also have

$$\mathbb{P}_1\{\overline{U}(\tau_{1+\varepsilon}^U) > b\} \geq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\},$$

with $\tau_{1+\varepsilon}^U := \inf\{t \geq 0 : \overline{U}(t) - U(t) \geq 1 + \varepsilon\}$. By scaling, $\overline{U}(\tau_{1+\varepsilon}^U)$ has the same distribution as $(1 + \varepsilon)\overline{U}(\tau_1^U)$. So, as long as $\max_{1 \leq i \leq n} (b_i - b_{i-1}) < \varepsilon$, we have

$$\mathbb{P}_1\{\overline{U}(\tau_1^U) > b\} \leq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} \leq \mathbb{P}_1\{\overline{U}(\tau_1^U) > \frac{b}{1 + \varepsilon}\}.$$

Since $\frac{1}{x}$ is a scale function for U , we have

$$\mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} = \frac{\frac{1}{b_i-1} - \frac{1}{b_i}}{\frac{1}{b_i-1} - \frac{1}{b_{i+1}}} = 1 - \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{\frac{1}{b_i-1} - \frac{1}{b_{i+1}}}.$$

If $\lim_{n \rightarrow \infty} \max_{0 \leq i \leq n-1} (b_{i+1} - b_i) = 0$, then for $n \rightarrow \infty$,

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{\frac{1}{b_i-1} - \frac{1}{b_{i+1}}} &= \sum_{i=1}^{n-1} \frac{b_i - 1}{b_i} (b_{i+1} - b_i) + o(1) \\ &\rightarrow \int_1^b \frac{r-1}{r} dr \\ &= b - 1 - \log b. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} = e^{-(b-1-\log b)} = b e^{-(b-1)}.$$

Consequently,

$$\mathbb{P}\{\bar{U}(\tau_1^U) > b\} = b e^{-(b-1)}, \quad \forall b > 1.$$

We have already noted that $U(\tau_1^U) = \bar{U}(\tau_1^U) - 1$. This completes the proof of Lemma 2.2.

□

Proof of (2.1). The Brownian motion B being the Dambis–Dubins–Schwarz Brownian motion associated with the continuous martingale $(U(t)-4t, t \geq 0)$, it is a $(\mathcal{G}_r)_{r \geq 0}$ -Brownian motion (Revuz and Yor [9], Theorem V.1.6), where, for $r \geq 0$,

$$\mathcal{G}_r := \mathcal{F}_{C(r)}, \quad C(r) := A^{-1}(r), \quad A(t) := \int_0^t U(s) ds,$$

and A^{-1} denotes the inverse of A . [We mention that $\mathcal{F}_{C(r)}$ is well defined because $C(r)$ is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time.] As such,

$$\int_0^{\tau_1^U} U(s) ds = A(\tau_1^U).$$

For all $r \geq 0$, $\{A(\tau_1^U) > r\} = \{\tau_1^U > C(r)\} \in \mathcal{F}_{C(r)} = \mathcal{G}_r$ (observing that τ_1^U is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time), which means that $A(\tau_1^U)$ is a $(\mathcal{G}_r)_{r \geq 0}$ -stopping time. If $A(\tau_1^U) = \int_0^{\tau_1^U} U(s) ds$ has a finite expectation, then we are entitled to apply the (first) Wald identity to see that $\mathbb{E}[B(A(\tau_1^U))] = 0$ as claimed in (2.1).

It remains to prove that $\mathbb{E}[A(\tau_1^U)] < \infty$.

Recall that U is the square of the Euclidean modulus of an \mathbb{R}^4 -valued Brownian motion. By considering only the first coordinate of this Brownian motion, say β , we have

$$\mathbb{P}\left\{\sup_{s \in [0, a]} U(s) < a^{1-\varepsilon}\right\} \leq \mathbb{P}\left\{\sup_{s \in [0, a]} |\beta(s)| < a^{(1-\varepsilon)/2}\right\} = \mathbb{P}\left\{\sup_{s \in [0, 1]} |\beta(s)| < a^{-\varepsilon/2}\right\};$$

so by the small ball probability for Brownian motion, we obtain:

$$\mathbb{P}\left\{\sup_{s \in [0, a]} U(s) < a^{1-\varepsilon}\right\} \leq \exp(-c_1 a^\varepsilon),$$

for all $a \geq 1$ et all $\varepsilon \in (0, 1)$, with some constant $c_1 = c_1(\varepsilon) > 0$. On the event $\{\sup_{s \in [0, a]} U(s) \geq a^{1-\varepsilon}\}$, if $\tau_1^U > a$, then for all $i \in [1, a^{1-\varepsilon} - 1] \cap \mathbb{Z}$, the squared Bessel process U , starting from i , must first hit position $i + 1$ before hitting $i - 1$ (which, for each i , can be realized with probability $\leq 1 - c_2$, where $c_2 \in (0, 1)$ is a constant that does not depend on i , nor on a). Accordingly,¹

$$\mathbb{P}\left\{\sup_{s \in [0, a]} U(s) \geq a^{1-\varepsilon}, \tau_1^U > a\right\} \leq (1 - c_2)^{\lfloor a^{1-\varepsilon} - 1 \rfloor} \leq \exp(-c_3 a^{1-\varepsilon}),$$

with some constant $c_3 > 0$, uniformly in $a \geq 2$. We have thus proved that for all $a \geq 2$ and all $\varepsilon \in (0, 1)$,

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_3 a^{1-\varepsilon}) + \exp(-c_1 a^\varepsilon).$$

Taking $\varepsilon := \frac{1}{2}$, we see that there exists a constant $c_4 > 0$ such that

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_4 a^{1/2}), \quad \forall a \geq 2.$$

On the other hand, U being a squared Bessel process, we have, for all $a > 0$ and all $b \geq a^2$,

$$\mathbb{P}\{A(a) \geq b\} = \mathbb{P}\{A(1) \geq \frac{b}{a^2}\} \leq \mathbb{P}\left\{\sup_{s \in [0, 1]} U(s) \geq \frac{b}{a^2}\right\} \leq e^{-c_5 b/a^2},$$

for some constant $c_5 > 0$. Hence, for $b \geq a^2$ and $a \geq 2$,

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \mathbb{P}\{\tau_1^U > a\} + \mathbb{P}\{A(a) \geq b\} \leq \exp(-c_4 a^{1/2}) + e^{-c_5 b/a^2}.$$

Taking $a := b^{2/5}$ gives that

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \exp(-c_6 b^{1/5}),$$

for some constant $c_6 > 0$ and all $b \geq 4$. In particular, $\mathbb{E}[A(\tau_1^U)] < \infty$ as desired. \square

¹This is the special case $b_i := i$ of the argument we have used to obtain (2.3).

References

- [1] Carraro, L., El Karoui, N. and Obłój, J. (2012). On Azéma-Yor processes, their optimal properties and the Bachelier-drawdown equation. *Ann. Probab.* **40**, 372–400.
- [2] Cheridito, P., Nikeghbali, A. and Platen, E. (2012). Processes of class sigma, last passage times, and drawdowns. *SIAM J. Financial Math.* **3**, 280–303.
- [3] Cherny, V. and Obłój, J. (2013). Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model. *Finance Stoch.* **17**, 771–800.
- [4] Hu, Y. and Shi, Z. (2015+). The slow regime of randomly biased walks on trees. [arXiv:1501.07700](https://arxiv.org/abs/1501.07700)
- [5] Imhof, J.-P. (1984). Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.* **21**, 500–510.
- [6] Lehoczky, J.P. (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Probab.* **5**, 601–607.
- [7] Mijatović, A. and Pistorius, M.R. (2012). On the drawdown of completely asymmetric Lévy processes. *Stoch. Proc. Appl.* **122**, 3812–3836.
- [8] Nikeghbali, A. (2006). A class of remarkable submartingales. *Stoch. Proc. Appl.* **116**, 917–938.
- [9] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*. (Third edition). Springer, Berlin.
- [10] Rieder, U. and Wittlinger, M. (2014). On optimal terminal wealth problems with random trading times and drawdown constraints. *Adv. Appl. Probab.* **46**, 121–138.
- [11] Yen, J.-Y. and Yor, M. (2013). *Local Times and Excursion Theory for Brownian Motion. A Tale of Wiener and Itô Measures*. Lecture Notes in Mathematics **2088**. Springer, Berlin.
- [12] Zhang, H. and Hadjiladis, O. (2012). Drawdowns and the speed of market crash. *Methodol. Comput. Appl. Probab.* **12**, 293–308.

Yueyun Hu
 Département de Mathématiques
 Université Paris XIII
 99 avenue J-B Clément
 F-93430 Villetaneuse
 France
yueyun@math.univ-paris13.fr

Zhan Shi
 LPMA, Case 188
 Université Paris VI
 4 place Jussieu
 F-75252 Paris Cedex 05
 France
zhan.shi@upmc.fr

Marc Yor[†]
 LPMA, Case 188 & IUF
 Université Paris VI
 4 place Jussieu
 F-75252 Paris Cedex 05
 France
my19492014@gmail.com