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# AUTOMORPHISMS OF RATIONAL SURFACES WITH POSITIVE ENTROPY

*by*

Julie Déserti & Julien Grivaux

***Abstract.*** — A complex compact surface which carries an automorphism of positive topological entropy has been proved by Cantat to be either a torus, a K3 surface, an Enriques surface or a rational surface. Automorphisms of rational surfaces are quite mysterious and have been recently the object of intensive studies. In this paper, we construct several new examples of automorphisms of rational surfaces with positive topological entropy. We also explain how to count parameters in families of rational surfaces.
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## Introduction

If $X$ is a topological space and $f$ is a homeomorphism of $X$, the topological entropy of $f$, denoted by $\mathrm{h_{top}}(f)$, is a nonnegative number measuring the complexity of the dynamical system $(X,f)$. If $X$ is a compact Kähler manifold and $f$ is a biholomorphism, then $\mathrm{h_{top}}(f)=\sup_{1\leq p\leq \dim X}\delta_p(f)$, where $\delta_p(f)$ is the $p$-th dynamical degree of $f$, *i.e.* the spectral radius of $f^*$ acting on $\mathrm{H}^{p,p}(X)$ (*see* [**Gro03, Gro87, Yom87**]). When $X$ is a complex compact surface (Kähler or not) carrying a biholomorphism of positive topological entropy, Cantat has proved in [**Can99**] that $X$ is either a complex torus, a K3 surface, an Enriques surface or a nonminimal rational surface. Although automorphisms of complex tori are easy to describe, it is rather difficult to construct automorphisms on K3 surfaces or rational surfaces (constructions and dynamical properties of automorphisms of K3 surfaces can be found in [**Can99**] and [**McM02**]). The aim of this paper is to develop a general method to construct explicit examples of automorphisms of rational surfaces.

The first examples of rational surfaces endowed with biholomorphisms of positive entropy are due to Kummer and Coble ([**Cob61**]). The Coble surfaces are obtained by blowing up the ten nodes of a nodal sextic in $\mathbb{P}^2(\mathbb{C})$, the Kummer surfaces are desingularizations of quotients of complex 2-tori by involutions with fixed points. Obstructions to the existence of such biholomorphisms on rational surfaces are also known: if $X$ is a rational surface and $f$ is a biholomorphism of $X$ such that $\mathrm{h_{top}}(f)>0$, then the representation of the automorphisms group of $X$ in $\mathrm{GL}(\mathrm{Pic}(X))$ given by $g\mapsto g^*$ has infinite image. This implies by a result of Harbourne ([**Har87**]) that its kernel is finite, so that $X$ has no nonzero holomorphic vector field. A second consequence which follows from ([**Nag60**], Theorem 5) is that $X$ is *basic*, *i.e.* can be obtained by successive blowups from the projective plane $\mathbb{P}^2(\mathbb{C})$; furthermore, the number of blowups must be at least ten.

The first infinite families of examples have been constructed independently in [**McM07**] and [**BK09a**] by different methods. The rational surfaces are obtained by blowing up distinct points of $\mathbb{P}^2(\mathbb{C})$. The corresponding automorphisms come from birational quadratic maps of $\mathbb{P}^2(\mathbb{C})$ which are of the form $A\sigma$, where $A$ is in $\mathrm{PGL}(3;\mathbb{C})$ and $\sigma$ is the Cremona involution. These constructions yield a countable family of examples.

More recently, Bedford and Kim constructed arbitrary big holomorphic families of rational surfaces endowed with biholomorphisms of positive entropy. These families are explicitly given as follows:

***Theorem 1*** **([BK])**. — *Consider two integers $n\geq 3$ and $k\geq 2$ such that $n$ is odd and $(n,k)\neq(3,2)$. There exists a nonempty subset $C_k$ of $\mathbb{R}$ such that, if $c\in C_k$ and $a=(a_2,a_4,\dots,a_{[\frac{n-3}{2}]})\in\mathbb{C}^{[\frac{n-3}{2}]}$, the map*

$$f_a\colon (x:y:z)\to\Big(xz^{n-1}:z^n:x^n-yz^{n-1}+cz^n+\sum_{\substack{\ell=2\\ \ell\ even}}^{n-3}a_\ell x^{\ell+1}z^{n-\ell-1}\Big) \tag{0.1}$$

*can be lifted to an automorphism of a rational surface $X_a$, with positive topological entropy. The surfaces $X_a$ are obtained by blowing up $k$ infinitely near points of length $2n-1$ on the invariant line $\{x=0\}$ and form a holomorphic family over the parameter space given by the $a_j$'s. If $k=2$ and $n\geq 4$ is even, then, for $a$ and $a'$ near $0$, $X_a$ and $X_{a'}$ are biholomorphic if and only if $a=a'$.*

These examples are generalizations of the birational cubic map introduced by [**HV00b, HV00a**] and studied by [**Tak01a, Tak01b, Tak01c**].

Our paper is devoted to the construction of examples of rational surfaces with biholomorphisms of positive entropy in a more systematic way than what has been done before. Our strategy is the following: start with $f$ a birational map of $\mathbb{P}^2(\mathbb{C})$. By the standard factorization theorem for birational maps on surfaces as a composition of blow up and blow down, there exist two sets of (possibly infinitely near) points $\widehat{P}_1$ and $\widehat{P}_2$ in $\mathbb{P}^2(\mathbb{C})$ such that $f$ can be lifted to an automorphism between $\mathbb{P}^2(\mathbb{C})$ blown up in $\widehat{P}_1$ and $\mathbb{P}^2(\mathbb{C})$ blown up in $\widehat{P}_2$. The data of $\widehat{P}_1$ and $\widehat{P}_2$ allow to get automorphisms of rational surfaces in the left $\mathrm{PGL}(3;\mathbb{C})$-orbit of $f$ : assume that $k\in\mathbb{N}$ is fixed and let $\varphi$ be an element of $\mathrm{PGL}(3;\mathbb{C})$ such that $\widehat{P}_1$, $\varphi\widehat{P}_2$, $(\varphi f)\varphi\widehat{P}_2$, $\dots$, $(\varphi f)^{k-1}\varphi\widehat{P}_2$ have pairwise disjoint supports in $\mathbb{P}^2(\mathbb{C})$ and that $(\varphi f)^k\varphi\widehat{P}_2=\widehat{P}_1$. Then $\varphi f$ can be lifted to an automorphism of $\mathbb{P}^2(\mathbb{C})$ blown up at $\widehat{P}_1$, $\varphi\widehat{P}_2$, $(\varphi f)\varphi\widehat{P}_2$, $\dots$, $(\varphi f)^{k-1}\varphi\widehat{P}_2$. Furthermore, if the conditions above are satisfied for a holomorphic family of $\varphi$, we get a holomorphic family of rational surfaces

(whose dimension is at most eight). Therefore, we see that the problem of lifting an element in the $\mathrm{PGL}(3;\mathbb{C})$-orbit of $f$ to an automorphism is strongly related to the equation $u(\widehat{P}_2)=\widehat{P}_1$, where $u$ is a germ of biholomorphism of $\mathbb{P}^2(\mathbb{C})$ mapping the support of $\widehat{P}_2$ to the support of $\widehat{P}_1$. In concrete examples, when $\widehat{P}_1$ and $\widehat{P}_2$ are known, this equation can actually be solved and involves polynomial equations in the Taylor expansions of $u$ at the various points of the support of $\widehat{P}_2$. It is worth pointing out that in the generic case, $\widehat{P}_1$ and $\widehat{P}_2$ consist of the same number $d$ of distinct points in the projective plane, and the equation $u(\widehat{P}_2)=\widehat{P}_1$ gives $2d$ independent conditions on $u$ (which is the maximum possible number if $\widehat{P}_1$ and $\widehat{P}_2$ have length $d$). Conversely, infinitely near points can considerably decrease the number of conditions on $u$ as shown in our examples. This explains why holomorphic families of automorphisms of rational surfaces occur when multiple blowups are made.

Let us describe the examples we obtain. We do not deal with the case of the Cremona involution $\sigma$ because birational maps of the type $A\sigma$, with $A$ in $\mathrm{PGL}(3;\mathbb{C})$, are linear fractional recurrences studied in [**BK06, BK09a**], and our approach does not give anything new in this case. Our first examples proceed from a family $(\Phi_n)_{n\geq 2}$ of birational maps of $\mathbb{P}^2(\mathbb{C})$ given by $\Phi_n(x:y:z)=(xz^{n-1}+y^n:yz^{n-1}:z^n)$. These birational maps are very special because their exceptional locus is a single line, the line $\{z=0\}$; and their indeterminacy locus is a single point on this line, the point $P=(1:0:0)$. We compute explicitly $\widehat{P}_1$ and $\widehat{P}_2$ and obtain sequences of blowups already done in [**BK**]. Then we exhibit various families of solutions of the equation $(\varphi\Phi_n)^{k-1}\varphi\widehat{P}_2=\widehat{P}_1$ for $k=2,3$ and $\varphi$ in $\mathrm{PGL}(3;\mathbb{C})$; this yields automorphisms of rational surfaces with positive topological entropy. Many of our examples are similar or even sometimes linearly conjugate to those constructed in [**BK**]: they have an invariant line and the sequences of blowups are of the same type. However, for $(n,k)=(3,2)$ we have found an example of a different type satisfying that $P$, $\varphi P$ and $\varphi\Phi_3\varphi P$ are not on the same line:

***Theorem 2***. — *If $\alpha$ is a complex number in $\mathbb{C}\setminus\{0,1\}$, let $\varphi_\alpha$ be the element of $\mathrm{PGL}(3;\mathbb{C})$ given by*

$$\varphi_\alpha=\begin{bmatrix}\alpha & 2(1-\alpha) & 2+\alpha-\alpha^2\\ -1 & 0 & \alpha+1\\ 1 & -2 & 1-\alpha\end{bmatrix}.$$

*The map $\varphi_\alpha\Phi_3$ has no invariant line and is conjugate to an automorphism of $\mathbb{P}^2(\mathbb{C})$ blown up in 15 points; its first dynamical degree is $\frac{3+\sqrt{5}}{2}$.*

Before going further, let us introduce the following terminology: let $U$ be an open set of $\mathbb{C}^n$, $(\varphi_\alpha)_{\alpha\in U}$ be a holomorphic family of matrices in $\mathrm{PGL}(3;\mathbb{C})$ parameterized by $U$ and $f$ be a birational map of the projective plane. We say that the family $(\varphi_\alpha f)_{\alpha\in U}$ is locally holomorphically trivial if for every parameter $\alpha_0$ in $U$ there exists a family of matrices $M_\alpha$ of $\mathrm{PGL}(3;\mathbb{C})$ parameterized by a neighborhood of $\alpha_0$ such that $M_{\alpha_0}=\mathrm{Id}$ and for all $\alpha$ near $\alpha_0$, $\varphi_\alpha f=M_\alpha^{-1}(\varphi_{\alpha_0}f)M_\alpha$. The family of birational maps of Theorem 2, as well as our other examples, are all locally holomorphically trivial. After many attempts to produce nontrivial holomorphic families, we have been led to conjecture that all families $\varphi_\alpha\Phi_n$ which yield automorphisms of positive topological entropy are locally holomorphically trivial. Nevertheless this phenomenon is not a generality. Theorem 1 gives, for $n\geq 5$, examples of families of birational maps of the type $\varphi_\alpha f$ conjugate to families of automorphisms of positive entropy on rational surfaces which are not locally holomorphically trivial (*see* section 3.4).

We finally carry out our method for another birational cubic map, namely $f(x:y:z)=\Big(y^2z:x(xz+y^2):y(xz+y^2)\Big)$. This map blows down a conic and a line intersecting transversally the conic along the two indeterminacy points. We obtain the following result:

***Theorem 3***. — *Let $\alpha$ be a nonzero complex number and*

$$\varphi_\alpha=\begin{bmatrix}\frac{2\alpha^3}{343}(37\mathrm{i}\sqrt{3}+3) & \alpha & -\frac{2\alpha^2}{49}(5\mathrm{i}\sqrt{3}+11)\\ \frac{\alpha^2}{49}(-15+11\mathrm{i}\sqrt{3}) & 1 & -\frac{\alpha}{14}(5\mathrm{i}\sqrt{3}+11)\\ -\frac{\alpha}{7}(2\mathrm{i}\sqrt{3}+3) & 0 & 0\end{bmatrix}$$

*The map $\varphi_\alpha f$ is conjugate to an automorphism of $\mathbb{P}^2(\mathbb{C})$ blown up in 15 points, its first dynamical degree is $\frac{3+\sqrt{5}}{2}$. Besides, the family $\varphi_\alpha f$ is locally holomorphically trivial.*

This example seems completely new, the configuration of exceptional curves shows that it is not linearly conjugate to any of the already known examples (although we do not know if it is the case when linear conjugacy is replaced by birational conjugacy).

In the last part of the paper, we propose an approach in order to define and count the generic numbers of parameters in a given family of rational surfaces. The good setting for this study is the theory of deformations of complex compact manifolds of Kodaira and Spencer. A deformation is a triplet $(\mathfrak{X},\pi,B)$ such that $\mathfrak{X}$ and $B$ are complex manifolds and $\pi\colon \mathfrak{X}\to B$ is a proper holomorphic submersion. If $X$ is a complex compact manifold, a deformation of $X$ is a deformation $(\mathfrak{X},\pi,B)$ such that for a specific $b$ in $B$, $\mathfrak{X}_b$ is biholomorphic to $X$.

If $(\mathfrak{X},\pi,B)$ is a deformation and $X$ is a fiber of $\mathfrak{X}$, Ehresmann's fibration theorem implies that $\mathfrak{X}$ is diffeomorphic to $X\times B$ over $B$. Therefore, a deformation can also be seen as a family of integrable complex structures $(J_b)_{b\in B}$ on a fixed differentiable manifold $X$, varying holomorphically with $b$. The main tool of the theory is the Kodaira-Spencer map: if $(\mathfrak{X},\pi,B)$ is a deformation and $b_0$ is a point of $B$, the Kodaira-Spencer map of $\mathfrak{X}$ at $b_0$ is a linear map $\mathrm{KS}_{b_0}(\mathfrak{X})\colon \mathrm{T}_{b_0}B\to \mathrm{H}^1(\mathfrak{X}_{b_0},\mathrm{T}\mathfrak{X}_{b_0})$, which is intuitively the differential of the map $b\mapsto J_b$ at $b_0$. If $(\mathfrak{X},\pi,B)$ is a deformation of projective varieties, we prove that the kernels of $\mathrm{KS}_b(\mathfrak{X})$ have generically the same dimension and define a holomorphic subbundle $E_{\mathfrak{X}}$ of $\mathrm{T}B$. This leads to a natural definition of the generic number $m(\mathfrak{X})$ of parameters of a deformation $\mathfrak{X}$ as $m(\mathfrak{X})=\dim B-\operatorname{rank}E_{\mathfrak{X}}$. In the case $m(\mathfrak{X})=\dim B$, we say that $\mathfrak{X}$ is generically effective; this means that for $b$ generic in $B$, $\mathrm{KS}_b(\mathfrak{X})$ is injective. This is slightly weaker than requiring that different fibers of $\mathfrak{X}$ are not biholomorphic (as in [**BK**]), but much easier to verify in concrete examples.

From a theoretical point of view, deformations of basic rational surfaces are easy to understand. The moduli space of ordered (possibly infinitely near) points in the projective plane $\mathbb{P}^2$ of length $N$ is a smooth projective variety $S_N$ of dimension $2N$ obtained by blowing up successive incidence loci. There exists a natural deformation $\mathfrak{X}_N$ over $S_N$ whose fibers are rational surfaces: if $\widehat{P}$ is in $S_N$, then $(\mathfrak{X}_N)_{\widehat{P}}$ is equal to $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$. This deformation is complete at any point of $S_N$, *i.e.* every deformation of a fiber of $\mathfrak{X}_N$ is locally induced by $\mathfrak{X}_N$ up to holomorphic base change. Therefore, if $(\mathfrak{X},\pi,B)$ is a deformation of a basic rational surface, all the fibers in a small neighborhood of the central fiber remain rational and basic (this is no longer the case for nonbasic rational surfaces).

There is a natural $\mathrm{PGL}(3;\mathbb{C})$-action on $S_N$ which can be lifted on $\mathfrak{X}_N$. This action can be used to describe the Kodaira-Spencer map of $\mathfrak{X}_N$ : if $\widehat{P}$ is in $S_N$, $\mathrm{KS}_{\widehat{P}}(\mathfrak{X}_N)$ is surjective and its kernel is the tangent space at $\widehat{P}$ of the $\mathrm{PGL}(3;\mathbb{C})$-orbit of $\widehat{P}$ in $S_N$. If $N\geq 4$, let $S_N^\dagger$ be the Zariski-dense open set of points $\widehat{P}$ in $S_N$ such that $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$ has no nonzero holomorphic vector field. Since $\mathfrak{X}_N$ is complete, families of rational surfaces with no nonzero holomorphic vector fields can locally be described as the pullback of $\mathfrak{X}_N$ by a holomorphic map from the parameter space to $S_N^\dagger$. We provide a practical way to count the generic number of parameters in such families:

***Theorem 4***. — *Let $U$ be an open set in $\mathbb{C}^n$, $N$ be an integer greater than or equal to 4 and $\psi\colon U\to S_N^\dagger$ be a holomorphic map. Then $m(\psi^*\mathfrak{X}_N)$ is the smallest integer $d$ such that for all generic $\alpha$ in $U$, there exist a neighborhood $\Omega$ of 0 in $\mathbb{C}^{n-d}$ and two holomorphic maps $\gamma\colon\Omega\to U$ and $M\colon\Omega\to\mathrm{PGL}(3;\mathbb{C})$ such that:*
- *$\gamma_*(0)$ is injective,*
- *$\gamma(0)=\alpha$ and $M(0)=\mathrm{Id}$,*
- *for all $t$ in $\Omega$, $\psi(\gamma(t))=M(t)\,\psi(\alpha)$.*

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## 1. Algebraic and dynamical properties of birational maps

A *rational map* from $\mathbb{P}^2(\mathbb{C})$ into itself is a map of the following type

$$f\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C}), \qquad (x:y:z) \mapsto \Big(f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)\Big)$$

where the $f_i$'s are homogeneous polynomials of the same degree without common factor. The *degree* of $f$ is equal to the degree of the $f_i$'s. A *birational map* is a rational map whose inverse is also rational. The birational maps of $\mathbb{P}^2(\mathbb{C})$ into itself form a group which is called the *Cremona group* and denoted $\mathrm{Bir}(\mathbb{P}^2)$.

If $f$ is a birational map, $\mathrm{Ind}\, f$ denotes the finite set of points blown up by $f$; this is the set of the common zeroes of the $f_i's$. We say that $\mathrm{Ind}\, f$ is the *indeterminacy locus* of $f$. The set of curves collapsed by $f$, called *exceptional set* of $f$, is denoted $\mathrm{Exc}\, f$; it can be obtained by computing the jacobian determinant of $f$.

The degree is not a birational invariant; if $f$ and $g$ are in $\mathrm{Bir}(\mathbb{P}^2)$, then usually $\deg(gfg^{-1}) \neq \deg f$. Nevertheless there exist two strictly positive constants $\alpha$ and $\beta$ such that for all integer $n$ the following holds:

$$\alpha \deg f^n \leq \deg(gf^ng^{-1}) \leq \beta \deg f^n.$$

This means that the degree growth is a birational invariant.

Let us recall the notion of first dynamical degree introduced in [**Fri95, RS97**]: if $f$ is in $\mathrm{Bir}(\mathbb{P}^2)$, the *first dynamical degree* of $f$ is defined by

$$\lambda(f) = \lim(\deg f^n)^{1/n}.$$

More generally we can define this notion for bimeromorphic maps of a Kähler surface. A bimeromorphic map $f$ on a Kähler surface $X$ induces a map $f^*$ from $\mathrm{H}^{1,1}(X,\mathbb{R})$ into itself. The first dynamical degree of $f$ is given by

$$\lambda(f) = \lim(|(f^n)^*|)^{1/n}.$$

Let $f$ be a map on a complex compact Kähler surface; the notions of first dynamical degree and topological entropy $\mathrm{h_{top}}(f)$ are related by the following formula: $\mathrm{h_{top}}(f) = \log \lambda(f)$ (*see* [**Gro03, Gro87, Yom87**]).
Diller and Favre characterize the birational maps of $\mathbb{P}^2(\mathbb{C})$ up to birational conjugacy; the case of automorphisms with quadratic growth is originally due to Gizatullin.

***Theorem 1.1*** ([**DF01, Giz80**]). — *Let $f$ be a bimeromorphic map of a Kähler surface. Up to bimeromorphic conjugacy, one and only one of the following holds.*

- *The sequence $(|(f^n)^*|)_{n\in\mathbb{N}}$ is bounded, $f$ is an automorphism on some rational surface and an iterate of $f$ is an automorphism isotopic to the identity.*
- *The sequence $(|(f^n)^*|)_{n\in\mathbb{N}}$ grows linearly and $f$ preserves a rational fibration; in this case $f$ is not an automorphism.*
- *The sequence $(|(f^n)^*|)_{n\in\mathbb{N}}$ grows quadratically and $f$ is an automorphism preserving an elliptic fibration.*
- *The sequence $(|(f^n)^*|)_{n\in\mathbb{N}}$ grows exponentially.*

*In the second (resp. third) case, the invariant fibration is unique. In the three first cases $\lambda(f)$ is equal to $1$, in the last case $\lambda(f)$ is strictly larger than $1$.*

***Examples 1.2***. —

- If $f$ is an automorphism of $\mathbb{P}^2(\mathbb{C})$ or a birational map of finite order, then $(\deg f^n)_n$ is bounded.
- The map $f\colon (x:y:z) \dashrightarrow (xy : yz : z^2)$ satisfies that $(\deg f^n)_n$ grows linearly.
- A Hénon map, *i.e.* an automorphism of $\mathbb{C}^2$ of the form

$$f\colon (x,y) \mapsto (y, P(y) - \delta x), \qquad \delta \in \mathbb{C}^*, P \in \mathbb{C}[y], \deg P \geq 2,$$

can be viewed as a birational map of $\mathbb{P}^2(\mathbb{C})$ and $\lambda(f) = \deg P > 1$.

Let $f$ be a bimeromorphic map on a Kähler surface $X$. To relate $\lambda(f)$ to the spectral radius of $f^*$ we need the equality $(f^*)^n = (f^n)^*$ for all $n$. When it occurs we say that $f$ is *analytically stable* ([**FS95, Sib99**]). An other characterization can be found in ([**DF01**], Theorem 1.14): the map $f$ is analytically stable if and only if there is no curve $\mathcal{C} \subset X$ such that $f^k(\mathcal{C}) \subset \operatorname{Ind} f$ for some integer $k \geq 0$. Up to a change of coordinates, one can always assume that a bimeromorphic map of a Kähler surface is analytically stable (*see* [**DF01**]). For instance, if $f$ is an automorphism, then $f$ is analytically stable and $\lambda(f)$ is the spectral radius of $f^*$. Besides, since $f^*$ is defined over $\mathbb{Z}$, $\lambda(f)$ is also the spectral radius of $f_*$.

Let us recall some properties about blowups of the complex projective plane. Let $p_1, \ldots, p_n$ be $n$ (possibly infinitely near) points in $\mathbb{P}^2(\mathbb{C})$ and $\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2$ denote the complex manifold obtained by blowing up $\mathbb{P}^2(\mathbb{C})$ at $p_1, \ldots, p_n$.

- We may identify $\mathrm{Pic}(\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2)$ and $\mathrm{H}^2(\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2, \mathbb{Z})$ so we won't make any difference in using them.
- If $\pi\colon \mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2 \to \mathbb{P}^2(\mathbb{C})$ is the sequel of blowups of the $n$ points $p_1, \ldots, p_n$, H the class of a generic line and $\mathrm{E}_j = \pi^{-1}(p_j)$ the exceptional fibers, then $\{\mathrm{H}, \mathrm{E}_1, \ldots, \mathrm{E}_n\}$ is a basis of $\mathrm{Pic}(\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2)$.
- Assume that $n \leq 9$ and that $f\colon \mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2 \to \mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2$ is an automorphism. Then the topological entropy of $f$ vanishes. If $n \leq 8$ then there exists an integer $k$ such that $f^k$ descends to a linear map of $\mathbb{P}^2(\mathbb{C})$ (*see* [**Dil**]).

In the sequel, $\mathbb{P}^2$ will denote the complex projective plane.

## 2. Birational maps whose exceptional locus is a line, I

Let us consider the sequence of birational maps defined by $\Phi_n = (xz^{n-1} + y^n : yz^{n-1} : z^n)$, with $n \geq 3$. In this section we first construct for every integer $n \geq 3$ two infinitely near points $\widehat{P}_1$ and $\widehat{P}_2$ such that $\Phi_n$ induces an isomorphism between $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$. Then we give theoric conditions to produce automorphisms $\varphi$ of $\mathbb{P}^2$ such that $\varphi\Phi_n$ is conjugate to an automorphism on a surface obtained from $\mathbb{P}^2$ by successive blowups.

In the affine chart $z = 1$, $\Phi_n$ is a polynomial automorphism which preserves the fibration $y = \mathrm{cte}$, *i.e.* it is an elementary automorphism; thus $\lambda(\Phi_n) = 1$. More precisely the sequence $(\deg \Phi_n^k)_{k\in\mathbb{N}}$ is bounded, so $\Phi_n$ is conjugate to an automorphism on some rational surface $X$ and an iterate of $\Phi_n$ is conjugate to an automorphism isotopic to the identity. The map $\Phi_n$ blows up one point $P = (1:0:0)$ and blows down one curve $\Delta = \{z = 0\}$.

**2.1. First step: description of the sequence of blowups. —** We start by the description of two points $\widehat{P}_1$ and $\widehat{P}_2$ infinitely near $P$ of length $2n-1$ such that $\Phi_n$ can be lifted to an automorphism between $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$.

*Convention*: if $\mathcal{D}$ (resp. $\mathcal{D}_i$) is a curve on a surface $X$, we will denote by $\mathcal{D}_1$ (resp. $\mathcal{D}_{i+1}$) the strict transform of this curve in $X$ blown up at a point.

Let us blow up $P$ at left and right; set $y = u_1$, $z = u_1v_1$ then $\Delta_1 = \{v_1 = 0\}$ and the exceptional divisor E is given by $\{u_1 = 0\}$. We can also set $y = r_1s_1$, $z = s_1$; in these coordinates $\mathrm{E} = \{s_1 = 0\}$. We get

$$\Phi_n\colon (u_1, v_1) \to (u_1, u_1v_1)_{(y,z)} \to \left(v_1^{n-1} + u_1 : u_1v_1^{n-1} : u_1v_1^n\right) = \left(\frac{u_1v_1^{n-1}}{v_1^{n-1}+u_1}, \frac{u_1v_1^n}{v_1^{n-1}+u_1}\right)_{(y,z)} \to \left(\frac{u_1v_1^{n-1}}{v_1^{n-1}+u_1}, v_1\right)_{(u_1,v_1)};$$

hence E is fixed, $\Delta_1$ is blown down to $P_1 = (0,0)_{(u_1,v_1)} = \mathrm{E} \cap \Delta_1$ and $P_1$ is an indeterminacy point. One also can see that

$$\Phi_n\colon (r_1, s_1) \to (r_1s_1, s_1)_{(y,z)} \to \left(1 + r_1^ns_1 : r_1s_1 : s_1\right) \to \left(\frac{r_1s_1}{1+r_1^ns_1}, \frac{s_1}{1+r_1^ns_1}\right)_{(y,z)} \to \left(r_1, \frac{s_1}{1+r_1^ns_1}\right)_{(r_1,s_1)}.$$

Then we blow up $P_1$ at left and right. Set $u_1 = u_2$, $v_1 = u_2v_2$ so the exceptional divisor F is given by $\{u_2 = 0\}$ and $\Delta_2$ by $\{v_2 = 0\}$. There is an other system of coordinates $(r_2, s_2)$ with $u_1 = r_2s_2$, $v_1 = s_2$; in this system, $\mathrm{F} = \{s_2 = 0\}$

and $\mathrm{E}_1 = \{r_2 = 0\}$. On the one hand

$$\Phi_n\colon (u_2,v_2) \to (u_2,u_2v_2)_{(u_1,v_1)} \to \left(u_2^{n-2}v_2^{n-1}+1 : u_2^{n-1}v_2^{n-1} : u_2^n v_2^n\right) = \left(\frac{u_2^{n-1}v_2^{n-1}}{u_2^{n-2}v_2^{n-1}+1}, \frac{u_2^n v_2^n}{u_2^{n-2}v_2^{n-1}+1}\right)_{(y,z)}$$
$$\to \left(\frac{u_2^{n-1}v_2^{n-1}}{u_2^{n-2}v_2^{n-1}+1}, u_2v_2\right)_{(u_1,v_1)} \to \left(\frac{u_2^{n-2}v_2^{n-2}}{u_2^{n-2}v_2^{n-1}+1}, u_2v_2\right)_{(r_2,s_2)};$$

on the other hand

$$\Phi_n\colon (r_2,s_2) \to (r_2s_2,s_2)_{(u_1,v_1)} \to \left(s_2^{n-2}+r_2 : r_2s_2^{n-1} : r_2s_2^n\right) \to \left(\frac{r_2s_2^{n-1}}{s_2^{n-2}+r_2}, \frac{r_2s_2^n}{s_2^{n-2}+r_2}\right)_{(y,z)}$$
$$\to \left(\frac{r_2s_2^{n-1}}{s_2^{n-2}+r_2}, s_2\right)_{(u_1,v_1)} \to \left(\frac{r_2s_2^{n-2}}{s_2^{n-2}+r_2}, s_2\right)_{(r_2,s_2)}.$$

So $A_1 = (0,0)_{(r_2,s_2)}$ is an indeterminacy point and F is blown down to $A_1$.

Moreover

$$\Phi_n\colon (y,z) \to (z^{n-1}+y^n : yz^{n-1} : z^n) \to \left(\frac{yz^{n-1}}{y^n+z^{n-1}}, \frac{z^n}{y^n+z^{n-1}}\right)_{(y,z)} \to \left(\frac{yz^{n-1}}{y^n+z^{n-1}}, \frac{z}{y}\right)_{(u_1,v_1)} \to \left(\frac{y^2z^{n-2}}{y^n+z^{n-1}}, \frac{z}{y}\right)_{(r_2,s_2)};$$

hence $\Delta_2$ is blown down to $A_1$.

The $(n-3)$ next steps are of the same type, so we will write it one time with some indice $k$, $1 \le k \le n-3$.
We will blow up $A_k = (0,0)_{(r_{k+1},s_{k+1})}$ at left and right. Set

$$r_{k+1} = u_{k+2}, \qquad s_{k+1} = u_{k+2}v_{k+2} \qquad \& \qquad r_{k+1} = r_{k+2}s_{k+2}, \qquad s_{k+1} = s_{k+2}.$$

Let us remark that $(u_{k+2},v_{k+2})$ (resp. $(r_{k+2},s_{k+2})$) is a system of coordinates in which the exceptional divisor $\mathrm{G}^k$ is given by $\mathrm{G}^k = \{u_{k+2} = 0\}$ and $\mathrm{G}_1^{k-1} = \{v_{k+2} = 0\}$ (resp. $\mathrm{G}^k = \{s_{k+2} = 0\}$, $\mathrm{E}_{k+1} = \{r_{k+2} = 0\}$). We have

$$\Phi_n\colon (u_{k+2},v_{k+2}) \to (u_{k+2},u_{k+2}v_{k+2})_{(r_{k+1},s_{k+1})} \to \left(1+u_{k+2}^{n-k-2}v_{k+2}^{n-k-1} : u_{k+2}^{n-1}v_{k+2}^{n-1} : u_{k+2}^n v_{k+2}^n\right)$$
$$\to \left(\frac{u_{k+2}^{n-k-2}v_{k+2}^{n-k-2}}{1+u_{k+2}^{n-k-2}v_{k+2}^{n-k-1}}, u_{k+2}v_{k+2}\right)_{(r_{k+2},s_{k+2})}$$

and

$$\Phi_n\colon (r_{k+2},s_{k+2}) \to (r_{k+2}s_{k+2},s_{k+2})_{(r_{k+1},s_{k+1})} \to \left(r_{k+2}+s_{k+2}^{n-k-2} : r_{k+2}s_{k+2}^{n-1} : r_{k+2}s_{k+2}^n\right)$$
$$\to \left(\frac{r_{k+2}s_{k+2}^{n-k-2}}{r_{k+2}+s_{k+2}^{n-k-2}}, s_{k+2}\right)_{(r_{k+2},s_{k+2})}.$$

So $A_{k+1} = \mathrm{G}^k \cap \mathrm{E}_{k+1} = (0,0)_{(r_{k+2},s_{k+2})}$ is an indeterminacy point and $\mathrm{G}^k$, $\mathrm{G}_1^{k-1}$ are blown down to $A_{k+2}$. Therefore

$$\Phi_n\colon (y,z) \to \left(z^{n-1}+y^n : yz^{n-1} : z^n\right) = \left(\frac{yz^{n-1}}{z^{n-1}+y^n}, \frac{z^n}{z^{n-1}+y^n}\right)_{(y,z)} \to \left(\frac{y^{k+2}z^{n-k-2}}{z^{n-1}+y^n}, \frac{z}{y}\right)_{(r_{k+2},s_{k+2})}$$

so $\Delta_{k+2}$ is also blown down to $A_{k+1}$.
One can remark that

$$\Phi_n\colon (u_2,v_2) \to \left(\frac{u_2^{n-2}v_2^{n-2}}{1+u_2^{n-2}v_2^{n-1}}, u_2v_2\right)_{(r_2,s_2)} \to \left(\frac{u_2^{n-k-2}v_2^{n-k-2}}{1+u_2^{n-2}v_2^{n-1}}, u_2v_2\right)_{(r_{k+2},s_{k+2})},$$

hence $\mathrm{F}_k$ is blown down to $A_{k+1}$.

Let us now blow up $A_{n-2}$ at left and right. Set $r_{n-1}=u_n$, $s_{n-1}=u_nv_n$, and $r_{n-1}=r_ns_n$, $s_{n-1}=s_n$. Let us remark that $(u_n,v_n)$ (resp. $(r_n,s_n)$) is a system of coordinates in which the exceptional divisor is given by $\mathrm{G}^{n-2}=\{u_n=0\}$ and $\mathrm{G}_1^{n-3}=\{v_n=0\}$ (resp. $\mathrm{G}^{n-2}=\{s_n=0\}$). We compute

$$\Phi_n\colon (u_n,v_n)\to (u_n,u_nv_n)_{(r_{n-1},s_{n-1})}\to \left(1+v_n : u_n^{n-1}v_n^{n-1} : u_n^nv_n^n\right)\to \left(\frac{1}{1+v_n},u_nv_n\right)_{(r_n,s_n)}$$

and

$$\Phi_n\colon (r_n,s_n)\to (r_ns_n,s_n)_{(r_{n-1},s_{n-1})}\to \left(1+r_n : r_ns_n^{n-1} : r_ns_n^n\right)\to \left(\frac{r_n}{1+r_n},s_n\right)_{(r_n,s_n)}.$$

This implies that $\mathrm{G}^{n-2}$ is fixed, $\mathrm{G}_1^{n-3}$ is blown down to the point $S=(1,0)_{(r_n,s_n)}$ of $\mathrm{G}^{n-2}$ and the point $T=(-1,0)_{(r_n,s_n)}$ of $\mathrm{G}^{n-2}$ is an indeterminacy point.

On the one hand

$$\Phi_n\colon (y,z)\to \left(\frac{y^{n-1}z}{z^{n-1}+y^n},\frac{z}{y}\right)_{(r_{n-1},s_{n-1})}\to \left(\frac{y^n}{z^{n-1}+y^n},\frac{z}{y}\right)_{(r_n,s_n)}$$

so $\Delta_n$ is blown down to $S$; on the other hand

$$\Phi_n\colon (u_2,v_2)\to \left(\frac{u_2^{n-2}v_2^{n-2}}{1+u_2^{n-2}v_2^{n-1}},u_2v_2\right)_{(r_2,s_2)}\to \left(\frac{u_2v_2}{1+u_2^{n-2}v_2^{n-1}},u_2v_2\right)_{(r_{n-1},s_{n-1})}\to \left(\frac{1}{1+u_2^{n-2}v_2^{n-1}},u_2v_2\right)_{(r_n,s_n)},$$

hence $\mathrm{F}_{n-2}$ is blown down to $S$.

Now we blow up $T$ at left and $S$ at right

$$\begin{cases} r_n=u_{n+1}-1\\ s_n=u_{n+1}v_{n+1}\end{cases}\qquad \mathrm{H}=\{u_{n+1}=0\}\qquad\qquad \begin{cases} r_n=a_{n+1}+1\\ s_n=a_{n+1}b_{n+1}\end{cases}\qquad \mathrm{K}=\{a_{n+1}=0\}$$

$$\begin{cases} r_n=r_{n+1}s_{n+1}-1\\ s_n=s_{n+1}\end{cases}\qquad \mathrm{H}=\{s_{n+1}=0\}\qquad\qquad \begin{cases} r_n=c_{n+1}d_{n+1}+1\\ s_n=d_{n+1}\end{cases}\qquad \mathrm{K}=\{d_{n+1}=0\}$$

We obtain

$$\begin{aligned}\Phi_n\colon (u_{n+1},v_{n+1})&\to (u_{n+1}-1,u_{n+1}v_{n+1})_{(r_n,s_n)}\to \left(1:(u_{n+1}-1)u_{n+1}^{n-2}v_{n+1}^{n-1}:(u_{n+1}-1)u_{n+1}^{n-1}v_{n+1}^n\right)\\ &=\left((u_{n+1}-1)u_{n+1}^{n-2}v_{n+1}^{n-1},(u_{n+1}-1)u_{n+1}^{n-1}v_{n+1}^n\right)_{(y,z)}\to \left((u_{n+1}-1)u_{n+1}^{n-2}v_{n+1}^{n-1},u_{n+1}v_{n+1}\right)_{(u_1,v_1)}\\ &\to \left((u_{n+1}-1)u_{n+1}^{n-3}v_{n+1}^{n-2},u_{n+1}v_{n+1}\right)_{(r_2,s_2)}\to\ldots\to \left((u_{n+1}-1)v_{n+1},u_{n+1}v_{n+1}\right)_{(r_{n-1},s_{n-1})}\end{aligned}$$

and

$$\begin{aligned}\Phi_n\colon (r_{n+1},s_{n+1})&\to (r_{n+1}s_{n+1}-1,s_{n+1})_{(r_n,s_n)}\to \left(r_{n+1}:(r_{n+1}s_{n+1}-1)s_{n+1}^{n-2}:(r_{n+1}s_{n+1}-1)s_{n+1}^{n-1}\right)\\ &=\left(\frac{(r_{n+1}s_{n+1}-1)s_{n+1}^{n-2}}{r_{n+1}},\frac{(r_{n+1}s_{n+1}-1)s_{n+1}^{n-1}}{r_{n+1}}\right)_{(y,z)}\to \left(\frac{(r_{n+1}s_{n+1}-1)s_{n+1}^{n-2}}{r_{n+1}},s_{n+1}\right)_{(u_1,v_1)}\\ &\to \left(\frac{(r_{n+1}s_{n+1}-1)s_{n+1}^{n-3}}{r_{n+1}},s_{n+1}\right)_{(r_2,s_2)}\to\ldots\to \left(\frac{r_{n+1}s_{n+1}-1}{r_{n+1}},s_{n+1}\right)_{(r_{n-1},s_{n-1})}.\end{aligned}$$

Thus H is sent on $\mathrm{G}_2^{n-3}$ and $B_1=(0,0)_{(r_{n+1},s_{n+1})}$ is an indeterminacy point. Moreover,

$$\Phi_n\colon (u_n,v_n)\to \left(\frac{1}{1+v_n},u_nv_n\right)_{(r_n,s_n)}\to \left(-\frac{v_n}{1+v_n},-u_n(1+v_n)\right)_{(a_{n+1},b_{n+1})},$$

$$\Phi_n\colon (y,z) \to \left(\frac{y^n}{z^{n-1}+y^n}, \frac{z}{y}\right)_{(r_n,s_n)} \to \left(-\frac{yz^{n-2}}{z^{n-1}+y^n}, \frac{z}{y}\right)_{(c_{n+1},d_{n+1})}$$

and

$$\Phi_n\colon (u_2,v_2) \to \left(\frac{1}{1+u_2^{n-2}v_2^{n-1}}, u_2v_2\right)_{(r_n,s_n)} \to \left(-\frac{u_2^{n-3}v_2^{n-2}}{1+u_2^{n-2}v_2^{n-1}}, u_2v_2\right)_{(c_{n+1},d_{n+1})}.$$

Therefore $\mathrm{G}_2^{n-3}$ is sent on K and $\Delta_{n+1}$, $\mathrm{F}_{n-1}$ are blown down to $C_1=(0,0)_{(c_{n+1},d_{n+1})}$.

The $(n-3)$ following steps are the same, so we will write it one time with some indice $\ell$, $1\le\ell\le n-3$.
We blow up $B_\ell=(0,0)_{(r_{n+\ell},s_{n+\ell})}$ at left and $C_\ell=(0,0)_{(c_{n+\ell},d_{n+\ell})}$ at right

$$\begin{cases} r_{n+\ell}=u_{n+\ell+1} \\ s_{n+\ell}=u_{n+\ell+1}v_{n+\ell+1} \end{cases} \qquad \mathrm{L}^\ell=\{u_{n+\ell+1}=0\} \qquad\qquad \begin{cases} c_{n+\ell}=a_{n+\ell+1} \\ d_{n+\ell}=a_{n+\ell+1}b_{n+\ell+1} \end{cases} \qquad \mathrm{M}^\ell=\{a_{n+\ell+1}=0\}$$

$$\begin{cases} r_{n+\ell}=r_{n+\ell+1}s_{n+\ell+1} \\ s_{n+\ell}=s_{n+\ell+1} \end{cases} \qquad \mathrm{L}^\ell=\{s_{n+\ell+1}=0\} \qquad\qquad \begin{cases} c_{n+\ell}=c_{n+\ell+1}d_{n+\ell+1} \\ d_{n+\ell}=d_{n+\ell+1} \end{cases} \qquad \mathrm{M}^\ell=\{d_{n+\ell+1}=0\}$$

On the one hand

$$\begin{aligned}\Phi_n\colon (u_{n+\ell+1},v_{n+\ell+1}) &\to (u_{n+\ell+1},u_{n+\ell+1}v_{n+\ell+1})_{(r_{n+\ell},s_{n+\ell})} \\ &\to \left(1:(u_{n+\ell+1}^{\ell+1}v_{n+\ell+1}^{\ell}-1)u_{n+\ell+1}^{n-\ell-2}v_{n+\ell+1}^{n-\ell-1}:(u_{n+\ell+1}^{\ell+1}v_{n+\ell+1}^{\ell}-1)u_{n+\ell+1}^{n-\ell-1}v_{n+\ell+1}^{n-\ell}\right) \\ &\to \left((u_{n+\ell+1}^{\ell+1}v_{n+\ell+1}^{\ell}-1)v_{n+\ell+1},u_{n+\ell+1}v_{n+\ell+1}\right)_{(r_{n-1-\ell},s_{n-1-\ell})}\end{aligned}$$

and on the other hand

$$\begin{aligned}\Phi_n\colon (r_{n+\ell+1},s_{n+\ell+1}) &\to (r_{n+\ell+1}s_{n+\ell+1},s_{n+\ell+1})_{(r_{n+\ell},s_{n+\ell})} \\ &\to \left(r_{n+\ell+1}:(r_{n+\ell+1}s_{n+\ell+1}^{\ell+1}-1)s_{n+\ell+1}^{n-\ell-2}:(r_{n+\ell+1}s_{n+\ell+1}^{\ell+1}-1)s_{n+\ell+1}^{n-\ell-1}\right) \\ &\to \left(\frac{r_{n+\ell+1}s_{n+\ell+1}^{\ell+1}-1}{r_{n+\ell+1}},s_{n+\ell+1}\right)_{(r_{n-1-\ell},s_{n-1-\ell})}.\end{aligned}$$

So $B_{\ell+1}=(0,0)_{(r_{n+\ell+1},s_{n+\ell+1})}$ is an indeterminacy point, $\mathrm{L}^\ell$ is sent on $\mathrm{G}_{2\ell+2}^{n-3-\ell}$ if $1\le\ell\le n-4$ and $\mathrm{L}^{n-3}$ is sent on $\mathrm{F}_{2n-4}$.
Remark that $B_{\ell+1}$ is on $\mathrm{L}^\ell$ but not on $\mathrm{L}_1^{\ell-1}$. Besides one can verify that

$$\Phi_n\colon (y,z) \to \left(-\frac{y^{\ell+1}z^{n-\ell-2}}{z^{n-1}+y^n}, \frac{z}{y}\right)_{(c_{n+\ell+1},d_{n+\ell+1})} \quad \text{and that} \quad \Phi_n\colon (u_2,v_2) \to \left(-\frac{u_2^{n-\ell-3}v_2^{n-\ell-2}}{1+u_2^{n-2}v_2^{n-1}}, u_2v_2\right)_{(c_{n+\ell+1},d_{n+\ell+1})};$$

thus $\Delta_{n+\ell+1}$ and $\mathrm{F}_{n+\ell-1}$ are blown down to $C_{\ell+1}=(0,0)_{(c_{n+\ell+1},d_{n+\ell+1})}$ for every $\ell<n-3$. The situation is different for $\ell=n-3$ : whereas $\Delta_{2n-2}$ is still blown down to $C_{n-2}=(0,0)_{(c_{2n-2},d_{2n-2})}$, the divisor $\mathrm{F}_{2n-4}$ is sent on $\mathrm{M}^{n-3}$.
One can also note that

$$\begin{aligned}(u_{k+2},v_{k+2}) &\to \left(\frac{u_{k+2}^{n-k-2}v_{k+2}^{n-k-2}}{1+u_{k+2}^{n-k-2}v_{k+2}^{n-k-1}}, u_{k+2}v_{k+2}\right)_{(r_{k+2},s_{k+2})} \to \ldots \to \left(\frac{1}{1+u_{k+2}^{n-k-2}v_{k+2}^{n-k-1}}, u_{k+2}v_{k+2}\right)_{(r_n,s_n)} \\ &\to \left(-\frac{u_{k+2}^{n-k-3}v_{k+2}^{n-k-2}}{1+u_{k+2}^{n-k-2}v_{k+2}^{n-k-1}}, u_{k+2}v_{k+2}\right)_{(c_{n+1},d_{n+1})} \to \ldots \to \left(-\frac{v_{k+2}}{1+u_{k+2}^{n-k-2}v_{k+2}^{n-k-1}}, u_{k+2}v_{k+2}\right)_{(c_{2n-k-2},d_{2n-k-2})}\end{aligned}$$

so $\mathrm{G}_{2n-4-k}^k$ is sent on $\mathrm{M}_k^{n-k-3}$ for all $1\le k\le n-4$.

Finally we blow up $B_{n-2}$ at right and $C_{n-2}$ at left.

$$\begin{cases} r_{2n-2} = u_{2n-1} \\ s_{2n-2} = u_{2n-1}v_{2n-1} \end{cases} \qquad \mathrm{L}^{n-2} = \{u_{2n-1} = 0\} \qquad\qquad \begin{cases} c_{2n-2} = a_{2n-1} \\ d_{2n-2} = a_{2n-1}b_{2n-1} \end{cases} \qquad \mathrm{M}^{n-2} = \{a_{2n-1} = 0\}$$

$$\begin{cases} r_{2n-2} = r_{2n-1}s_{2n-1} \\ s_{2n-2} = s_{2n-1} \end{cases} \qquad \mathrm{L}^{n-2} = \{s_{2n-1} = 0\} \qquad\qquad \begin{cases} c_{2n-2} = c_{2n-1}d_{2n-1} \\ d_{2n-2} = d_{2n-1} \end{cases} \qquad \mathrm{M}^{n-2} = \{d_{n-2} = 0\}$$

This yields

$$\Phi_n\colon (u_{2n-1}, v_{2n-1}) \to (u_{2n-1}, u_{2n-1}v_{2n-1})_{(r_{2n-2}, s_{2n-2})} \to \Big(1 : (u_{2n-1}^{n-1}v_{2n-1}^{n-2} - 1)v_{2n-1} : (u_{2n-1}^{n-1}v_{2n-1}^{n-2} - 1)u_{2n-1}v_{2n-1}^{2}\Big)$$

and

$$\Phi_n\colon (r_{2n-1}, s_{2n-1}) \to (r_{2n-1}s_{2n-1}, s_{2n-1})_{(r_{2n-2}, s_{2n-2})} \to \Big(r_{2n-1} : r_{2n-1}s_{2n-1}^{n-1} - 1 : (r_{2n-1}s_{2n-1}^{n-1} - 1)s_{2n-1}\Big).$$

Thus there is no indeterminacy point and $\mathrm{L}^{n-2}$ is sent on $\Delta_{2n-1}$. Furthermore,

$$(y,z) \to \left(-\frac{y^{n-1}}{z^{n-1} + y^n}, \frac{z}{y}\right)_{(c_{2n-1}, d_{2n-1})}$$

so $\Delta_{2n-1}$ is sent on $\mathrm{M}^{n-2}$.

All these computations yield the following result:

***Proposition 2.1***. — *Let $\widehat{P}_1$ (resp. $\widehat{P}_2$) denote the point infinitely near $P$ obtained by blowing up $P$, $P_1$, $A_1$, …, $A_{n-2}$, $T$, $B_1$, …, $B_{n-3}$ and $B_{n-2}$ (resp. $P$, $P_1$, $A_1$, …, $A_{n-2}$, $S$, $C_1$, …, $C_{n-3}$ and $C_{n-2}$). The map $\Phi_n$ induces an isomorphism between $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$. The different components are swapped as follows:*

$$\Delta \to \mathrm{M}^{n-2}, \quad \mathrm{E} \to \mathrm{E}, \quad \mathrm{F} \to \mathrm{M}^{n-3}, \quad \mathrm{G}^{n-3} \to \mathrm{K}, \quad \mathrm{G}^{n-2} \to \mathrm{G}^{n-2}, \quad \mathrm{H} \to \mathrm{G}^{n-3}, \quad \mathrm{L}^{n-3} \to \mathrm{F}, \quad \mathrm{L}^{n-2} \to \Delta,$$

$$\mathrm{G}^k \to \mathrm{M}^{n-k-3} \ \text{ for } 1 \le k \le n-4, \qquad\qquad \mathrm{L}^{\ell} \to \mathrm{G}^{n-3-\ell} \ \text{ for } 1 \le \ell \le n-4.$$

This result is close to ([**BK**], Proposition 1).

**2.2. Second step: gluing conditions. —** The gluing conditions reduce to the following problem: if $u$ is a germ of biholomorphism in a neighborhood of $P$, find the conditions on $u$ in order that $u(\widehat{P}_2) = \widehat{P}_1$.

Consider a neighborhood of $(0,0)$ in $\mathbb{C}^2$ with the coordinates $\eta_1$, $\mu_1$. For every integer $d \ge 1$, we introduce an infinitely near point $\widehat{\Omega}_d$ of length $d$ centered at $(0,0)$ by blowing up successively $\omega_1$, …, $\omega_d$ where $\omega_i = (0,0)_{(\eta_i, \mu_i)}$, the coordinates $(\eta_i, \mu_i)$ being given by the formulae $\eta_i = \eta_{i+1}\mu_{i+1}$ and $\mu_i = \mu_{i+1}$.

Let $g(\eta_1, \mu_1) = \left(\sum_{(i,j)\in\mathbb{N}^2} \alpha_{i,j}\eta_1^i\mu_1^j, \sum_{(i,j)\in\mathbb{N}^2} \beta_{i,j}\eta_1^i\mu_1^j\right)_{(\eta_1,\mu_1)}$ be a germ of biholomorphism at $(0,0)_{(\eta_1,\mu_1)}$. If $d$ is a positive integer, we define the subset $I_d$ of $\mathbb{N}^2$ by $I_d = \{(0,0), (0,1), \ldots, (0,d-1)\}$.

***Lemma 2.2***. — *If $d$ is in $\mathbb{N}^*$, $g$ can be lifted to a biholomorphism $\widetilde{g}$ in a neighborhood of the exceptional components in $\mathrm{Bl}_{\widehat{\Omega}_d}\mathbb{C}^2$ if and only if $\alpha_{0,0} = \beta_{0,0} = 0$ and $\alpha_{0,1} = \ldots = \alpha_{0,d-1} = 0$. If these conditions are satisfied, $\beta_{0,1} \neq 0$ and $\widetilde{g}$ is given in the coordinates $(\eta_{d+1}, \mu_{d+1})$ by the formula*

$$\widetilde{g}(\eta_{d+1}, \mu_{d+1}) = \left(\frac{\sum_{(i,j)\in I_d} \alpha_{i,j}\eta_{d+1}^i\mu_{d+1}^{d(i-1)+j}}{\sum_{(i,j)\in I_1} \beta_{i,j}\eta_{d+1}^i\mu_{d+1}^{di+j-1}}, \sum_{(i,j)\in I_1} \beta_{i,j}\eta_{d+1}^i\mu_{d+1}^{di+j}\right)_{(\eta_{d+1},\mu_{d+1})}, \qquad |\mu_{d+1}| < \varepsilon, |\eta_{d+1}\mu_{d+1}| < \varepsilon.$$

*Proof*. — This is straightforward by induction on $d$. □

Fix $n \ge 3$, then $\mathrm{Bl}_{\widehat{P}_1}\mathbb{C}^2$ can be obtained as follows:

– blow up $P$ ;
– blow up $\widehat{\Omega}_{n-1}$ centered at $P_1$ (*i.e.* $\eta_1 = u_1, \mu_1 = v_1$) ;
– blow up $\widehat{\Omega}_{n-1}$ centered at $T$ (*i.e.* $\eta_1 = r_n + 1, \mu_1 = s_n$).

The same holds with $\widehat{P}_2$, the point $T$ being replaced by $S$.

***Proposition 2.3***. — *Let* $u(y,z) = \left( \sum_{(i,j)\in\mathbb{N}^2} m_{i,j} y^i z^j, \sum_{(i,j)\in\mathbb{N}^2} n_{i,j} y^i z^j \right)$ *be a germ of biholomorphism at P.*

– *If* $n = 3$, *then u can be lifted to a germ of biholomorphism between* $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$ *and* $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ *if and only if*
  – $m_{0,0} = n_{0,0} = 0$;
  – $n_{1,0} = 0$;
  – $m_{1,0}^3 + n_{0,1}^2 = 0$;
  – $n_{2,0} = \frac{3m_{0,1}n_{0,1}}{2m_{1,0}}$.
– *If* $n \geq 4$, *then u can be lifted to a germ of biholomorphism between* $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$ *and* $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ *if and only if*
  – $m_{0,0} = n_{0,0} = 0$;
  – $n_{1,0} = 0$;
  – $m_{1,0}^n + n_{0,1}^{n-1} = 0$;
  – $m_{0,1} = n_{2,0} = 0$.

*Proof*. — The first condition is $u(P) = P$, *i.e.* $m_{0,0} = n_{0,0} = 0$. The associated lift $\widetilde{u}_1$ is given by

$$\widetilde{u}_1(u_1,v_1) = \left( \sum_{(i,j)\in I_1} m_{i,j} u_1^{i+j} v_1^j, \frac{\sum_{(i,j)\in I_1} n_{i,j} u_1^{i+j-1} v_1^j}{\sum_{(i,j)\in I_1} m_{i,j} u_1^{i+j-1} v_1^j} \right)_{(u_1,v_1)}.$$

We must now verify the gluing conditions of Lemma 2.2 for $g = \widetilde{u}_1$ with $d = n-1$. This implies only the condition $n_{1,0} = 0$, since $\alpha_{0,j}(\widetilde{u}_1) = 0$ for $j \geq 0$. After blowing up $\widehat{\Omega}_{n-1}$ we get

$$\widetilde{u}_n(r_n,s_n) = \left( \frac{\left( \sum_{(i,j)\in I_1} m_{i,j} r_n^{i+j} s_n^{(n-1)(i+j-1)+j} \right)^n}{\left( \sum_{\substack{(i,j)\in I_1 \\ (i,j)\neq(1,0)}} m_{i,j} r_n^{i+j-1} s_n^{(n-1)(i+j-1)+j-1} \right)^{n-1}}, \frac{\sum_{\substack{(i,j)\in I_1 \\ (i,j)\neq(1,0)}} n_{i,j} r_n^{i+j-1} s_n^{(n-1)(i+j-1)+j}}{\sum_{(i,j)\in I_1} m_{i,j} r_n^{i+j-1} s_n^{(n-1)(i+j-1)+j}} \right)_{(r_n,s_n)}.$$

The condition $\widetilde{u}_n(S) = T$ is equivalent to $\frac{m_{1,0}^n}{n_{0,1}^{n-1}} = -1$. In the coordinates $(\eta_1,\mu_1)$ centered at $S$ and $T$,

$$\widetilde{u}_n(\eta_1,\mu_1) = (\Gamma_1(\eta_1,\mu_1), \Gamma_2(\eta_1,\mu_1))_{(\eta_1,\mu_1)}$$

where

$$\Gamma_1(\eta_1,\mu_1) = \frac{\left( \sum_{(i,j)\in I_1} m_{i,j} (1+\eta_1)^{i+j} \mu_1^{(n-1)(i+j-1)+j} \right)^{n-1}}{\left( \sum_{(i,j)\in I_1} n_{i,j} (1+\eta_1)^{i+j-1} \mu_1^{(n-1)(i+j-1)+j-1} \right)^{n-1}} + 1.$$

Thus

$$\Gamma_1(0,\mu_1)=\frac{\Big(m_{1,0}+m_{0,1}\mu_1+o(\mu_1^{n-2})\Big)^n}{\Big(n_{0,1}+n_{2,0}\mu_1^{n-2}+o(\mu_1^{n-2})\Big)^{n-1}}+1.$$

The higher gluing conditions of Lemma 2.2 for $g=\widetilde{u}_n$ with $d=n-1$ are given by $\frac{\partial^\ell\Gamma_1}{\partial\mu_1^\ell}(0,0)=0,\ 1\leq\ell\leq n-2$.
If $n=3$, then

$$\Gamma_1(0,\mu_1)=\frac{\Big(m_{1,0}+m_{0,1}\mu_1+o(\mu_1)\Big)^3}{\Big(n_{0,1}+n_{2,0}\mu_1+o(\mu_1)\Big)^2}+1=\frac{m_{1,0}^2}{n_{0,1}^2}\left(3m_{0,1}-2m_{1,0}\frac{n_{2,0}}{n_{0,1}}\right)\mu_1+o(\mu_1);$$

hence the condition is given by $n_{2,0}=\frac{3m_{0,1}n_{0,1}}{2m_{1,0}}$.

If $n\geq 4$, then $\frac{\partial\Gamma_1}{\partial\mu_1}(0,0)=n\frac{m_{1,0}^{n-1}m_{0,1}}{n_{0,1}^{n-1}}$. Since the coefficient $m_{1,0}$ is nonzero, $m_{0,1}=0$. This implies

$$\Gamma(0,\mu_1)=\frac{n_{2,0}}{n_{0,1}}\mu_1^{n-2}+o(\mu_1^{n-2}),$$

so the last condition is $n_{2,0}=0$.

The other computations are quite similar. □

**2.3. Remarks on degenerate birational quadratic maps. —** We can do this construction for $n=2$ and find gluing conditions: a germ of biholomorphism $g$ of $\mathbb{C}^2$ around 0 given by

$$g(y,z)=\Big(\sum_{0\leq i,j\leq 4}m_{i,j}y^iz^j,\sum_{0\leq i,j\leq 4}n_{i,j}y^iz^j\Big)$$

sends $\widehat{P}_2$ on $\widehat{P}_1$ if and only if $m_{0,0}=n_{0,0}=0$, $n_{1,0}=0$ and $m_{1,0}^2=n_{2,0}-n_{0,1}$.
As we have to blow up $\mathbb{P}^2$ at least ten times to get automorphisms with nonzero entropy, we want to find an automorphism $\varphi$ of $\mathbb{P}^2$ such that $(\varphi\Phi_2)^k\varphi(\widehat{P}_2)=\widehat{P}_1$ with $k\geq 4$ and $(\varphi\Phi_2)^i\varphi(P)\neq P$ for $0\leq i\leq k-1$. The Taylor series of $(\varphi\Phi_2)^k\varphi$ is of the form

$$\Big(\sum_{0\leq i,j\leq 4}m_{i,j}y^iz^j,\sum_{0\leq i,j\leq 4}n_{i,j}y^iz^j\Big)+o\Big(||(y,z)||^4\Big)$$

in the affine chart $x=1$. The degrees of the equations increase exponentially with $k$ so even for $k=4$ it is not easy to explicit a family. However we can verify that if

$$\varphi=\begin{bmatrix}0&0&-\frac{\alpha^2}{2}\\0&1&0\\1&0&\alpha\end{bmatrix}\qquad\text{with }\alpha\text{ in }\mathbb{C}\text{ such that }\alpha^8+2\alpha^6+4\alpha^4+8\alpha^2+16=0,$$

then $(\varphi\Phi_2)^4\varphi(\widehat{P}_2)=\widehat{P}_1$. These examples are conjugate to those studied in [**BK09b**].

## 3. Birational maps whose exceptional locus is a line, II

In this section, we apply the results of §2 to produce explicit examples of automorphisms of rational surfaces obtained from birational maps in the $\mathrm{PGL}(3;\mathbb{C})$-orbit of the $\Phi_n$. As we have to blow up $\mathbb{P}^2$ at least ten times to have nonzero entropy, we want to find an automorphism $\varphi$ of $\mathbb{P}^2$ and a positive integer $k$ such that

$$(k+1)(2n-1)\geq 10,\ \ (\varphi\Phi_n)^k\varphi(\widehat{P}_2)=\widehat{P}_1\ \text{ and }\ (\varphi\Phi_n)^i\varphi(P)\neq P\ \text{ for }0\leq i\leq k-1. \tag{3.1}$$

First of all, let us introduce the following definition.

***Definition 3.1***. — Let $U$ be an open set of $\mathbb{C}^d$, $\varphi\colon U \to \mathrm{PGL}(3;\mathbb{C})$ be a holomorphic map and $f$ be a birational map of the projective plane. We say that the family of birational maps $(\varphi_{\alpha_1,\dots,\alpha_n} f)_{(\alpha_1,\dots,\alpha_n)\in U}$ is *locally holomorphically trivial* if for every $\alpha_0$ in $U$ there exists a holomorphic map $M$ from a neighborhood $U_{\alpha_0}$ of $\alpha_0$ to $\mathrm{PGL}(3;\mathbb{C})$ such that $M_{\alpha_0} = \mathrm{Id}$ and for all $\alpha$ in $U_{\alpha_0}$, $\varphi_\alpha f = M_\alpha^{-1}(\varphi_{\alpha_0} f)M_\alpha$.

**3.1. Families of birational maps of degree $n$ with exponential growth conjugate to automorphisms of $\mathbb{P}^2$ blown up in $6n-3$ points. —** Let $\varphi$ be an automorphism of $\mathbb{P}^2$. We will find solutions of (3.1) for $n \geq 3$ and $k = 2$.

Remark that the Taylor series of $(\varphi\Phi_n)^2\varphi$ is of the form

$$\Big(\sum_{0\leq i,j\leq 2n-2} m_{i,j}y^iz^j, \sum_{0\leq i,j\leq 2n-2} n_{i,j}y^iz^j\Big) + o\Big(||(y,z)||^{2n-2}\Big)$$

in the affine chart $x = 1$. Assume that $(\varphi\Phi_n)^2\varphi(P) = P$; one can show that this is the case when

$$\varphi_{\alpha,\beta} = \begin{bmatrix} 1 & \gamma & -\frac{1+\delta+\delta^2}{\alpha} \\ 0 & -1 & 0 \\ \alpha & \beta & \delta \end{bmatrix}.$$

One can verify that the conditions of the Proposition 2.3 are satisfied if $\beta = \dfrac{\alpha\gamma}{2}$ and $(1+\delta)^{3n} = -1$.

For $0 \leq k \leq 3n-1$, let $\delta_k = \exp\Big(\frac{(2k+1)\mathrm{i}\pi}{3n}\Big) - 1$. If

$$\varphi_{\alpha,\beta} = \begin{bmatrix} 1 & \frac{2\beta}{\alpha} & -\frac{1+\delta_k+\delta_k^2}{\alpha} \\ 0 & -1 & 0 \\ \alpha & \beta & \delta_k \end{bmatrix},$$

then $(\varphi_{\alpha,\beta}\Phi_n)^2\varphi(\widehat{P}_2) = \widehat{P}_1$.

***Theorem 3.2***. — *Assume that $n \geq 3$ and that*

$$\varphi_{\alpha,\beta} = \begin{bmatrix} 1 & \frac{2\beta}{\alpha} & -\frac{1+\delta_k+\delta_k^2}{\alpha} \\ 0 & -1 & 0 \\ \alpha & \beta & \delta_k \end{bmatrix}, \qquad \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}, \delta_k = \exp\left(\frac{(2k+1)\mathrm{i}\pi}{3n}\right) - 1, 0 \leq k \leq 3n-1.$$

*The map $\varphi_{\alpha,\beta}\Phi_n$ is conjugate to an automorphism of $\mathbb{P}^2$ blown up in $3(2n-1)$ points.*

*The first dynamical degree of $\varphi_{\alpha,\beta}\Phi_n$ is strictly larger than 1; more precisely $\lambda(\varphi_{\alpha,\beta}\Phi_n) = \frac{n+\sqrt{n^2-4}}{2}$.*

*The family $\varphi_{\alpha,\beta}\Phi_n$ is locally holomorphically trivial.*

*Proof*. — Let $\varphi$ denote $\varphi_{\alpha,\beta}$. In the basis

$$\{\Delta, \mathrm{E}, \mathrm{F}, \mathrm{G}^1, \dots, \mathrm{G}^{n-2}, \mathrm{H}, \mathrm{L}^1, \dots, \mathrm{L}^{n-2}, \varphi\mathrm{E}, \varphi\mathrm{F}, \varphi\mathrm{G}^1, \dots, \varphi\mathrm{G}^{n-2}, \varphi\mathrm{K}, \varphi\mathrm{M}^1 \dots, \varphi\mathrm{M}^{n-2},$$
$$\varphi\Phi_n\varphi\mathrm{E}, \varphi\Phi_n\varphi\mathrm{F}, \varphi\Phi_n\varphi\mathrm{G}^1, \dots, \varphi\Phi_n\varphi\mathrm{G}^{n-2}, \varphi\Phi_n\varphi\mathrm{K}, \varphi\Phi_n\varphi\mathrm{M}^1, \dots, \varphi\Phi_n\varphi\mathrm{M}^{n-2}\}$$

the matrix $M$ of $(\varphi\Phi_n)_*$ is

$$\left[\begin{array}{c|c|c|c} 0_1 & {}^tA & 0_{1,2n-1} & 0_{1,2n-1} \\ \hline 0_{2n-1,1} & B & 0_{2n-1} & \mathrm{Id}_{2n-1} \\ \hline A & C & 0_{2n-1} & 0_{2n-1} \\ \hline 0_{2n-1,1} & 0_{2n-1} & \mathrm{Id}_{2n-1} & 0_{2n-1} \end{array}\right] \in \mathcal{M}_{6n-2}$$

with

$$A=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\in\mathcal{M}_{2n-1,1},\qquad B=\begin{bmatrix}0&0&\cdots&\cdots&0&0&1\\0&0&\cdots&\cdots&0&0&2\\ \vdots&\vdots&&&\vdots&\vdots&\vdots\\ \vdots&\vdots&&&\vdots&\vdots&n\\ \vdots&\vdots&&&\vdots&\vdots&n\\0&0&\cdots&\cdots&0&0&\vdots\\0&0&\cdots&\cdots&0&0&n\end{bmatrix}\in\mathcal{M}_{2n-1},\qquad C=\begin{bmatrix}1&0&\cdots&\cdots&0&0&-1\\0&0&\cdots&\cdots&0&1&-2\\ \vdots&\vdots&&\iddots&\iddots&0&\vdots\\ \vdots&\vdots&\iddots&\iddots&\iddots&\vdots&-n\\0&0&\iddots&\iddots&&\vdots&-n\\0&1&0&\cdots&\cdots&0&\vdots\\0&0&0&\cdots&\cdots&0&-n\end{bmatrix}\in\mathcal{M}_{2n-1}.$$

Its characteristic polynomial is $(X^2-nX+1)(X^2-X+1)^{n-2}(X+1)^{n-1}(X^2+X+1)^n(X-1)^{n+1}$. Hence

$$\lambda(\varphi\Phi_n)=\frac{n+\sqrt{n^2-4}}{2}$$

which is larger than 1 as $n\geq 3$.

Fix a point $(\alpha_0,\beta_0)$ in $\mathbb{C}^*\times\mathbb{C}$. We can find locally around $(\alpha_0,\beta_0)$ a matrix $M_{\alpha,\beta}$ depending holomorphically on $(\alpha,\beta)$ such that for all $(\alpha,\beta)$ near $(\alpha_0,\beta_0)$, we have $\varphi_{\alpha,\beta}\Phi_n=M_{\alpha,\beta}^{-1}\varphi_{\alpha_0,\beta_0}\Phi_n M_{\alpha,\beta}$ : if $\mu$ is a local holomorphic solution of the equation $\alpha=\mu^n\alpha_0$ such that $\mu_0=1$ we can take

$$M_{\alpha,\beta}=\begin{bmatrix}1&\frac{\beta-\beta_0\mu}{\mu^n\alpha_0}&0\\0&\frac{1}{\mu^{n-1}}&0\\0&0&\frac{1}{\mu^n}\end{bmatrix}.$$

□

***Remark 3.3***. — Assume that $\delta_k=-2$ and $n$ is odd. Consider the automorphism $A$ of $\mathbb{P}^2$ given by

$$A=(uy:\alpha x+\beta y-z:z),\qquad\qquad \alpha\in\mathbb{C}^*,\ \beta\in\mathbb{C},\ u^n=\alpha.$$

One can verify that $A(\varphi_{\alpha,\beta}\Phi_n)A^{-1}=(xz^{n-1}:z^n:x^n+z^n-yz^{n-1})$ which is of the form of (0.1).

**3.2. Families of birational maps of degree $n$ with exponential growth conjugate to automorphisms of $\mathbb{P}^2$ blown up in $4n-2$ points. —** In this section we will assume that $n$ is larger than 4. In that case, we succeed in providing solutions of (3.1) for $k=1$.

Proposition 2.3 allows us to establish the following statement.

***Theorem 3.4***. — *Assume that $n\geq 4$ and*

$$\varphi_{\alpha,\beta,\gamma,\delta}=\begin{bmatrix}\alpha&\beta&\frac{\beta(\gamma^2\varepsilon_k-\alpha^2)}{\delta(\alpha-\gamma)}\\0&\gamma&0\\\frac{\delta(\alpha-\gamma)}{\beta}&\delta&-\alpha\end{bmatrix},\qquad \alpha,\beta\in\mathbb{C},\ \gamma,\delta\in\mathbb{C}^*,\ \alpha\neq\gamma,\ \varepsilon_k=\exp\left(\frac{(2k+1)\mathrm{i}\pi}{n}\right),\ 0\leq k\leq n-1.$$

*The map $\varphi_{\alpha,\beta,\gamma,\delta}\Phi_n$ is conjugate to an automorphism of $\mathbb{P}^2$ blown up in $4n-2$ points.*

*The first dynamical degree of $\varphi_{\alpha,\beta,\gamma,\delta}\Phi_n$ is strictly larger than 1; more precisely $\lambda(\varphi_{\alpha,\beta,\gamma,\delta}\Phi_n)=\frac{(n-1)+\sqrt{(n-1)^2-4}}{2}$.*

*The family $\varphi_{\alpha,\beta,\gamma,\delta}\Phi_n$ is locally holomorphically trivial.*

*Proof*. — Let $\varphi$ denote $\varphi_{\alpha,\beta,\gamma,\delta}$.
In the basis

$$\{\Delta,\mathrm{E},\mathrm{F},\mathrm{G}^1,\dots,\mathrm{G}^{n-2},\mathrm{H},\mathrm{L}^1,\dots,\mathrm{L}^{n-2},\varphi\mathrm{E},\varphi\mathrm{F},\varphi\mathrm{G}^1,\dots,\varphi\mathrm{G}^{n-2},\varphi\mathrm{K},\varphi\mathrm{M}^1\dots,\varphi\mathrm{M}^{n-2}\}$$

the matrix $M$ of $(\varphi\Phi_n)_*$ is

$$\left[\begin{array}{c|c|c}0_1&{}^tA&0_{1,2n-1}\\\hline 0_{2n-1,1}&B&\mathrm{Id}_{2n-1}\\\hline A&C&0_{2n-1}\end{array}\right]\in\mathcal{M}_{4n-1}$$

where

$$A=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\in\mathcal{M}_{2n-1,1},\qquad B=\begin{bmatrix}0&0&\cdots&\cdots&0&0&1\\0&0&\cdots&\cdots&0&0&2\\ \vdots&\vdots&&&\vdots&\vdots&\vdots\\ \vdots&\vdots&&&\vdots&\vdots&n\\ \vdots&\vdots&&&\vdots&\vdots&n\\0&0&\cdots&\cdots&0&0&\vdots\\0&0&\cdots&\cdots&0&0&n\end{bmatrix}\in\mathcal{M}_{2n-1},\qquad C=\begin{bmatrix}1&0&\cdots&\cdots&0&0&-1\\0&0&\cdots&\cdots&0&1&-2\\ \vdots&\vdots&&\iddots&\iddots&0&\vdots\\ \vdots&\vdots&\iddots&\iddots&\iddots&\vdots&-n\\0&0&\iddots&\iddots&&\vdots&-n\\0&1&0&\cdots&\cdots&0&\vdots\\0&0&0&\cdots&\cdots&0&-n\end{bmatrix}\in\mathcal{M}_{2n-1}.$$

Its characteristic polynomial is $(X^2-(n-1)X+1)(X^2+1)^{n-2}(X+1)^{n-1}(X-1)^{n+1}$. Hence

$$\lambda(\varphi\Phi_n)=\frac{(n-1)+\sqrt{(n-1)^2-4}}{2}$$

which is larger than 1 as $n\geq 4$.

Fix a point $(\alpha_0,\beta_0,\gamma_0,\delta_0)$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^*$ such that $\alpha_0\neq\gamma_0$. We can find locally around $(\alpha_0,\beta_0,\gamma_0,\delta_0)$ a matrix $M_{\alpha,\beta,\gamma,\delta}$ depending holomorphically on $(\alpha,\beta,\gamma,\delta)$ such that for all $(\alpha,\beta,\gamma,\delta)$ near $(\alpha_0,\beta_0,\gamma_0,\delta_0)$, we have

$$\varphi_{\alpha,\beta,\gamma,\delta}\Phi_n=M^{-1}_{\alpha,\beta,\gamma,\delta}\varphi_{\alpha_0,\beta_0,\gamma_0,\delta_0}\Phi_n M_{\alpha,\beta,\gamma,\delta}:$$

if $\mu$ is a local holomorphic solution of the equation $\beta=\dfrac{\mu^n\beta_0\gamma_0\,\delta(\gamma-\alpha)}{\gamma\delta_0(\gamma_0-\alpha_0)}$ such that $\mu_0=0$, we can take

$$M_{\alpha,\beta,\gamma,\delta}=\begin{bmatrix}1&A&B\\0&\mu^{n-1}&0\\0&0&\mu^n\end{bmatrix},\qquad\text{where}\qquad A=\frac{\beta_0\mu^{n-1}(\gamma\delta_0-\mu\gamma_0\,\delta)}{\gamma\delta_0(\gamma_0-\alpha_0)}\qquad\text{and}\qquad B=\frac{\beta_0\mu^n(\alpha_0\gamma-\alpha\gamma_0)}{\gamma\delta_0(\alpha_0-\gamma_0)}.$$

□

***Remark 3.5***. — Consider the automorphism $A$ of $\mathbb{P}^2$ given by

$$A=(uy:\delta(\alpha-\gamma)x+\beta\delta y-\alpha\beta z:\varepsilon_k\beta\gamma z),\qquad \alpha,\,\beta\in\mathbb{C},\,\gamma,\,\delta\in\mathbb{C}^*,\,\alpha\neq\gamma,\,u^n=(\alpha-\gamma)\varepsilon_k^n\gamma^{n-1}\beta^{n-1}\delta.$$

One can verify that $A(\varphi_{\alpha,\beta,\gamma,\delta}\Phi_n)A^{-1}=(xz^{n-1}:z^n:x^n+\varepsilon_k yz^{n-1})$ which is of the form of (0.1).

### 3.3. An example in degree 3 with indeterminacy points not aligned. —

***Theorem 3.6***. — *Let $\varphi_\alpha$ be the automorphism of the complex projective plane given by*

$$\varphi_\alpha=\begin{bmatrix}\alpha&2(1-\alpha)&2+\alpha-\alpha^2\\-1&0&\alpha+1\\1&-2&1-\alpha\end{bmatrix},\qquad\qquad \alpha\in\mathbb{C}\setminus\{0,1\}.$$

*The map $\varphi_\alpha\Phi_3$ has no invariant line and is conjugate to an automorphism of $\mathbb{P}^2$ blown up in 15 points.*
*The first dynamical degree of $\varphi_\alpha\Phi_3$ is $\frac{3+\sqrt{5}}{2}>1$.*
*The family $\varphi_\alpha\Phi_3$ is locally holomorphically trivial.*

***Remark 3.7***. — The three points $P$, $\varphi_\alpha(P)$ and $\varphi_\alpha\Phi_3\varphi_\alpha(P)$ are not aligned in the complex projective plane. Indeed, $P=(1:0:0)$, $\varphi_\alpha(P)=(\alpha:-1:1)$ and $\varphi_\alpha\Phi_3\varphi_\alpha(P)=(\alpha:1:1)$.

*Proof.* — The first assertion is given by a direct computation and Proposition 2.3, the second by Theorem 3.2.

Fix a point $\alpha_0$ in $\mathbb{C}\setminus\{0,1\}$. We can find locally around $\alpha_0$ a matrix $M_\alpha$ depending holomorphically on $\alpha$ such that for all $\alpha$ near $\alpha_0$, we have $\varphi_\alpha\Phi_3 = M_\alpha^{-1}\varphi_{\alpha_0}\Phi_3 M_\alpha$ : it suffices to take

$$M_\alpha = \begin{bmatrix} 1 & 0 & \alpha_0-\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

**3.4. A conjecture. —** Let us recall a question which was communicated to the first author by E. Bedford:

*Does there exist a birational map of the projective plane $f$ such that for all $\varphi$ in $\mathrm{PGL}(3;\mathbb{C})$, the map $\varphi f$ is not birationally conjugate to an automorphism with positive entropy?*

We do not know at the present time the answer to this question. However, after a long series of examples, it seems that the birational maps $\Phi_n$ satisfy a rigidity property:

***Conjecture***. — *Let $U$ be an open set of $\mathbb{C}^d$, $n$ be an integer greater than or equal to three and $\varphi_\alpha$ be a holomorphic family of matrices in $\mathrm{PGL}(3;\mathbb{C})$ parameterized by $U$. Assume that there exists a positive integer $k$ such that*

$$(k+1)(2n-1)\geq 10,\ (\varphi_\alpha\Phi_n)^i\varphi_\alpha(P)\neq P \text{ for } 0\leq i\leq k-1 \text{ and } (\varphi_\alpha\Phi_n)^k\varphi_\alpha(\widehat{P}_2)=\widehat{P}_1.$$

*Then $(\varphi_\alpha\Phi_n)_{\alpha\in U}$ is holomorphically trivial.*

Let us remark that the maps of the form (0.1) don't satisfy this conjecture: for $n=6$ the family $f_a$ is not holomorphically trivial and one can verify that for any nonzero complex number $s$,

$$Af_a = f_{a/s^3}B, \qquad \text{where} \qquad A=\Big(sx : y : c(1-s^6)y+s^6z\Big) \quad \text{and} \quad B=(sx : s^6y : z).$$

## 4. A birational cubic map blowing down one conic and one line

Let $f$ denote the following birational map

$$f=\Big(y^2z : x(xz+y^2) : y(xz+y^2)\Big);$$

it blows up two points and blows down two curves, more precisely

$$\mathrm{Ind}\, f=\{R=(1:0:0), P=(0:0:1)\}, \qquad \mathrm{Exc}\, f=\Big(\mathcal{C}=\{xz+y^2=0\}\Big)\cup\Big(\Delta'=\{y=0\}\Big).$$

One can verify that $f^{-1}=\Big(y(z^2-xy) : z(z^2-xy) : xz^2\Big)$ and

$$\mathrm{Ind}\, f^{-1}=\{Q=(0:1:0), R\}, \qquad \mathrm{Exc}\, f^{-1}=\Big(\mathcal{C}'=\{z^2-xy=0\}\Big)\cup\Big(\Delta''=\{z=0\}\Big).$$

First we blow up $R$ at left and right. We set

$$\begin{cases} y=u_1 & \mathrm{E}=\{u_1=0\} \\ z=u_1v_1 & \end{cases} \qquad\qquad \begin{cases} y=r_1s_1 & \mathrm{E}=\{s_1=0\} \\ z=s_1 & \Delta_1'=\{r_1=0\} \end{cases}$$

where E is the exceptional divisor.

We have

$$f\colon (u_1,v_1)\to(u_1,u_1v_1)_{(y,z)}\to\Big(u_1^2v_1 : u_1+v_1 : u_1(u_1+v_1)\Big)\to\left(\frac{u_1+v_1}{u_1^2v_1},\frac{u_1+v_1}{u_1v_1}\right)_{(y,z)}\to\left(\frac{u_1+v_1}{u_1^2v_1},u_1\right)_{(u_1,v_1)}$$

and

$$f\colon (r_1,s_1)\to(r_1s_1,s_1)_{(y,z)}\to\Big(r_1^2s_1^2 : 1+r_1^2s_1 : r_1s_1(1+r_1^2s_1)\Big)\to\left(\frac{1+r_1^2s_1}{r_1^2s_1^2},\frac{1+r_1^2s_1}{r_1s_1}\right)_{(y,z)}\to\left(\frac{1}{r_1s_1},\frac{1+r_1^2s_1}{r_1s_1}\right)_{(r_1,s_1)}.$$

Hence $\mathcal{C}_1 = \{u_1 + v_1 = 0\}$ is sent on E, E is blown down to $Q = (0:1:0)$ and $S = (0,0)_{(u_1,v_1)}$ is an indeterminacy point. Next we blow up $P$ at left and $Q$ at right:

$$\begin{cases} x = u_2 \\ y = u_2 v_2 \end{cases} \quad \mathrm{F} = \{u_2 = 0\} \qquad\qquad \begin{cases} x = a_2 \\ z = a_2 b_2 \end{cases} \quad \mathrm{G} = \{a_2 = 0\}$$

$$\begin{cases} x = r_2 s_2 \\ y = s_2 \end{cases} \quad \mathrm{F} = \{s_2 = 0\} \qquad\qquad \begin{cases} x = c_2 d_2 \\ z = d_2 \end{cases} \quad \mathrm{G} = \{d_2 = 0\}$$

One can verify that

$$f\colon (u_2, v_2) \to (u_2, u_2 v_2)_{(x,y)} \to \left(v_2^2 : 1 + u_2 v_2^2 : v_2(1 + u_2 v_2^2)\right) = \left(\frac{v_2}{1 + u_2 v_2^2}, \frac{1}{v_2}\right)_{(x,y)},$$

$$f\colon (r_2, s_2) \to (r_2 s_2, s_2)_{(x,y)} \to \left(1 : r_2(r_2 + s_2) : r_2 + s_2\right)$$

and

$$f\colon (u_1, v_1) \to \left(u_1^2 v_1 : u_1 + v_1 : u_1(u_1 + v_1)\right) = \left(\frac{u_1^2 v_1}{u_1 + v_1}, u_1\right)_{(x,z)} \to \left(\frac{u_1 v_1}{u_1 + v_1}, u_1\right)_{(c_2,d_2)}.$$

In particular F is sent on $\mathcal{C}_2'$ and $\mathrm{E}_1$ is blown down to $T = (0,0)_{(c_2,d_2)}$. Moreover,

$$(x,y) \to \left(y^2 : x(x + y^2) : y(x + y^2)\right) \to \left(\frac{y^2}{x(x + y^2)}, \frac{y}{x}\right)_{(x,z)} \to \left(\frac{y}{x + y^2}, \frac{y}{x}\right)_{(c_2,d_2)}$$

so that $\Delta_2'$ is blown down to $T$.

Then we blow up $S$ at left and $T$ at right

$$\begin{cases} u_1 = u_3 \\ v_1 = u_3 v_3 \end{cases} \quad \mathrm{H} = \{u_3 = 0\} \qquad\qquad \begin{cases} c_2 = a_3 \\ d_2 = a_3 b_3 \end{cases} \quad \mathrm{K} = \{a_3 = 0\}$$

$$\begin{cases} u_1 = r_3 s_3 \\ v_1 = s_3 \end{cases} \quad \mathrm{H} = \{s_3 = 0\} \qquad\qquad \begin{cases} c_2 = c_3 d_3 \\ d_2 = d_3 \end{cases} \quad \mathrm{K} = \{d_3 = 0\}$$

We compute:

$$f\colon (u_3, v_3) \to (u_3, u_3 v_3)_{(u_1,v_1)} \to \left(u_3^2 v_3 : 1 + v_3 : u_3(1 + v_3)\right) \to \left(\frac{u_3^2 v_3}{1 + v_3}, u_3\right)_{(x,z)}$$
$$\to \left(\frac{u_3 v_3}{1 + v_3}, u_3\right)_{(c_2,d_2)} \to \left(\frac{v_3}{1 + v_3}, u_3\right)_{(c_3,d_3)}$$

and

$$f\colon (u_1, v_1) \to \left(\frac{u_1 v_1}{u_1 + v_1}, u_1\right)_{(c_1,d_1)} \to \left(\frac{v_1}{u_1 + v_1}, u_1\right)_{(c_3,d_3)}.$$

Thus H is sent on K, $\mathrm{E}_2$ is blown down to $V = (1,0)_{(c_3,d_3)}$ and $U = (0,-1)_{(u_3,v_3)}$ is an indeterminacy point. One can also remark that

$$(x,y) \to \left(\frac{y}{x + y^2}, \frac{y}{x}\right)_{(c_2,d_2)} \to \left(\frac{x}{x + y^2}, \frac{y}{x}\right)_{(c_3,d_3)}$$

so $\Delta_3'$ is blown down to $V$.

We will now blow up $U$ at left and $V$ at right

$$\begin{cases} u_3 = u_4 \\ v_3 = u_4 v_4 - 1 \end{cases} \quad \mathrm{L} = \{u_4 = 0\} \qquad\qquad \begin{cases} c_3 = a_4 + 1 \\ d_3 = a_4 b_4 \end{cases} \quad \mathrm{M} = \{a_4 = 0\}$$

$$\begin{cases} u_3 = r_4 s_4 \\ v_3 = s_4 - 1 \end{cases} \qquad \mathrm{L} = \{s_4 = 0\} \qquad\qquad \begin{cases} c_3 = c_4 d_4 + 1 \\ d_3 = d_4 \end{cases} \qquad \mathrm{M} = \{d_4 = 0\}$$

We get

$$f\colon (u_4, v_4) \to (u_4, u_4 v_4 - 1)_{(u_3,v_3)} \to \Big(u_4(u_4 v_4 - 1) : v_4 : u_4 v_4\Big) \to \left(\frac{u_4(u_4 v_4 - 1)}{v_4}, u_4\right)_{(x,z)} \to \left(\frac{u_4 v_4 - 1}{v_4}, u_4\right)_{(c_2,d_2)},$$

$$f\colon (r_4, s_4) \to (r_4 s_4, s_4 - 1)_{(u_3,v_3)} \to \Big(r_4^2 s_4(s_4 - 1) : 1 : r_4 s_4\Big)$$

and

$$f\colon (u_1, v_1) \to \left(\frac{v_1}{u_1 + v_1}, u_1\right)_{(c_3,d_3)} \to \left(-\frac{1}{u_1 + v_1}, u_1\right)_{(c_4,d_4)};$$

therefore $Y = (0,0)_{(u_4,v_4)}$ is an indeterminacy point, L is sent on $\mathrm{G}_2$ and $\mathrm{E}_3$ on M.
Furthermore

$$f\colon (x, y) \to \left(\frac{x}{x + y^2}, \frac{y}{x}\right)_{(c_3,d_3)} \to \left(-\frac{xy}{x + y^2}, \frac{y}{x}\right)_{(c_4,d_4)}$$

*i.e.* $\Delta_4'$ is blown down to $Z = (0,0)_{(c_4,d_4)}$.
Finally we blow up $Y$ at left and $Z$ at right

$$\begin{cases} u_4 = u_5 \\ v_4 = u_5 v_5 \end{cases} \qquad \mathrm{N} = \{u_5 = 0\} \qquad\qquad \begin{cases} c_4 = a_5 \\ d_4 = a_5 b_5 \end{cases} \qquad \Omega = \{a_5 = 0\}$$

$$\begin{cases} u_4 = r_5 s_5 \\ v_4 = s_5 \end{cases} \qquad \mathrm{N} = \{s_5 = 0\} \qquad\qquad \begin{cases} c_4 = c_5 d_5 \\ d_4 = d_5 \end{cases} \qquad \Omega = \{d_5 = 0\}$$

Hence $\Delta_5'$ is sent on $\Omega$ and N on $\Delta_5''$.

***Proposition 4.1***. — *Let $\widehat{P}_1$ (resp. $\widehat{P}_2$) denote the point infinitely near $R$ (resp. $Q$) obtained by blowing up $R$, $S$, $U$ and $Y$ (resp. $Q$, $T$, $V$ and $Z$). The map $f$ induces an isomorphism between $\mathrm{Bl}_{\widehat{P}_1,P}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2,R}\mathbb{P}^2$. The different components are swapped as follows:*

$$\mathcal{C} \to \mathrm{E}, \qquad \mathrm{F} \to \mathcal{C}', \qquad \mathrm{H} \to \mathrm{K}, \qquad \mathrm{L} \to \mathrm{G}, \qquad \mathrm{E} \to \mathrm{M}, \qquad \Delta' \to \Omega, \qquad \mathrm{N} \to \Delta''.$$

The following statement gives the gluing conditions.

***Proposition 4.2***. — *Let $u(x,z) = \left(\sum_{(i,j)\in\mathbb{N}^2} m_{i,j} x^i z^j, \sum_{(i,j)\in\mathbb{N}^2} n_{i,j} x^i z^j\right)$ be a germ of biholomorphism at $Q$.*
*Then $u$ can be lifted to a germ of biholomorphism between $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ if and only if*
- $m_{0,0} = n_{0,0} = 0$;
- $n_{0,1} = 0$;
- $n_{0,2} + n_{1,0} + m_{0,1}^2 = 0$;
- $n_{0,3} + n_{1,1} + 2m_{0,1}(m_{0,2} + m_{1,0}) = 0$.

Let $\varphi$ be an automorphism of $\mathbb{P}^2$. We will adjust $\varphi$ such that $(\varphi f)^k \varphi$ sends $\widehat{P}_2$ onto $\widehat{P}_1$ and $R$ onto $P$. As we have to blow up $\mathbb{P}^2$ at least ten times to have nonzero entropy, $k$ must be larger than two, $\{\widehat{P}_1, \varphi\widehat{P}_2, \varphi f \varphi \widehat{P}_2, (\varphi f)^2 \varphi \widehat{P}_2, \ldots,$

$(\varphi f)^{k-1}\varphi\widehat{P}_2\}$ must all have distinct supports and $(\varphi f)^k\varphi\widehat{P}_2=\widehat{P}_1$. We provide such matrices for $k=3$ : by Proposition 4.2 one can verify that for every nonzero complex number $\alpha$,

$$\varphi_\alpha=\begin{bmatrix} \frac{2\alpha^3}{343}(37\mathrm{i}\sqrt{3}+3) & \alpha & -\frac{2\alpha^2}{49}(5\mathrm{i}\sqrt{3}+11) \\ \frac{\alpha^2}{49}(-15+11\mathrm{i}\sqrt{3}) & 1 & -\frac{\alpha}{14}(5\mathrm{i}\sqrt{3}+11) \\ -\frac{\alpha}{7}(2\mathrm{i}\sqrt{3}+3) & 0 & 0 \end{bmatrix}$$

is such a $\varphi$.

***Theorem 4.3***. — *Assume that* $f=\Big(y^2z : x(xz+y^2) : y(xz+y^2)\Big)$ *and that*

$$\varphi_\alpha=\begin{bmatrix} \frac{2\alpha^3}{343}(37\mathrm{i}\sqrt{3}+3) & \alpha & -\frac{2\alpha^2}{49}(5\mathrm{i}\sqrt{3}+11) \\ \frac{\alpha^2}{49}(-15+11\mathrm{i}\sqrt{3}) & 1 & -\frac{\alpha}{14}(5\mathrm{i}\sqrt{3}+11) \\ -\frac{\alpha}{7}(2\mathrm{i}\sqrt{3}+3) & 0 & 0 \end{bmatrix}, \qquad \alpha\in\mathbb{C}^*.$$

*The map* $\varphi_\alpha f$ *is conjugate to an automorphism of* $\mathbb{P}^2$ *blown up in* 15 *points.*
*The first dynamical degree of* $\varphi_\alpha f$ *is* $\lambda(\varphi_\alpha f)=\frac{3+\sqrt{5}}{2}$.
*The family* $\varphi_\alpha f$ *is locally holomorphically trivial.*

*Proof.* — Set $\varphi=\varphi_\alpha$. In the basis

$$\{\Delta', \mathrm{E}, \mathrm{F}, \mathrm{H}, \mathrm{L}, \mathrm{N}, \varphi\mathrm{E}, \varphi\mathrm{G}, \varphi\mathrm{K}, \varphi\mathrm{M}, \varphi\Omega, \varphi f\varphi\mathrm{E}, \varphi f\varphi\mathrm{G}, \varphi f\varphi\mathrm{K}, \varphi f\varphi\mathrm{M}, \varphi f\varphi\Omega\}$$

the matrix $M$ of $(\varphi f)_*$ is

$$\begin{bmatrix}
0&0&2&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&2&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&2&0&0&1&0&0&0&0&0&0&0&1&0&0\\
0&0&2&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&2&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&-1&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&1&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&-2&1&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&1&-3&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
1&0&-4&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0
\end{bmatrix}.$$

Its characteristic polynomial is $(X-1)^4(X+1)^2(X^2-X+1)(X^2+X+1)^3(X^2-3X+1)$. Hence $\lambda(\varphi f)=\frac{3+\sqrt{5}}{2}$.

Fix a point $\alpha_0$ in $\mathbb{C}^*$. We can find locally around $\alpha_0$ a matrix $M_\alpha$ depending holomorphically on $\alpha$ such that for all $\alpha$ near $\alpha_0$, we have $\varphi_\alpha f=M_\alpha^{-1}\varphi_{\alpha_0}fM_\alpha$ : take

$$M_\alpha=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha}{\alpha_0} & 0 \\ 0 & 0 & \frac{\alpha^2}{\alpha_0^2} \end{bmatrix}.$$

□

## 5. Families of rational surfaces

Families of rational surfaces are usually constructed by blowing up $\mathbb{P}^2$ (or a Hirzebruch surface $\mathbb{F}_n$) successively at $N$ points $p_1,\dots,p_N$ and then by deforming the points $p_i$. Such deformations can be holomorphically trivial: the simplest example is given by the family $\mathrm{Bl}_{M_t p_1,\dots,M_t p_N}\mathbb{P}^2$, where $p_1,\dots,p_N$ are $N$ distinct points in $\mathbb{P}^2$ and $t\mapsto M_t$ is a holomorphic curve in $\mathrm{PGL}(3;\mathbb{C})$ such that $M_0=\mathrm{Id}$. In this section, our first aim is to describe deformations of rational surfaces, using the general theory of Kodaira and Spencer ([**Kod86**]). Then, after a general digression about the generic numbers of parameters of an algebraic deformation, we will give a practical way to count the generic number of parameters of a given family of rational surfaces with no holomorphic vector field.

**5.1. Deformations of basic rational surfaces. —** Recall that every rational surface can be obtained by blowing up finitely many times $\mathbb{P}^2$ of a Hirzebruch surface $\mathbb{F}_n$ (*see* [**GH94**]). A rational surface is called *basic* if it is a blowup of $\mathbb{P}^2$. By ([**Nag60**], Theorem 5), if $f$ is an automorphism of a rational surface $X$ such that $f^*$ is of infinite order on $\mathrm{Pic}(X)$, then $X$ is basic. Furthermore, by the main result of [**Har87**], $X$ carries no nonzero holomorphic vector field.

For each integer $N$, let us define a sequence of deformations $\pi_N\colon \mathfrak{X}_N\to S_N$ as follows:
- $S_0$ is a point and $\mathfrak{X}_0=\mathbb{P}^2$.
- $S_{N+1}=\mathfrak{X}_N$; $\mathfrak{X}_{N+1}=\mathrm{Bl}_{\mathfrak{X}_N}(\mathfrak{X}_N\times_{S_N}\mathfrak{X}_N)$, where $\mathfrak{X}_N$ is diagonally embedded in $\mathfrak{X}_N\times_{S_N}\mathfrak{X}_N$; and $\pi_{N+1}$ is obtained by composing the blow up morphism from $\mathfrak{X}_{N+1}$ to $\mathfrak{X}_N\times_{S_N}\mathfrak{X}_N$ with the first projection.

The varieties $S_N$ and $\mathfrak{X}_N$ are smooth and projective, they can be given the following geometric interpretation:
- For $N\geq 1$, $S_N$ is the set of ordered lists of (possibly infinitely near) points of $\mathbb{P}^2$ of length $N$. This means that
$$S_N=\{p_1,\dots,p_N \text{ such that } p_1\in\mathbb{P}^2 \text{ and if } 2\leq i\leq N,\ p_i\in \mathrm{Bl}_{p_{i-1}}\mathrm{Bl}_{p_{i-2}}\dots\mathrm{Bl}_{p_1}\mathbb{P}^2\}.$$
Elements of $S_N$ will be denoted by $\widehat{P}$.
- For $N\geq 1$, $\mathfrak{X}_N$ is the universal family of rational surfaces over $S_N$ : for every $\widehat{P}$ in $S_N$, the fiber $\pi_N^{-1}(\widehat{P})$ of $\widehat{P}$ in $\mathfrak{X}_N$ is the rational surface $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$ parameterized by $\widehat{P}$.

The group $\mathrm{PGL}(3;\mathbb{C})$ of biholomorphisms of $\mathbb{P}^2$ acts naturally on the configuration spaces $S_N$. If $g$ is an element of $\mathrm{PGL}(3;\mathbb{C})$ and $\widehat{P}$ lies in $S_N$, $g.\widehat{P}$ is the unique element of $S_N$ such that $g$ induces an isomorphism between $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$ and $\mathrm{Bl}_{g.\widehat{P}}\mathbb{P}^2$. Furthermore, if $\widehat{P}$ is a point in $S_N$ and $G_{\widehat{P}}$ is the stabilizer of $\widehat{P}$ in $\mathrm{PGL}(3;\mathbb{C})$, the Lie algebra of $G_{\widehat{P}}$ is the vector space of holomorphic vector fields on $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$.

In the sequel, for every integer $N\geq 4$, we will denote by $S_N^\dagger$ the Zariski-dense open subset of $S_N$ consisting of points $\widehat{P}$ in $S_N$ such that $G_{\widehat{P}}$ is trivial. The associated rational surfaces $\{\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2,\widehat{P}\in S_N^\dagger\}$ are rational surfaces in the family $\mathfrak{X}_N$ carrying no nonzero holomorphic vector field. Besides, the action of $\mathrm{PGL}(3;\mathbb{C})$ defines a regular foliation on $S_N^\dagger$.

For any point $\widehat{P}$ in $S_N$, let $O_{\widehat{P}}$ be the $\mathrm{PGL}(3;\mathbb{C})$-orbit of $\widehat{P}$ in $S_N$.

***Theorem 5.1***. — *For any point $\widehat{P}$ in $S_N$, the kernel of the Kodaira-Spencer map of $\mathfrak{X}_N$ at $\widehat{P}$ is equal to $\mathrm{T}_{\widehat{P}}O_{\widehat{P}}$.*

Before giving the proof, we start by some generalities. Let $(\mathfrak{X},\pi,B)$ be a deformation and $b$ be a point in $B$. Recall that $\mathfrak{X}$ is complete at $b$ if any small deformation of $\mathfrak{X}_b$ is locally induced by $\mathfrak{X}$ via a holomorphic map. Let us quote two fundamental results in deformation theory (*see* [**Kod86**], p. 270 and 284):

(i) *Theorem of existence.* Let $X$ be a complex compact manifold such that $\mathrm{H}^2(X,\mathrm{T}X)=0$. Then there exists a deformation $(\mathfrak{X},\pi,B)$ of $X$ such that $\mathfrak{X}_0=X$ and $\mathrm{KS}_0(\mathfrak{X})\colon \mathrm{T}_0B\to\mathrm{H}^1(X,\mathrm{T}X)$ is an isomorphism.

(ii) *Theorem of completeness.* Let $(\mathfrak{X},\pi,B)$ be a deformation and $b$ be in $B$ such that $\mathrm{KS}_b(\mathfrak{X})\colon \mathrm{T}_bB\to\mathrm{H}^1(\mathfrak{X}_b,\mathrm{T}\mathfrak{X}_b)$ is surjective. Then $\mathfrak{X}$ is complete at $b$.

As a consequence, if $(\mathfrak{X},\pi,B)$ is a deformation which is complete at a point $b$ of $B$ and such that $\mathrm{H}^2(\mathfrak{X}_b,\mathrm{T}\mathfrak{X}_b)=0$, then $\mathrm{KS}_b(\mathfrak{X})$ is surjective.

***Definition 5.2***. — Let $(\mathfrak{X},\pi,B)$ be a deformation. The *blown up deformation* $\widehat{\mathfrak{X}}$ is a deformation over $\mathfrak{X}$ defined by $\widehat{\mathfrak{X}} = \mathrm{Bl}_{\mathfrak{X}}(\mathfrak{X}\times_B\mathfrak{X})$, where $\mathfrak{X}$ is diagonally embedded in $\mathfrak{X}\times_B\mathfrak{X}$. The projection from $\widehat{\mathfrak{X}}$ to $\mathfrak{X}$ is induced by the projection on the first factor.

Thus, for any $x$ in $\mathfrak{X}$, $\widehat{\mathfrak{X}}_x = \mathrm{Bl}_x(\mathfrak{X}_b)$, where $b=\pi(x)$. The following result is originally due to Fujiki and Nakano and in a more general setting to Horikawa:

***Proposition 5.3*** **([FN72, Hor76])**. — *Let $(\mathfrak{X},\pi,B)$ be a deformation, $b$ be a point of $B$ and assume that $\mathfrak{X}$ is complete at $b$. Then the blown up deformation $\widehat{\mathfrak{X}}$ is complete at any point of $\mathfrak{X}_b$.*

Remark that for every integer $N$, $\widehat{\mathfrak{X}}_N = \mathfrak{X}_{N+1}$. Since $\mathfrak{X}_0$ is complete, it follows by induction that for every integer $N$, $\mathfrak{X}_N$ is complete at any point of $S_N$.

***Lemma 5.4***. — *Let $X$ be a rational surface obtained from the projective plane $\mathbb{P}^2$ via $N_+$ blow up and $N_-$ blow down. If $N=N_+-N_-$, then*
- $\mathrm{h}^1(X,\mathrm{T}X) = \mathrm{h}^0(X,\mathrm{T}X)+2N-8$;
- $\mathrm{h}^2(X,\mathrm{T}X)=0$.

*Proof*. — *See* [**Kod86**], p. 220. □

The second statement of the previous lemma together with the completeness of $\mathfrak{X}_N$ implies that the Kodaira-Spencer map of $\mathfrak{X}_N$ is surjective at any point of $S_N$.

*Proof of Theorem 5.1*. — Let $N$ be a positive integer and $\widehat{P}$ a point in $S_N$. Since the restriction of $\mathfrak{X}_N$ on $O_{\widehat{P}}$ is trivial, $\ker \mathrm{KS}_{\widehat{P}}(\mathfrak{X}_N)$ contains $\mathrm{T}_{\widehat{P}}O_{\widehat{P}}$. Let us compute the dimension of $\mathrm{T}_{\widehat{P}}O_{\widehat{P}}$. If $G_{\widehat{P}}$ is the stabilizer of $\widehat{P}$ in $\mathrm{PGL}(3;\mathbb{C})$, one has an exact sequence

$$0\longrightarrow \mathrm{Lie}(G_{\widehat{P}})\longrightarrow \mathrm{Lie}(\mathrm{PGL}(3;\mathbb{C}))\longrightarrow \mathrm{T}_{\widehat{P}}O_{\widehat{P}}\longrightarrow 0.$$

Thus $\dim(\mathrm{T}_{\widehat{P}}O_{\widehat{P}}) = 8-\mathrm{h}^0(X,\mathrm{T}X)$. Otherwise, since $\mathrm{KS}_{\widehat{P}}(\mathfrak{X}_N)$ is surjective, we get

$$\dim(\ker \mathrm{KS}_{\widehat{P}}(\mathfrak{X}_N)) = 2N-\mathrm{h}^1(X,\mathrm{T}X) = 8-\mathrm{h}^0(X,\mathrm{T}X)$$

by the first assertion of Lemma 5.4. □

Remark that if $N\geq 4$, the kernels of the Kodaira-Spencer maps of $\mathfrak{X}_N$ define a holomorphic vector bundle of rank eight on $S_N^\dagger$, which is the tangent bundle of the regular foliation defined by the $\mathrm{PGL}(3;\mathbb{C})$-action on $S_N^\dagger$.

We will also discuss deformations of nonbasic rational surfaces in §5.4.

**5.2. Generic numbers of parameters of an algebraic deformation. —** In the section, we define the generic numbers of parameters of an algebraic deformation. Recall that a deformation $(\mathfrak{X},\pi,B)$ is algebraic if there exists an embedding $i\colon \mathfrak{X}\to B\times\mathbb{P}^N$ such that $\pi$ is induced by the first projection of $B\times\mathbb{P}^N$. If $\mathfrak{X}$ is algebraic, the fibers $(\mathfrak{X}_b)_{b\in B}$ are complex projective varieties.

***Proposition 5.5***. — *Let $(\mathfrak{X},\pi,B)$ be an algebraic deformation. Then there exist a proper analytic subset $Z$ of $B$ and a holomorphic vector bundle $E$ on $U=B\setminus Z$ such that:*
- *$E$ is a holomorphic subbundle of $\mathrm{T}U$;*
- *the function $b\mapsto \mathrm{h}^1(\mathfrak{X}_b,\mathrm{T}\mathfrak{X}_b)$ is constant on $U$;*
- *for all $b$ in $B$, $E_{|b}$ is the kernel of $\mathrm{KS}_b(\mathfrak{X})$.*

*Proof*. — Let $\mathrm{T}^{\mathrm{rel}}\mathfrak{X}$ be the relative tangent bundle of $\mathfrak{X}$ defined by the exact sequence

$$0\to \mathrm{T}^{\mathrm{rel}}\mathfrak{X}\to \mathrm{T}\mathfrak{X}\to \pi^*\mathrm{T}B\to 0,$$

where the last map is the differential of $\pi$. The connection morphism $\mu\colon \mathrm{T}B\simeq \mathrm{R}^0\pi_*(\pi^*\mathrm{T}B)\to \mathrm{R}^1\pi_*\mathrm{T}^{\mathrm{rel}}\mathfrak{X}$ induces for every $b$ in $B$ a map

$$\mu_b\colon \mathrm{T}_bB\longrightarrow (\mathrm{R}^1\pi_*\mathrm{T}\mathfrak{X}^{\mathrm{rel}})_{|b}\longrightarrow \mathrm{H}^1(\mathfrak{X}_b,\mathrm{T}\mathfrak{X}_b)$$

which is exactly the Kodaira-Spencer map of $\mathfrak{X}$ at $b$ (*see* [**Voi07**], p. 219). Since the deformation $\mathfrak{X}$ is algebraic, there exists a complex $\mathcal{E}^\bullet$ of vector bundles on $B$ such that for every $b$ in $B$, $\mathrm{H}^1(\mathfrak{X}_b, \mathrm{T}\mathfrak{X}_b)$ is the cohomology in degree one of the complex $\mathcal{E}_{|b}$ (*see* [**Voi07**], p. 220). This implies that the function $b \mapsto \dim \mathrm{H}^1(\mathfrak{X}_b, \mathrm{T}\mathfrak{X}_b)$ is constant outside a proper analytic subset $Z$ of $B$. By Grauert's theorem ([**Har77**], p. 288), $\mathrm{R}^1\pi_*\mathrm{T}\mathfrak{X}^{\mathrm{rel}}$ is locally free on $U = B \setminus Z$ and for every $b$ in $U$, the base change morphism from $\mathrm{R}^1\pi_*\mathrm{T}\mathfrak{X}^{\mathrm{rel}}_{|b}$ to $\mathrm{H}^1(\mathfrak{X}_b, \mathrm{T}\mathfrak{X}_b)$ is an isomorphism. After removing again a proper analytic subset in $U$, we can assume that $\mu$ has constant rank on $U$, so that its kernel is a holomorphic vector bundle. □

This being done, the definition of the generic number of parameters of an algebraic deformation runs as follows:

***Definition 5.6***. — The number $m(\mathfrak{X}) = \dim B - \operatorname{rank} E$ is called the *generic number of parameters of* $\mathfrak{X}$.

***Remark 5.7***. — (i) Recall that a deformation $(\mathfrak{X}, \pi, B)$ is called *effectively parameterized* (resp. *generically effectively parameterized*) if for every $b$ in $B$ (resp. for every generic $b$ in B), the Kodaira-Spencer map $\mathrm{KS}_b(\mathfrak{X}) : \mathrm{T}_b B \to \mathrm{H}^1(\mathfrak{X}_b, \mathrm{T}\mathfrak{X}_b)$ is injective (*see* [**Kod86**], p. 215). By Proposition 5.5, an algebraic deformation $(\mathfrak{X}, \pi, B)$ is generically effectively parameterized if and only if $m(\mathfrak{X}) = \dim B$.

(ii) By Theorem 5.1, for any integer $N \geq 4$, $m(\mathfrak{X}_N) = 2N - 8$.

**5.3. How to count parameters in a family of rational surfaces?—** Let $\mathfrak{Y}$ be a family of rational surfaces parameterized by an open set $U$ of $\mathbb{C}^n$. Since the deformations $\mathfrak{X}_N$ are complete, we can suppose that $\mathfrak{Y}$ is obtained by pulling back the deformation $\mathfrak{X}_N$ by a holomorphic map $\psi \colon U \to S_N$. We will make the assumption that the fibers of $\mathfrak{Y}$ have no holomorphic vector field, so that $\psi$ takes its values in $S_N^\dagger$. In this situation, we are able to compute the numbers of parameters of such a family quite simply:

***Theorem 5.8***. — *Let $U$ be an open set in $\mathbb{C}^n$, $N$ be an integer greater than or equal to 4 and $\psi \colon U \to S_N^\dagger$ be a holomorphic map. Then $m(\psi^*\mathfrak{X}_N)$ is the smallest integer $d$ such that for all generic $\alpha$ in $U$, there exist a neighborhood $\Omega$ of 0 in $\mathbb{C}^{n-d}$ and two holomorphic maps $\gamma \colon \Omega \to U$ and $M \colon \Omega \to \mathrm{PGL}(3;\mathbb{C})$ such that:*
- *$\gamma_*(0)$ is injective,*
- *$\gamma(0) = \alpha$ and $M(0) = \mathrm{Id}$,*
- *for all $t$ in $\Omega$, $\psi(\gamma(t)) = M(t)\,\psi(\alpha)$.*

*Proof*. — Let $\alpha$ be a generic point in $U$, $U_\alpha$ be a small neighborhood of $\alpha$ and $Z_\alpha = \psi(U_\alpha)$; $Z_\alpha$ is a smooth complex submanifold of $S_N^\dagger$ passing through $\psi(\alpha)$. The rank of $\psi$ is generically constant, so that after a holomorphic change of coordinates, we can suppose that $U_\alpha = V_\alpha \times Z_\alpha$ and that $\psi$ is the projection on the second factor. If $(v,z)$ is a point of $V_\alpha \times Z_\alpha$, the kernel of $\mathrm{KS}_{(v,z)}(\psi^*\mathfrak{X}_N)$ is the set of vectors $(h,k)$ in $\mathrm{T}_v V_\alpha \oplus \mathrm{T}_z Z_\alpha$ such that $k$ is tangent to the orbit $O_z$. If $\alpha$ is sufficiently generic, these kernels define a holomorphic subbundle of $\mathrm{T}(V_\alpha \times Z_\alpha)$ of rank $n - m(\psi^*\mathfrak{X}_N)$, which is obviously integrable because the $\mathrm{PGL}(3;\mathbb{C})$-orbits in $S_N^\dagger$ define a regular foliation. Let $V_\alpha \times T_\alpha$ be the associated germ of integral manifold passing through $\alpha$. For every point $z$ in $T_\alpha$, $\mathrm{T}_z T_\alpha$ is included in $\mathrm{T}_z O_z$. Thus $T_\alpha$ is completely included in the orbit $O_{\psi(\alpha)}$. Let $\gamma$ be a local parametrization of $V_\alpha \times T_\alpha$. As the natural orbit map from $\mathrm{PGL}(3;\mathbb{C})$ to $O_{\psi(\alpha)}$ is a holomorphic submersion, we can choose locally around $\psi(\alpha)$ a holomorphic section $s$ such that $s(\psi(\alpha)) = \mathrm{Id}$. If we define $M(t) = s[\gamma(t)]$, then $\gamma(t) = M(t)\,\psi(\alpha)$.

Conversely, let $\alpha$ be a generic point in $U$, $d$ be an integer and $(\gamma, M)$ satisfying the hypotheses of the theorem. The image of $\gamma$ defines a germ of smooth subvariety $Y_\alpha$ in $U$ passing through $\alpha$, and its image by $\psi$ is entirely contained in the orbit $O_{\psi(\alpha)}$. This implies that the restriction of $\psi^*(\mathfrak{X}_N)$ to $Y_\alpha$ is holomorphically trivial. Thus $\mathrm{T}_\alpha Y_\alpha$ is contained in the kernel of $\mathrm{KS}_\alpha(\psi^*\mathfrak{X}_N)$. Since $\dim Y_\alpha = n - d$, we obtain $m(\psi^*\mathfrak{X}_N) \leq d$. □

**5.4. Nonbasic rational surfaces. —** We will briefly explain how to adapt the methods developed above to nonbasic rational surfaces, although we won't need it in the paper. The situation is more subtle, even for Hirzebruch surfaces. Indeed, if $n \geq 2$, $\mathrm{Aut}(\mathbb{F}_n)$ has dimension $n+5$ (*see* [**Bea78**]) so that $\mathrm{h}^1(\mathbb{F}_n, \mathrm{T}\mathbb{F}_n) = n-1$ and $\mathrm{h}^2(\mathbb{F}_n, \mathrm{T}\mathbb{F}_n) = 0$ by Lemma 5.4. Therefore the Hirzebruch surfaces $\mathbb{F}_n$ are not rigid if $n \geq 2$. Complete deformations of Hirzebruch surfaces $(\mathbb{F}_n)_{n\geq 2}$ are known and come from flat deformations of rank-two holomorphic bundles on $\mathbb{P}^1(\mathbb{C})$ (*see* [**Man04**],

Chap. II). These deformations $(\mathfrak{F}_n)_{n\geq 2}$ are highly noneffective because their generic number of parameters is zero. The fibers of $\mathfrak{F}_n$ are Hirzebruch surfaces $\mathbb{F}_{n-2k}$ of smaller index.

The deformations of nonbasic rational surfaces can be explicitly described using the same method as in §5.1: for every integer $n \geq 2$, define inductively a sequence of deformations $\tilde{\pi}_{N,n}\colon \mathfrak{F}_{N,n} \to S_{N,n}$ by $\mathfrak{F}_{0,n} = \mathfrak{F}_n$ and $\mathfrak{F}_{N+1,n} = \widetilde{\mathfrak{F}_{N,n}}$ (*cf* Definition 5.2). This means that

$$S_{N,n} = \{a, p_1, \dots, p_N \,|\, a \in \mathbb{F}_n,\, p_1 \in (\mathfrak{F}_n)_a,\, p_2 \in \mathrm{Bl}_{p_1}(\mathfrak{F}_n)_a, \dots, p_N \in \mathrm{Bl}_{p_{N-1}} \dots \mathrm{Bl}_{p_1}(\mathfrak{F}_n)_a\}$$

and that $(\mathfrak{F}_{N,n})_{a,\,p_1,\dots,p_N} = \mathrm{Bl}_{p_N} \dots \mathrm{Bl}_{p_1}(\mathfrak{F}_n)_a$.

If $X = \mathrm{Bl}_{\widehat{P}}\mathbb{F}_n$ is a nonbasic rational surface, then $\widehat{P}$ defines a point in $S_{N,n}$ for a certain integer $N$. By Proposition 5.3, $\mathfrak{F}_{N,n}$ is complete at $\widehat{P}$. Therefore small deformations of a nonbasic rational surface can be parameterized by (possibly infinitely near) points on Hirzebruch surfaces $\mathbb{F}_n$, but $n$ can jump with the deformation parameters.

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