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# VARIATION OF THE HOLOMORPHIC DETERMINANT BUNDLE

JULIEN GRIVAUX

ABSTRACT. In this paper, we prove that the Grothendieck-Riemann-Roch formula in Deligne cohomology computing the determinant of the cohomology of a holomorphic vector bundle on the fibers of a proper submersion between abstract complex manifolds is invariant by deformation of the bundle.

## 1. INTRODUCTION

The Grothendieck-Riemann-Roch theorem is one of the cornerstones of modern algebraic geometry, and can be stated in its initial form as follows:

**Theorem 1.1.** [8] *For any smooth quasi-projective variety  $X$  over a field of characteristic zero, the morphism  $\mathcal{F} \rightarrow \text{ch}(\mathcal{F}) \text{Td}(X)$  from the Grothendieck group  $K(X)$  of coherent sheaves on  $X$  to the Chow ring  $\text{CH}(X)$  of  $X$  commutes with proper push-forward.*

Since Serre's fundamental papers on coherent sheaves [29] [28], it has become natural and useful to translate results from algebraic to analytic geometry. Concerning the Grothendieck-Riemann-Roch theorem, this has been the object of many researches from early sixties till eighties, starting with the case of analytic immersions [2] and pursuing with the index theorem for vector bundles and coherent analytic sheaves (see [3], [25], [33], [24]). The outcome of these works is O'Brian-Toledo-Tong's proof of the Grothendieck-Riemann-Roch theorem in Hodge cohomology for arbitrary proper holomorphic maps between complex manifolds [23]. By completely different methods, Levy [22] succeeded in proving the analogous statement in De Rham cohomology, where the Chern classes are constructed by means of locally-free resolutions in the category of real-analytic coherent sheaves.

From the middle of the eighties, some new ideas about the Grothendieck-Riemann-Roch theorem emerged after the seminal article of Quillen [26] introducing canonical hermitian metrics on determinant bundles associated with the cohomology of a vector bundle on the fibers of a holomorphic submersion (see [30, Chap. VI]). Building on initial results in the case of families of curves (see [26], [5], [9]), Bismut, Gillet and Soulé [6] proved that, for locally Kähler fibrations, the curvature of this determinant bundle is exactly given by the component of degree two of the Grothendieck-Riemann-Roch theorem at the level of differential forms. This theorem has been extended to all degrees in [7] provided that the higher direct images of the bundle are locally free. Quite recently, Bismut succeeded in removing the kählerianity assumption on the morphism and obtained the following result:

**Theorem 1.2.** [4, Thm 0.1.1] *Let  $f: X \rightarrow Y$  be a proper holomorphic submersion between complex manifolds and  $\mathcal{E}$  be a holomorphic vector bundle on  $X$ . Then the following identities hold in the Bott-Chern cohomology ring of  $Y$ :*

- (i)  $c_1^{\text{BC}}[\det Rf_* \mathcal{E}] = \left( f_* \{ \text{ch}^{\text{BC}}(\mathcal{E}) \text{td}^{\text{BC}}(\text{T}_{X/Y}) \} \right)^{(1,1)}$  in  $H_{\text{BC}}^{1,1}(Y)$ .
- (ii) *If all the sheaves  $R^i f_* \mathcal{E}$  are locally free, then  $\text{ch}^{\text{BC}}(Rf_* \mathcal{E}) = f_* \{ \text{ch}^{\text{BC}}(\mathcal{E}) \text{td}^{\text{BC}}(\text{T}_{X/Y}) \}$  in  $\bigoplus_{p \geq 0} H_{\text{BC}}^{p,p}(Y)$ .*

On abstract complex manifolds, one of the finest known cohomology theory where Chern classes exist for holomorphic vector bundles is Deligne-Beilinson cohomology. The ultimate goal of our program would be to prove the Grothendieck-Riemann-Roch theorem in this cohomology. The statement does not immediately make sense even for holomorphic vector bundles, because Chern classes must be defined for the direct images sheaves  $R^i f_* \mathcal{E}$ . The problem of defining Chern classes of coherent sheaves in Deligne cohomology is solved on compact manifolds in [14]. The corresponding Grothendieck-Riemann-Roch theorem is proved only for projective morphisms between complex compact manifolds.

In this paper, we focus only on the component of degree two on the base of the Grothendieck-Riemann-Roch theorem in Deligne cohomology for holomorphic vector bundles. Our main result describes completely the variation of the determinant bundle, first in the rational Deligne cohomology group  $H_D^2(Y, \mathbb{Q}(1))$  and then in  $\text{Pic}(Y)$  under mild assumptions on  $Y$ :

**Theorem 1.3.** *Let  $f: X \rightarrow Y$  be a proper holomorphic submersion between complex manifolds  $X, Y$  and let  $(\mathcal{E}_t)_{t \in \Delta}$  be a holomorphic family of holomorphic vector bundles on  $X$  parameterized by the complex unit disc  $\Delta$ .*

(i) **Rational variation formula**

For any  $s$  and  $t$  in  $\Delta$ ,

$$c_1^D[\det Rf_* \mathcal{E}_s] - c_1^D[\det Rf_* \mathcal{E}_t] = \left( f_* \{ [\text{ch}^D(\mathcal{E}_s) - \text{ch}^D(\mathcal{E}_t)] \text{td}^D(\mathbb{T}_{X/Y}) \} \right)^{(2)}$$

in the rational Deligne cohomology group  $H_D^2(Y, \mathbb{Q}(1))$ .

(ii) **Integral variation formula**

If  $H^1(Y, \mathcal{O}_Y)$  is separated and if  $H^1(Y, \mathbb{Q}_Y)$  is countable, then there exists a unique analytic curve  $\gamma$  from  $\Delta$  to  $\text{Pic}^0(Y)$  vanishing at 0 such that for any  $t$  in  $\Delta$ ,

$$c_1^D(\gamma(t)) = \left( f_* \{ [\text{ch}^D(\mathcal{E}_t) - \text{ch}^D(\mathcal{E}_0)] \text{td}^D(\mathbb{T}_{X/Y}) \} \right)^{(2)}$$

in  $H_D^2(Y, \mathbb{Q}(1))$ . For any  $s$  and  $t$  in  $\Delta$ ,

$$[\det Rf_* \mathcal{E}_s] - [\det Rf_* \mathcal{E}_t] = \gamma(s) - \gamma(t)$$

Remark that since  $H^1(Y, \mathbb{Q})$  is isomorphic to  $\text{Hom}_{\mathbb{Z}}(H_1(Y, \mathbb{Z}), \mathbb{Q})$ , it is sufficient to require that the torsion-free part of  $H_1(Y, \mathbb{Z})$  is finitely generated to ensure that  $H^1(Y, \mathbb{Q})$  is countable. This condition can be easier to check in concrete situations.

Even if  $\text{Pic}(Y)$  is not always a complex manifold, it is possible to give a precise definition of an analytic curve with values in  $\text{Pic}(Y)$  that matches with the usual one when the image of  $H^1(Y, \mathbb{Z})$  is discrete in  $H^1(Y, \mathcal{O}_Y)$ .

On the one hand, Theorem 1.3 is motivated by Teleman's program on the classification of class VII surfaces (see [31], [32]). On the other hand, it is a little step towards the general Grothendieck-Riemann-Roch theorem in Deligne cohomology (which is currently out of reach in full generality). For instance, we have the following result:

**Theorem 1.4.** *Let  $Y$  and  $F$  be complex manifolds such that  $F$  is compact of complex dimension  $n$ , and let  $p: Y \times F \rightarrow Y$  and  $q: Y \times F \rightarrow F$  be the two natural projections. Then, for any  $\mathcal{L}$  in  $\text{Pic}^0(Y \times F)$ ,*

$$c_1^D[\det Rp_* \mathcal{L}] = p_* \{ c_1^D(\mathcal{L}) q^* \text{td}_n^D(F) \}$$

in the rational Deligne cohomology group  $H_D^2(Y, \mathbb{Q}(1))$ .

The paper is organized as follows: in §2 we provide a short recollection of operations on derived categories in the analytic setting. Then we recall the theory of determinants for coherent analytic sheaves (see [19], [20, Chap. 5 §6] and [6, §3]) and we prove in Proposition 2.4 a base change formula for determinant bundles. We also prove a folklore result (Lemma 2.2) saying that the first Chern class of a coherent sheaf in Hodge cohomology is the same as the first Chern class of its determinant. In §3, we recall the basics of Deligne cohomology (see [11] and [34, Chap. 12]). Proposition 3.1 is the main ingredient of the proof of Theorem 1.3. Then we discuss analytic curves with values in the Picard group of a complex manifold. Lastly, §4 is devoted to the proofs of Theorems 1.3 and 1.4.

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## 2. DERIVED CATEGORIES AND DETERMINANT BUNDLES

**2.1. Derived categories in the analytic setting.** For generalities on sheaves and derived categories, we refer the reader to [17], [12] and [10]. For any connected complex manifold  $X$ , let  $D^+(X)$  (resp.  $D^-(X)$ , resp.  $D^b(X)$ ) denote the right-bounded (resp. left-bounded, resp. bounded) derived category of analytic sheaves on  $X$ ; and let  $D_{\text{coh}}^b(X)$  be the full subcategory of  $D^b(X)$  consisting of bounded complexes with coherent cohomology. Recall that the category of sheaves over any sheaf of rings has enough injectives and flat objects [17, Prop. 2.4.3 and 2.4.12]. Let  $f: X \rightarrow Y$  be a holomorphic map. Then there exists a derived pullback morphism

$$Lf^*: D^-(Y) \rightarrow D^-(X)$$

given by  $Lf^* \mathcal{F} = f^{-1} \mathcal{F} \overset{L}{\otimes}_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$ , where  $\overset{L}{\otimes}$  denotes the derived tensor product. If  $f$  is flat, we write  $f^*$  instead of  $Lf^*$ . Since  $\mathcal{O}_X$  is quasi-isomorphic to a bounded complex of flat  $f^{-1} \mathcal{O}_Y$ -modules,  $Lf^*$  maps  $D^b(Y)$  to  $D^b(X)$ . Besides,  $Lf^*$  maps  $D_{\text{coh}}^b(Y)$  to  $D_{\text{coh}}^b(X)$ . On the other hand, there is a derived push-forward morphism

$$Rf_*: D^+(X) \rightarrow D^+(Y).$$

If  $f$  is a proper holomorphic map, thanks to Grauert's finiteness theorem [13],  $Rf_*$  maps  $D_{\text{coh}}^b(X)$  to  $D_{\text{coh}}^b(Y)$ . The derived pullback and push forward functors are related via the projection formula: if  $\mathcal{F}$  and  $\mathcal{G}$  are elements of  $D_{\text{coh}}^b(Y)$  and  $D_{\text{coh}}^b(X)$  respectively and if  $f$  is proper, there is a canonical isomorphism in  $D_{\text{coh}}^b(Y)$ :

$$(1) \quad Rf_* \mathcal{G} \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{F} \simeq Rf_* \left( \mathcal{G} \overset{L}{\otimes}_{\mathcal{O}_X} Lf^* \mathcal{F} \right).$$

This isomorphism is called the projection formula. We refer the reader to the proof of [17, Prop. 2.6.6] which is valid in this context.

**2.2. Determinants for coherent sheaves.** Let  $\mathcal{F}$  be a coherent analytic sheaf on a connected complex manifold  $X$ . The determinant of  $\mathcal{F}$ , denoted by  $\det \mathcal{F}$ , is a holomorphic line bundle on  $X$  defined as follows:

– If  $\mathcal{F}$  is torsion-free, there exists a Zariski-open subset  $U$  of  $X$  such that  $\mathcal{F}$  is locally free on  $U$  and  $X \setminus U$  has codimension at least two in  $X$ . Then the top exterior power of  $\mathcal{F}|_U$  is a holomorphic line bundle on  $U$ , which extends uniquely to a line bundle  $\det \mathcal{F}$  on  $X$ .

– If  $\mathcal{F}$  is a torsion sheaf, let  $Z$  be the maximal closed hypersurface contained in  $\text{supp } \mathcal{F}$ , let  $(Z_i)_{i \in I}$  be the irreducible components of  $Z$ , and let  $\mathfrak{S}_i$  be the ideal sheaf defining  $Z_i$ . For any integer  $k$ , the sheaf  $\mathfrak{S}_i^k \mathcal{F} / \mathfrak{S}_i^{k+1} \mathcal{F}$  is the push-forward of a coherent sheaf on  $Z_i$ , we denote its generic rank by  $r_i(k)$ . If  $m_i = \sum_k r_i(k)$ , then the determinant of  $\mathcal{F}$  is given by the formula

$$\det \mathcal{F} = \bigotimes_i O_X(m_i Z_i).$$

– If  $\mathcal{F}$  is arbitrary and  $\mathcal{F}_{\text{tors}}$  is its maximal torsion subsheaf, then  $\mathcal{F} / \mathcal{F}_{\text{tors}}$  is torsion-free and

$$\det \mathcal{F} = \det(\mathcal{F} / \mathcal{F}_{\text{tors}}) \otimes \det \mathcal{F}_{\text{tors}}.$$

Recall that it is also possible to construct determinants of coherent sheaves using local free resolutions, we refer the reader to [20, Chap. 5 §6] for this approach. The main property of determinants is the following [20, Chap. 5 Prop. 6.9]: for any bounded complex  $\mathcal{K}^\bullet$  of coherent sheaves on  $X$ , using additive notation for line bundles, we have a canonical isomorphism

$$\sum_i (-1)^i \det \mathcal{K}^i \simeq \sum_i (-1)^i \det \mathcal{H}^i(\mathcal{K}^\bullet).$$

For any bounded complex  $\mathcal{K}^\bullet$  on  $X$  with coherent cohomology, the determinant of  $\mathcal{K}^\bullet$  is defined by the formula  $\det \mathcal{K}^\bullet = \sum_i (-1)^i \det \mathcal{H}^i(\mathcal{K}^\bullet)$ . Two quasi-isomorphic bounded complexes with coherent cohomology have canonically isomorphic determinants. Therefore, the determinant of an object in  $D_{\text{coh}}^b(X)$  is well-defined, and two isomorphic objects have the same determinant. Besides, the determinant map factorizes through the Grothendieck group  $K(X)$  of the category of coherent sheaves on  $X$ .

**Lemma 2.1.** *Let  $\varphi: X \rightarrow Y$  be a holomorphic map between connected complex manifolds. Then for any element  $\mathcal{K}$  in  $D_{\text{coh}}^b(Y)$ ,  $\varphi^* \det \mathcal{K} \simeq \det(\mathbb{L}\varphi^* \mathcal{K})$ .*

*Proof.* It is enough to prove the lemma when  $\mathcal{K}$  is a single coherent sheaf in degree zero. For any Stein subset  $U$  of  $Y$ , let  $\mathcal{E}^\bullet$  be a locally free resolution of  $\mathcal{K}|_U$ . Then we have canonical isomorphisms

$$\varphi^* \det \mathcal{K}|_U \simeq \varphi^* \left[ \sum_i (-1)^i \det(\mathcal{E}^i) \right] = \sum_i (-1)^i \det(\varphi^* \mathcal{E}^i) = \det(\mathbb{L}\varphi^* \mathcal{K}|_U)$$

which can be glued together by a standart procedure (see e.g. [20, p. 163–165]) to give a global isomorphism on  $X$  between  $\varphi^* \det \mathcal{K}$  and  $\det(\mathbb{L}\varphi^* \mathcal{K})$ .  $\square$

For any coherent sheaf  $\mathcal{F}$  on  $X$ , we denote by  $c_i^H(\mathcal{F})$  the Chern classes of  $\mathcal{F}$  in  $H^i(X, \Omega_X^i)$  and by  $\text{ch}^H(\mathcal{F})$  its Chern character in the total Hodge cohomology ring of  $X$ .

**Lemma 2.2.** *For any complex manifold  $X$  and any coherent analytic sheaf  $\mathcal{F}$  on  $X$ , we have  $c_1^H(\mathcal{F}) = c_1^H(\det \mathcal{F})$  in  $H^1(X, \Omega_X^1)$ .*

*Proof.* We can assume without loss of generality that  $\mathcal{F}$  is either a torsion sheaf or a torsion-free sheaf. Thanks to the vanishing theorem for local cohomology [8, Thm 3.6], for any analytic set  $W$  of codimension at least two in  $X$ , the natural map from  $H^1(X, \Omega_X^1)$  to  $H^1(X \setminus W, \Omega_{X \setminus W}^1)$  is injective. Thus, if  $U$  is a Zariski open subset of  $X$  such that  $\text{codim}_X(X \setminus U) \geq 2$ , it is enough to prove that  $c_1^H(\mathcal{F}|_U) = c_1^H(\det \mathcal{F}|_U)$ . Now we have to deal with two different cases:

– First case: the sheaf  $\mathcal{F}$  is a torsion sheaf whose support is a smooth hypersurface. Since  $\mathcal{F} = \sum_k \mathfrak{S}_Z^k \mathcal{F} / \mathfrak{S}_Z^{k+1} \mathcal{F}$  in  $K(X)$ , we can assume without loss of generality that  $\mathfrak{S}_Z \mathcal{F} = 0$ . After

removing again an analytic subset of  $X$  of codimension at least two, we can even assume that  $\mathcal{F}$  is the push-forward of a locally-free sheaf of rank  $r$  on  $Z$ . This implies  $\det \mathcal{F} = \mathcal{O}_X(rZ)$ . Then, using the Grothendieck-Riemann-Roch theorem in Hodge cohomology for immersions [23], we get  $c_1^H(\mathcal{F}) = r[Z]_H = c_1^H(\mathcal{O}_X(rZ)) = c_1^H(\det \mathcal{F})$ .

– Second case: the sheaf  $\mathcal{F}$  is locally free. Then it is well-know that  $c_1^H(\mathcal{F}) = c_1^H(\det \mathcal{F})$ .  $\square$

**2.3. Base change.** Let  $f: X \rightarrow Y$  be a proper holomorphic submersion between two connected complex manifolds  $X$  and  $Y$ , and let  $\mathcal{E}$  be a locally free sheaf on  $X$ .

**Definition 2.3.** The determinant of the cohomology<sup>1</sup>  $\lambda(\mathcal{E}, f)$  attached to the couple  $(\mathcal{E}, f)$  is the class of  $\det(\mathbf{R}f_* \mathcal{E})$  in  $\text{Pic}(Y)$ .

We now state and prove a base change theorem for the determinant of the cohomology. Let  $T$  be a complex manifold and  $u: T \rightarrow Y$  be a closed immersion of complex manifolds, and consider the cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{v} & X \\ g \downarrow & \square & \downarrow f \\ T & \xrightarrow{u} & Y \end{array}$$

**Proposition 2.4.** For any vector bundle  $\mathcal{E}$  on  $X$ ,  $u^* \lambda(\mathcal{E}, f) = \lambda(v^* \mathcal{E}, g)$  in  $\text{Pic}(T)$ .

*Proof.* Since  $f$  and  $g$  are holomorphic submersions, they are flat morphisms. Using the projection formula (1) twice, we have in  $D_{\text{coh}}^b(Y)$

$$\begin{aligned} u_*(Lu^* \mathbf{R}f_* \mathcal{E}) &\simeq \mathbf{R}f_* \mathcal{E} \otimes_{\mathcal{O}_Y}^L u_* \mathcal{O}_T \simeq \mathbf{R}f_* (\mathcal{E} \otimes_{\mathcal{O}_X}^L f^* u_* \mathcal{O}_T) \simeq \mathbf{R}f_* (\mathcal{E} \otimes_{\mathcal{O}_X}^L v_* g^* \mathcal{O}_T) \\ &\simeq \mathbf{R}f_* \mathbf{R}v_* (v^* \mathcal{E} \otimes_{\mathcal{O}_Z}^L g^* \mathcal{O}_T) \simeq \mathbf{R}u_* \mathbf{R}g_* (v^* \mathcal{E} \otimes_{\mathcal{O}_Z}^L g^* \mathcal{O}_T) \simeq u_* [\mathbf{R}g_* (v^* \mathcal{E})]. \end{aligned}$$

This proves that for any integer  $i$ ,

$$\mathcal{H}^i(Lu^* \mathbf{R}f_* \mathcal{E}) \simeq \mathcal{R}^i g_*(v^* \mathcal{E}).$$

Then we can conclude using Lemma 2.1.  $\square$

### 3. DELIGNE COHOMOLOGY AND PICARD GROUP

**3.1. Deligne classes of degree 2.** For any complex manifold  $X$  and any nonnegative integer  $p$ , the Deligne complex  $\mathbb{Z}_{D,X}(p)$  is the complex

$$\mathbb{Z}_X \xrightarrow{(2i\pi)^p} \mathcal{O}_X \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_X^{p-1},$$

where the constant sheaf  $\mathbb{Z}_X$  sits in degree zero. The integral Deligne cohomology groups of  $X$  are defined by the formula  $H_D^k(X, \mathbb{Z}(p)) = H^k(X, \mathbb{Z}_{D,X}(p))$ . Similar definitions hold for the rational Deligne complex  $\mathbb{Q}_{D,X}(p)$  as well as for the rational Deligne cohomology groups  $H_D^k(X, \mathbb{Q}(p))$ . For any locally-free sheaf  $\mathcal{E}$  on  $X$ , we will denote by  $c_i^D(\mathcal{E})$  the rational Chern classes of  $\mathcal{E}$  in  $H_D^{2i}(X, \mathbb{Q}(i))$ , and by  $\text{ch}^D(\mathcal{E})$  the Chern character of  $\mathcal{E}$ .

<sup>1</sup>It is also called Knudsen-Mumford determinant in the literature.

Let  $\text{Pic}(X)$  be the Picard group of  $X$  classifying isomorphism classes of line bundles on  $X$ , it is isomorphic to  $H^1(X, \mathcal{O}_X^*)$ . Since  $\mathbb{Z}_{D,X}(1)$  is quasi-isomorphic to  $\mathcal{O}_X^*[-1]$ ,  $H_D^2(X, \mathbb{Z}(1))$  is isomorphic to  $\text{Pic}(X)$ . Note that the kernel of

$$(2) \quad c_1^D: \text{Pic}(X) \simeq H_D^2(X, \mathbb{Z}(1)) \rightarrow H_D^2(X, \mathbb{Q}(1))$$

is exactly the maximal torsion subgroup of  $\text{Pic}(X)$ .

There is a natural cup-product in Deligne cohomology (cf [11]). Besides, for any nonnegative integer  $p$ , the morphism  $\partial: \Omega_X^{p-1} \rightarrow \Omega_X^p$  induces a morphism from  $\mathbb{Q}_{D,X}(p)$  to  $\Omega_X^p[-p]$ . Hence for any nonnegative integer  $k$ , we obtain a natural map from  $H_D^{k+p}(X, \mathbb{Q}(p))$  to  $H^k(X, \Omega_X^p)$  which is compatible with cup-products on both sides.

Let us now give the key ingredient of the proof of Theorem 1.3.

**Proposition 3.1.** *Let  $\Delta$  be the complex unit disc, let  $X$  be a complex manifold and let  $\alpha$  be a class in  $H_D^2(X \times \Delta, \mathbb{Q}(1))$ . The class  $\alpha$  maps to 0 via the morphism*

$$H_D^2(X \times \Delta, \mathbb{Q}(1)) \rightarrow H^1(X \times \Delta, \Omega_{X \times \Delta}^1) \rightarrow H^1(X \times \Delta, \mathcal{O}_X \boxtimes \Omega_\Delta^1)$$

*if and only if there exists a class  $\beta$  in  $H_D^2(X, \mathbb{Q}(1))$  such that  $\alpha = \text{pr}_1^* \beta$ .*

*Proof.* Let  $p: X \times \Delta \rightarrow X$  be the first projection. By the projection formula,

$$R p_* \{p^{-1} \mathbb{Q}_{D,X}(1)\} \simeq \mathbb{Q}_{D,X}(1) \otimes_{\mathbb{Q}_X}^L R p_* \mathbb{Q}_{X \times \Delta} \simeq \mathbb{Q}_{D,X}(1)$$

so that we obtain the isomorphisms

$$R\Gamma(X \times \Delta, p^{-1} \mathbb{Q}_{D,X}(1)) \simeq R\Gamma(X, R p_* \{p^{-1} \mathbb{Q}_{D,X}(1)\}) \simeq R\Gamma(X, \mathbb{Q}_{D,X}(1)).$$

We have a natural exact sequence of complexes

$$0 \longrightarrow p^{-1} \mathbb{Q}_{D,X}(1) \xrightarrow{p^*} \mathbb{Q}_{D,X \times \Delta}(1) \longrightarrow \mathcal{O}_X \boxtimes \Omega_\Delta^1[-1] \longrightarrow 0.$$

which is given by the isomorphism  $p^{-1} \mathbb{Q}_X \xrightarrow{\sim} \mathbb{Q}_{X \times \Delta}$  in degree zero and by the sequence

$$0 \longrightarrow p^{-1} \mathcal{O}_X \longrightarrow \mathcal{O}_{X \times \Delta} \xrightarrow{d_t} \mathcal{O}_X \boxtimes \Omega_\Delta^1 \longrightarrow 0$$

in degree one. Writing down the long cohomology exact sequence, we get the following short exact sequence:

$$H_D^2(X, \mathbb{Q}(1)) \xrightarrow{p^*} H_D^2(X \times \Delta, \mathbb{Q}(1)) \longrightarrow H^1(X \times \Delta, \mathcal{O}_X \boxtimes \Omega_\Delta^1)$$

This gives the result. □

**Remark 3.2.**

- (i) Proposition 3.1 is also valid for integral Deligne cohomology.
- (ii) It is possible to prove Proposition 3.1 by a purely geometric argument. We can assume without loss of generality that  $\alpha$  is the class of a vector bundle  $\mathcal{L}$  in  $\text{Pic}(X \times \Delta)$ . The vanishing of the relative Atiyah class of  $\mathcal{L}$  in  $H^1(X \times \Delta, \mathcal{O}_X \boxtimes \Omega_\Delta^1)$  means that there exists a relative holomorphic connection  $\nabla: \mathcal{L} \rightarrow \mathcal{L} \otimes \text{pr}_2^* \Omega_\Delta^1$  on  $\mathcal{L}$ . This relative connexion defines a true holomorphic connexion on each slice  $\{x\} \times \Delta$  which is automatically flat; so that the restriction of  $\mathcal{L}$  to this slice is canonically trivial. This proves that  $\mathcal{L}$  is globally isomorphic to  $p^* \mathcal{L}_{|X \times \{0\}}$ , and we take for  $\beta$  the class of  $\mathcal{L}_{|X \times \{0\}}$  in  $\text{Pic}(X)$ .
- (iii) For a geometric interpretation of (i) using analytic curves with values in the Picard group, we refer the reader to Proposition 3.5 (iii).

**3.2. Analytic curves in the Picard group.** In this section, we discuss the notion of analytic curve with values in the Picard group of a complex manifold. This problem must be dealt with great care. Let us first recall that there is an exact sequence

$$(3) \quad 0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

where  $\text{Pic}^0(X) = H^1(X, \mathcal{O}_X) / 2i\pi H^1(X, \mathbb{Z}) \simeq H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z})$ , the last isomorphism being given by  $\alpha \rightarrow \alpha / 2i\pi$ . Remark that  $H^1(X, \mathcal{O}_X)$  carries a natural topology from its representation using Dolbeault cohomology. Let us put the discrete topology on  $H^2(X, \mathbb{Z})$ . Then we can endow  $\text{Pic}(X)$  with a canonical topology, which is defined by requiring that all arrows in (3) are continuous.

If  $X$  is compact, the group  $H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z}_X)$  is a genuine complex Lie group. Indeed, if  $\alpha$  is a De Rham representative of a cohomology class in  $H^1(X, \mathbb{R})$ , then  $\alpha^{0,1}$  is a Dolbeault representative of the image of  $[\alpha]$  in  $H^1(X, \mathcal{O}_X)$ . Therefore, if  $\alpha^{0,1} = \bar{\partial}f$  where  $f = u + iv$  is a smooth complex-valued function on  $X$ , then  $\alpha = du + d^c v$ . It follows that  $dd^c v = 0$  and since  $X$  is compact,  $v$  is constant. Hence  $H^1(X, \mathbb{R})$  injects in  $H^1(X, \mathcal{O}_X)$ . Since  $H^1(X, \mathbb{R})$  and  $H^1(X, \mathcal{O}_X)$  are finite-dimensional,  $H^1(X, \mathbb{Z})$  is a discrete subgroup of  $H^1(X, \mathcal{O}_X)$ .

If  $X$  is not compact the natural topology on  $\text{Pic}(X)$  is not Hausdorff in general. This comes from the fact that the Dolbeault cohomology groups of a complex manifold are no longer separated. For instance, there is a counterexample due to Rossi [27] of a 2-dimensional complex manifold  $X$  such that  $H^1(X, \mathcal{O}_X)$  is not separated (see [21] for more details on this topic). And of course, even if  $H^1(X, \mathcal{O}_X)$  is separated, the image of  $H^1(X, \mathbb{Z})$  is not a priori discrete in  $H^1(X, \mathcal{O}_X)$ <sup>2</sup>.

Let  $\Delta$  denote the complex unit disk. We make the following definition, which is intuitively very natural:

**Definition 3.3.** For any complex manifold  $X$ , a curve  $\gamma: \Delta \rightarrow \text{Pic}(X)$  is analytic if there exists an element  $\alpha$  in  $\text{Pic}(X \times \Delta)$  such that for all  $t$  in  $\Delta$ ,  $\gamma(t) = \alpha|_{X \times \{t\}}$ .

This definition is of little use if  $H^1(X, \mathcal{O}_X)$  is not separated, we will avoid this pathological case. In what follows, we will use a little bit of the theory of holomorphic functions of one complex variable with values in a Fréchet space. We refer the interested reader to the papers [15] and [16] for a complete account of this theory. If  $X$  is compact, these subtleties are no longer needed.

**Proposition 3.4.** *Let  $X$  be a complex manifold, assume that  $H^1(X, \mathcal{O}_X)$  is separated, and let  $\gamma: \Delta \rightarrow \text{Pic}(X)$  be a curve.*

- (i) *The curve  $\gamma$  is analytic if and only if  $\gamma - \gamma(0)$  can be lifted to an holomorphic curve with values in the Fréchet space  $H^1(X, \mathcal{O}_X)$ .*
- (ii) *Assume that  $H^1(X, \mathbb{Q})$  is countable. If  $\gamma$  is analytic, it is entirely determined by  $\gamma(0)$  and by the curve  $c_1^D \circ \gamma$  from  $\Delta$  to  $H_D^2(X, \mathbb{Q}(1))$ , where  $c_1^D$  is given by (2).*

*Proof.* (i) Since  $H^1(X, \mathcal{O}_X)$  is separated, we can apply the Künneth formula for Fréchet spaces (cf. [18] and [1, exposé 24]): we obtain that  $H^1(X \times \Delta, \mathcal{O}_{X \times \Delta}) = H^1(X, \mathcal{O}_X) \widehat{\otimes} \mathcal{O}(\Delta)$ , where  $\widehat{\otimes}$  denotes the topological tensor product. Besides, thanks to [16, §7.1 Prop. 6], the natural map from  $H^1(X, \mathcal{O}_X) \otimes \mathcal{O}(\Delta)$  to  $\mathcal{O}\{\Delta, H^1(X, \mathcal{O}_X)\}$  given by  $\alpha \otimes f \rightarrow \{t \rightarrow f(t)\alpha\}$  extends to an isomorphism between  $H^1(X, \mathcal{O}_X) \widehat{\otimes} \mathcal{O}(\Delta)$  and  $\mathcal{O}\{\Delta, H^1(X, \mathcal{O}_X)\}$ . Therefore, we get a natural isomorphism between  $H^1(X \times \Delta, \mathcal{O}_{X \times \Delta})$  and  $\mathcal{O}\{\Delta, H^1(X, \mathcal{O}_X)\}$  obtained by attaching to each class  $\beta$  in  $H^1(X \times \Delta, \mathcal{O}_{X \times \Delta})$  the function  $t \rightarrow \beta|_{X \times \{t\}}$ . It is easy to conclude from there: if  $\gamma - \gamma(0)$

<sup>2</sup>The author was told by M. Verbitsky that a well-chosen nilmanifold would probably provide such an example.



can be lifted to an holomorphic curve with values in  $H^1(X, \mathcal{O}_X)$ , then there exists a class  $\beta$  in  $H^1(X \times \Delta, \mathcal{O}_{X \times \Delta})$  such that for all  $t$  in  $\Delta$ , the curve  $t \rightarrow \beta|_{X \times \{t\}}$  lifts  $\gamma - \gamma(0)$ . Therefore the curve  $\gamma$  comes from the class  $\beta + \text{pr}_1^* \gamma(0)$ , so it is analytic. The converse implication is proved in the same way.

(ii) Assume that  $\gamma$  is analytic, that  $\gamma(0) = 0$ , and that for any  $t$  in  $\Delta$ , the image of  $\gamma(t)$  in  $H^1_D(X, \mathbb{Q}(1))$  vanishes. If  $\tilde{\gamma}$  is a holomorphic lift of  $\gamma$  from  $\Delta$  to  $H^1(X, \mathcal{O}_X)$  vanishing at the origin, then  $\tilde{\gamma}(\Delta)$  lies in the image of  $H^1(X, \mathbb{Q}_X)$  in  $H^1(X, \mathcal{O}_X)$ . Therefore  $\tilde{\gamma}$  vanishes since it is continuous with values in a Hausdorff space and has countable image.  $\square$

We can define the derivative of an analytic curve with values in  $\text{Pic}(X)$ , it is a holomorphic curve with values in  $H^1(X, \mathcal{O}_X)$ . Also not used elsewhere in the paper, the next proposition will provide another geometric interpretation of Proposition 3.1.

**Proposition 3.5.** *Let  $X$  be a complex manifold and assume that  $H^1(X, \mathcal{O}_X)$  is separated and that  $H^1(X, \mathbb{Q})$  is countable. Let  $\gamma: \Delta \rightarrow \text{Pic}(X)$  be an analytic curve given by a class  $\alpha$  in  $\text{Pic}(X \times \Delta)$  and let  $\hat{\gamma}$  be a holomorphic lift of  $\gamma - \gamma(0)$  in  $H^1(X, \mathcal{O}_X)$ . For any  $t$ , we define  $\gamma'(t)$  in  $H^1(X, \mathcal{O}_X)$  by putting  $\gamma'(t) = \hat{\gamma}'(t)$ . Then:*

- (i) *The derivative of  $\gamma$  is independant of  $\hat{\gamma}$ .*
- (ii) *For any  $t$  in  $\Delta$ ,  $\gamma'(t)$  is the image of  $\alpha$  by the composition*

$$H^1_D(X \times \Delta, \mathbb{Q}(1)) \longrightarrow H^1(X \times \Delta, \Omega_{X \times \Delta}^1) \longrightarrow H^1(X \times \Delta, \mathcal{O}_X \boxtimes \Omega_\Delta^1) \longrightarrow H^1(X \times \{t\}, \mathcal{O}_X).$$

- (iii) *If  $\gamma'$  vanishes identically then  $\gamma$  is constant.*

*Proof.* The argument of the proof of Proposition 3.4 (ii) shows that two holomorphic lifts of  $\gamma$  differ by a constant. This gives the statement of point (i). Point (ii) results of the isomorphism between  $H^1(X \times \Delta, \mathcal{O}_{X \times \Delta})$  and  $H^1(X, \mathcal{O}_X) \widehat{\otimes} \mathcal{O}(\Delta)$ . For the point (iii), we use this last isomorphism to conclude that the vanishing of  $\gamma'$  implies that the image of  $\alpha$  in  $H^1(X \times \Delta, \mathcal{O}_X \boxtimes \Omega_\Delta^1)$  is zero. Then we can apply Proposition 3.1 for integral Deligne cohomology.  $\square$

#### 4. PROOF OF THE MAIN RESULTS

**4.1. Proof of Theorem 1.3.** Let us consider the line bundle  $\lambda(\mathcal{E}, \hat{f})$  on  $Y \times \Delta$ , where  $\hat{f} = f \times \text{id}$ . By Proposition 2.4, its restriction to a slice  $Y \times \{t\}$  is  $\lambda(\mathcal{E}_t, f)$ . Therefore the curve  $t \rightarrow \lambda(\mathcal{E}_t, f)$  is analytic. Let  $\alpha$  be the class in  $H^1_D(Y \times \Delta, \mathbb{Q}(1))$  defined by

$$\alpha = c_1^D(\lambda(\mathcal{E}, \hat{f})) - \left( \hat{f}_* [\text{ch}^D(\mathcal{E}) \text{td}^D(\text{T}_{X \times \Delta / Y \times \Delta})] \right)^{(2)}.$$

By Lemma 2.2,

$$c_1^H(\lambda(\mathcal{E}, \hat{f})) = \sum_p (-1)^p c_1^H(\mathbb{R}^p \hat{f}_* \mathcal{E})$$

in  $H^1(Y \times \Delta, \Omega_{Y \times \Delta}^1)$ . Therefore, thanks to the Grothendieck-Riemann-Roch theorem in Hodge cohomology [23],  $\alpha$  maps to zero in  $H^1(Y \times \Delta, \Omega_{Y \times \Delta}^1)$ . By Proposition 3.1, we obtain that for all  $t$  in  $\Delta$ ,  $\alpha|_{Y \times \{t\}}$  doesn't depend on  $t$ .

It remains to compute  $\alpha_{|Y \times \{t\}}$ . If we denote by  $[\cdot]_{\mathbb{D}}$  the Deligne cohomology class of an analytic cycle, for any class  $\beta$  in the rational Deligne cohomology ring of  $X \times \Delta$ , we have <sup>3</sup>

$$\begin{aligned} \{\hat{f}_* \beta\}_{|Y \times \{t\}} &= \text{pr}_{1*} \{\hat{f}_* \beta \cdot [Y \times \{t\}]_{\mathbb{D}}\} = \text{pr}_{1*} \hat{f}_* \{\beta \cdot \hat{f}^* [Y \times \{t\}]_{\mathbb{D}}\} \\ &= f_* \text{pr}_{1*} \{\beta \cdot [X \times \{t\}]_{\mathbb{D}}\} = f_* (\beta_{|X \times \{t\}}). \end{aligned}$$

Then  $\alpha_{|Y \times \{t\}} = c_1^{\mathbb{D}}(\lambda(\mathcal{E}_t, f)) - (f_* \{\text{ch}^{\mathbb{D}}(\mathcal{E}_t) \text{td}^{\mathbb{D}}(\mathbb{T}_{X/Y})\})^{(2)}$ . This gives (i). Remark now that for all  $t$  in  $\Delta$ , we have

$$c_1^{\mathbb{D}}(\lambda(\mathcal{E}_t, f)) - c_1^{\mathbb{D}}(\lambda(\mathcal{E}_0, f)) = \left( f_* \left\{ [\text{ch}^{\mathbb{D}}(\mathcal{E}_t) - \text{ch}^{\mathbb{D}}(\mathcal{E}_0)] \text{td}^{\mathbb{D}}(\mathbb{T}_{X/Y}) \right\} \right)^{(2)}.$$

Thus the curve  $t \rightarrow \left( f_* \left\{ [\text{ch}^{\mathbb{D}}(\mathcal{E}_t) - \text{ch}^{\mathbb{D}}(\mathcal{E}_0)] \text{td}^{\mathbb{D}}(\mathbb{T}_{X/Y}) \right\} \right)^{(2)}$  in  $H_{\mathbb{D}}^2(Y, \mathbb{Q}(1))$  can be lifted to the analytic curve  $\gamma: t \rightarrow \lambda(\mathcal{E}_t, f) - \lambda(\mathcal{E}_0, f)$  in  $\text{Pic}^0(Y)$ . Thanks to Proposition 3.4 (ii), this lift is unique. □

**4.2. Proof of Theorem 1.4.** Since  $\mathcal{L}$  lies in  $\text{Pic}^0(Y \times F)$ ,  $c_1^{\mathbb{D}}(\mathcal{L})^i = 0$  for  $i \geq 2$  so that

$$p_* \{c_1^{\mathbb{D}}(\mathcal{L}) q^* \text{td}_n^{\mathbb{D}}(F)\} = \left( p_* \{\text{ch}^{\mathbb{D}}(\mathcal{L}) \text{td}^{\mathbb{D}}(\mathbb{T}_{Y \times F/Y})\} \right)^{(2)}.$$

Then we can conclude using Theorem 1.3. Indeed, since the line bundle  $\mathcal{L}$  lies in  $\text{Pic}^0(Y \times F)$ , there exists a holomorphic family of holomorphic line bundles joining  $\mathcal{O}_{Y \times F}$  to  $\mathcal{L}$ . Thanks to Theorem 1.3, we are reduced to the case  $\mathcal{L} = \mathcal{O}_{Y \times F}$ . In this case, Theorem 1.4 is straightforward since

$$\det \text{Rp}_*(\mathcal{O}_{Y \times F}) \simeq \det [\mathcal{O}_Y \otimes_{\mathbb{C}_Y}^{\mathbb{L}} \text{R}\Gamma(F, \mathcal{O}_F)] \simeq \mathcal{O}_Y. \quad \square$$

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<sup>3</sup>This computation is a particular case of a base-change type formula in Deligne cohomology, which is currently not available in the literature for arbitrary base change morphisms.

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