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SMOOTH FAMILY OF TORI AND LINEAR KÄHLER GROUPS

BENOÎT CLAUDON

ABSTRACT. This short note, meant as an *addendum* to [CCE14], enhances the results contained in *loc. cit.* In particular it is proven here that a linear Kähler group is already the fundamental group of a smooth complex projective variety. This is achieved studying the relative deformation of the total space of a smooth family of tori in an equivariant context.

1. INTRODUCTION

In his seminal paper on compact complex surfaces [Kod63], Kodaira proved that a compact Kähler surface can be deformed to an algebraic one (Theorem 16.1 in *loc. cit.*). However since the groundbreaking works of Voisin [Voi04, Voi06] we know that this is specific to the surface case: in dimension at least 4, there exists compact Kähler manifolds which do not have the cohomology algebra of a projective manifold (and in particular cannot be deformed to such an algebraic manifold). The examples of Voisin being bimeromorphic to a torus (or to a projective bundle over a torus), it leaves open the following question concerning the fundamental groups of compact Kähler manifolds (known as Kähler groups).

Question 1. *Can any Kähler group be realized as the fundamental group of a smooth complex projective variety? In other terms, is any Kähler group already a projective one?*

Going back to Kodaira's Theorem, Buchdahl gave another proof of this result in [Buc06, Buc08], providing by the way a useful criterion ensuring that a compact Kähler manifold can be approximated by projective ones. This criterion applies nicely to the case of smooth family¹ of tori (this was already observed in [CCE14]) and can even be used when the family is equivariant under the action of a finite group.

Theorem 1.1.

Let $f : X \rightarrow B$ be a smooth family of tori whose total space is Kähler and let us assume that f is equivariant with respect to the action of a finite group Γ on both X and B . Then there exists a smooth family of tori of the form

$$\mathcal{X} \xrightarrow{\pi} T \times B \xrightarrow{p_1} T$$

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¹As recalled in Paragraph 2.1, it is simply a holomorphic proper submersion whose fibres are complex tori.

with T connected and a point $t_0 \in T$ such that the family $\mathcal{X}_{t_0} := (\pi \circ p_T)^{-1}(t_0)$ is (isomorphic to) the initial one. This family has moreover the following properties:

- (i) the group Γ acts on \mathcal{X} ,
- (ii) the projection π is equivariant with respect to this action on \mathcal{X} and to the action on $T \times B$ induced by the trivial one on T ,
- (iii) there exists a point $t \in T$ such that $\mathcal{X}_t \rightarrow B$ has a multisection and its fibres are abelian varieties.

If the base B is compact (and thus Kähler) then the set T_{alg} of points $t \in T$ such that $\mathcal{X}_t \rightarrow B$ has a multisection and its fibres are abelian varieties is dense near t_0 .

The last sentence means that the closure of T_{alg} contains an open neighbourhood of t_0 . Up to shrinking T we can assume that $\overline{T_{alg}}$ is thus the whole of T .

In particular, Theorem 1.1 shows that the problem of approximating compact Kähler manifolds with projective ones has a positive answer in the case of smooth tori families.

Corollary 1.2.

Let X be a compact Kähler manifold and let us assume that there is a finite étale Galois cover $\tilde{X} \rightarrow X$ which is the total space of a smooth family of tori over a projective base (equivariant under the action of the Galois group). Then X can be approximated by projective manifolds: it is the central fibre of a smooth morphism $(\mathcal{X}, X) \rightarrow (T, t_0)$ (with T smooth) and the set of $t \in T$ such that \mathcal{X}_t is projective is dense near t_0 .

Proof. We can apply Theorem 1.1 to the smooth family of tori $f : \tilde{X} \rightarrow B$ and to action of $\Gamma := \text{Gal}(\tilde{X}/X)$. We get a smooth deformation $\tilde{\mathcal{X}} \rightarrow T \times B$ of the initial family (over t_0) and we can assume the set of points $t \in T$ such that $\tilde{\mathcal{X}}_t \rightarrow B$ has a multisection and its fibres are abelian varieties is dense in T . The manifolds $\tilde{\mathcal{X}}_t$ having these properties are thus projective according to [Cam81]. Since the action of Γ is free on $\tilde{\mathcal{X}}_{t_0}$ we can assume that it is also free on $\tilde{\mathcal{X}}$ (up to shrinking T). The family $\mathcal{X} := \tilde{\mathcal{X}}/\Gamma \rightarrow T$ is thus a smooth deformation of $\mathcal{X}_{t_0} \simeq \tilde{X}/\Gamma \simeq X$ and the set of points $t \in T$ such that \mathcal{X}_t is projective is dense in T . \square

Mixing Theorem 1.1 with the structure results obtained in [CCE15], we get a definitive answer to Question 1 in the linear case.

Corollary 1.3.

A Kähler group which is linear is also a projective one: the fundamental group of a compact Kähler manifold can be realised as the fundamental group of a smooth projective variety if it is a linear group.

Proofs of Theorem 1.1 and Corollary 1.3 will be given in Paragraph 3.3.

Before presenting the ingredients involved in these proofs, let us give a word of explanation on the relative deformation constructed in Theorem 1.1 (the reader is advised to consult [Ser06, §3.4.2, p. 191] for the notions concerning relative deformations). The infinitesimal relative deformations of

a smooth morphism $f : X \rightarrow B$ are described by the space $H^1(X, T_{X|B})$ (cf. Lemma 3.4.7 in *loc. cit.*) and the Leray spectral sequence for $T_{X|B}$ and f gives a piece of exact sequence:

$$(1) \quad 0 \longrightarrow H^1(B, f_*T_{X|B}) \longrightarrow H^1(X, T_{X|B}) \longrightarrow H^0(B, R^1f_*T_{X|B}).$$

In our situation (smooth families of tori), both sides of (1) corresponds to a different type of relative deformation. The left-hand side parametrizes relative deformation using translations in the fibres of f (see the content of Proposition 2.4) whereas the right-hand side has to do with deformation of the variation of Hodge structures induced by f (these deformations are identified in Paragraph 3.2). In a sense, the strategy of the proof is thus dictated by the terms appearing in (1).

2. SMOOTH FAMILY OF TORI

We recall here some basic facts about smooth families of tori: their description as torsors and their deformations. We then put this study in an equivariant framework. Some facts recalled in Paragraph 2.1 already appear in [Nak99, §2]. Our reference concerning Hodge theory is [Voi02]. For more advanced stuffs on Jacobian fibrations, the reader is referred to [Sch12, BP13] and to the references therein.

2.1. Jacobian fibrations. Let $f : X \rightarrow B$ be a proper submersion between complex manifolds. We assume moreover that the fibres of f are complex tori. We shall call such a fibration a *smooth family of tori*. This fibration determines² a variation of Hodge structures \mathcal{H} (vhs for short) of weight -1 and rank $2g$ where $g := \dim(f)$ is the relative dimension of f : the underlying local system is

$$\mathbb{H}_{\mathbb{Z}} := \text{Hom}(R^1f_*\mathbb{Z}_X, \mathbb{Z}_B),$$

the Hodge filtration being given by

$$\mathcal{F} := \text{Hom}(R^1f_*\mathcal{O}_X, \mathcal{O}_B) \subset \mathbb{H}_{\mathbb{Z}} \otimes \mathcal{O}_B.$$

Let us remark that the duality

$$R^1f_*\mathbb{Z}_X \otimes R^{2g-1}f_*\mathbb{Z}_X \longrightarrow R^{2g}f_*\mathbb{Z}_X \simeq \mathbb{Z}_B$$

shows that $\mathbb{H}_{\mathbb{Z}}$ is isomorphic to $R^{2g-1}f_*\mathbb{Z}_X$.

With these data we can associate a particular family of tori. Let us consider the injection

$$\mathbb{H}_{\mathbb{Z}} \hookrightarrow \mathcal{E} := \mathbb{H}_{\mathbb{Z}} \otimes \mathcal{O}_B / \mathcal{F} \simeq f_*T_{X|B}$$

to this end. It can be used to define an action of $\mathbb{H}_{\mathbb{Z}}$ on the total space of \mathcal{E} and the quotient gives rise to a smooth family of tori which will be denoted

$$p : J(\mathcal{H}) \longrightarrow B$$

and called the *Jacobian fibration* associated with \mathcal{H} . This fibration comes endowed with a natural section (the image of the zero section of \mathcal{E}) and using it as the origins of the fibres we can define an abelian group law on

²Since in the sequel we will have to change the Hodge structure keeping the local system fixed, we will use calligraphic letters when referring to a vhs and straight ones to denote the underlying local system.

the sections of p . We have thus a short exact sequence of sheaves of abelian groups:

$$(2) \quad 0 \longrightarrow \mathbb{H}_{\mathbb{Z}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}(\mathcal{H}) \longrightarrow 0.$$

Let us say a word about polarizations (inspired from [Nak99, p. 15-17]). A real polarization of \mathcal{H} is a flat non degenerate skew-symmetric bilinear form

$$q : \mathbb{H}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}} \longrightarrow \mathbb{R}_B$$

satisfying the Hodge-Riemann relations:

$$q(\mathcal{F}, \mathcal{F}) = 0 \quad \text{and} \quad \forall 0 \neq x \in \mathcal{F}, \quad iq(x, \bar{x}) > 0.$$

The polarization is said to be rational if it defined on $\mathbb{H}_{\mathbb{Q}}$ (with values in \mathbb{Q}_B). If such a rational polarization exists, we shall say that \mathcal{H} is \mathbb{Q} -polarizable. In this case, the corresponding tori are abelian varieties.

Once such a polarization is fixed, the period domain of $(\mathbb{H}_{\mathbb{Z}}, q)$ can be identified with the Siegel half space

$$\mathbf{H}_g := \{ \tau \in M_g(\mathbb{C}) \mid \tau^t = \tau \text{ and } \Im m(\tau) > 0 \}$$

and the representation associated with the local system has its value in the symplectic group

$$\pi_1(B) \longrightarrow \mathrm{Sp}_g(\mathbb{Z}).$$

This can then be used to define an action of $\pi_1(B) \ltimes \mathbb{Z}^{2g}$ on $\tilde{B} \times \mathbb{C}^g$, the resulting quotient being another realization of the Jacobian fibration. In this case, the Jacobian fibration is endowed with a relative Kähler form ω_q : its restriction to any fiber is a Kähler metric. If q is rational, the fibration $J(\mathcal{H}) \rightarrow B$ is then a locally projective morphism.

In the reverse direction, starting from a smooth family of tori $f : X \rightarrow B$ inducing \mathcal{H} , it is obvious that a (relative) Kähler metric ω on X induces a real polarization q_ω on $\mathbb{H}_{\mathbb{R}}$.

2.2. Smooth families of tori as torsors. Now it is well known that the initial family $f : X \rightarrow B$ can be seen as a torsor under the Jacobian fibration and as such can be described by an element

$$\eta(f) \in H^1(B, \mathcal{J}(\mathcal{H})).$$

Here is a simple description of the class $\eta(f)$. If (U_i) is an open cover of B such that $f^{-1}(U_i) \rightarrow U_i$ has a section σ_i then the quantity $\eta_{ij} := \sigma_i - \sigma_j$ is a perfectly well defined cocycle with values in $\mathcal{J}(\mathcal{H})$. Reciprocally, given a cohomology class η represented by a cocycle (η_{ij}) , we can look at the isomorphisms induced by the sections η_{ij} (translations in the fibres):

$$\mathrm{tr}(\eta_{ij}) : p^{-1}(U_{ij}) \xrightarrow{\sim} p^{-1}(U_{ij})$$

define by the formulas:

$$\mathrm{tr}(\eta_{ij})(x) = x + \eta_{ij}(p(x))$$

(the addition referring to the one in $J(\mathcal{H})$). The isomorphisms $\mathrm{tr}(\eta_{ij})$ satisfy a cocycle relation and we can use them to glue the fibrations $p^{-1}(U_i) \rightarrow U_i$ into a new family of tori $J(\mathcal{H})^\eta \rightarrow B$ (and both mechanisms are inverse one to each other).

Proposition 2.1.

There is a one-to-one correspondence between smooth families of tori $f : X \rightarrow B$ inducing \mathcal{H} and the cohomology classes $\eta \in H^1(B, \mathcal{J}(\mathcal{H}))$. In particular, if $f : X \rightarrow B$ and $g : Y \rightarrow B$ are smooth families of tori inducing the same VHS on B , we can glue them over B to get a new family $h : Z \rightarrow B$ such that $\eta(h) = \eta(f) + \eta(g)$.

With this mind it is obvious that there always exist an étale morphism

$$J(\mathcal{H})^\eta \rightarrow J(\mathcal{H})^{m \cdot \eta}$$

for $\eta \in H^1(B, \mathcal{J}(\mathcal{H}))$ and $m \geq 1$ an integer (obtained by gluing the multiplication by m defined on the Jacobian fibration). In particular, this proves at least one implication of the following proposition.

Proposition 2.2.

Let $f : X \rightarrow B$ be a smooth family of tori (inducing the VHS \mathcal{H}). The class $\eta(f)$ is torsion in $H^1(B, \mathcal{J}(\mathcal{H}))$ if and only if f has a multisection. If it is the case, the multisection can be chosen étale over B .

Remark 2.3. Using relative Deligne groups (as in [Nak99, §2]), we can give an intrinsic definition of the class $\eta(f)$ associated with a family of tori $f : X \rightarrow B$. Let us look at the following complex:

$$(3) \quad \mathbb{Z}_{\mathcal{D}}^\bullet(X/B) : 0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{d_{X|B}} \Omega_{X|B}^1 \xrightarrow{d_{X|B}} \dots \xrightarrow{d_{X|B}} \Omega_{X|B}^{g-1}$$

where $d_{X|B}$ denotes the relative differential. The complex (3) sits obviously in the exact sequence

$$(4) \quad 0 \longrightarrow \Omega_{X|B}^{\leq g-1}[-1] \longrightarrow \mathbb{Z}_{\mathcal{D}}^\bullet(X/B) \longrightarrow \mathbb{Z}_X \longrightarrow 0$$

where the last term is the complex given by the constant sheaf concentrated in degree 0. Taking derived direct image of (4) yields a triangle:

$$(5) \quad \mathbb{R}f_* \Omega_{X|B}^{\leq g-1}[-1] \longrightarrow \mathbb{R}f_* \mathbb{Z}_{\mathcal{D}}^\bullet(X/B) \longrightarrow \mathbb{R}f_* \mathbb{Z}_X \xrightarrow{+1}.$$

On the other hand, we also have another triangle:

$$(6) \quad \mathbb{R}f_* \Omega_{X|B}^{\geq g} \longrightarrow \mathbb{R}f_* \Omega_{X|B}^\bullet \longrightarrow \mathbb{R}f_* \Omega_{X|B}^{\leq g-1} \xrightarrow{+1}$$

and the long exact sequence of cohomology associated with (6) shows that

$$(7) \quad H^k \mathbb{R}f_* \Omega_{X|B}^{\leq g-1} \simeq \left(R^k f_* \mathbb{C}_X \otimes \mathcal{O}_B \right) / F^g$$

where F^g is the g^{th} -step of the Hodge filtration on the VHS $R^k f_* \mathbb{C}_X$. Now looking at the long exact sequence associated with (5), we get:

$$R^{2g-1} f_* \mathbb{Z}_X \rightarrow H^{2g-1} \mathbb{R}f_* \Omega_{X|B}^{\leq g-1} \rightarrow H^{2g} \mathbb{R}f_* \mathbb{Z}_{\mathcal{D}}^\bullet(X/B) \rightarrow R^{2g} f_* \mathbb{Z}_X \rightarrow 0.$$

We can identify several terms in the sequence above: $R^{2g} f_* \mathbb{Z}_X$ is the constant sheaf \mathbb{Z}_B and $R^{2g-1} f_* \mathbb{Z}_X$ is nothing but $\mathbb{H}_{\mathbb{Z}}$. Using the isomorphism (7), the last piece of exact sequence reads as:

$$(8) \quad 0 \longrightarrow \mathcal{J}(\mathcal{H}) \longrightarrow \mathcal{D}_0(X/B) := H^{2g} \mathbb{R}f_* \mathbb{Z}_{\mathcal{D}}^\bullet(X/B) \longrightarrow \mathbb{Z}_B \longrightarrow 0$$

which is nothing but a relative version of [Voi02, cor. 12.27, p. 285]. So we have just associated with $f : X \rightarrow B$ an extension of the sheaf $\mathcal{J}(\mathcal{H})$ by \mathbb{Z}_B

and it is fairly clear that the cohomology class $\eta(f)$ is obtained as the image of 1 under the connecting morphism

$$\delta_f : H^0(B, \mathbb{Z}_B) \longrightarrow H^1(B, \mathcal{J}(\mathcal{H}))$$

coming from (8).

As the name suggests, the sheaf $\mathcal{D}_0(X/B)$ should be thought as a sheaf of relative 0-cycles of X/B . With this mind, we see that a multisection of f determines a global section of $\mathcal{D}_0(X/B)$ which is sent to some non zero integer in $H^0(B, \mathbb{Z}_B)$ (the relative degree of the corresponding cycle) and description of $\eta(f)$ we got above implies that this class should be a torsion one, thus proving the second implication in Proposition 2.2.

Now we can use the exact sequence (2) to define a topological invariant of a smooth family of tori. The long exact sequence associated with (2) reads as

$$H^1(B, \mathcal{E}) \xrightarrow{\text{exp}} H^1(B, \mathcal{J}(\mathcal{H})) \xrightarrow{c} H^2(B, \mathbb{H}_{\mathbb{Z}}).$$

The following was first observed by Kodaira in his study of elliptic surfaces [Kod63, Theorem 11.3].

Proposition 2.4.

Let us fix η a class in $H^1(B, \mathcal{J}(\mathcal{H}))$. Then any finite dimensional vector space $V \subset H^1(B, \mathcal{E})$ appears as the base space of a smooth deformation

$$\pi : \mathcal{X}_V^\eta \rightarrow V \times B$$

such that if $v \in V$ the smooth family of tori

$$\pi_v : \mathcal{X}_v^\eta := \pi^{-1}(\{v\} \times B) \rightarrow B$$

is such that $\eta(\pi_v) = \text{exp}(v) + \eta$.

In particular, if $c(\eta)$ is torsion, $\mathcal{J}(\mathcal{H})^\eta$ can be deformed (over B) to a smooth family of tori having a multisection.

Proof. There is a tautological vector bundle \mathcal{E}_V which is an extension:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_V \longrightarrow \underline{V} \longrightarrow 0$$

where \underline{V} is the trivial vector bundle. Its extension class is given by

$$\text{Id}_V \in \text{End}(V) \subset V^* \otimes H^1(B, \mathcal{E}) \simeq H^1(B, \underline{V}^* \otimes \mathcal{E}).$$

The local system $\mathbb{H}_{\mathbb{Z}}$ acts on the total space of \mathcal{E}_V by translations and we can form the quotient. The manifold \mathcal{Y} we obtained has a natural projection to the total space of \underline{V} . This is thus a smooth family of tori

$$\rho : \mathcal{Y} \longrightarrow V \times B$$

and over a point $v \in V$ we get from the construction that $\eta(\rho_v) := \text{exp}(v)$. Now we can glue the trivial family $V \times X \rightarrow V \times B$ and $\mathcal{Y} \rightarrow V \times B$ over $V \times B$ to get the sought family

$$\pi : \mathcal{X} \longrightarrow V \times B.$$

If $c(\eta)$ is torsion then there exists an integer $m \geq 1$ such that $m \cdot \eta = \text{exp}(v)$ for some $v \in H^1(B, \mathcal{E})$. The latter being a vector space we can rewrite this equality as $m \cdot (\eta - \text{exp}(v/m)) = 0$. The construction explained above with $\mathbb{C} \cdot v \subset H^1(B, \mathcal{E})$ gives a smooth family of tori $\mathcal{X} \rightarrow \mathbb{C} \times B$ such that $\eta(\mathcal{X}_0) = \eta$

and $\eta(\mathcal{X}_{-1/m})$ is torsion. The family $\mathcal{X}_{-1/m} \rightarrow B$ has thus a multisection according to Proposition 2.2. \square

Let us remark that the situation corresponding to the second part of the preceding proposition occurs in the Kähler case.

Proposition 2.5.

Let $f : X \rightarrow B$ be a smooth family of tori inducing \mathcal{H} . If X is Kähler, the class $c(\eta(f))$ is torsion in $H^2(B, \mathbb{H}_{\mathbb{Z}})$.

Proof. Using the description of the class $\eta(f)$ given in Remark 2.3, we readily infer that there is a commutative diagram

$$(9) \quad \begin{array}{ccc} H^0(B, \mathbb{Z}_B = R^{2g} f_* \mathbb{Z}_X) & \xrightarrow{d_2} & H^2(B, R^{2g-1} f_* \mathbb{Z}_X = \mathbb{H}_{\mathbb{Z}}) \\ & \searrow \delta_f & \uparrow c \\ & & H^1(B, \mathcal{J}(\mathcal{H})) \end{array}$$

where d_2 is the differential appearing in the Leray spectral sequence associated with f and \mathbb{Z}_X . But it is well known that this spectral sequence degenerates at E_2 for a Kähler morphism and when it is computed using real coefficients [Del68, Prop. 2.4] (see also [Voi02, Th. 16.15, p. 379]). The diagram (9) is translated into the equality

$$c(\eta(f)) = c(\delta_f(1)) = d_2(1)$$

and the vanishing of $d_{2, \mathbb{R}}$ exactly means that $c(\eta(f))$ is torsion. \square

Remark 2.6. Obviously a relative Kähler class is enough to get the same conclusion as above. It is quite surprising that $c(\eta)$ being torsion is in fact equivalent to the fibration $J(\mathcal{H})^\eta \rightarrow B$ being cohomologically Kähler (meaning that there is a class of degree 2 on $J(\mathcal{H})^\eta$ whose restriction to the fibres is a Kähler class). This is the content of [Nak99, Proposition 2.17].

2.3. Equivariant cohomology. In this paragraph, we intend to recall some facts about equivariant cohomology with respect to the action of a (finite) group Γ . This formalism was also used in the study of elliptic surfaces [Kod63, §13-14]. Here are the settings: we consider a finite group Γ acting on a complex manifold B and we look at sheaves of abelian groups \mathcal{F} over B endowed³ with an action of Γ compatible with the one on B : it means that for any $\gamma \in \Gamma$, there exists an isomorphism

$$i_\gamma : \gamma_* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

or, even more concretely, for any open subset $U \subset B$, there is an isomorphism

$$i_\gamma : H^0(U, \mathcal{F}) \xrightarrow{\sim} H^0(\gamma^{-1}(U), \mathcal{F}).$$

The collection of these isomorphisms have to satisfy the cocycle relation:

$$i_{\gamma g} = i_\gamma \circ (\gamma_* i_g).$$

³This is equivalent to giving an action on the étalé space \mathbb{F} associated with \mathcal{F} such that the natural projection $\mathbb{F} \rightarrow B$ is Γ -equivariant.

If \mathcal{F} is such a Γ -sheaf, the group Γ acts on the space of global sections and we can define the following functor:

$$F_\Gamma : \begin{cases} Sh_\Gamma(B) & \longrightarrow & \mathcal{A}b \\ \mathcal{F} & \longmapsto & H^0(B, \mathcal{F})^\Gamma \end{cases}$$

from the category of Γ -sheaves (of abelian groups) to the category of abelian groups.

Definition 2.7.

The equivariant cohomology groups of a Γ -sheaf \mathcal{F} are defined using the (right) derived functors of F_Γ :

$$H_\Gamma^i(B, \mathcal{F}) := R^i F_\Gamma(\mathcal{F}).$$

The functor F_Γ being expressed as the composition of two functors (taking first the global sections and then the invariants under Γ), the equivariant cohomology groups can be computed using the spectral sequence of a composed functor (see [Voi02, th. 16.9, p. 371]).

Proposition 2.8.

For any Γ -sheaf \mathcal{F} , there is a spectral sequence

$$(10) \quad E_2^{p,q} := H^p(\Gamma, H^q(B, \mathcal{F})) \implies H_\Gamma^{p+q}(B, \mathcal{F})$$

abutting to the equivariant cohomology of \mathcal{F} .

Remark 2.9. *It is well known that the higher cohomology groups $H^p(\Gamma, M)$ are torsion groups for any Γ -module M and for any $p > 0$ when Γ is finite (see [Bro82, Chapter III, Corollary 10.2]). In particular, if M is in addition a vector space, then the groups $H^i(\Gamma, M)$ vanish for $p > 0$. It applies for instance when $M = H^q(B, \mathcal{F})$ for \mathcal{F} a Γ -sheaf which is at the same time a coherent sheaf. In this case, the spectral sequence from the preceding proposition degenerates and the equivariant cohomology is nothing but the invariants:*

$$H_\Gamma^i(B, \mathcal{F}) = H^q(B, \mathcal{F})^\Gamma.$$

2.4. Smooth family of tori endowed with a group action. We now aim at applying results from the previous paragraph to the following situation: $f : X \rightarrow B$ is smooth family of tori endowed with an action of a finite group Γ . The fibration f is equivariant with respect to both actions of Γ on X and B . In particular, all the natural objects arising in this situation (the local system $\mathbb{H}_\mathbb{Z}$, the vhs, the Jacobian fibration as well as its sheaf of sections) are endowed with compatible actions of Γ . In particular, the sequence (2) is an exact sequence of Γ -sheaves and using Remark 2.9 the long exact sequence reads now as:

$$(11) \quad H^1(B, \mathcal{E})^\Gamma \xrightarrow{\text{exp}} H_\Gamma^1(B, \mathcal{J}(\mathcal{H})) \xrightarrow{-c_\Gamma} H_\Gamma^2(B, \mathbb{H}_\mathbb{Z}) \dots$$

As in Paragraph 2.1, we can naturally identify a Γ -equivariant smooth family of tori $f : X \rightarrow B$ with its cohomology class

$$\eta_\Gamma(f) \in H_\Gamma^1(B, \mathcal{J}(\mathcal{H})).$$

This can be done as in Kodaira's work [Kod63, Theorem 14.1] or using relative Deligne groups. The exact sequence

$$0 \longrightarrow \mathcal{J}(\mathcal{H}) \longrightarrow \mathcal{D}_0(X/B) \longrightarrow \mathbb{Z}_B \longrightarrow 0$$

is indeed an exact sequence of Γ -sheaves and the connecting morphism

$$\delta_f^\Gamma : H^0(B, \mathbb{Z}_B)^\Gamma = \mathbb{Z} \longrightarrow H_\Gamma^1(B, \mathcal{J}(\mathcal{H}))$$

enables us to define $\eta_\Gamma(f) := \delta_f^\Gamma(1)$ in the group $H_\Gamma^1(B, \mathcal{J}(\mathcal{H}))$.

We can now promote Propositions 2.4 and 2.5 to the Γ -equivariant realm. The proof of Proposition 2.4 applies verbatim to give the following result.

Proposition 2.10.

Let us fix η a class in $H_\Gamma^1(B, \mathcal{J}(\mathcal{H}))$. Then any finite dimensional vector space $V \subset H^1(B, \mathcal{E})^\Gamma$ appears then as the base space of a smooth Γ -equivariant deformation

$$\pi : \mathcal{X}_V^\eta \rightarrow V \times B.$$

Precisely: the group Γ acts on \mathcal{X} and the morphism π is equivariant for the trivial action of Γ on V . If $v \in V$ the smooth family of tori

$$\pi_v : \mathcal{X}_v^\eta := \pi^{-1}(\{v\} \times B) \rightarrow B$$

has the following cohomology class

$$\eta_\Gamma(\pi_v) = \exp(v) + \eta \in H_\Gamma^1(B, \mathcal{J}(\mathcal{H})).$$

Proposition 2.11.

Let $f : X \rightarrow B$ be a Γ -equivariant smooth family of tori and let us assume that X is Kähler. Then the class

$$c_\Gamma(\eta_\Gamma(f)) \in H_\Gamma^2(B, \mathbb{H}_\mathbb{Z})$$

is torsion and $f : X \rightarrow B$ can be deformed (over B) to another smooth family of tori having a multisection and acting on by Γ .

Proof. Since the $E_\infty^{0,2}$ coming from the spectral sequence (10) is a subgroup of $E_2^{0,2}$, we have a natural morphism:

$$H_\Gamma^2(B, \mathbb{H}_\mathbb{Z}) \xrightarrow{\pi^{0,2}} H^2(B, \mathbb{H}_\mathbb{Z})^\Gamma.$$

The following relation is clear:

$$\pi^{0,2}(c_\Gamma(\eta_\Gamma(f))) = c(\eta(f))$$

and consists in ignoring the Γ -action. Now we can use Proposition 2.5 to infer that $\pi^{0,2}(c_\Gamma(\eta_\Gamma(f)))$ is torsion. Finally the kernel of $\pi^{0,2}$ is an extension of $E_\infty^{2,0}$ by $E_\infty^{1,1}$ and these groups are torsion according to Remark 2.9. It is enough to conclude that $c_\Gamma(\eta_\Gamma(f))$ is a torsion class in $H_\Gamma^2(B, \mathbb{H}_\mathbb{Z})$.

Since $c_\Gamma(\eta_\Gamma(f))$ is torsion, we can mimic the proof of Proposition 2.4: it produces a deformation

$$\mathcal{X} \longrightarrow \mathbb{C} \times B \longrightarrow \mathbb{C}$$

endowed with an action of Γ , the group acting fibrewise over \mathbb{C} . Moreover there is a point in the base space $t \in \mathbb{C}$ such that

$$\eta_\Gamma(\mathcal{X}_t \rightarrow B) \in H_\Gamma^1(B, \mathcal{J}(\mathcal{H}))$$

is torsion and it implies that $\mathcal{X}_t \rightarrow B$ has a multisection (look at the natural projection $H_\Gamma^1(B, \mathcal{J}(\mathcal{H})) \longrightarrow H^1(B, \mathcal{J}(\mathcal{H}))^\Gamma$). \square

3. FROM KÄHLER FIBRATIONS TO PROJECTIVE ONES

3.1. Deforming the vhs. In this section we show how to deform a smooth family of tori once a deformation of the vhs is fixed. Let us make this more precise. We consider $f : X \rightarrow B$ a smooth family of tori and as before we denote by \mathcal{H} the vhs induced on the local system $\mathbb{H}_{\mathbb{Z}}$. We aim at considering small deformation of \mathcal{H} in the following sense.

Definition 3.1.

A small deformation of \mathcal{H} is a vhs \mathcal{H}_U on $\mathbb{H}_{\mathbb{Z}}$ seen as a local system on $U \times B$ where U is a polydisk around $o \in U$ and such that the restriction of \mathcal{H}_U to $\{o\} \times B \simeq B$ is the given \mathcal{H} . We shall denote by \mathcal{E}_U the holomorphic vector bundle $\mathcal{H}_U/\mathcal{H}_U^{1,0}$.

With this definition we have the following deformation process.

Proposition 3.2.

Let $f : X \rightarrow B$ be a smooth family of tori inducing \mathcal{H} and \mathcal{H}_U a small deformation of \mathcal{H} . If $c(\eta(f))$ is torsion (in particular if X is Kähler) then there exists

$$\mathcal{X}_U \xrightarrow{\pi_U} U \times B \xrightarrow{p_1} U$$

a smooth family of tori over $U \times B$ inducing \mathcal{H}_U and such that the family of tori $(\pi_U \circ p_1)^{-1}(o) \rightarrow B$ is isomorphic to $X \rightarrow B$.

Proof. The vhs \mathcal{H}_U being fixed we can consider the Jacobian fibration

$$\mathcal{J}(\mathcal{H}_U) \rightarrow U \times B$$

associated with it and the corresponding long exact sequences:

(12)

$$\begin{array}{ccccccc} \mathrm{H}^1(U \times B, \mathcal{E}_U) & \xrightarrow{\exp} & \mathrm{H}^1(U \times B, \mathcal{J}(\mathcal{H}_U)) & \xrightarrow{c} & \mathrm{H}^2(U \times B, \mathbb{H}_{\mathbb{Z}}) & \longrightarrow & \mathrm{H}^2(U \times B, \mathcal{E}_U) \\ \downarrow & & \downarrow & & \parallel & & \\ \mathrm{H}^1(B, \mathcal{E}) & \xrightarrow{\exp} & \mathrm{H}^1(B, \mathcal{J}(\mathcal{H})) & \xrightarrow{c} & \mathrm{H}^2(B, \mathbb{H}_{\mathbb{Z}}) & & \end{array}$$

The vertical arrows in the preceding diagram are induced by the restriction to $B \simeq \{0\} \times B$. Since $c(\eta(f))$ is torsion, its image in the vector space $\mathrm{H}^2(U \times B, \mathcal{E}_U)$ vanishes and thus there exists a class $\eta^1 \in \mathrm{H}^1(U \times B, \mathcal{J}(\mathcal{H}_U))$ whose restriction to B satisfies $c(\eta^1|_B) = c(\eta(f))$. It means that there exists a class $\alpha \in \mathrm{H}^1(B, \mathcal{E})$ such that $\eta^1|_B - \eta(f) = \exp(\alpha)$. To conclude it is enough to observe that the first vertical arrow is surjective and if $\alpha_U \in \mathrm{H}^1(U \times B, \mathcal{E}_U)$ is such that $(\alpha_U)|_B = \alpha$ then the class $\eta_U := \eta^1 - \exp(\alpha_U)$ restricts to B as the given $\eta(f)$. The class η_U corresponds thus to a smooth family of tori over $U \times B$ inducing \mathcal{H}_U and whose restriction to B is isomorphic to the fibration $f : X \rightarrow B$ we started with. \square

Remark 3.3. The last proposition holds also in the equivariant setting (we wrote down the proof without a group acting to keep the notation readable). It is enough to use equivariant cohomology and it gives the following conclusion.

Proposition 3.4.

Let $f : X \rightarrow B$ be a smooth family of tori equivariant under the action of a finite group Γ on both X and B . Let us moreover consider \mathcal{H}_U a small deformation of \mathcal{H} which at the same time a Γ -VHS for the action on $U \times B$ given by the trivial one on U . If $c_\Gamma(\eta(f))$ is torsion (in particular if X is Kähler) then there exists

$$\mathcal{X}_U \xrightarrow{\pi_U} U \times B \xrightarrow{p_1} U$$

a smooth family of tori over $U \times B$ as in Proposition 3.2 such that Γ is acting on \mathcal{X}_U and π_U is equivariant for the trivial action on U .

3.2. Buchdahl's criterion for family of tori. We now recall the relative Buchdahl criterion we obtained in [CCE14, Th. 1.1] and explain how to make it equivariant (adapting Graf's arguments from [Gra16, §9]).

Proposition 3.5.

Let \mathcal{H} be a weight -1 and rank $2g$ VHS over B (whose underlying local system is denoted $\mathbb{H}_{\mathbb{Z}}$). Let us assume moreover that \mathcal{H} admits a real polarization q . Then there exists a small deformation \mathcal{H}_V of \mathcal{H} such that the set

$$\{v \in V \mid \mathcal{H}_{V,v} \text{ admits a rational polarization}\}$$

is dense near $o \in V$: its closure contains an open neighbourhood of o .

Since we need to check that the construction of [CCE14] can be made in an equivariant framework, we recall how the proof goes.

Sketch of proof. Let us consider the \mathbb{R} -algebra

$$A_{\mathbb{R}} := H^0(B, \text{End}(\mathbb{H}_{\mathbb{R}})).$$

The VHS \mathcal{H} is nothing but an element $I \in A_{\mathbb{R}}$ such that $I^2 = -1$ and as such determines a complex structure on $A_{\mathbb{R}}$. This structure can be enriched as follows. Let us consider the decomposition

$$A_{\mathbb{R}} = A^I \oplus A^{-I}$$

where A^I (resp. A^{-I}) consists in elements of $A_{\mathbb{R}}$ commuting with I (resp. anti-commuting with I). Multiplication by I respects the decomposition and thus induces a complex structure on each piece. If we let

$$A_{\mathbb{C}}^{-I} := A_{\mathbb{C}}^{1,-1} \oplus A_{\mathbb{C}}^{-1,1},$$

we then have a weight 0 Hodge structure on $A_{\mathbb{R}}$ whose $(0,0)$ part is just A^I .

Let G be the group of invertible elements of $A_{\mathbb{R}}$: it acts on $A_{\mathbb{R}}$ by conjugation. The orbit through I is G/G° where G° is the group of invertible elements commuting with I . The space G/G° inherits a complex structure from the local diffeomorphism

$$G/G^\circ \longrightarrow G_{\mathbb{C}}/G_{\mathbb{C}}^\circ.$$

Let us consider V a small neighbourhood of o the class of the identity in G/G° : it is the base of a tautological family of complex structures on $\mathbb{H}_{\mathbb{R}}$,

i.e. it carries a small deformation \mathcal{H}_V of \mathcal{H} . Now we can consider the following⁴ weight 2 vhs on V : the local system is given by

$$\mathbb{W}_{\mathbb{Q}} := \mathbb{H}^0(B, \Lambda^2 \mathbb{H}_{\mathbb{Q}}^{\vee})$$

and the Hodge structure on \mathcal{W}_v is induced by \mathcal{H}_v for $v \in V$. Now we aim at applying [Voi02, Proposition 17.20] and we first remark that the polarization can be seen as an element $q \in \mathcal{W}_o^{1,1}$. Moreover such an element induces in particular an morphism (of bidegree $(1, 1)$) of Hodge structure

$$q \circ \cdot : A_{\mathbb{R}} \longrightarrow \mathcal{W}_o$$

which is clearly surjective (q is an isomorphism between \mathbb{H} and \mathbb{H}^{\vee}). It implies that the following component of the differential of the period map

$$(13) \quad \bar{\nabla}_o(q) : T_{V,o} = A^{-I} = A_{\mathbb{R}} \cap \left(A_{\mathbb{C}}^{1,-1} \oplus A_{\mathbb{C}}^{-1,1} \right) \longrightarrow \mathcal{W}_o^{0,2}$$

is surjective. The statement of [Voi02, Proposition 17.20] ensures that the set of $v \in V$ such that \mathcal{H}_v is \mathbb{Q} -polarizable is dense in V . \square

From the proof we get the following equivariant version.

Corollary 3.6.

Let us assume that a finite group Γ acts on B and that the vhs \mathcal{H} is a Γ -vhs. Then there exists a small deformation \mathcal{H}_U of \mathcal{H} on $U \times B$ which is at the same time a Γ -vhs over $U \times B$ for the trivial action on U and such that the set of points $u \in U$ corresponding to \mathbb{Q} -polarizable complex structures is dense in U .

Proof. Let us consider the small deformation \mathcal{H}_V constructed in the above proof. It is obvious from the construction that Γ acts on V and that \mathcal{H}_V is a Γ -vhs for the diagonal action of Γ on $V \times B$. Now let us restrict it to the set $U := V^{\Gamma}$ of fixed points of Γ in V . Since we saw that the space V can be identify with an open neighbourhood of $0 \in A_{\mathbb{R}} \cap \left(A_{\mathbb{C}}^{1,-1} \oplus A_{\mathbb{C}}^{-1,1} \right)$ and since Γ acts linearly on the latter vector space, we see that U is smooth⁵ near the point o . Replacing the polarization q with its average over the group Γ we can assume that q is Γ -invariant. Finally, since we are dealing with vector spaces, taking the invariants under the group Γ preserves surjectivity in (13):

$$\bar{\nabla}_o(q)^{\Gamma} : T_{V,o}^{\Gamma} = T_{U,o} \longrightarrow (\mathcal{W}_o^{0,2})^{\Gamma}.$$

The use of [Voi02, Proposition 17.20] in this invariant context (we apply it to the vhs \mathcal{W}^{Γ}) shows that we can endowed $\mathcal{H}_U := (\mathcal{H}_V)_{|U}$ with a Γ -invariant

⁴Here a remark is in order. Usually to be able to endow the cohomology of a vhs with a Hodge structure, the base manifold needs to be compact Kähler or at least a Zariski open subset of a compact Kähler manifold (in the latter case we end up with a mixed Hodge structure). But in our situation we only have to handle Hodge structure coming from weight one vhs. Since the complex structure I commutes with the monodromy of the underlying local system, the Hodge decompositions induced on tensor products are compatible with the action of the monodromy group and these decompositions are preserved when taking the invariants. That is the reason why no assumption is needed on B in our study.

⁵It is a general fact: the set of fixed points X^{Γ} of a finite group acting on a complex manifold is smooth, see [Car57].

polarization q_U such that q_u is a rational polarization of \mathcal{H}_u for a dense set of points $u \in U$. \square

3.3. Proofs of main statements. We will use the following lemma in the proof of Theorem 1.1.

Lemma 3.7.

Let $\mathbb{V}_{\mathbb{R}}$ be a local system underlying to a VHS \mathcal{V} of weight w defined on a compact Kähler manifold B . Then for any $k \geq 0$, the natural map

$$\mathrm{H}^k(B, \mathbb{V}_{\mathbb{R}}) \longrightarrow \mathrm{H}^k(B, \mathcal{V}/F^1)$$

induced by $\mathbb{V}_{\mathbb{R}} \rightarrow \mathbb{V}_{\mathbb{C}} \rightarrow \mathcal{V} \rightarrow \mathcal{V}/F^1$ is surjective.

Proof. The vector space $\mathrm{H}^k(B, \mathbb{V}_{\mathbb{R}})$ carries a natural Hodge structure of weight $k + w$. This is Deligne's construction explained in [Zuc79, Theorem 2.9] (see also [Még10, §4.3]). From the construction itself, the (P, Q) part of this Hodge structure is given the hypercohomology of a certain complex

$$\mathrm{H}^k(B, \mathbb{V}_{\mathbb{C}})^{P, Q} = \mathbb{H}^k(K_{P, Q}^{\bullet}).$$

It happens that when $(P, Q) = (0, k + w)$ this complex reduces to the Dolbeault complex

$$K_{0, k+w}^{\bullet} = \mathcal{A}^{0, \bullet}(\mathcal{V}^{0, w})$$

and its hypercohomology is thus the usual one of the holomorphic vector bundle $\mathcal{V}^{0, w} = \mathcal{V}/F^1$. The $(0, k + w)$ part of this Hodge structure is then given by

$$\mathrm{H}^k(B, \mathbb{V}_{\mathbb{C}})^{0, k+w} \simeq \mathrm{H}^k(B, \mathcal{V}/F^1).$$

Now it is an easy observation that the real vector space underlying a weight n Hodge structure always surjects onto its $(0, n)$ Hodge component. \square

Since the case when the base is compact needs special attention, we postpone it a little and begin with the proof of the general case.

Proof of Theorem 1.1. Let $f : X \rightarrow B$ be a smooth family of tori with X Kähler and assumed to be equivariant under the action of a finite group Γ . We denote by \mathcal{H} the VHS induced on the local system $\mathbb{H}_{\mathbb{Z}}$. We first apply Corollary 3.6: it produces a small deformation \mathcal{H}_U (over a polydisk U) of \mathcal{H} as a Γ -VHS and such that the VHS \mathcal{H}_u is \mathbb{Q} -polarizable for a dense subset of U . Since X is Kähler, we can apply Proposition 3.4: there exists a smooth family of tori $f_U : \mathcal{X}_U \rightarrow U \times B$ inducing \mathcal{H}_U and such that Γ acts equivariantly on $\mathcal{X}_U \rightarrow U \times B$ (with the trivial action on U). The invariant $c_{\Gamma}(\eta_{\Gamma}(f_U))$ being equal to $c_{\Gamma}(\eta_{\Gamma}(f))$ it is still torsion in $\mathrm{H}_{\Gamma}^2(U \times B, \mathbb{H}_{\mathbb{Z}})$ and we can resort to Proposition 2.11 to get a new family

$$f_U^c : \mathcal{X} \rightarrow \mathbb{C} \times U \times B$$

such that $\mathcal{X}_0 \rightarrow \{0\} \times U \times B$ is the family $\mathcal{X}_U \rightarrow U \times B$ and such that there exists a point $t \in \mathbb{C}$ with the property that $\mathcal{X}_t \rightarrow \{t\} \times U \times B$ has a multisection. This new family is naturally endowed with a Γ -action. To conclude, we choose $u \in U$ a point such that \mathcal{H}_u is \mathbb{Q} -polarizable and the corresponding deformation

$$\mathcal{X}_{t, u} \rightarrow \{t\} \times \{u\} \times B$$

has both a multisection and abelian varieties as fibres. \square

We finally come to the case of a compact base.

Proof of Theorem 1.1 when the base B is compact. Let us first observe that if B is compact then X has to be so and B has then to be Kähler as the target of a smooth morphism $f : X \rightarrow B$ from a compact Kähler manifold [Var84].

We can also proceed as above to get a family $f_U : \mathcal{X}_U \rightarrow U \times B$ endowed with a Γ -action and such that $\mathcal{X}_{U,u} \rightarrow \{u\} \times B$ has abelian fibres for a dense set of points $u \in U$. Now we need to use a bigger space of deformation than in the previous proof to get the density statement. First we apply Lemma 3.7 to infer that

$$H^1(B, \mathbb{H}_{\mathbb{R}}) \longrightarrow H^1(B, \mathcal{E})$$

is surjective. Since we want to use it as a space of deformation for the family f_U , we look at the following commutative diagram:

$$\begin{array}{ccc} H^1(U \times B, \mathbb{H}_{\mathbb{R}}) & \longrightarrow & H^1(U \times B, \mathcal{E}_U) \\ \parallel & & \downarrow \\ H^1(B, \mathbb{H}_{\mathbb{R}}) & \longrightarrow & H^1(B, \mathcal{E}). \end{array}$$

We can remark that taking the invariants yields a diagram of the same shape:

$$\begin{array}{ccc} H^1(U \times B, \mathbb{H}_{\mathbb{R}})^{\Gamma} & \longrightarrow & H^1(U \times B, \mathcal{E}_U)^{\Gamma} \\ \parallel & & \downarrow \\ H^1(B, \mathbb{H}_{\mathbb{R}})^{\Gamma} & \longrightarrow & H^1(B, \mathcal{E})^{\Gamma}. \end{array}$$

Let us consider V the image of $H^1(B, \mathbb{H}_{\mathbb{R}})^{\Gamma}$ in $H^1(U \times B, \mathcal{E}_U)^{\Gamma}$. We can use it in Proposition 2.10 to construct a Γ -equivariant deformation:

$$f_{U,V} : \mathcal{X}_{V,U} \rightarrow V \times U \times B$$

such that $\mathcal{X}_{U,0} \rightarrow \{0\} \times U \times B$ is the previous f_U . Moreover, up to a translation, the points of $V_{\mathbb{Q}}$ the image of $H^1(B, \mathbb{H}_{\mathbb{Q}})^{\Gamma}$ corresponds to torsion points in

$$H_{\Gamma}^1(U \times B, \mathcal{J}(\mathcal{H}_U))$$

and thus to smooth families of tori with multisections. Finally if we denote by U_{alg} the set of points $u \in U$ such that \mathcal{H}_u is \mathbb{Q} -polarizable, the set

$$T_{alg} := V_{\mathbb{Q}} \times U_{alg} \subset T := V \times U$$

is dense in T and parametrizes (up to a translation in the first factor) families $\mathcal{X}_{v,u}$ having multisections and abelian varieties as fibres. \square

Proof of Corollary 1.3. In [CCE15, Théorème 4] we proved the following structure result for linear representations of the fundamental group of a compact Kähler manifold X . If $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is a representation then let us choose $H < \pi_1(X)$ a finite index subgroup such that the image $\rho(H)$ is torsion free (such a subgroup exists thanks to Selberg's lemma). This subgroup corresponds to a finite étale cover $Y \rightarrow X$ which can be assumed Galois, its Galois group to be denoted Γ . Then (up to bimeromorphic transformation), the base of the Shafarevich morphism $sh_{\rho} : Y \rightarrow Sh_{\rho}(Y) := W$ is such that the Iitaka fibration of W is a smooth fibration $f : W \rightarrow B$ (onto

a projective manifold). From now on let us assume that ρ is injective. We make the following observations:

- (1) the Shafarevich and Iitaka fibrations being functorial, the group Γ acts on W , B and the fibration f is equivariant to both actions. Let us note however that the action is in general no longer free on both W and B .
- (2) the fundamental group of W is isomorphic to the one of Y .

The last assertion is a consequence of the torsion freeness of $\pi_1(Y)$. We have indeed an exact sequence

$$1 \longrightarrow \pi_1(F)_Y \longrightarrow \pi_1(Y) \longrightarrow \pi_1^{orb}(W) \longrightarrow 1$$

where F is the general fibre of sh_ρ and the orbifold structure on W is induced by the fibration sh_ρ . The defining property of F being the finiteness of $\pi_1(F)_Y$, we infer that this group is trivial. Finally the orbifold fundamental group $\pi_1^{orb}(W)$ is an extension

$$1 \longrightarrow K \longrightarrow \pi_1^{orb}(W) \longrightarrow \pi_1(W) \longrightarrow 1$$

where K is a group generated by torsion elements. As before, it implies that $K = 1$ and that $\pi_1(Y) \simeq \pi_1(W)$.

We can now apply Theorem 1.1: W can be deformed to a projective manifold W_{alg} on which the group Γ acts. To deal with the lack of freeness of the action of Γ on W_{alg} , let us introduce a simply connected projective manifold \mathbb{P} on which Γ acts freely: such a manifold exists according to [Ser58] (see also [Sha13, Chapter IX, §4.2]). We can finally consider the quotient $X_{alg} := (W_{alg} \times \mathbb{P})/\Gamma$. This is a smooth projective variety whose fundamental group is the extension

$$(14) \quad 1 \longrightarrow \pi_1(W_{alg}) \longrightarrow \pi_1(X_{alg}) \longrightarrow \Gamma \longrightarrow 1.$$

Using the isomorphisms $\pi_1(W_{alg}) \simeq \pi_1(W) \simeq \pi_1(Y)$, we infer that $\pi_1(X_{alg}) \simeq \pi_1(X)$. It is indeed quite easy to see that the group extension (14) and the one corresponding to the Galois étale cover $Y \rightarrow X$ are isomorphic. For instance we can look at the Γ -equivariant morphism $sh_\rho : Y \rightarrow W$ (inducing an isomorphism between fundamental groups). Taking the product with \mathbb{P} and then the quotient under the action of Γ , the map

$$(sh_\rho \times \text{id}_{\mathbb{P}})/\Gamma : (Y \times \mathbb{P})/\Gamma \longrightarrow (W \times \mathbb{P})/\Gamma$$

also induces an isomorphism at the level of fundamental groups. This is the sought isomorphism between $\pi_1(X_{alg})$ and $\pi_1(X)$ and it ends the proof of Corollary 1.3. \square

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