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Topological properties of Hilbert schemes of almost-complex four-manifolds (II)

JULIEN GRIVAUX

In this article, we study the rational cohomology rings of Voisin's Hilbert schemes $X^{[n]}$ associated with a symplectic compact four-manifold X. We prove that these rings can be universally constructed from $H^*(X,\mathbb{Q})$ and $c_1(X)$, and that Ruan's crepant resolution conjecture holds if $c_1(X)$ is a torsion class. Next, we prove that for any almost-complex compact four-manifold X, the complex cobordism class of $X^{[n]}$ depends only on the complex cobordism class of X.

32Q60; 14C05,14J35

1 Introduction

The Hilbert schemes of points $X^{[n]}$ of a smooth projective complex surface X are moduli spaces for finite subschemes of length n on X. By a result of Fogarty [17], the varieties $X^{[n]}$ are smooth crepant resolutions of the n–fold symmetric powers $X^{(n)}$ of X, so that they present a strong geometric interest. Hilbert schemes of points have been intensively studied in the past twenty years, and this has led to important developments in algebraic and differential geometry as well as in theoretical physics (see [25], [20] for an overview).

Among these various studies, we will recall here what concerns the cohomology rings $H^*\big(X^{[n]},\mathbb{Q}\big)$ of Hilbert schemes.

The first step towards the understanding of the vector spaces $H^*\big(X^{[n]},\mathbb{Q}\big)$ was achieved by Göttsche [18] with the computation of the generating series for the Betti numbers $b_i(X^{[n]})$ in terms of the Betti numbers of the surface X. Then Wafa and Witten pointed out that the infinite-dimensional vector space $\mathbb{H}=\oplus_{n\geq 0}H^*\big(X^{[n]},\mathbb{Q}\big)$ was (by Göttsche's formula) an abstract highest-weight representation of the Heisenberg super-algebra modeled on $H^*\big(X,\mathbb{Q}\big)$. Such a construction was geometrically realized independently by Nakajima [35] and by Grojnowski [24], using correspondences given by incidence varieties. The additive structure of the cohomology rings of Hilbert schemes of points was thus given a precise geometric description.

Then Lehn [26] obtained a decisive result in the study of the multiplicative structure. He computed explicitly, when E is an algebraic vector bundle on X and the $E^{[n]}$ are the associated tautological bundles on the $X^{[n]}$, the cup product by $\mathrm{ch}(E^{[n]})$ on $H^*\big(X^{[n]},\mathbb{Q}\big)$, and described in this way the subring of $\mathbb{H}$ generated by the Chern classes of all the tautological bundles. This study was completed by Li, Qin and Wang [30], who constructed virtual tautological Chern characters $G(\alpha,n)$ attached to each cohomology class α on X, and computed the cup product by each $G(\alpha,n)$ on $\mathbb{H}$. The classes $G(\alpha,n)$ extend the Chern characters of the tautological bundles in the following way: if α is the Chern character of an algebraic vector bundle E, then $G(\alpha,n)$ is the Chern character of $E^{[n]}$. Besides, it is shown in [30] that the components of $\big(G_i(\alpha,n)\big)_{0\leq i<n}$ of the classes $G(\alpha,n)$ generate the ring $H^*\big(X^{[n]},\mathbb{Q}\big)$ when α runs through a basis of $H^*\big(X,\mathbb{Q}\big)$. Using these generators, the authors obtained in [31] a universal description of $H^*\big(X^{[n]},\mathbb{Q}\big)$ from the ring $H^*\big(X,\mathbb{Q}\big)$ and the first Chern class of X in $H^2(X,\mathbb{Q})$.

This study led to interesting consequences in the particular case where X is a $K3$–surface. Indeed, at the same time, Chen and Ruan [10] developed the theory of orbifold cohomology and at the existence of strong relations between the orbifold cohomology ring of an orbifold and the cohomology ring of a crepant resolution: this is known as the cohomological crepant resolution conjecture (see [39]). If X is a $K3$–surface, the Hilbert schemes $X^{[n]}$ are hyperkähler by a result of Beauville [5] and in this case the cohomological crepant resolution conjecture predicts that for every positive integer n, the ring $H^*\big(X^{[n]},\mathbb{C}\big)$ and the orbifold cohomology ring $H^*_{\mathrm{orb}}\big(X^{(n)},\mathbb{C}\big)$ of the n–fold symmetric product of X are isomorphic [1, Conj. 4.24]. The above recalled description of the cohomology ring of Hilbert schemes made it possible to prove this prediction (see [28]) by putting together results of Lehn–Sorger [28] for the Hilbert schemes part with the computations performed independently by Fantechi–Göttsche [16] and Uribe [43] for the orbifold part.

If we leave the algebraic setting and consider abstract compact complex surfaces instead of projective ones, Hilbert schemes of points still exist (they are usually called Douady schemes). If X is a compact Kähler surface, it can be deformed to a projective surface, so that the general description given for the cohomology rings of $X^{[n]}$ when X is projective remains valid. In the general case of possibly non-Kähler compact complex surfaces, Göttsche's formula has been proved by De Cataldo–Migliorini [8], but the study of the multiplicative structure cannot be performed as in the projective case. Indeed, Lehn [26] uses in an essential way the fact that the cohomological cycle classes of smooth algebraic curves on a smooth projective surface X span $H^{1,1}(X)$. This property fails for abstract complex compact surfaces.

In a still more general context, Voisin defined in [44] Hilbert schemes $X^{[n]}$ associated

with any almost-complex four-manifold (X, J): for every positive integer n, $X^{[n]}$ is a stably almost-complex differentiable manifold of real dimension $4n$. These almost-complex Hilbert schemes are symplectic if X is symplectic (see [45]), and are still crepant resolutions of the n–fold symmetric products of X. This construction sheds a new light on the results we have mentioned about the cohomology rings of Hilbert schemes: indeed, Voisin's results imply that, for any projective surface X and any positive integer n, the underlying differentiable manifold of $X^{[n]}$ depends only on the underlying differentiable manifold of X and on the deformation class of the complex structure of X in the space of almost-complex structures. This explains why the ring $H^*\big(X, \mathbb{Q}\big)$ depends only on almost-complex invariants of X. On the other hand, it is worth noticing that orbifold cohomology is naturally defined for almost-complex orbifolds, which includes n–fold symmetric products of any almost-complex manifold.

Our paper is the second part of a program, the aim of which is the study of Voisin's almost-complex Hilbert schemes. The first part [23] has been devoted to the additive structure of their cohomology rings: Göttsche's formula has been proved and Nakajima operators have been constructed. The first main concern here is the study of the ring structure of the almost-complex Hilbert schemes. We prove the analog of Li–Qin–Wang's result quoted above under a symplectic hypothesis:

Theorem 1.1 *If (X, J) is a symplectic four-manifold, the rings $H^*\big(X^{[n]}, \mathbb{Q}\big)$ can be constructed by universal formulae from the ring $H^*\big(X, \mathbb{Q}\big)$ and the first Chern class of X in $H^2(X, \mathbb{Q})$.*

This theorem is proved in § 5.3. Let us have a glimpse at the strategy of the proof. The techniques developed in [23] allow us to adapt in the almost-complex case the quasi-totality of the proof of Lehn's main formula [26, Th 3.10], except for the very argument which has already been pointed out for non algebraic complex surfaces: if (X, J) is an almost-complex compact four-manifold, the homology classes of smooth J–holomorphic curves in X do not span $H_2(X, \mathbb{Q})$ in general. To overcome this difficulty, we use the symplectic assumption: if X is symplectic and if J is an adapted almost-complex structure, Donaldson's theorem on symplectic divisors [12] makes it possible to span $H_2(X, \mathbb{Q})$ by pseudo-holomorphic curves for small perturbations of the almost-complex structure J.

The second important problem in the proof of Theorem 1.1, which occurs only if the first Betti number of X is nonzero, is to construct virtual tautological characters for almost-complex Hilbert schemes. Indeed, in the classical situation, if X is a projective surface, and if Y_n is the incidence locus in $X^{[n]} \times X$, the virtual tautological Chern

character $G(\alpha, n)$ is equal to $\mathrm{pr}_{1*}\left[\mathrm{ch}(\mathcal{O}_{Y_n}) \,.\, \mathrm{pr}_2^*(\alpha \,.\, \mathrm{td}(X))\right]$. The problem in the almost-complex setting lies in the term $\mathrm{ch}(\mathcal{O}_{Y_n})$. Unlike in the case of vector bundles, there is no tractable analog of coherent analytic sheaves on almost-complex Hilbert schemes: the differentiable structure of $X^{[n]}$ is pretty hard to deal with; the topological structure of $X^{[n]}$ (as a C^0–manifold) would do better, but sheaves of continuous functions are generally ill-behaved. To explain the means used to cope with this problem, we have to recall the basics of the construction of $X^{[n]}$ when X is an almost-complex four-manifold: it relies on the choice of a relative integrable complex structure J_n^{rel}, which is essentially a smooth family $J_{n,\underline{x}}^{\mathrm{rel}}$ of integrable complex structures parameterized by $X^{(n)}$ such that for each $\underline{x}$ in $X^{(n)}$, $J_{n,\underline{x}}^{\mathrm{rel}}$ is an integrable complex structure in a neighbourhood $W_{\underline{x}}$ of the points of $\underline{x}$. If $W_{\mathrm{rel}}^{[n]}$ is the disjoint union of the Hilbert–Douady schemes $W_{\underline{x}}^{[n]}$, where each $W_{\underline{x}}$ is endowed with the integrable structure $J_{n,\underline{x}}^{\mathrm{rel}}$, then $X^{[n]}$ is a subset of $W_{\mathrm{rel}}^{[n]}$. Our main idea is to replace $X^{[n]}$ by $W_{\mathrm{rel}}^{[n]}$, the latter having a much better structure: it is a differentiable orbifold fibred in smooth analytic sets over $X^{(n)}$. We develop in a systematic way a theory for these spaces, which we call relative analytic spaces, and for a particular class of sheaves on them, the relatively coherent sheaves. These sheaves are locally an extension of classical coherent analytical sheaves by $\mathcal{C}^\infty$ parameters. If T is a differentiable orbifold chosen as parameter space and if $\mathcal{F}$ is a coherent analytic sheaf on an analytic set Z, then it is possible to define a sheaf $\mathcal{C}^\infty(T, \mathcal{F})$ of smooth sections of $\mathcal{F}$ with parameters in T as follows: if $\mathcal{F}$ is the sheaf $\mathcal{O}_Z$, then $\mathcal{C}^\infty(T, \mathcal{O}_Z)$ is the subsheaf of $\mathcal{C}^\infty_{Z\times T}$ consisting of smooth functions holomorphic in the first variables, and for an arbitrary $\mathcal{F}$, the sheaf $\mathcal{C}^\infty(T, \mathcal{F})$ is equal to $\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{C}^\infty(T, \mathcal{O}_Z)$. Besides, this construction can be sheafified in the space of parameters: if V is a differentiable orbifold, there exists a sheaf $\widetilde{\mathcal{F}}$ on $Z \times V$ such that for all open subsets U and T of Z and V respectively, $\Gamma(U \times T, \widetilde{\mathcal{F}})$ is equal to $\Gamma(U, \mathcal{C}^\infty(T, \mathcal{F}))$. In our construction, $Z \times V$ is a local model for a relative analytic space and $\widetilde{\mathcal{F}}$ is a local model for a relatively coherent sheaf on $Z \times V$. A relative analytic space is obtained by gluing together a family of such local models and so is a relatively coherent sheaf.

The formalism of relative analytic spaces and relatively coherent sheaves allows us to use tools of algebraic and analytic geometry in the almost-complex setting. It does not only solve our present problem, but will be essential in the last section of the paper (§ 6). We give an independent exposition of this formalism, as it may be useful in other situations.

As soon as Lehn's formula is obtained and virtual Chern characters are constructed, Theorem 1.1 follows from formal combinatorial arguments. We also prove (in § 5.3) that Ruan's cohomological crepant resolution conjecture is valid for Hilbert schemes of symplectic compact four-manifolds with torsion first Chern class:

Theorem 1.2 *Let (X, ω) be a symplectic compact four-manifold with vanishing first Chern class in $H^2(X, \mathbb{Q})$. Then, for every positive integer n, the rings $H^*\big(X^{[n]}, \mathbb{C}\big)$ and $H^*_{\mathrm{orb}}\big(X^{(n)}, \mathbb{C}\big)$ are isomorphic.*

Our second object in this paper is the description of the complex cobordism classes of almost-complex Hilbert schemes. We obtain:

Theorem 1.3 *Let (X, J) be an almost-complex compact four-manifold. For any positive integer n, the complex cobordism class of $X^{[n]}$ given by its stable almost-complex structure depends only on the complex cobordism class of X.*

This theorem is proved in § 6. Our interest in this problem goes back to Voisin's original motivation for constructing Hilbert schemes in the almost-complex setting: it is the computation of the cobordism classes of Hilbert schemes of points for projective surfaces, achieved by Ellingsrud–Göttsche–Lehn in [15]. The authors proved that these classes can be universally computed from the cobordism classes of the surface X itself. Their result shows that Hilbert schemes of points for projective surfaces can be interpreted as modifications at the level of complex cobordism. Voisin's idea was that this modification of the complex cobordism could be lifted at the level of almost-complex manifolds. Although she actually constructed Hilbert schemes for almost-complex four-manifolds, it is not at all clear that they actually lift the classical Hilbert schemes at the cobordism level. Our theorem means that it is indeed the case. The proof relies heavily on the use of relatively coherent sheaves to adapt the argument of [15] in the almost-complex setting.

Let us now describe the organization of the paper.

Two distinct aims are pursued in § 2. The first one is to define relative analytic spaces, which occur in particular in Voisin's construction of almost-complex Hilbert schemes. The second one is to recall this construction as well as related results. This section, which is mainly expository, will be used throughout the paper. In § 2.1, we recall classical results about Hilbert schemes of points. In § 2.2, we introduce relative and differentiable analytic spaces. This section consists mostly of definitions. The link between relative analytic spaces and relative integrable structures originally used in [44] is given by Proposition 2.12. We also state a general existence result for relative integrable structures (Proposition 2.13), which is proved in § 7. In § 2.3, we recall the construction of almost-complex Hilbert schemes only as topological spaces. For the construction of their differentiable structures, we refer the reader to [45, § 2] for an outline and to [44, § 3] for a detailed exposition. Then we state the main results of [44], [45] and [23] about almost-complex and symplectic Hilbert schemes. In § 2.4, we

recall briefly the construction of incidence varieties carried out in [23, § 4], mainly to fix the notations. The compatibility conditions (A) and (B) introduced on page 18 will appear several times in § 3.5, § 5.2 and § 6.

In § 3, we present the general theory of relatively coherent sheaves on the relative analytic spaces introduced in § 2.2. This formalism is rather heavy and will be used in § 5.2 to construct virtual tautological Chern characters and in § 6 to compute the complex cobordism class of almost-complex Hilbert schemes. The reader interested only in the description of the cohomology ring of $X^{[n]}$ when X is an almost-complex compact four-manifold with vanishing first Betti number may skip this section except for Definition 3.1, and go directly to § 4. In § 3.1, we define relatively coherent sheaves as well as related operations: pullback, internal Hom, tensor product and the corresponding derived operations. In § 3.2, we define relative analytic subspaces and prove in Proposition 3.11 that their structure sheaves are relatively coherent. The proof relies on deep properties concerning ideals of differentiable functions. We also prove in Proposition 3.13 that relatively coherent sheaves are stable under push-forward by finite maps. The aim of § 3.3 is to generalize the formalism of analytic K–theory of [6] for relative analytic spaces. Suitable morphisms are introduced in Definition 3.14, so that the relatively coherent sheaves on a relative analytic space $\mathfrak{X}$ becomes an abelian category. The associated Grothendieck group is by definition the relative analytic K–theory of $\mathfrak{X}$. Then the operations defined in § 3.2 and § 3.3 induce operations in relative K–theory. Various formulae relating these operations are grouped in Proposition 3.16; they will be frequently used in § 3.5, § 5.2 and § 6. The aim of § 3.4 is to construct a map from relative analytic K–theory to topological K–theory with complex coefficients. In the case of usual analytic K–theory, such a map can be obtained via global real analytic locally free resolutions of coherent analytic sheaves (see [2, Prop. 2.6]). It is also possible to use differentiable resolutions instead of real analytic ones, as explained in [3, § 6]; this is the method we adopt in the case of relatively coherent sheaves. The important point is that, although differentiable coherent sheaves as introduced by Atiyah and Hirzebruch in [3, Def. 6.1] are defined by a global condition, it turns out that they can be characterized by a local condition: it is the object of Proposition 3.17. This allows us to prove that any relatively coherent sheaf $\mathcal{F}$ on a relative analytic space $\mathfrak{X}$ admits a finite locally free resolution over $\mathcal{C}^\infty_{\mathfrak{X}}$ in a neighbourhood of every compact subset of $\mathfrak{X}$ (Proposition 3.19), and then to associate with $\mathcal{F}$ a well defined element in topological K–theory, called the topological class of $\mathcal{F}$. Next, we prove two important results about this class, namely the functoriality by pullback (Proposition 3.20) and the homotopy invariance (Proposition 3.21). In § 3.5 we apply the formalism of § 3.1–3.4 and associate with any almost-complex compact four-manifold X various incidence

sheaves on relative analytic spaces built from X via suitable relative integrable complex structures. This defines the geometric setting which will be used independently in § 5.2 and § 6. The last result of the section (Proposition 3.27), which is the analog of [15, Lemma 1.1] in a relative setting, will be used only in the proof of Proposition 6.8.

The object of § 4 is to carry out for symplectic four-manifolds Lehn's computation of the boundary operator [26, Th. 3.10]. In § 4.1 we adapt the first part of Lehn's argument to the almost-complex case as we did for the Nakajima relations in [23]. This yields half of Lehn's formula (Theorem 4.2). In § 4.2, we use Donaldson's theorem on symplectic divisors [12] to establish a general result (Proposition 4.6) concerning pseudo-holomorphic curves on symplectic four-manifolds. This result allows us to obtain in § 4.3 the other half of Lehn's formula when X is symplectic: this is the object of Theorem 4.7.

In § 5, we deal with the cohomology rings of Hilbert schemes of points for symplectic compact four-manifolds. In § 5.1, we prove an induction relation for the Chern characters of the tautological vector bundles constructed in [23] (Lemma 5.1). In § 5.2, we construct virtual Chern characters satisfying the same induction relation (Proposition 5.2). As we already mentioned, we use the machinery of relatively coherent sheaves, and especially § 3.5. Note that § 5.2 can be skipped if the first Betti number of X vanishes. In § 5.3, we state and prove our main results about the cohomology rings of symplectic Hilbert schemes which are Theorems 5.6, 5.7 and 5.9.

Finally, § 6 is entirely devoted to the computation of the cobordism class of Hilbert schemes of an almost-complex compact four-manifold (Theorem 6.1). We combine the strategy of [15, § 1-3] with the use of the relative incidence sheaves of § 3.5. In § 6.1, § 6.2 and § 6.3 respectively, we extend to the relative setting the results of [15, Prop. 2.2, Prop. 2.3 and Prop. 3.1]. This is the object of Propositions 6.4, 6.5 and 6.8.

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2 Almost-complex Hilbert schemes and relative analytic spaces

2.1 Hilbert schemes, incidence varieties and tautological bundles

Let X be a smooth complex manifold, $\mathcal{O}_X$ be its sheaf of holomorphic functions and n be a positive integer.

Definition 2.1 The *Hilbert–Douady* scheme $X^{[n]}$ of n–points in X is the set of zero-dimensional subschemes of length n in X, ie the set of ideal sheaves $\mathcal{J}$ of $\mathcal{O}_X$ such that $\sum_{p\in X} \dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{J}_p$ is equal to n.

Let $\mathfrak{S}_n$ be the symmetric group on n symbols and let $X^{(n)} := X^n/\mathfrak{S}_n$ denote the n–fold symmetric product of X.

Definition 2.2 The *Hilbert–Chow morphism* $\Gamma\colon X^{[n]} \longrightarrow X^{(n)}$ is defined by the formula $\Gamma(\xi) = \sum_{p\in X} l_p(\xi)\, p$, where $l_p(\xi)$ is the length of ξ at p.

Some basic properties of Hilbert schemes of points are:

– $X^{[n]}$ is a complex analytic space and Γ is a bimeromorphic map.
– If X is compact, so is $X^{[n]}$.
– The fibers of Γ are projective, and irreducible if $\dim X = 2$ [7].
– If X is a complex curve, then Γ is an isomorphism and $X^{[n]}$ is smooth.
– If X is a complex surface, then $X^{[n]}$ is smooth of dimension $2n$ and is irreducible if X is connected [17].

For a thorough study of Hilbert schemes of points, we refer the reader to [19] and [35].

In this section, we only consider the case $\dim X = 2$. The Hilbert schemes corresponding to different values of n are related through the incidence varieties:

Definition 2.3 For all positive integers m and n such that $m > n$, the *incidence variety* $X^{[m,n]}$ is the set of couples (ξ, ξ') in $X^{[n]} \times X^{[m]}$ such that ξ is a subscheme of ξ'.

The incidence varieties $X^{[m,n]}$ are analytic subvarieties of $X^{[n]} \times X^{[m]}$. The case $m = n+1$ appears as particularly interesting in the theory:

– $X^{[n+1,n]}$ is smooth and irreducible [9], [42], [Ellingsrud (unpublished)].
– If Y_n is the *incidence locus*, defined as

$$Y_n = \{(\xi, x) \text{ in } X^{[n]} \times X \text{ such that } x \in \operatorname{supp}(\xi)\}, \tag{2–1}$$

then $X^{[n+1,n]} \simeq \mathbb{P}(\mathcal{J}_n)$ [11, Prop. 3.3]. As a consequence, $X^{[n+1,n]}$ is isomorphic to the schematic blowup of Y_n in $X^{[n]} \times X$. The *exceptional divisor* associated with this blowup is

$$D = \{(\xi, \xi') \text{ in } X^{[n+1,n]} \text{ such that } \operatorname{supp}(\xi) = \operatorname{supp}(\xi')\}. \tag{2–2}$$

These properties show that the incidence variety $X^{[n+1,n]}$ is closely related to the incidence locus Y_n. The latter satisfies the following properties:

– The morphism $\mathrm{pr}_{1\,|Y_n} : Y_n \longrightarrow X^{[n]}$ is flat and finite.
– The ideal sheaf $\mathcal{J}_{Y_n}$ admits a global locally free resolution of length 2 on $X^{[n]} \times X$ [11, Lemma 3.2].
– If $\phi\colon X^{[n+1,n]} \longrightarrow X^{[n]}$ and $\psi\colon X^{[n+1,n]} \longrightarrow X^{[n+1]}$ are the natural morphisms induced by the projections, if $\rho\colon X^{[n+1,n]} \longrightarrow X$ is the *residual map* defined by the formula $\rho(\xi, \xi') = \operatorname{supp}\big(\xi/\xi'\big)$, if $j = (\mathrm{id}, \rho)\colon X^{[n+1,n]} \longrightarrow X^{[n+1,n]} \times X$ and if $\mathcal{L} = \mathcal{O}_{X^{[n+1,n]}}(-D)$, then there exists a natural exact sequence on $X^{[n+1,n]} \times X$ [15, § 1]

$$0 \longrightarrow j_*\mathcal{L} \longrightarrow \psi_X^*\mathcal{O}_{Y_{n+1}} \longrightarrow \phi_X^*\mathcal{O}_{Y_n} \longrightarrow 0. \tag{2–3}$$

The variety $X^{[n+1,n]}$ can be constructed explicitly via a global locally free resolution

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{J}_{Y_n} \longrightarrow 0$$

of $\mathcal{J}_{Y_n}$: if $\mathbb{P}(\mathcal{B})$ is the projective bundle of $\mathcal{B}$ (using Grothendieck's convention for projective bundles), if $\pi\colon \mathbb{P}(\mathcal{B}) \longrightarrow X^{[n]} \times X$ is the associated projection and if s is the section of $\pi^*\mathcal{A}^*(1)$ given by the morphism $\pi^*\mathcal{A} \to \pi^*\mathcal{B} \to \mathcal{O}_{\mathcal{B}}(1)$, then s is transverse to the zero section and its vanishing locus is isomorphic to $X^{[n+1,n]}$.

We end this section with tautological bundles. Let pr_1 and pr_2 be the projections from $X^{[n]} \times X$ on the first and second factors.

Definition 2.4 Let E be a holomorphic vector bundle on X and n be a positive integer. The *tautological vector bundle* $E^{[n]}$ is a holomorphic vector bundle on $X^{[n]}$ defined by the formula $E^{[n]} = \mathrm{pr}_{1*}\big(\mathcal{O}_{Y_n} \,.\, \mathrm{pr}_2^* E\big)$.

If E is a holomorphic vector bundle on X and n is a positive integer, the tautological vector bundles $E^{[n]}$ and $E^{[n+1]}$ are related through an exact sequence on $X^{[n+1,n]}$ (see [26, p. 193]):

$$0 \longrightarrow \rho^*E \otimes \mathcal{L} \longrightarrow \psi^*E^{[n+1]} \longrightarrow \phi^*E^{[n]} \longrightarrow 0. \tag{2–4}$$

2.2 Relative spaces and relative integrable complex structures

The geometric structure underlying the construction of the almost-complex Hilbert scheme of [44] is that of relative integrable structure, or in an almost equivalent way that of relative analytic space. In this article, this point of view is systematically expanded in order to study coherent sheaves in relative analytic spaces, which is done in § 3.

Throughout this section, B and B' denote compact differentiable effective orbifolds (see [40], [1, § 1.1]). Recall that a map $f\colon B \longrightarrow B'$ is smooth if for any b in B there exist two orbifold charts $(V_b,\ G_b,\ U_b)$ and $(V_{f(b)},\ G_{f(b)},\ U_{f(b)})$ near b and $f(b)$, a group morphism $\lambda\colon G_b \longrightarrow G_{f(b)}$ and a smooth λ–equivariant map from V_b to $V_{f(b)}$ inducing f on U_b.

Definition 2.5 Let $\mathfrak{X}$ be a separated topological space and $\pi\colon \mathfrak{X} \longrightarrow B$ be a continuous surjective map.

(i) A *relative chart* (resp. relative holomorphic chart) on $\mathfrak{X}$ is given by a homeomorphism $\phi\colon U \xrightarrow{\sim} Y \times V$ such that $\pi_{|U} = \mathrm{pr}_2 \circ \phi$, where U is an open subset of $\mathfrak{X}$, V is an open subset of B and Y is a differentiable manifold (resp. Y is a smooth analytic space).

(ii) Let $\phi\colon U \xrightarrow{\sim} Y \times V$ and $\psi\colon U' \xrightarrow{\sim} Y' \times V'$ be two relative charts (resp. relative holomorphic charts) and let $(y, v) \longmapsto (\gamma(y, v), v)$ be the associated transition function $\psi \circ \phi^{-1}\colon \phi(U \cap U') \longrightarrow \psi(U \cap U')$. The charts ϕ and ψ are *compatible* if γ is smooth (resp. γ is smooth and for all v in $\mathrm{pr}_2(\phi(U \cap U'))$ the function $y \longmapsto \gamma(y, v)$ is holomorphic).

(iii) A *relative atlas* (resp. *relative holomorphic atlas*) on $\mathfrak{X}$ is a collection of compatible relative charts (resp. relative holomorphic charts) on X whose domains cover $\mathfrak{X}$.

(iv) A relative atlas (resp. relative holomorphic atlas) $\mathcal{A}$ on $\mathfrak{X}$ is *maximal* if every relative chart (resp. relative holomorphic chart) on $\mathfrak{X}$ compatible with all the charts of $\mathcal{A}$ belongs to $\mathcal{A}$.

(v) If $\mathcal{A}$ is a relative atlas (resp. relative holomorphic atlas) on $\mathfrak{X}$, the *saturated atlas* of $\mathcal{A}$ is the smallest maximal atlas containing $\mathcal{A}$.

To be able to define and study relatively coherent sheaves in § 3, we introduce the notion of complete relative holomorphic atlas (cf. Remark 3.3 (iii)).

Definition 2.6 Let $\mathfrak{X}$ be a separated topological space and let $\pi\colon \mathfrak{X} \longrightarrow B$ be a continuous surjective map.

(i) If $\phi\colon U \xrightarrow{\sim} Y \times V$ is a relative holomorphic chart, if Y' (resp. V') is an open subset of Y (resp. V) and if $U' = \phi^{-1}(Y' \times V')$, then $\phi_{|U'}\colon U' \xrightarrow{\sim} Y' \times V'$ is a relative holomorphic chart called *the restriction* of ϕ to U'.

(ii) A relative holomorphic atlas $\mathcal{A}$ on $\mathfrak{X}$ is *complete* if
– for all relative holomorphic chart ϕ in $\mathcal{A}$, all the restrictions of ϕ are in $\mathcal{A}$.
– for all finite family $\{\phi_i\colon U_i \xrightarrow{\sim} Y_i \times V_i\}_{1 \leq i \leq r}$ of relative holomorphic charts in $\mathcal{A}$ such that the open sets U_i are pairwise disjoint, then the relative holomorphic chart $\coprod_{i=1}^{r} \phi_i\colon \coprod U_i \xrightarrow{\sim} \coprod (Y_i \times V_i)$ is in $\mathcal{A}$.

(iii) If $\mathcal{A}$ is a relative holomorphic atlas on $\mathfrak{X}$, the *completed atlas* of $\mathcal{A}$ is the smallest complete relative holomorphic atlas containing $\mathcal{A}$.

(iv) If $\mathcal{A}$ and $\mathcal{A}'$ are two relative holomorphic atlases on $\mathfrak{X}$, we say $\mathcal{A}$ *refines* $\mathcal{A}'$ if for any relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Y \times V$ in $\mathcal{A}'$ and any x in U, there exists a neighbourhood U_x of x in U such that $\phi_{|U_x}$ is in $\mathcal{A}$.

(v) Two relative holomorphic atlases $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent* if $\mathcal{A}$ refines $\mathcal{A}'$ and $\mathcal{A}'$ refines $\mathcal{A}$.

We define now relative differentiable spaces and relative analytic spaces.

Definition 2.7 (i) A *relative differentiable space* (resp. *relative analytic space*) over B is the data of a separated topological space $\mathfrak{X}$ endowed with a continuous surjective map $\pi\colon \mathfrak{X} \longrightarrow B$ and with a maximal relative atlas (resp. an equivalent class of complete relative holomorphic atlas) over B.

(ii) If $\mathfrak{X}$ is a separated topological space, if $\pi\colon \mathfrak{X} \longrightarrow B$ is a continuous surjective map and if $\mathcal{A}$ is a maximal relative atlas (resp. a complete relative holomorphic atlas) on $\mathfrak{X}$, we denote by $(\mathfrak{X}, \mathcal{A})$ the associated relative differentiable space (resp. relative analytic space).

Remark 2.8 (i) If $(\mathfrak{X}, \mathcal{A})$ is a relative differentiable space (resp. relative analytic space) over B, the fibers $\left(\mathfrak{X}_b\right)_{b \in B}$ defined by $\mathfrak{X}_b := \pi^{-1}(b)$ are differentiable (resp. complex) manifolds, but they do not form in general a fibration over B, since the projection map π is not assumed to be proper.

(ii) The connected components of a relative differentiable space (resp. relative analytic space) $(\mathfrak{X}, \mathcal{A})$ are still relative differentiable spaces (resp. relative analytic spaces). If $\mathfrak{X}$ is connected, the dimension (resp. complex dimension) of $\mathfrak{X}_b$ is independent of b. We call it the *relative real dimension* (resp. *relative complex dimension*) of $\mathfrak{X}$.

(iii) If $(\mathfrak{X}, \mathcal{A})$ is a relative differentiable space over B, then $\mathfrak{X}$ is a differentiable orbifold and the projection $\pi\colon \mathfrak{X} \longrightarrow B$ is smooth.

Let us introduce some natural operations on relative analytic spaces.

Definition 2.9 (i) BASE CHANGE. Let $(\mathfrak{X}, \mathcal{A})$ be a relative differentiable space (resp. relative analytic space) over B and $u\colon B' \longrightarrow B$ be a smooth map. If $\mathcal{A} = \{\phi_i\colon U_i \xrightarrow{\sim} Y_i \times V_i\}_{i\in I}$, if $\mathcal{A}' = \{\phi_i \times_B \mathrm{id}_{B'}\colon U_i \times_B B' \xrightarrow{\sim} Y_i \times u^{-1}(V_i)\}_{i\in I}$ and if $u^*(\mathcal{A})$ is the saturated (resp. completed) atlas of $\mathcal{A}'$, then $\big(\mathfrak{X} \times_B B', u^*(\mathcal{A})\big)$ is a relative differentiable space (resp. relative analytic space) over B'.

(ii) FIBER PRODUCT. Let $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{X}', \mathcal{A}')$ be two relative differentiable spaces (resp. relative analytic spaces) over the same base B. If $\phi\colon U \xrightarrow{\sim} Z \times V$ and $\phi'\colon U' \xrightarrow{\sim} Z' \times V$ are two relative charts (resp. relative holomorphic charts) over the same open set V, then $\phi \times_B \phi'\colon U \times_B U' \xrightarrow{\sim} (Z \times Z') \times V$ is a relative chart (resp. relative holomorphic chart) on $\mathfrak{X} \times_B \mathfrak{X}'$. If $\mathcal{A}''$ is the relative atlas (resp. relative holomorphic atlas) on $\mathfrak{X} \times_B \mathfrak{X}'$ consisting of such charts $\phi \times_B \phi'$ and if $\mathcal{A} \times_B \mathcal{A}'$ is the saturated (resp. completed) atlas of $\mathcal{A}''$, then $\big(\mathfrak{X} \times_B \mathfrak{X}', \mathcal{A} \times_B \mathcal{A}'\big)$ is a relative differentiable space (resp. relative analytic space).

(iii) RELATIVE TANGENT BUNDLE. Let $(\mathfrak{X}, \mathcal{A})$ be a relative differentiable space (resp. relative analytic space) over B, where $\mathcal{A} = \{\phi_i\colon U_i \xrightarrow{\sim} Y_i \times V_i\}_{i\in I}$. We define a set $T^{\mathrm{rel}}\mathfrak{X}$ by $T^{\mathrm{rel}}\mathfrak{X} = \coprod_{b\in B} T\mathfrak{X}_b$, and a relative differentiable (resp. relative holomorphic) atlas $\mathcal{A}'$ on $T^{\mathrm{rel}}\mathfrak{X}$ by $\mathcal{A}' = \{d^{\mathrm{rel}}\phi_i\colon T^{\mathrm{rel}}U_i \xrightarrow{\sim} TY_i \times V_i\}_{i\in I}$, where $d^{\mathrm{rel}}\phi_i$ is the relative differential of ϕ_i. If $T^{\mathrm{rel}}\mathcal{A}$ is the saturated (resp. completed) atlas of $\mathcal{A}'$, then $\big(T^{\mathrm{rel}}\mathfrak{X}, T^{\mathrm{rel}}\mathcal{A}\big)$ is a relative differentiable space (resp. relative analytic space) called the *relative tangent bundle* of $\mathfrak{X}$. As a topological space, $T^{\mathrm{rel}}\mathfrak{X}$ is a topological vector bundle over $\mathfrak{X}$.

Relative analytic spaces are introduced in [44] by means of relative integrable structures:

Definition 2.10 Let $\mathfrak{X}$ be a relative differentiable space over B. A *relative integrable complex structure* J^{rel} on $\mathfrak{X}$ is a continuous section of $\mathrm{End}\,(T^{\mathrm{rel}}\mathfrak{X})$ satisfying the following conditions:

– If $\phi\colon U \xrightarrow{\sim} Y \times V$ is any relative chart of $\mathfrak{X}$, the map $J^{\mathrm{rel}}\colon Y \times V \longrightarrow TY$ is smooth.
– For every b in B, the map $J^{\mathrm{rel}}_b\colon \mathfrak{X}_b \longrightarrow \mathrm{End}\,(T\mathfrak{X}_b)$ defines an integrable complex structure on $\mathfrak{X}_b$.

The next proposition allows us to construct local holomorphic trivializations for relative integrable complex structures.

Proposition 2.11 *Let $(\mathfrak{X}, \mathcal{A})$ be a relative differentiable space over B and J^{rel} be a relative integrable complex structure on $\mathfrak{X}$. For every x in $\mathfrak{X}$, there exist a neighbourhood U_x of x and a relative chart $\phi\colon U_x \xrightarrow{\sim} W \times V$ in $\mathcal{A}$ such that:*

– W is an open subset of $\mathbb{C}^N$, where $2N$ is the relative real dimension of the connected component of x in $\mathfrak{X}$.
– If J_{st} is the standard complex structure on $\mathbb{C}^N$ and if b is any point in V, then

$$\phi_b\colon \left(U_x \cap \mathfrak{X}_b, J_b^{\mathrm{rel}}\right) \longrightarrow \left(W, J_{\mathrm{st}}\right)$$

is a biholomorphism.

Proof Let $\widetilde{U}_x$ be a neighbourhood of x, $\widetilde{\phi}\colon \widetilde{U}_x \xrightarrow{\sim} Y \times V$ be a relative chart, and put $\widetilde{\phi}(x) = (y_0, v_0)$. The relative integrable complex structure J^{rel} defines a smooth family $(J_v)_{v \in V}$ of integrable complex structures on Y. By the Newlander–Nirenberg theorem with parameters, there exist a neighbourhood $U_{y_0} \times U_{v_0}$ of (y_0, v_0) and smooth complex-valued functions $z^1, \dots, z^N$ on $U_{y_0} \times U_{v_0}$ such that for every v in U_{v_0}, $(z_v^1, \dots, z_v^N)$ are holomorphic coordinates on $\left(U_{y_0}, J_v\right)$ (see [45, p. 271]). If $U_x = \widetilde{\phi}^{-1}\left(U_{y_0} \times U_{v_0}\right)$, we define ϕ on U_x by $\phi = \left((z^1, \dots, z^N) \circ \widetilde{\phi}, \pi\right)$. □

Proposition 2.11 enables us to relate relative complex structures and relative analytic spaces.

Proposition 2.12 *Let $(\mathfrak{X}, \mathcal{A})$ be a relative differentiable space over B. There is a natural bijection between relative integrable complex structures on $\mathfrak{X}$ and maximal relative holomorphic atlases contained in $\mathcal{A}$.*

Proof If $\mathfrak{X}$ is endowed with a structure of relative analytic space over B, Proposition 2.11 allows us to construct a relative holomorphic atlas on $\mathfrak{X}$ and then the corresponding saturated atlas. Conversely, if $\{\phi_i\colon U_i \xrightarrow{\sim} Z_i \times V_i\}_{i \in I}$ is a relative holomorphic atlas on $\mathfrak{X}$, the complex structures of the Z_i's define a relative integrable complex structure on $\mathfrak{X}$. □

As a corollary, if $\mathfrak{X}$ is a relative analytic space over B, there exists a canonical relative integrable complex structure on the underlying relative differentiable space.

Relative integrable structures are introduced in [44] to deal with problems in almost-complex geometry. Let us give some general results about relative integrable complex structures associated with an almost-complex manifold.

Let (X, J) be an almost-complex compact manifold and B be a compact connected differentiable orbifold. Recall that B carries a stratification with finitely many strata. If Z is a closed subset of $X \times B$, we say that Z is an *incidence set* if the following conditions are satisfied:

– The map $\mathrm{pr}_{2\,|Z} \colon Z \longrightarrow B$ is surjective and finite.
– For each stratum B_λ of B, if $Z_\lambda = \mathrm{pr}_2^{-1}(B_\lambda) \cap Z$, then Z_λ is a submanifold of $X \times B_\lambda$ and the map $\mathrm{pr}_{2\,|Z_\lambda} \colon Z_\lambda \longrightarrow B_\lambda$ is a covering map.

Let us introduce now some notations. Let g be a Riemannian metric on X, ε be a positive integer and Z be an incidence set of $X \times B$.

– If W is neighbourhood of Z in $X \times B$ (considered as a relative differentiable space over B), $\mathcal{B}_{g,\varepsilon}(W)$ will denote the set of relative integrable complex structures J^{rel} on W such that $\|J^{\mathrm{rel}} - J\|_{C^0, g, W} < \varepsilon$.
– We put $\mathcal{B}_{g,\varepsilon} = \varinjlim_{W,\, Z \subseteq W} \mathcal{B}_{g,\varepsilon}(W)$.

Then we have:

Proposition 2.13 *Let B be a compact differentiable orbifold, (X, J) be an almost-complex compact manifold and Z be an incidence set in $X \times B$. If g is a Riemannian metric on X, there exists a positive real number ε_0 such that for any positive ε smaller than ε_0, $\mathcal{B}_{g,\varepsilon}$ is nonempty and weakly contractible in the following sense: for every nonnegative integer p and every pair of smooth families $\left(J^{\mathrm{rel}}_{0,s}\right)_{s \in \mathbb{S}^p}$ and $\left(J^{\mathrm{rel}}_{1,s}\right)_{s \in \mathbb{S}^p}$ in $\mathcal{B}_{g,\varepsilon}$ parameterized by the sphere $\mathbb{S}^p$, there exists a smooth family $\left(J^{\mathrm{rel}}_{t,s}\right)_{(t,s) \in [0,1] \times \mathbb{S}^p}$ parameterized by $[0,1] \times \mathbb{S}^p$ joining $\left(J^{\mathrm{rel}}_{0,s}\right)$ and $\left(J^{\mathrm{rel}}_{1,s}\right)$.*

This result is implicit in [45], although not stated in this degree of generality (see [45, Prop. 4]). We provide a proof in § 7.

2.3 Construction of the almost-complex Hilbert scheme

First of all, we define relative Hilbert schemes.

Definition 2.14 Let $(\mathfrak{X}, \mathcal{A})$ be a relative analytic space over B of relative complex dimension two and J^{rel} be the associated relative integrable complex structure. For

any positive integer n, we define a set $\mathfrak{X}^{[n]}_{\mathrm{rel}}$ by $\mathfrak{X}^{[n]}_{\mathrm{rel}} = \coprod_{b\in B} \mathfrak{X}^{[n]}_b$, where $\mathfrak{X}^{[n]}_b$ is the Hilbert scheme of n–points of $\mathfrak{X}_b$ endowed with the integrable complex structure J^{rel}_b. If $\mathcal{A} = \{\phi_i\colon U_i \xrightarrow{\sim} Y_i \times V_i\}_{i\in I}$, if $\mathcal{A}' = \{\phi^{[n]}_{i,\mathrm{rel}}\colon U^{[n]}_{i,\mathrm{rel}} \longrightarrow Y^{[n]}_i \times V_i\}_{i\in I}$ and if $\mathcal{A}^{[n]}$ is the completed atlas of $\mathcal{A}'$, then $\left(\mathfrak{X}^{[n]}_{\mathrm{rel}}, \mathcal{A}^{[n]}\right)$ is a relative analytic space called the *relative Hilbert scheme* of $\mathfrak{X}$.

Let us now recall the definition of Voisin's almost-complex Hilbert schemes. Let (X, J) be an almost-complex compact manifold of dimension four, g be a Riemannian metric on X, n be a positive integer and Z_n be the incidence set in $X \times X^{(n)}$ defined by

$$Z_n = \{(p; \underline{x}) \text{ in } X \times X^{(n)} \text{ such that } p \in \underline{x}\}. \tag{2–5}$$

We use the notations introduced at the end of § 2.2. Let ε be any positive real number smaller than the bound ε_0 of Proposition 2.13. Then $\mathcal{B}_{g,\varepsilon}$ is nonempty.

Definition 2.15 Let W be a small neighbourhood of Z_n in $X \times X^{(n)}$ and J^{rel} be a relative integrable complex structure in $\mathcal{B}_{g,\varepsilon}(W)$. The *topological Hilbert scheme* $X^{[n]}_{J^{\mathrm{rel}}}$ is the subset of the relative Hilbert scheme $W^{[n]}_{\mathrm{rel}}$ defined by

$$X^{[n]}_{J^{\mathrm{rel}}} = \{(\xi;\, \underline{x}) \text{ in } W^{[n]}_{\mathrm{rel}} \text{ such that } \underline{x} = \Gamma_{\underline{x}}(\xi)\},$$

where $\Gamma_{\underline{x}}\colon W^{[n]}_{\underline{x}} \longrightarrow W^{(n)}_{\underline{x}}$ is the Hilbert–Chow morphism associated with the integrable complex structure $J^{\mathrm{rel}}_{\underline{x}}$.

Remark 2.16 (i) The topological Hilbert scheme $X^{[n]}_{J^{\mathrm{rel}}}$ depends only on the germ of J^{rel} along Z_n, ie of the image of J^{rel} in $\mathcal{B}_{g,\varepsilon}$.

(ii) If W is a small neighbourhood of Z_n, let $\{J^{\mathrm{rel}}_t\}_{t\in B(0,r)\subseteq\mathbb{R}^d}$ be a smooth family in $\mathcal{B}_{g,\varepsilon}(W)$. This family defines a relative integrable complex structure $\widetilde{J}^{\mathrm{rel}}$ on the relative analytic space $\widetilde{W} = W \times_{X^{(n)}} (X^{(n)} \times B(0,r))$. Then there exists a natural relative topological Hilbert scheme over $B(0,r)$ whose fibers are the $\left(X^{[n]}_{J^{\mathrm{rel}}_t}\right)_{t\in B(0,r)}$, namely:

$$\left(X^{[n]}, \{J^{\mathrm{rel}}_t\}_{t\in B(0,r)}\right) = \{(\xi;\, \underline{x},\, t) \text{ in } \widetilde{W}^{[n]}_{\mathrm{rel}} \text{ such that } \underline{x} = \Gamma_{\underline{x},t}(\xi)\},$$

where $\Gamma_{\underline{x},t}\colon W^{[n]}_{\underline{x}} \longrightarrow W^{(n)}_{\underline{x}}$ is the Hilbert–Chow morphism associated with the complex structure $\widetilde{J}^{\mathrm{rel}}_{\underline{x},t}$.

To obtain a differentiable structure on $X^{[n]}_{J^{\mathrm{rel}}}$, Voisin uses relative integrable structures in a contractible subset $\mathcal{B}'$ of $\mathcal{B}_{g,\varepsilon}$ satisfying some additional geometric conditions (see [44, p. 711]). The main results she obtains are:

Theorem 2.17 [44, Th. 5, Th. 6, Th. 3], [45, Th. 3] *Let (X, J) be an almost-complex compact four-manifold, J^{rel} be a relative integrable structure in $\mathcal{B}'$ and n be a positive integer. Then*

(i) *$X^{[n]}_{J^{\mathrm{rel}}}$ has a natural differentiable structure. Furthermore, if J'^{rel} is another relative integrable structure in $\mathcal{B}'$, there is a diffeomorphism between $X^{[n]}_{J^{\mathrm{rel}}}$ and $X^{[n]}_{J'^{\mathrm{rel}}}$ which is uniquely defined up to isotopy.*

(ii) *There is a canonical Hilbert–Chow map $\Gamma\colon X^{[n]}_{J^{\mathrm{rel}}} \longrightarrow X^{(n)}$ satisfying the following property: for any $\underline{x}$ in $X^{(n)}$ and any integrable complex structure in a neighbourhood $U_{\underline{x}}$ of $\mathrm{supp}(\underline{x})$, $\Gamma^{-1}(\underline{x})$ is homeomorphic to the fiber at x of the usual Hilbert–Chow morphism from $U^{[n]}_{\underline{x}}$ to $U^{(n)}_{\underline{x}}$.*

(iii) *$X^{[n]}_{J^{\mathrm{rel}}}$ can be endowed with a stable almost-complex structure, and the associated complex cobordism class of $X^{[n]}_{J^{\mathrm{rel}}}$ depends only on the deformation class of J.*

(iv) *If X is symplectic and J is compatible with the symplectic structure, $X^{[n]}_{J^{\mathrm{rel}}}$ is also symplectic.*

For arbitrary relative integrable structures, this theorem has the following topological form:

Theorem 2.18 [23, Prop. 3.4, Prop. 3.10, Rem. 3.5]

(i) *Let J^{rel} be a relative integrable complex structure in $\mathcal{B}_{g,\varepsilon}$. Then $X^{[n]}_{J^{\mathrm{rel}}}$ is a topological manifold of real dimension $4n$.*

(ii) *If W is a neighbourhood of Z_n in $X \times X^{(n)}$ and if $\{J^{\mathrm{rel}}_t\}_{t\in B(0,r)\subseteq\mathbb{R}^d}$ is a smooth path in $\mathcal{B}_{g,\varepsilon}(W)$, then the associated relative topological Hilbert scheme $\big(X^{[n]}, \{J^{\mathrm{rel}}_t\}_{t\in B(0,r)}\big)$ over $B(0,r)$ is a topological fibration (cf Remark 2.16 (ii)).*

(iii) *For any $\underline{x}$ in $X^{(n)}$ and any integrable structure J in a neighbourhood $U_{\underline{x}}$ of $\mathrm{supp}(\underline{x})$, the Hilbert–Chow morphism $\Gamma\colon X^{[n]}_{J^{\mathrm{rel}}} \longrightarrow X^{(n)}$ is locally homeomorphic over a neighbourhood of $\mathrm{supp}(\underline{x})$ to the classical Hilbert–Chow morphism from $U^{[n]}_{\underline{x}}$ to $U^{(n)}_{\underline{x}}$.*

We can compare almost-complex Hilbert schemes corresponding to different relative integrable complex structures. Let g and g' be two Riemannian metrics on X, ε_0 and ε'_0 be the bounds given by Proposition 2.13, ε and ε' be positive real numbers smaller than ε_0 and ε'_0 respectively and J^{rel}, J'^{rel} be relative integrable complex structures in $\mathcal{B}_{g,\varepsilon}$ and $\mathcal{B}_{g',\varepsilon'}$. Then there exists a positive real number ε'' smaller than ε such that $\mathcal{B}_{g,\varepsilon''}$ is included in $\mathcal{B}_{g',\varepsilon'}$. Since, by Proposition 2.13, $\mathcal{B}_{g,\varepsilon}$, $\mathcal{B}_{g',\varepsilon'}$ and $\mathcal{B}_{g,\varepsilon''}$

are nonempty, connected and simply connected, $X^{[n]}_{J^{\mathrm{rel}}}$ and $X^{[n]}_{J'^{\mathrm{rel}}}$ are homeomorphic by Theorem 2.18 (ii); and this homeomorphism is canonical up to isotopy.

Therefore, for every positive integer n, there exists a canonical ring $H^*\big(X^{[n]},\mathbb{Q}\big)$ (resp. $K\big(X^{[n]}\big)$) such that for every relative integrable complex structure J^{rel} in $\mathcal{B}_{g,\varepsilon}$, the ring $H^*\big(X^{[n]}_{J^{\mathrm{rel}}},\mathbb{Q}\big)$ (resp. the ring $K\big(X^{[n]}_{J^{\mathrm{rel}}}\big)$) is canonically isomorphic to $H^*\big(X^{[n]},\mathbb{Q}\big)$ (resp. to $K\big(X^{[n]}\big)$).

Theorem 2.18 (iii) implies that Göttsche's classical formula for the Betti numbers of Hilbert schemes of points also holds in the almost-complex case (see [22, Th. 3.9]).

2.4 Incidence varieties and Nakajima operators

If m and n are two positive integers, let

$$Z_{n\times m} = \{(p;\, \underline{x},\, \underline{y}) \text{ in } X\times X^{(n)}\times X^{(m)} \text{ such that } p\in \underline{x}\cup\underline{y}\,\}\cdot \tag{2–6}$$

Relative integrable structures in a neighbourhood of $Z_{n\times m}$ are denoted by $J^{\mathrm{rel}}_{n\times m}$.

Definition 2.19

(i) If $J^{\mathrm{rel}}_{1,n\times m}$ and $J^{\mathrm{rel}}_{2,n\times m}$ are two relative integrable structures in neighbourhoods W_1 and W_2 of $Z_{n\times m}$, the *product Hilbert scheme* $\big(X^{[n]\times[m]}, J^{\mathrm{rel}}_{1,n\times m}, J^{\mathrm{rel}}_{2,n\times m}\big)$ is defined by

$$\big(X^{[n]\times[m]}, J^{\mathrm{rel}}_{1,n\times m}, J^{\mathrm{rel}}_{2,n\times m}\big) = \{(\xi,\, \xi';\, \underline{x},\, \underline{z}\,) \text{ in } W^{[n]}_{1,\mathrm{rel}} \times_{X^{(n)}\times X^{(m)}} W^{[m]}_{2,\mathrm{rel}}$$
$$\text{such that } \Gamma_{1,\underline{x},\underline{z}}(\xi) = \underline{x} \text{ and } \Gamma_{2,\underline{x},\underline{z}}(\xi') = \underline{z}\}\cdot$$

(ii) If $m>n$ and if $J^{\mathrm{rel}}_{n\times(m-n)}$ is a relative integrable structure in a neighbourhood W of $Z_{n\times(m-n)}$, the *incidence variety* $\big(X^{[m,n]}, J^{\mathrm{rel}}_{n\times(m-n)}\big)$ is defined by

$$\big(X^{[m,n]}, J^{\mathrm{rel}}_{n\times(m-n)}\big) = \{(\xi,\, \xi';\, \underline{x},\, \underline{y}\,) \text{ in } W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X^{(m-n)}} W^{[m]}_{\mathrm{rel}}$$
$$\text{such that } \xi\subset\xi',\ \Gamma_{\underline{x},\underline{y}}(\xi) = \underline{x} \text{ and } \Gamma_{\underline{x},\underline{y}}(\xi') = \underline{x}\cup\underline{y}\}\cdot$$

As it is the case for topological Hilbert schemes, the product Hilbert schemes and the incidence varieties are canonically defined up to homeomorphisms isotopic to the identity if the relative integrable structures used to define them are chosen close enough to J in C^0–norm.

From now on, we fix a Riemannian metric g on X and assume that all relative integrable structures are sufficiently close to J in C^0–norm.

Let $J^{\mathrm{rel}}_{n\times(m-n)}$, J^{rel}_{n}, J^{rel}_{m} and $J^{\mathrm{rel}}_{n\times m}$ be relative integrable structures in respective neighbourhoods W, W', W'' and $\widetilde{W}$ of $Z_{n\times(m-n)}$, Z_n, Z_m and $Z_{n\times m}$. We consider the following compatibility conditions of relative analytic spaces:

(A) For every $(\underline{x},\underline{y})$ in $X^{(n)}\times X^{(m-n)}$, $W'_{\underline{x}}\subseteq W_{\underline{x},\underline{y}}$ and $J^{\mathrm{rel}}_{n\times(m-n),\,\underline{x},\,\underline{y}\,|W'_{\underline{x}}}=J^{\mathrm{rel}}_{n,\,\underline{x}}$; ie $W'\times_{X^{(n)}}(X^{(n)}\times X^{(m-n)})\subseteq W$, where the base change map is the first projection. If this condition holds, there is a natural morphism λ from $\left(X^{[m,\,n]},J^{\mathrm{rel}}_{n\times(m-n)}\right)$ to $\left(X^{[n]},J^{\mathrm{rel}}_{n}\right)$.

(B) For every $(\underline{x},\underline{y})$ in $X^{(n)}\times X^{(m-n)}$, $W''_{\underline{x}\cup\underline{y}}=W_{\underline{x},\,\underline{y}}$ and $J^{\mathrm{rel}}_{n\times(m-n),\,\underline{x},\,\underline{y}}=J^{\mathrm{rel}}_{m,\,\underline{x}\cup\underline{y}}$; ie $W''\times_{X^{(m)}}(X^{(n)}\times X^{(m-n)})=W$, where the base change map is $(\underline{x},\underline{y})\longmapsto\underline{x}\cup\underline{y}$. If this condition holds, there is a canonical morphism ν from $\left(X^{[m,\,n]},J^{\mathrm{rel}}_{n\times(m-n)}\right)$ to $\left(X^{[m]},J^{\mathrm{rel}}_{m}\right)$.

(C) For every $(\underline{x},\underline{y})$ in $X^{(n)}\times X^{(m-n)}$, $\widetilde{W}_{\underline{x},\,\underline{x}\cup\underline{y}}=W_{\underline{x},\,\underline{y}}$ and $J^{\mathrm{rel}}_{n\times(m-n),\,\underline{x},\,\underline{y}}=J^{\mathrm{rel}}_{n\times m,\,\underline{x},\,\underline{x}\cup\underline{y}}$; ie $\widetilde{W}\times_{X^{(n)}\times X^{(m)}}(X^{(n)}\times X^{(m-n)})=W$, where the base change map is $(\underline{x},\underline{y})\longmapsto(\underline{x},\underline{x}\cup\underline{y})$. If this condition holds, there is a natural embedding of $\left(X^{[m,\,n]},J^{\mathrm{rel}}_{n\times(m-n)}\right)$ into the product Hilbert scheme $\left(X^{[n]\times[m]},J^{\mathrm{rel}}_{n\times m},J^{\mathrm{rel}}_{n\times m}\right)$.

Each of these conditions can be satisfied for a suitable choice of relative integrable complex structures (this is obvious for conditions (B) and (C); for condition (A), it is necessary to use the gluing method developed in § 7). Unfortunately, conditions (A) and (B) cannot hold at the same time, unless X carries an integrable complex structure. Indeed, if $n=1$ and $m=2$, assume that we are given three relative integrable complex structures $J^{\mathrm{rel}}_{1\times1}$, J^{rel}_{1} and J^{rel}_{2} such that $\left(J^{\mathrm{rel}}_{1},J^{\mathrm{rel}}_{1\times1}\right)$ satisfy (A) and $\left(J^{\mathrm{rel}}_{2},J^{\mathrm{rel}}_{1\times1}\right)$ satisfy (B). Then, for all x and y in X,

$$J^{\mathrm{rel}}_{1,\,x\,|W'_x\cap W'_y}=J^{\mathrm{rel}}_{1\times1,\,x,\,y\,|W'_x\cap W'_y}=J^{\mathrm{rel}}_{2,\,\{x,\,y\}\,|W'_x\cap W'_y}=J^{\mathrm{rel}}_{1\times1,\,y,\,x\,|W'_x\cap W'_y}=J^{\mathrm{rel}}_{1,\,y\,|W'_x\cap W'_y}$$

so that J^{rel}_{1} defines a *global* integrable complex structure on X.

If $J^{\mathrm{rel}}_{n\times(m-n)}$ is a relative integrable complex structure in a neighbourhood of $Z_{n\times(m-n)}$, let us fix four relative integrable complex structures J^{rel}_{n}, $J^{\mathrm{rel}}_{1,\,n\times(m-n)}$, J^{rel}_{m} and $J^{\mathrm{rel}}_{2,\,n\times(m-n)}$ in respective neighbourhoods of Z_n, $Z_{n\times(m-n)}$, Z_m and $Z_{n\times(m-n)}$ such that $\left(J^{\mathrm{rel}}_{n},J^{\mathrm{rel}}_{1,n\times(m-n)}\right)$ satisfies the compatibility condition (A) and $\left(J^{\mathrm{rel}}_{m},J^{\mathrm{rel}}_{2,\,n\times(m-n)}\right)$ satisfies the compatibility condition (B). For $i=1,2$, there is a homeomorphism between $\left(X^{[m,\,n]},J^{\mathrm{rel}}_{m\times(m-n)}\right)$ and $\left(X^{[m,\,n]},J^{\mathrm{rel}}_{i,\,m\times(m-n)}\right)$ which is canonical up to isotopy. In this way, we get two continuous maps from $\left(X^{[m,\,n]},J^{\mathrm{rel}}_{m\times(m-n)}\right)$ to $\left(X^{[n]}_{J^{\mathrm{rel}}_{n}}\right)$ and $\left(X^{[m]}_{J^{\mathrm{rel}}_{m}}\right)$. Their homotopy classes are canonical and are still denoted by λ and ν.

The incidence varieties $X^{[m,n]}$ are locally homeomorphic to the integrable model $U^{[m,n]}$ where U is an open set of $\mathbb{C}^2$; this allows us to put a stratification on each $X^{[m,n]}$. In this way, the $X^{[m,n]}$ are stratified topological spaces locally homeomorphic to analytic spaces endowed with their natural stratifications, and so each of them has a fundamental homology class.

The construction by Nakajima and Grojnowski of representations of the Heisenberg superalgebra $\mathcal{H}(H^*(X,\mathbb{Q}))$ of $H^*(X,\mathbb{Q})$ into $\mathbb{H} = \oplus_{n\in\mathbb{N}} H^*\big(X^{[n]},\mathbb{Q}\big)$ via correspondence actions of incidence varieties done in [35] and [24] also holds in the almost-complex setting:

Theorem 2.20 [23, Def. 4.3, Th. 4.5] *If (X,J) is an almost-complex compact four-manifold, Nakajima operators $\{\mathfrak{q}_i(\alpha),\ i\in\mathbb{Z},\ \alpha\in H^*(X,\mathbb{Q})\}$ can be constructed. They depend only on the deformation class of J and satisfy the Heisenberg commutation relations*:

$$\forall i,j\in\mathbb{Z},\ \forall\alpha,\beta\in H^*(X,\mathbb{Q}),\qquad \big[\mathfrak{q}_i(\alpha),\mathfrak{q}_j(\beta)\big] = i\,\delta_{i+j,0}\left(\int_X \alpha\beta\right)\mathrm{id}_{\mathbb{H}}\,.$$

Furthermore, these operators induce an irreducible representation of $\mathcal{H}\big(H^*(X,\mathbb{Q})\big)$ in $\mathbb{H}$ with highest weight vector 1.

Finally, we introduce relative incidence varieties, which are essential in § 4.1, 5.2, 6.2 and 6.3.

Definition 2.21 If m, n are two positive integers with $m>n$, if W is a neighbourhood of $Z_{n\times(m-n)}$ and if $J^{\mathrm{rel}}_{n\times(m-n)}$ is a relative integrable complex structure on W, $W^{[m,n]}_{\mathrm{rel}}$ is the subset of $W^{[n]}_{\mathrm{rel}}$ defined by

$$W^{[m,n]}_{\mathrm{rel}} = \{(\xi,\ \xi';\ \underline{x},\ \underline{y})\ \text{ in }\ W^{[n]}_{\mathrm{rel}}\times_{X^{(n)}\times X^{(m-n)}} W^{[m]}_{\mathrm{rel}}\ \text{ such that }\ \xi\subseteq\xi'\}\cdot$$

If $\{\phi_i\colon\ U_i\xrightarrow{\sim}\Omega_i\times V_i\ \}_{i\in I}$ is a maximal relative holomorphic atlas on W, then for all i in I we have $\big(\phi_i^{[n]}\times_{V_i}\phi_i^{[m]}\big)\ \big[\big(U^{[n]}_{i,\mathrm{rel}}\times_{V_i}U^{[m]}_{i,\mathrm{rel}}\big)\cap W^{[m,n]}_{\mathrm{rel}}\big] = \Omega_i^{[m,n]}\times V_i$. Thus, for $m=n+1$, the relative incidence varieties $W^{[m,n]}_{\mathrm{rel}}$ are relative analytic spaces.

3 Coherent sheaves on relative analytic spaces

In this part, the letters B and B' always denote compact differentiable effective orbifolds.

3.1 Operations on relatively coherent sheaves

We start by defining relative holomorphic functions.

Definition 3.1 Let $(\mathfrak{X}, \mathcal{A})$ be a relative analytic space over B. A continuous complex-valued function f defined on an open subset Ω of $\mathfrak{X}$ is *relatively holomorphic* if for any x in Ω and for any relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Z \times V$ of $\mathcal{A}$ in a neighbourhood of x, the function $f \circ \phi^{-1}$ is smooth and holomorphic in the variables of Z in a neighbourhood of $\phi(x)$.

The sheaf $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ of relatively holomorphic functions on $\mathfrak{X}$ is a sheaf of rings on $\mathfrak{X}$.

Definition 3.2 Let $(\mathfrak{X}, \mathcal{A})$ be a relative analytic space over B. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$–modules is *relatively coherent* if there exists a relative holomorphic atlas $\widetilde{\mathcal{A}}$ equivalent to $\mathcal{A}$ such that for any relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Z \times V$ in $\widetilde{\mathcal{A}}$, there exists a coherent analytic sheaf $\overline{\mathcal{F}}$ on Z such that $\mathcal{F}_{|U}$ and $\phi^{-1}\big(\mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\big)$ are isomorphic as sheaves of $\mathcal{O}_{Z\times V}^{\mathrm{rel}}$–modules.

An equivalent definition of relatively coherent sheaves can be stated using gluing conditions: if $\widetilde{\mathcal{A}} = \{\phi_i\colon U_i \xrightarrow{\sim} Z_i \times V_i\}_{i\in I}$ and if $\phi_{ij} := \phi_i \circ \phi_j^{-1}$ are the associated transition functions, a relatively coherent sheaf on $\mathfrak{X}$ is given by a family of coherent sheaves $\{\overline{\mathcal{F}}_i\}_{i\in I}$ on the smooth analytic sets $\{Z_i\}_{i\in I}$ and a family of isomorphisms of sheaves of $\mathcal{O}_{\phi_j(U_{ij})}^{\mathrm{rel}}$–modules between $\phi_{ij}^{-1}\Big[\big(\overline{\mathcal{F}}_i \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z_i}} \mathcal{O}_{Z_i\times V_i}^{\mathrm{rel}}\big)_{|\phi_i(U_{ij})}\Big]$ and $\big(\overline{\mathcal{F}}_j \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z_j}} \mathcal{O}_{Z_j\times V_j}^{\mathrm{rel}}\big)_{|\phi_j(U_{ij})}$ satisfying the usual cocycle condition.

Remark 3.3 (i) Let $\mathcal{F}$ be a relatively coherent sheaf on $\mathfrak{X}$ given by a family of sheaves $\{\overline{\mathcal{F}}_i\}_{i\in I}$. Then, for any b in B, if J is the set of the indices i in I such that b belongs to V_i, the sheaves $\{\overline{\mathcal{F}}_i\}_{i\in J}$ on $\{Z_i \times b\}_{i\in J}$ patch together into a coherent analytic sheaf on $\mathfrak{X}_b$, which we denote by $\mathcal{F}_b$.

(ii) If $(\mathfrak{X}, \mathcal{A})$ is a relative analytic space and if $\mathcal{E}$ is a locally-free sheaf of $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$–modules, then $\mathcal{E}$ is relatively coherent. In particular $T^{\mathrm{rel}}\mathfrak{X}$ is relatively coherent on $\mathfrak{X}$.

(iii) If $(\mathfrak{X}, \mathcal{A})$ is a *maximal* relative analytic space, it is not difficult to prove that any relatively coherent sheaf on $\mathfrak{X}$ is in fact locally $\mathcal{O}_{\mathfrak{X}}$–free. This fact justifies the use of non-maximal atlases.

Let us now introduce a class of morphisms between relative analytic spaces that is well-adapted to relatively coherent sheaves.

Definition 3.4 (i) Let $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{X}', \mathcal{A}')$ be two relative analytic spaces over B and let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a continuous map over B. Two relative holomorphic charts $\phi\colon U \xrightarrow{\sim} Z \times V$ and $\phi'\colon U' \xrightarrow{\sim} Z' \times V$ in $\mathcal{A}$ and $\mathcal{A}'$ are *f-compatible* if $f(U) \subseteq U'$ and if there exists a holomorphic map $g\colon Z \longrightarrow Z'$ such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "f_{|U}"] \arrow[d, "\phi"', "\sim"] & U' \arrow[d, "\sim"', "\phi'"] \\ Z \times V \arrow[r, "(g,\,\mathrm{id})"'] & Z' \times V \end{tikzcd}$$

(ii) Let $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{X}', \mathcal{A}')$ be two relative analytic spaces over B and let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a continuous map over B. We say that f is a *morphism* when there exists a relative atlas $\widetilde{\mathcal{A}}$ equivalent to $\mathcal{A}$ such that for any relative holomorphic chart ϕ in $\widetilde{\mathcal{A}}$, there exists a relative holomorphic chart ϕ' in $\mathcal{A}'$ such that and ϕ and ϕ' are f–compatible.

(iii) Let $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{X}', \mathcal{A}')$ be two relative analytic spaces over B and B' and $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a continuous map. We say that f is a *weak morphism* if there exist a smooth orbifold map $u\colon B \longrightarrow B'$ and a morphism $\widetilde{f}\colon \mathfrak{X} \longrightarrow \mathfrak{X}' \times_{B'} B$ such that f is obtained by composing $\widetilde{f}$ with the base change map from $\mathfrak{X}' \times_{B'} B$ to $\mathfrak{X}'$ induced by u.

If $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ is a weak morphism, then the sheaf $f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}$ is a subsheaf of $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}$. Therefore we get a pullback functor $f^*\colon \mathrm{Mod}\left(\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}\right) \longrightarrow \mathrm{Mod}\left(\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\right)$ given by the formula $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}} \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}$.

To prove that many usual operations on $\mathrm{Mod}\,(\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}})$ induce operations on relatively coherent sheaves, we use a flatness lemma:

Lemma 3.5 *Let W be an open subset of $\mathbb{R}^n$, G be a finite group of diffeomorphisms of W and Z be a smooth analytic set. If $V = W/G$, then $\mathcal{O}^{\mathrm{rel}}_{Z\times V}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$.*

Proof Let $\delta\colon W \longrightarrow V$ be the projection and $\mathcal{M}$ be a sheaf of $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$–modules. Then

$$\begin{aligned}(\delta, \mathrm{id})^{-1}\left(\mathcal{M} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\right) &= (\delta, \mathrm{id})^{-1}\mathcal{M} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \left(\mathcal{O}^{\mathrm{rel}}_{Z\times W}\right)^G \\ &\simeq \left[(\delta, \mathrm{id})^{-1}\mathcal{M} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times W}\right]^G.\end{aligned}$$

Since the functor $\mathcal{F} \longmapsto \mathcal{F}^G$ from $\mathrm{Mod}_G\big(\mathcal{O}^{\mathrm{rel}}_{Z\times W}\big)$ to $\mathrm{Mod}\big[\big(\mathcal{O}^{\mathrm{rel}}_{Z\times W}\big)^G\big]$ is exact, it suffices to prove that $\mathcal{O}^{\mathrm{rel}}_{Z\times W}$ is smooth over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$. Let $k = \lfloor (n+1)/2 \rfloor$. Then $W \times \mathbb{R}^k$ can be seen as an open subset $\widetilde{W}$ in $\mathbb{C}^{(n+k)/2}$. By [32, Th. 2 bis] $\mathcal{O}^{\mathrm{rel}}_{Z\times\widetilde{W}}$ is flat over $\mathcal{O}_{Z\times\widetilde{W}}$, and $\mathcal{O}_{Z\times\widetilde{W}}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$. Thus $\mathcal{O}^{\mathrm{rel}}_{Z\times\widetilde{W}}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$. If $q\colon Z\times\widetilde{W} \longrightarrow Z\times W$ is the projection, $q^{-1}\mathcal{O}^{\mathrm{rel}}_{Z\times W}$ is a direct factor of $\mathcal{O}^{\mathrm{rel}}_{Z\times\widetilde{W}}$ in $\mathrm{Mod}\big(\mathrm{pr}_1^{-1}\,\mathcal{O}_Z\big)$, so that $\mathcal{O}^{\mathrm{rel}}_{Z\times W}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$. □

We obtain as a consequence:

Proposition 3.6 *(i) Let $(\mathfrak{X},\mathcal{A})$ be a relative analytic space over B and $\mathcal{F}$, $\mathcal{G}$ be relatively coherent sheaves on $\mathfrak{X}$. Then for every nonnegative integer k, $\mathcal{T}or^k_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})$ and $\mathcal{E}xt^k_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})$ are relatively coherent on $\mathfrak{X}$.*

(ii) Let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a weak morphism between two relative analytic spaces $(\mathfrak{X},\mathcal{A})$ and $(\mathfrak{X}',\mathcal{A}')$, and $\mathcal{G}$ be a relatively coherent sheaf on $\mathfrak{X}'$. Then for every nonnegative integer k, $\mathcal{T}or^k_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}(f^{-1}\mathcal{G},\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}})$ is relatively coherent on $\mathfrak{X}$.

(iii) Let $(\mathfrak{X},\mathcal{A})$ be a relative analytic space over B, $\mathcal{F}$ be a relatively coherent sheaf on $\mathfrak{X}$ and $u\colon B' \longrightarrow B$ be a smooth map. If $\widetilde{u}\colon \mathfrak{X}\times_B B' \longrightarrow \mathfrak{X}$ is the associated base change morphism, then $\mathcal{T}or^k_{\widetilde{u}^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\big(\widetilde{u}^{-1}\mathcal{F},\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}\times_B B'}\big)$ vanishes for every positive integer k.

Proof (i) Since two equivalent relative holomorphic atlases always admit a common equivalent refinement, there exists a complete relative holomorphic atlas $\widetilde{\mathcal{A}}$ on $\mathfrak{X}$ equivalent to $\mathcal{A}$ such that for every relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Z\times V$ in $\widetilde{\mathcal{A}}$,

$$\mathcal{F}_{|U} \simeq \phi^{-1}\big(\mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big) \quad \text{and} \quad \mathcal{G}_{|U} \simeq \phi^{-1}\big(\mathrm{pr}_1^{-1}\,\overline{\mathcal{G}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big)$$

where $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are coherent analytic sheaves on Z. Then

$$\big(\mathcal{F} \otimes_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}} \mathcal{G}\big)_{|U} \simeq \phi^{-1}\big(\mathrm{pr}_1^{-1}(\overline{\mathcal{F}} \otimes_{\mathcal{O}_Z} \overline{\mathcal{G}}) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big) \quad \text{and}$$
$$\mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})_{|U} \simeq \phi^{-1}\big(\mathrm{pr}_1^{-1}\,\mathcal{H}om_{\mathcal{O}_Z}(\overline{\mathcal{F}},\overline{\mathcal{G}}) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big).$$

Since $\mathcal{O}^{\mathrm{rel}}_{Z\times V}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$, for any nonnegative integer k, we obtain by derivation

$$\mathcal{T}or^k_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})_{|U} \simeq \phi^{-1}\big(\mathcal{T}or^k_{\mathcal{O}_Z}(\overline{\mathcal{F}},\overline{\mathcal{G}}) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big) \qquad \text{and}$$
$$\mathcal{E}xt^k_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})_{|U} \simeq \phi^{-1}\big(\mathcal{E}xt^k_{\mathcal{O}_Z}(\overline{\mathcal{F}},\overline{\mathcal{G}}) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big).$$

(ii) We can take refinements $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}'$ of $\mathcal{A}$ and $\mathcal{A}'$ such that for any relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Z \times V$ in $\widetilde{\mathcal{A}}$, there exists a relative holomorphic chart $\phi'\colon U' \xrightarrow{\sim} Z' \times V'$ in $\widetilde{\mathcal{A}}'$, a holomorphic map $g\colon Z \longrightarrow Z'$, a smooth map $u\colon V \longrightarrow V'$ and a coherent analytic sheaf $\overline{\mathcal{G}}$ on Z' such that $f(U) \subseteq U'$, $\phi' \circ f \circ \phi^{-1} = (g,u)$ and $\mathcal{G}_{|U'} \simeq \phi'^{-1}\big(\mathrm{pr}_1^{-1}\,\overline{\mathcal{G}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z'}} \mathcal{O}^{\mathrm{rel}}_{Z'\times V'}\big)$. Thus

$$\begin{aligned}\Big(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}} \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\Big)_{|U} &\simeq \phi^{-1}\Big((g,u)^{-1}\big[\mathrm{pr}_1^{-1}\,\overline{\mathcal{G}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z'}} \mathcal{O}^{\mathrm{rel}}_{Z'\times V'}\big] \otimes_{(g,u)^{-1}\mathcal{O}^{\mathrm{rel}}_{Z'\times V'}} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\Big)\\ &\simeq \phi^{-1}\Big(\mathrm{pr}_1^{-1}\big(g^{-1}\overline{\mathcal{G}} \otimes_{g^{-1}\mathcal{O}_{Z'}} \mathcal{O}_Z\big) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\Big),\end{aligned}$$

so that for any nonnegative integer k, Lemma 3.5 yields:

$$\mathcal{T}or^k_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}\Big(f^{-1}\mathcal{G}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\Big) \simeq \phi^{-1}\Big(\mathrm{pr}_1^{-1}\,\mathcal{T}or^k_{g^{-1}\mathcal{O}_{Z'}}\big(g^{-1}\overline{\mathcal{G}}, \mathcal{O}_Z\big) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}\Big).$$

(iii) We can assume that $\mathfrak{X} = Z \times B$, $\mathcal{F} = \mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times B}$ and $\widetilde{u} = (\mathrm{id}, u)$, where Z is a smooth analytic space and $\overline{\mathcal{F}}$ is a coherent analytic sheaf on Z. Let (z,b') be an element of $Z \times B'$, put $b = u(b')$, and let $E_\bullet$ be a free resolution of $\overline{\mathcal{F}}_z$. By Lemma 3.5, $E_\bullet \otimes_{\mathcal{O}_z} \mathcal{O}^{\mathrm{rel}}_{z,b}$ is a free resolution of $\mathcal{F}_{z,b}$. Therefore, the germ of $\mathcal{T}or^k_{\widetilde{u}^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\big(\widetilde{u}^{-1}\mathcal{F}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}\times_B B'}\big)$ at (z,b') is the k–th cohomology group of $\big(E_\bullet \otimes_{\mathcal{O}_z} \mathcal{O}^{\mathrm{rel}}_{z,b}\big) \otimes_{\mathcal{O}^{\mathrm{rel}}_{z,b}} \mathcal{O}^{\mathrm{rel}}_{z,b'}$. The latter complex being isomorphic to $E_\bullet \otimes_{\mathcal{O}_z} \mathcal{O}^{\mathrm{rel}}_{z,b'}$, the result is again a consequence of Lemma 3.5. □

Remark 3.7 Let $\mathcal{F}$ and $\mathcal{G}$ be two relatively coherent sheaves on a relative analytic space $\mathfrak{X}$ over B. Then for every b in B and for every nonnegative integer k, we have $\mathcal{T}or^k(\mathcal{F},\mathcal{G})_b = \mathcal{T}or^k_{\mathcal{O}_{\mathfrak{X}_b}}(\mathcal{F}_b,\mathcal{G}_b)$ and $\mathcal{E}xt^k(\mathcal{F},\mathcal{G})_b = \mathcal{E}xt^k_{\mathcal{O}_{\mathfrak{X}_b}}(\mathcal{F}_b,\mathcal{G}_b)$. A similar result holds for the $\mathcal{T}or$ sheaves appearing in (ii).

3.2 Relative analytic subspaces and direct image

The definition of a relative analytic subspace runs as follows:

Definition 3.8 Let $(\mathfrak{X},\mathcal{A})$ be a relative analytic space over B and $\mathfrak{Z}$ be a closed subset of $\mathfrak{X}$. We say that $\mathfrak{Z}$ is a *relative analytic subspace* (resp. *smooth relative analytic subspace*) of $\mathfrak{X}$ if there exists a relative holomorphic atlas $\widetilde{\mathcal{A}}$ equivalent to $\mathcal{A}$ such that for any holomorphic chart $\phi\colon U \xrightarrow{\sim} Z \times V$ in $\widetilde{\mathcal{A}}$ with $U \cap \mathfrak{Z} \neq \emptyset$, there exists a closed (resp. closed and smooth) analytic subspace Z' of Z satisfying $\phi(U \cap \mathfrak{Z}) = Z' \times V$.

For instance, the relative incidence varieties $W^{[m,n]}_{\mathrm{rel}}$ introduced in Definition 2.21 are relative analytic subspaces of $W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X^{(m-n)}} W^{[m]}_{\mathrm{rel}}$, they are smooth if $m = n+1$. We will study other examples in § 3.5.

Remark 3.9 If $\mathfrak{Z}$ is a smooth relative analytic subspace of a relative analytic space $(\mathfrak{X}, \mathcal{A})$, then $\mathfrak{Z}$ is also a relative analytic space: a complete relative holomorphic atlas on $\mathfrak{Z}$ is obtained by taking the restrictions to $\mathfrak{Z}$ of the charts of $\mathcal{A}$.

The forthcoming proposition is needed to associate relatively coherent sheaves with relative analytic subspaces:

Proposition 3.10 *Let n and k be positive integers, Ω and W be open subsets of $\mathbb{C}^n$ and $\mathbb{R}^k$ respectively, and T be a reduced analytic subset of Ω. Assume that the ideal sheaf of T is globally generated by holomorphic functions $f_1, \ldots, f_d$ on Ω.*

– If $g\colon \Omega \times W \longrightarrow \mathbb{C}$ is a relatively holomorphic function on $\Omega \times W$ such that $g_{|T\times W} \equiv 0$, then for any (z_0, w_0) in $\Omega \times W$, there exist relatively holomorphic functions $\phi_1, \ldots, \phi_d$ in a neighbourhood U_{z_0,w_0} of (z_0, w_0) such that $g = \sum_{i=1}^d \phi_i f_i$ on U_{z_0,w_0}.

– Furthermore, if G is a finite group of diffeomorphisms of W fixing w_0 and if g is G–invariant, then the functions ϕ_i can be chosen G-invariant too.

Proof Let (z_0, w_0) in $\Omega \times W$. For any z in Ω, let $\widehat{g}_{w_0}(z)$ be the formal Taylor expansion of the function $w \longmapsto g(z,w)$ at w_0. If we write $\widehat{g}_{w_0}(z) = \sum_{|I|=k} \alpha_I(z)\,(w-w_0)^I$, the hypotheses made on g imply that the functions α_I are holomorphic on Ω and vanish on T. Therefore for every multiindex I of length k, there exist holomorphic functions $\alpha_{I1}, \ldots, \alpha_{Id}$ in a Stein neighbourhood U_{z_0} of z_0 such that $\alpha_I = \sum_{i=1}^d \alpha_{Ii} f_i$ on U_{z_0}. Hence we get $\widehat{g}_{w_0} = \sum_{i=1}^d \Big(\sum_{|I|=k} \alpha_{Ii}\,(w-w_0)^I\Big) f_i$ in $\mathcal{O}(U_{z_0})\big[\big[w-w_0\big]\big]$. If S_i (resp. $\widehat{f}_{i,z_0}$) denotes the formal expansion of $z \longmapsto \sum_{|I|=k} \alpha_{Ii}(z)\,(w-w_0)^I$ (resp. f_i) at z_0 in $\mathbb{C}\big[\big[z-z_0, w-w_0\big]\big]$ (resp. in $\mathbb{C}\big[\big[z-z_0\big]\big]$) and if $\widehat{g}_{z_0,w_0}$ denotes the formal Taylor expansion of g at (z_0, w_0) in $\mathbb{C}\big[\big[z-z_0, w-w_0\big]\big]$, then $\widehat{g}_{z_0,w_0} = \sum_{i=1}^d S_i \widehat{f}_{i,z_0}$. Thus for any (z_0, w_0) in $\Omega \times W$, $\widehat{g}_{z_0,w_0}$ is divisible by $\widehat{f}_{1,z_0}, \ldots, \widehat{f}_{d,z_0}$ in $\mathbb{C}\big[\big[z-z_0, w-w_0\big]\big]$. Since the f_i's are analytic, it follows from [33, Th. 1.1' p. 82] that there exist $\phi_1, \ldots, \phi_d$ in $\mathcal{C}^\infty(\Omega)$ such that $g = \sum_{i=1}^d \phi_i f_i$. It remains to prove that the functions f_i can be chosen relatively holomorphic in a neighbourhood of any point in $\Omega \times W$.

If (z_0, w_0) is an element of $\Omega \times W$ and if r is an integer such that $0 \le r \le n$, let us consider the property (P_r):

(P_r) There exist a neighbourhood U_{z_0,w_0} of (z_0,w_0) in $\Omega\times W$ and smooth functions $\phi_1,\dots,\phi_d$ on U_{z_0,w_0} such that
– $g=\sum_{i=1}^d \phi_i f_i$ on U_{z_0,w_0}
– If $r\geq 1$, $1\leq i\leq d$ and $1\leq j\leq r$, $\dfrac{\partial\phi_i}{\partial\bar z_j}\equiv 0$.

We have seen that (P_0) is true. Fix r such that $0\leq r\leq n-1$, and assume that (P_r) holds. We consider a presentation $\mathcal{O}_{z_0}^q\longrightarrow\mathcal{O}_{z_0}^d\longrightarrow\big(\mathcal{J}_T\big)_{z_0}\longrightarrow 0$ of the ideal sheaf of T at z_0, where the first map is given by a matrix M in $\mathfrak{M}_{d,q}\big(\mathcal{O}_{z_0}\big)$ and the second one by $(f_1,\dots,f_d)$. If $\mathcal{O}_r^{\mathrm{rel}}$ denotes the sheaf of smooth functions on $\Omega\times W$ holomorphic in the first r variables, then $\mathcal{O}_r^{\mathrm{rel}}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_\Omega$ (this is proved exactly as in Lemma 3.5). Thus we get an exact sequence

$$\big(\mathcal{O}_r^{\mathrm{rel}}\big)^q_{z_0,w_0}\xrightarrow{\;M\;}\big(\mathcal{O}_r^{\mathrm{rel}}\big)^d_{z_0,w_0}\xrightarrow{(f_1,\dots,f_d)}\big(\mathcal{O}_r^{\mathrm{rel}}\big)_{z_0,w_0}\tag{3–1}$$

Pick $\phi_1.\dots,\phi_d$ in $\big(\mathcal{O}_r^{\mathrm{rel}}\big)_{z_0,w_0}$ such that $g=\sum_{i=1}^d\phi_i f_i$. Since g is relatively holomorphic, $\sum_{i=1}^d\frac{\partial\phi_i}{\partial\bar z_{r+1}}f_i\equiv 0$. By the exactness of (3–1), there exists an element $\big(\varphi_j\big)_{1\leq j\leq q}$ in $\big(\mathcal{O}_r^{\mathrm{rel}}\big)^q_{z_0,w_0}$ such that for every i with $1\leq i\leq d$, $\frac{\partial\phi_i}{\partial\bar z_{r+1}}=\sum_{j=1}^q M_{ij}\varphi_j$. Now the map $\frac{\partial}{\partial\bar z_{r+1}}:\big(\mathcal{O}_r^{\mathrm{rel}}\big)_{z_0,w_0}\longrightarrow\big(\mathcal{O}_r^{\mathrm{rel}}\big)_{z_0,w_0}$ is surjective, so that there exist functions $\big(\gamma_j\big)_{1\leq j\leq q}$ in $\big(\mathcal{O}_r^{\mathrm{rel}}\big)_{z_0,w_0}$ such that for $1\leq j\leq q$, $\frac{\partial\gamma_j}{\partial\bar z_{r+1}}=\varphi_j$. If $\widetilde\phi_i=\phi_i-\sum_{j=1}^d M_{ij}\gamma_j$ then $\sum_{i=1}^d\widetilde\phi_i f_i=\sum_{i=1}^d\phi_i f_i=g$ in a small neighbourhood of the point (z_0,w_0) and $\frac{\partial\widetilde\phi_i}{\partial\bar z_{r+1}}=\frac{\partial\phi_i}{\partial\bar z_{r+1}}-\sum_{i=1}^d M_{ij}\frac{\partial\gamma_j}{\partial\bar z_{r+1}}=0$, so that $\widetilde\phi_i$ is in $\big(\mathcal{O}_{r+1}^{\mathrm{rel}}\big)_{z_0,w_0}$. This proves that (P_{r+1}) holds. By a finite induction, we obtain that (P_n) holds. This means that $\phi_1,\dots,\phi_d$ are relatively holomorphic in a neighbourhood of (z_0,w_0).

To prove the last statement, it suffices to replace for each i the function ϕ_i by the G–invariant function $\widetilde\phi_i$ defined by $\widetilde\phi_i(z,w)=|G|^{-1}\sum_{u\in G}\phi_i(z,u\,.\,w)$. □

Proposition 3.11 *Let $\mathfrak{X}$ be a relative analytic space over B and $\mathfrak{Z}$ be a relative analytic subspace of $\mathfrak{X}$. If $\mathcal{J}_{\mathfrak{Z}}^{\mathrm{rel}}$ is the ideal sheaf of $\mathfrak{Z}$ in $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ consisting of the relative holomorphic functions vanishing on $\mathfrak{Z}$ and if $\mathcal{O}_{\mathfrak{Z}}^{\mathrm{rel}}=\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}/\mathcal{J}_{\mathfrak{Z}}^{\mathrm{rel}}$ is the structure sheaf of $\mathfrak{Z}$, then $\mathcal{J}_{\mathfrak{Z}}^{\mathrm{rel}}$ and $\mathcal{O}_{\mathfrak{Z}}^{\mathrm{rel}}$ are relatively coherent on $\mathfrak{X}$.*

Proof Since the result is local, we can assume that $\mathfrak{X}=Z\times V$ and that $\mathfrak{Z}=Z'\times V$, where V is a differentiable orbifold, Z is a smooth analytic space and Z' is an analytic subset of Z. If $\mathcal{J}_{Z'}$ is the ideal sheaf of Z' in Z, then $\mathcal{J}_{\mathfrak{Z}}^{\mathrm{rel}}=\mathrm{pr}_1^{-1}\,\mathcal{J}_{Z'}\,.\,\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ by Proposition 3.10. By Lemma 3.5, $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$, so that

the equality $\mathrm{pr}_1^{-1}\,\mathcal{J}_{Z'}\,.\,\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}} = \mathrm{pr}_1^{-1}\,\mathcal{J}_{Z'} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ holds. This implies that $\mathcal{J}_{\mathfrak{Z}}^{\mathrm{rel}}$ is relatively coherent. Using again the fact that $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$, we obtain that $\mathcal{O}_{\mathfrak{Z}}^{\mathrm{rel}} = \mathrm{pr}_1^{-1}\,\mathcal{O}_{Z'} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$, and thus the sheaf $\mathcal{O}_{\mathfrak{Z}}^{\mathrm{rel}}$ is relatively coherent too. □

Let us make an important remark:

Remark 3.12 Let Z be any reduced analytic set and V be a differentiable orbifold. Then it is possible to define a sheaf $\mathcal{O}_{Z\times V}^{\mathrm{rel}}$ of relative holomorphic functions on $Z \times V$ (which is a subsheaf of $C^0_{Z\times V}$) as follows: since we can argue locally, we assume that there exists an open set U in some $\mathbb{C}^n$ such that Z is a reduced analytic subset of U. Then we define $\mathcal{O}_{Z\times V}^{\mathrm{rel}}$ as the structure sheaf of $Z \times V$ in $U \times V$. Using Proposition 3.11, it is easy to prove that the definition in independent of U. This makes it possible to construct singular relative analytic spaces, although we will not go any further in this direction.

Following the strategy of [21, Ch. 1 § 3], we prove:

Proposition 3.13 *Let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a morphism between two relative analytic spaces $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{X}', \mathcal{A}')$, $\mathfrak{Z}$ be a relative analytic subspace of $\mathfrak{X}$ such that f is finite on $\mathfrak{Z}$ and $\mathcal{F}$ be a relatively coherent sheaf on $\mathfrak{X}$ supported in $\mathfrak{Z}$. Then $f_*\mathcal{F}$ is relatively coherent on $\mathfrak{X}'$.*

Proof For any point x' in $\mathfrak{X}'$, let $\phi'\colon U' \xrightarrow{\sim} Z' \times V$ be a relative holomorphic chart of $\mathcal{A}'$ in a neighbourhood U' of x'. Since f is finite on $\mathfrak{Z}$, there exists a relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Z \times V$ in a neighbourhood of $f^{-1}(x') \cap \mathfrak{Z}$ such ϕ and ϕ' are f–adapted. Up to a refinement of $\mathcal{A}$, we can assume that:

– $\phi(U) \subseteq U'$ and $\phi^{-1}(U') \cap \mathfrak{Z} \subseteq U$.
– $\phi' \circ f \circ \phi^{-1} = (g, \mathrm{id})$, where $g\colon Z \longrightarrow Z'$ is holomorphic.
– $\phi(U \cap \mathfrak{Z}) = Y \times V$, where Y is a reduced analytic subset of Z.
– The function $g_{|Y}\colon Y \longrightarrow Z'$ is finite.
– The sheaf $\mathcal{F}_{|U}$ is isomorphic to $\phi^{-1}\big(\mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\big)$, where $\overline{\mathcal{F}}$ is a coherent analytic sheaf on Z supported in Y.

Let us prove that the natural morphism

$$\mathrm{pr}_1^{-1}\big(g_*\overline{\mathcal{F}}\big) \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_{Z'}} \mathcal{O}_{Z'\times V}^{\mathrm{rel}} \longrightarrow (g, \mathrm{id})_* \Big(\mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\,\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\Big) \tag{3–2}$$

is an isomorphism. The function g being finite on Y and $\mathcal{O}_{Z\times V}^{\mathrm{rel}}$ (resp. $\mathcal{O}_{Z'\times V}^{\mathrm{rel}}$) being flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$ (resp. $\mathrm{pr}_1^{-1}\,\mathcal{O}_{Z'}$), both members of (3–2) define exact functors from

$\mathrm{Coh}_Y(Z)$ to $\mathrm{Mod}\left(\mathcal{O}^{\mathrm{rel}}_{Z'\times V}\right)$. Let $\mathcal{J}_Y$ be the ideal sheaf of Y. Since g is proper, we can assume, after shrinking Z' if necessary, that $\mathcal{J}_Y^N\overline{\mathcal{F}}=0$ for N large enough. Using the exact sequences

$$0\longrightarrow \mathcal{J}_Y^{N+1}\overline{\mathcal{F}}\longrightarrow \mathcal{J}_Y^{N}\overline{\mathcal{F}}\longrightarrow \mathcal{J}_Y^{N}\overline{\mathcal{F}}/\mathcal{J}_Y^{N+1}\overline{\mathcal{F}}\longrightarrow 0,$$

we see that it is sufficient to prove that (3–2) is an isomorphism when $\overline{\mathcal{F}}$ is an $\mathcal{O}_Y$–module. Under this assumption, for any z' in $g(Y)$, we can take local $\mathcal{O}_Y$–presentations of $\overline{\mathcal{F}}$ in a neighbourhood of the finite set $g^{-1}(z')\cap Y$. Thus it is enough to prove that (3–2) is an isomorphism when $\overline{\mathcal{F}}=\mathcal{O}_Y$. If $h=g_{|Y}$, this amounts to show that the natural morphism from $h_*\mathcal{O}_Y\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z'}}\mathcal{O}^{\mathrm{rel}}_{Z'\times V}$ to $(h,\,\mathrm{id})_*\,\mathcal{O}^{\mathrm{rel}}_{Y\times V}$ is an isomorphism. Since the problem is local, we can assume that there exist two positive integers m and n and an open subset Ω of $\mathbb{C}^m$ such that Z' is open in $\mathbb{C}^n$, Y is a closed analytic subset of $\Omega\times Z'$ and h is the restriction to Y of the projection from $\mathbb{C}^m\times\mathbb{C}^n$ to $\mathbb{C}^n$.

We deal at first with the case $m=1$. If (w,z') denotes the coordinates on $\mathbb{C}\times\mathbb{C}^n$, then for any z' in $h(Y)$, $\left(h_*\mathcal{O}_Y\right)_{z'}$ is free over $\mathcal{O}_{z'}$: a basis is given by the functions $1,w,\dots,w^d$, where d is the degree of h (see [21, I § 2]). Since the general Weierstrass division theorem remains valid for relative holomorphic functions, for any v in V, the module $\left((h,\mathrm{id})_*\,\mathcal{O}^{\mathrm{rel}}_{Y\times V}\right)_{z',v}$ is also free over $\mathcal{O}^{\mathrm{rel}}_{z',v}$ with basis $1,w,\dots,w^d$. This yields the required isomorphism.

To conclude, we argue by induction on m. Let us write $\Omega=\Omega''\times\Omega'$, where Ω'' and Ω' are open in $\mathbb{C}$ and $\mathbb{C}^{m-1}$ respectively, and let $p\colon\ \Omega\times Z'\longrightarrow\Omega'\times Z'$ and $q\colon\ \Omega'\times Z'\longrightarrow Z'$ be the natural projections. Then there exists an analytic hypersurface S in $\Omega\times Z'$ containing Y such that p is finite on S. If $\widetilde{Y}=p(Y)$, then $\widetilde{Y}$ is an analytic subset of $\Omega'\times Z'$ and q is finite on $\widetilde{Y}$. By Proposition 3.11, $\mathcal{O}^{\mathrm{rel}}_{Y\times V}$ is, as a sheaf of $\mathcal{O}^{\mathrm{rel}}_{(\Omega\times Z')\times V}$–modules, isomorphic to $\mathrm{pr}_1^{-1}\,\mathcal{O}_Y\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{\Omega\times Z'}}\mathcal{O}^{\mathrm{rel}}_{(\Omega\times Z')\times V}$. Therefore the result in the case $m=1$ yields the isomorphism

$$p_*\mathcal{O}_Y\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{\Omega'\times Z'}}\mathcal{O}^{\mathrm{rel}}_{(\Omega'\times Z')\times V}\simeq(p,\mathrm{id})_*\,\mathcal{O}^{\mathrm{rel}}_{Y\times V}.$$

Besides, we get another isomorphism by induction, namely

$$q_*\left(p_*\mathcal{O}_Y\right)\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z'}}\mathcal{O}^{\mathrm{rel}}_{Z'\times V}\simeq(q,\mathrm{id})_*\left(p_*\mathcal{O}_Y\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{\Omega'\times Z'}}\mathcal{O}^{\mathrm{rel}}_{(\Omega'\times Z')\times V}\right).$$

Putting these two isomorphisms together, we get the result. □

3.3 Relative analytic K-theory

We are now going to introduce morphisms of relatively coherent sheaves. A natural idea would be to consider relatively coherent sheaves on a relative analytic space $(\mathfrak{X},\mathcal{A})$ as

a full subcategory of the abelian category $\mathrm{Mod}\left(\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}\right)$ of sheaves of $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$–modules on $\mathfrak{X}$. Unfortunately, the resulting category would be non-abelian. Indeed, if $\mathfrak{X} = Z \times B$ is trivial, if $\mathcal{F}$ is relatively coherent on $\mathfrak{X}$ and if χ is a smooth cut-off function in B, the multiplication by χ defines an endomorphism of $\mathcal{F}$ over $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ whose kernel is far from being relatively coherent in general. Now, if $\mathcal{A}$ is given by the family of relative holomorphic charts $\{\phi_i\colon U_i \xrightarrow{\sim} Z_i \times V_i\}_{i\in I}$, another natural definition is to glue together the abelian categories $\left(\mathrm{Coh}(Z_i)\right)_{i\in I}$ of coherent analytic sheaves on Z_i. This is what we do.

Definition 3.14 (i) A *strict morphism* between to relatively coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a relative analytic space $(\mathfrak{X}, \mathcal{A})$ is a morphism u in $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}}(\mathcal{F}, \mathcal{G})$ satisfying the following condition: there exists a relative holomorphic atlas $\widetilde{\mathcal{A}}$ equivalent to $\mathcal{A}$ such that for every relative holomorphic chart $\phi\colon U \xrightarrow{\sim} Z \times V$ in $\widetilde{\mathcal{A}}$, there exist two coherent analytic sheaves $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ on Z as well as a morphism v in $\mathrm{Hom}_{\mathcal{O}_Z}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ such that $\mathcal{F}_{|U} \simeq \phi^{-1}\left(\mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\right)$, $\mathcal{G}_{|U} \simeq \phi^{-1}\left(\mathrm{pr}_1^{-1}\,\overline{\mathcal{G}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\right)$ and the following diagram commutes up to isomorphism:

$$\begin{tikzcd}
\mathcal{F}_{|U} \arrow[r, "u"] \arrow[d, "\sim"] & \mathcal{G}_{|U} \arrow[d, "\sim"] \\
\phi^{-1}\left(\mathrm{pr}_1^{-1}\,\overline{\mathcal{F}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\right) \arrow[r, "\phi^{-1}(v\otimes \mathrm{id})"] & \phi^{-1}\left(\mathrm{pr}_1^{-1}\,\overline{\mathcal{G}} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}\right)
\end{tikzcd}$$

(ii) If $\mathfrak{X}$ is a relative analytic space and $\mathfrak{Z}$ is a closed subspace of $\mathfrak{X}$, we call $\mathrm{Coh}^{\mathrm{rel}}(\mathfrak{X})$ (resp. $\mathrm{Coh}_{\mathfrak{Z}}^{\mathrm{rel}}(\mathfrak{X})$) the subcategory of $\mathrm{Mod}\left(\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}\right)$ whose objects are relatively coherent sheaves on $\mathfrak{X}$ (resp. relatively coherent sheaves on $\mathfrak{X}$ supported in $\mathfrak{Z}$) and whose morphisms are strict morphisms.

If $\mathfrak{X}$ is a relative analytic space and $\mathfrak{Z}$ is a closed subset of $\mathfrak{X}$, the categories $\mathrm{Coh}^{\mathrm{rel}}(\mathfrak{X})$ and $\mathrm{Coh}_{\mathfrak{Z}}^{\mathrm{rel}}(\mathfrak{X})$ are abelian subcategories of $\mathrm{Mod}\left(\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}\right)$ by Lemma 3.5.

In the sequel, the Grothendieck group of an abelian category $\mathcal{C}$ is denoted by $K(\mathcal{C})$.

Definition 3.15 Let $\mathfrak{X}$ be a relative analytic space and $\mathfrak{Z}$ be a closed subset of $\mathfrak{X}$. The *relative analytic K–theory of* $\mathfrak{X}$ (resp. *relative analytic K–theory of* $\mathfrak{X}$ *with support in* $\mathfrak{Z}$) is defined by $K^{\mathrm{rel}}(\mathfrak{X}) = K(\mathrm{Coh}^{\mathrm{rel}}(\mathfrak{X}))$ (resp. $K_{\mathfrak{Z}}^{\mathrm{rel}}(\mathfrak{X}) = K(\mathrm{Coh}_{\mathfrak{Z}}^{\mathrm{rel}}(\mathfrak{X}))$).

As for coherent sheaves, we can define usual operations on relative analytic K–theory. These definitions rely on Proposition 3.6 and Proposition 3.13.

– PRODUCT. If $\mathfrak{X}$ is a relative analytic space and $\mathfrak{Z}$ is a closed subset of $\mathfrak{X}$, a product from $K^{\mathrm{rel}}(\mathfrak{X}) \otimes_{\mathbb{Z}} K^{\mathrm{rel}}(\mathfrak{X})$ (resp. $K^{\mathrm{rel}}(\mathfrak{X}) \otimes_{\mathbb{Z}} K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$) to $K^{\mathrm{rel}}(\mathfrak{X})$ (resp. $K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$) is defined by

$$\mathcal{F} \,.\, \mathcal{G} = \sum_{k \geq 0} (-1)^k \mathcal{T}or^k_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}).$$

– DUAL MORPHISM. Let $\mathfrak{X}$ be a relative analytic space and $\mathfrak{Z}$ be a closed subset of $\mathfrak{X}$. The *dual morphism* $\mathcal{F} \longmapsto \mathcal{F}^{\vee}$ from $K^{\mathrm{rel}}(\mathfrak{X})$ to $K^{\mathrm{rel}}(\mathfrak{X})$ (resp. from $K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$ to $K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$) is given by

$$\mathcal{F}^{\vee} = \sum_{k \geq 0} (-1)^k \mathcal{E}xt^k_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\big(\mathcal{F}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\big).$$

– PULLBACK MORPHISM. Let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a weak morphism between relative analytic spaces and $\mathfrak{Z}'$ be a closed subset of $\mathfrak{X}'$. The *pullback morphism* (resp. *the pullback morphism with support*) $f^{!}\colon K^{\mathrm{rel}}(\mathfrak{X}') \to K^{\mathrm{rel}}(\mathfrak{X})$ (resp. $f^{!}\colon K^{\mathrm{rel}}_{\mathfrak{Z}'}(\mathfrak{X}') \to K^{\mathrm{rel}}_{f^{-1}(\mathfrak{Z}')}(\mathfrak{X}')$) is defined by

$$f^{!}\mathcal{G} = \sum_{k \geq 0} (-1)^k \mathcal{T}or^k_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}(f^{-1}\mathcal{G}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}).$$

– GYSIN MORPHISM. Let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a morphism between two relative analytic spaces and $\mathfrak{Z}$ be a relative analytic subset of $\mathfrak{X}$ such that f is finite on $\mathfrak{Z}$. The *Gysin morphism* f_* from $K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$ to $K^{\mathrm{rel}}(\mathfrak{X}')$ is induced by the exact functor $f_*\colon \mathrm{Coh}^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}(\mathfrak{X}')$.

We now list all the properties we need concerning the operations introduced above.

Proposition 3.16 *(i)* PRODUCT STRUCTURE. *If $\mathfrak{X}$ is a relative analytic space and if $\mathfrak{Z}$ is a closed subspace of $\mathfrak{X}$, then $K^{\mathrm{rel}}(\mathfrak{X})$ is a unitary ring and $K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$ is a module over $K^{\mathrm{rel}}(\mathfrak{X})$. Besides, if $\mathcal{F}$ and $\mathcal{G}$ are relatively coherent sheaves on $\mathfrak{X}$ and if $\mathcal{G}$ is supported in $\mathfrak{Z}$, then*

$$\mathcal{F}^{\vee} \,.\, \mathcal{G} = \sum_{i \geq 0} (-1)^i \, \mathcal{E}xt^i_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \ \textit{in} \ K^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X}).$$

(ii) FUNCTORIALITY. *The pullback morphism (resp. Gysin morphism) in relative K–theory is contravariant (resp. covariant) with respect to weak morphisms (resp. with respect to morphisms). Besides, the pullback and the dual morphism commute.*

(iii) PROJECTION FORMULA. *Let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a morphism between two relative analytic spaces and $\mathfrak{Z}$ be a relative analytic subspace of $\mathfrak{X}$ such that f is finite on $\mathfrak{Z}$. If $\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}$ supported in $\mathfrak{Z}$ and $\mathcal{G}$ is a relatively coherent sheaf on $\mathfrak{X}'$, then $f_*\big(\mathcal{F} \,.\, f^{!}\mathcal{G}\big) = f_*\mathcal{F} \,.\, \mathcal{G}$ in $K^{\mathrm{rel}}(\mathfrak{X}')$.*

(iv) BASE CHANGE I. *Let $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a morphism between two relative analytic spaces over B, Δ be a relative analytic space over B and $\mathfrak{Z}'$ be a relative analytic subspace of $\mathfrak{X}' \times_B \Delta$ such that the projection $q\colon \mathfrak{X}' \times_B \Delta \longrightarrow \mathfrak{X}'$ is finite on $\mathfrak{Z}'$. If $f_\Delta = f \times_B \mathrm{id}_\Delta$, we consider the cartesian diagram:*

$$\begin{tikzcd} \mathfrak{X} \times_B \Delta \arrow[r, "f_\Delta"] \arrow[d, "p"'] & \mathfrak{X}' \times_B \Delta \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, "f"'] & \mathfrak{X}' \end{tikzcd}$$

If $\mathfrak{Z} = f_\Delta^{-1}(\mathfrak{Z}')$, then $\mathfrak{Z}$ is a relative analytic subspace of $\mathfrak{X} \times_B \Delta$ and p is finite on $\mathfrak{Z}$. Besides, the pullback and Gysin morphisms

$$p_*\colon K^{\mathrm{rel}}_{\mathfrak{Z}}\big(\mathfrak{X} \times_B \Delta\big) \longrightarrow K^{\mathrm{rel}}(\mathfrak{X}), \quad q_*\colon K^{\mathrm{rel}}_{\mathfrak{Z}'}(\mathfrak{X}' \times_B \Delta) \longrightarrow K^{\mathrm{rel}}(\mathfrak{X}'),$$

$$f^!\colon K^{\mathrm{rel}}(\mathfrak{X}') \longrightarrow K^{\mathrm{rel}}(\mathfrak{X}) \text{ and } f_\Delta^!\colon K^{\mathrm{rel}}_{\mathfrak{Z}'}(\mathfrak{X}' \times_B \Delta) \longrightarrow K^{\mathrm{rel}}_{\mathfrak{Z}}\big(\mathfrak{X} \times_B \Delta\big)$$

are related through the formula $f^! q_* = p_* f_\Delta^!$.

(v) BASE CHANGE II. *Let Δ be a relative analytic space over B, $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ be a morphism between two relative analytic spaces over B and assume that $\mathfrak{X}'$ is a smooth relative analytic subspace of Δ. If $f_\Delta = f \times_B \mathrm{id}_\Delta$, consider the cartesian diagram*

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "(\mathrm{id},\, i\circ f)"] \arrow[d, "f"'] & \mathfrak{X} \times_B \Delta \arrow[d, "f_\Delta"] \\ \mathfrak{X}' \arrow[r, "(\mathrm{id},\, i)"'] & \mathfrak{X}' \times_B \Delta \end{tikzcd}$$

Then the following pullback and Gysin morphisms

$$(\mathrm{id}, i)_*\colon K^{\mathrm{rel}}(\mathfrak{X}') \longrightarrow K^{\mathrm{rel}}_{\mathfrak{X}'}(\mathfrak{X}' \times_B \Delta), \quad (\mathrm{id}, i \circ f)_*\colon K^{\mathrm{rel}}(\mathfrak{X}) \longrightarrow K^{\mathrm{rel}}_{\mathfrak{X}}(\mathfrak{X} \times_B \Delta),$$

$$f^!\colon K^{\mathrm{rel}}(\mathfrak{X}') \longrightarrow K^{\mathrm{rel}}(\mathfrak{X}) \text{ and } f_\Delta^!\colon K^{\mathrm{rel}}_{\mathfrak{X}'}(\mathfrak{X}' \times_B \Delta) \longrightarrow K^{\mathrm{rel}}_{\mathfrak{X}}(\mathfrak{X} \times_B \Delta)$$

are related through the formula $f_\Delta^!(\mathrm{id}, i)_* = (\mathrm{id}, i \circ f)_* f^!$.

(vi) BASE CHANGE III. *Let $(\mathfrak{X}, \mathcal{A})$ be a relative analytic space over B, $\mathfrak{Z}$ be a closed subset of $\mathfrak{X}$, $u\colon B \longrightarrow B'$ be a smooth map and $\widetilde{u}\colon \mathfrak{X} \times_B B' \longrightarrow \mathfrak{X}$ be the associated base change morphism. Then the pullback functor $\widetilde{u}^*$ from $\mathrm{Coh}^{\mathrm{rel}}(\mathfrak{X})$ to $\mathrm{Coh}^{\mathrm{rel}}\big(\mathfrak{X} \times_B B'\big)$ (resp. from $\mathrm{Coh}^{\mathrm{rel}}_{\mathfrak{Z}}(\mathfrak{X})$ to $\mathrm{Coh}^{\mathrm{rel}}_{\widetilde{u}^{-1}(\mathfrak{Z})}\big(\mathfrak{X} \times_B B'\big)$) is exact, and $\widetilde{u}^! = \widetilde{u}^*$.*

Proof (i) If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are relatively coherent sheaves on $\mathfrak{X}$, there is a spectral sequence (canonical from E_2) such that

$$E_2^{p,q} = \mathcal{T}or^p_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\big(\mathcal{T}or^q_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G}),\mathcal{H}\big), \quad E_\infty^{p,q} = \mathrm{Gr}_p\ \mathcal{T}or^{p+q}_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G},\mathcal{H})$$

and $E_2^{p,q}$ vanishes on each component of $\mathfrak{X}$ except for finitely many couples (p,q). Furthermore, by Proposition 3.6 (i), the sheaves $E_2^{p,q}$ are relatively coherent on $\mathfrak{X}$ and the morphisms $d_2^{p,q}$ are strict. Thus, for all $r \geq 2$, the sheaves $E_r^{p,q}$ are relatively coherent and the morphisms $d_r^{p,q}$ are strict, so that

$$\sum_{p,q\geq 0}(-1)^{p+q}E_2^{p,q} = \sum_{n\geq 0}(-1)^n\,\mathcal{T}or^n_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G},\mathcal{H})$$

in $K^{\mathrm{rel}}(\mathfrak{X})$. This yields the associativity of the product. The proofs of the remaining properties in (i) and of (ii) are essentially similar, using spectral sequences associated with the composition of two functors.

The proofs of (iii), (iv) and (v) are performed in the same way. We detail the proof of (iv).

(iv) For x in $\mathfrak{X}$, we take two relative holomorphic charts $\phi\colon U \xrightarrow{\sim} Z\times V$ and $\phi'\colon U' \xrightarrow{\sim} Z'\times V$ in neighbourhoods of x and $f(x)$ such that ϕ and ϕ' are $f-$compatible. Let us write $\phi'\circ f\circ\phi^{-1} = (g,\mathrm{id})$, where $g\colon Z \longrightarrow Z'$ is holomorphic. If $\delta_1,\dots,\delta_N$ are elements of Δ such that $q^{-1}(f(x))\cap\mathfrak{Z} = \cup_{i=1}^N\big(f(x),\delta_i\big)$, we choose a relative holomorphic chart $\psi\colon U_{\delta_1,\dots,\delta_N} \xrightarrow{\sim} Y\times V$ in a neighbourhood of the δ_i's. The local form of the diagram of (iv) is then:

$$\begin{tikzcd}
Z\times Y\times V \arrow[r, "(g,\mathrm{id},\mathrm{id})"] \arrow[d, "\mathrm{pr}_{13}"'] & Z'\times Y\times V \arrow[d, "\mathrm{pr}_{13}"] \\
Z\times V \arrow[r, "(g,\mathrm{id})"'] & Z'\times V
\end{tikzcd}$$

Furthermore, we can assume that $\big(\phi'\times_V\psi\big)\big[\big(U'\times U_{\delta_1,\dots,\delta_N}\big)\cap\mathfrak{X}\big] = \overline{\mathfrak{Z}}\times V$, where $\overline{\mathfrak{Z}}$ is an analytic subset of $Z'\times Y$ and pr_1 is finite on $\overline{\mathfrak{Z}}$. Then for any coherent analytic sheaf $\overline{\mathcal{G}}$ on $Z'\times Y$ supported in $\overline{\mathfrak{Z}}$, we use Proposition 3.13 and we get

$$\begin{aligned}
(g,\mathrm{id})^*\Big[\mathrm{pr}_{13*}\big(\mathrm{pr}_{12}^{-1}\,\overline{\mathcal{G}}\otimes_{\mathrm{pr}_{12}^{-1}\mathcal{O}_{Z'\times Y}}\mathcal{O}^{\mathrm{rel}}_{Z'\times Y\times V}\big)\Big] &\simeq (g,\mathrm{id})^*\big(\mathrm{pr}_{1*}\,\overline{\mathcal{G}}\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z'}}\mathcal{O}^{\mathrm{rel}}_{Z'\times V}\big)\\
&\simeq g^*\big(\mathrm{pr}_{1*}\,\overline{\mathcal{G}}\big)\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z}\mathcal{O}^{\mathrm{rel}}_{Z\times V} \simeq \mathrm{pr}_{1*}\big[(g,\mathrm{id})^*\,\overline{\mathcal{G}}\,\big]\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_Z}\mathcal{O}^{\mathrm{rel}}_{Z\times V}\\
&\simeq \mathrm{pr}_{13*}\big[\mathrm{pr}_{12}^{-1}(g,\mathrm{id})^*\,\overline{\mathcal{G}}\otimes_{\mathrm{pr}_{12}^{-1}\mathcal{O}_{Z\times Y}}\mathcal{O}^{\mathrm{rel}}_{Z\times Y\times V}\big]\\
&\simeq \mathrm{pr}_{13*}\Big[(g,\mathrm{id},\mathrm{id})^*\big(\mathrm{pr}_{12}^{-1}\,\overline{\mathcal{G}}\otimes_{\mathrm{pr}_{12}^{-1}\mathcal{O}_{Z'\times Y}}\mathcal{O}^{\mathrm{rel}}_{Z'\times Y\times V}\big)\Big].
\end{aligned}$$

Taking the derivative with respect to $\overline{\mathcal{G}}$ and using Lemma 3.5, we obtain the result.

(vi) This is an immediate consequence of Proposition 3.6 (iii). □

3.4 Topological K–theory for relatively coherent sheaves

In § 3.3, we have constructed a theory for relative coherent sheaves as well as associated operations. It remains to obtain cohomological information about these objects. To do so, we construct global resolutions by complex vector bundles for relatively coherent sheaves. We start with a general result:

Proposition 3.17 *Let Y be a differentiable orbifold.*

(i) Locally free sheaves of $\mathcal{C}_Y^\infty$–modules are projective elements in the category $\mathrm{Mod}\left(\mathcal{C}_Y^\infty\right)$. *In particular, if $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of sheaves of $\mathcal{C}_Y^\infty$–modules on Y and if $\mathcal{H}$ is locally free, then this sequence globally splits.*

(ii) If $\mathcal{H}$ is sheaf of $\mathcal{C}_Y^\infty$–modules admitting a finite free resolution in a neighbourhood of any point of Y, then $\mathcal{H}$ admits a finite locally free resolution in a neighbourhood of any compact subset of Y.

(iii) Two finite locally free resolutions of a sheaf of $\mathcal{C}_Y^\infty$–modules are sub-resolutions of a third one.

(iv) Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be an exact sequence of sheaves of $\mathcal{C}_Y^\infty$–modules such that $\mathcal{G}$ and $\mathcal{H}$ admit finite locally free resolutions on Y. Then $\mathcal{F}$ admits a finite locally free resolution on Y and we can find three such resolutions $F_\bullet$, $G_\bullet$ and $H_\bullet$ of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ related by an exact sequence $0 \longrightarrow F_\bullet \longrightarrow G_\bullet \longrightarrow H_\bullet \longrightarrow 0$.

Proof (i) Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$ be sheaves of $\mathcal{C}_Y^\infty$–modules such that $\mathcal{Q}$ is a quotient of $\mathcal{P}$ and $\mathcal{H}$ is locally free, and let π be in $\mathrm{Hom}_{\mathcal{C}_Y^\infty}(\mathcal{H}, \mathcal{Q})$. It is possible to lift π locally to a morphism from $\mathcal{H}$ to $\mathcal{P}$, and then globally using a partition of unity on Y.

(ii) Let K be a compact subset of Y. We choose a finite covering $\left(U_i\right)_{1\leq i\leq d}$ of K and open sets $\left(V_i\right)_{1\leq i\leq d}$ such that for $1 \leq i \leq d$, U_i is relatively compact in V_i and $\mathcal{H}$ admits a finite free resolution on V_i. For each i, we multiply this resolution by a smooth cut-off function equal to 1 on U_i and supported in V_i. We obtain in this way a complex of sheaves

$$0 \longrightarrow \left(\mathcal{C}_Y^\infty\right)^{n_{iN}} \longrightarrow \cdots\cdots \longrightarrow \left(\mathcal{C}_Y^\infty\right)^{n_{i1}} \xrightarrow{\pi_i} \mathcal{H} \longrightarrow 0$$

on Y, which is exact in U_i. If $E = \bigoplus_{i=1}^{d} \left(\mathcal{C}_Y^\infty\right)^{n_{i1}}$ and if $\pi = \bigoplus_{i=1}^{d} \pi_i \colon E \longrightarrow \mathcal{H}$ is the sum of the π_i's, then the morphism π is surjective in a neighbourhood of K. For $1 \leq i \leq d$, let $\mathcal{N}_i$ and $\mathcal{N}$ denote the kernels of π_i and π respectively. We have an exact sequence:

$$0 \longrightarrow \mathcal{N}_{i\,|U_i} \longrightarrow \mathcal{N}_{|U_i} \longrightarrow \bigoplus_{j\neq i} \left(\mathcal{C}_{U_i}^\infty\right)^{n_{j1}} \longrightarrow 0.$$

By (i), $\mathcal{N}_{|U_i}$ is isomorphic to $\mathcal{N}_{i\,|U_i} \oplus \left(\mathcal{C}_{U_i}^\infty\right)^{\sum_{j\neq i} n_{j1}}$. Furthermore $\mathcal{N}_{i\,|U_i}$ admits a finite free resolution of length $N-1$. Thus $\mathcal{N}$ admits a finite free resolution of length at most $N-1$ in a neighbourhood of every point in K and we can start the argument again. After at most N steps, we obtain a locally free kernel.

(iii) Let $\mathcal{H}$ be a sheaf of $\mathcal{C}_Y^\infty$–modules and $\left(E_i\right)_{0\leq i\leq N}$ and $\left(F_i\right)_{0\leq i\leq N}$ be two finite locally free resolutions of $\mathcal{H}$ on Y. Let us construct by induction a finite locally free resolution $G_\bullet$ of $\mathcal{H}$ on Y such that $E_\bullet$ and $F_\bullet$ are sub-resolutions of $G_\bullet$. We put $G_0 = E_0 \oplus F_0$, the map from G_0 to $\mathcal{H}$ being obtained by adding the two maps from E_0 and from F_0 to $\mathcal{H}$. If k is a positive integer smaller than or equal to $N-1$, assume that we have constructed $\left(G_i\right)_{0\leq i\leq k}$ as well as injections $E_\bullet \hookrightarrow G_\bullet$ and $F_\bullet \hookrightarrow G_\bullet$ in degrees at most k. Then in the two diagrams

$$\begin{tikzcd}
0 \arrow[r] & E_k \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & Q_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_{k-1} \arrow[r] \arrow[d] & G_{k-1} \arrow[r] \arrow[d] & Q_{k-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & E_0 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[r] & F_k \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & R_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{k-1} \arrow[r] \arrow[d] & G_{k-1} \arrow[r] \arrow[d] & R_{k-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_0 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & R_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

all the lines and the columns are exact and $Q_0, \ldots, Q_k, R_0, \ldots, R_k$ are locally free by induction. Let $N_k = \ker\left(E_k \to E_{k-1}\right)$, $N_k' = \ker\left(F_k \to F_{k-1}\right)$, $N_k'' = \ker\left(G_k \to G_{k-1}\right)$, $\widetilde{Q}_k = \ker\left(Q_k \to Q_{k-1}\right)$ and $\widetilde{R}_k = \ker\left(R_k \to R_{k-1}\right)$. By breaking the exact sequences of the two last columns into short exact sequences, we obtain that $\widetilde{Q}_k$ and $\widetilde{R}_k$ are locally free. The two sequences

$$0 \longrightarrow N_k \longrightarrow N_k'' \longrightarrow \widetilde{Q}_k \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N_k' \longrightarrow N_k'' \longrightarrow \widetilde{R}_k \longrightarrow 0$$

are exact. By (i), $N''_k \simeq N_k \oplus \widetilde{Q}_k \simeq N'_k \oplus \widetilde{R}_k$, and we can define G_{k+1} by the formula $G_{k+1} = \big(E_{k+1} \oplus \widetilde{Q}_k\big) \oplus \big(F_{k+1} \oplus \widetilde{R}_k\big)$.

If $k = N$, we end the resolution $G_\bullet$ by putting $G_{N+1} = N''_N$.

(iv) If $\big(\widetilde{G}_i\big)_{0\le i\le N}$ and $\big(H_i\big)_{0\le i\le N}$ are locally free resolutions of $\mathcal{G}$ and $\mathcal{H}$, let us construct by induction locally free resolutions $F_\bullet$ and $G_\bullet$ of $\mathcal{F}$ and $\mathcal{G}$ such that $\widetilde{G}_\bullet$ is a quotient of $G_\bullet$, together with an exact sequence $0 \longrightarrow F_\bullet \longrightarrow G_\bullet \longrightarrow H_\bullet \longrightarrow 0$.

Since H_0 (resp. $\widetilde{G}_0$) is a projective object by (i), we can lift the map from H_0 to $\mathcal{H}$ (resp. from $\widetilde{G}_0$ to $\mathcal{H}$) to a map π (resp. $\widetilde{\pi}$) from H_0 to $\mathcal{G}$ (resp. from $\widetilde{G}_0$ to H_0). If $G_0 = H_0 \oplus \widetilde{G}_0$, there is a natural surjective map from G_0 to $\mathcal{G}$ obtained by adding π and the map from $\widetilde{G}_0$ to $\mathcal{G}$. Besides, $(\mathrm{id}, \widetilde{\pi})\colon G_0 \longrightarrow H_0$ is surjective, we denote its kernel by F_0.

If k is a positive integer smaller than or equal to N, assume that we have constructed $\big(F_i\big)_{0\le i\le k}$ and $\big(G_i\big)_{0\le i\le k}$, an exact sequence $0 \longrightarrow F_\bullet \longrightarrow G_\bullet \longrightarrow H_\bullet \longrightarrow 0$ and a surjective morphism from $G_\bullet$ to $\widetilde{G}_\bullet$ in degrees at most k. Let $N_k = \ker\big(F_k \to F_{k-1}\big)$, $N'_k = \ker\big(G_k \to G_{k-1}\big)$, $N''_k = \ker\big(H_k \to H_{k-1}\big)$ and $\widetilde{N}_k = \ker\big(\widetilde{G}_k \to \widetilde{G}_{k-1}\big)$. We have an exact sequence $0 \longrightarrow N_k \longrightarrow N'_k \longrightarrow N''_k \longrightarrow 0$. If $R_\bullet = \ker\big(G_\bullet \to \widetilde{G}_\bullet\big)$ and $S_k = \ker\big(R_k \to R_{k-1}\big)$ the sequence $0 \longrightarrow S_k \longrightarrow N'_k \longrightarrow \widetilde{N}_k \longrightarrow 0$ is exact, so that $N'_k \simeq S_k \oplus \widetilde{N}_k$ by (i). As above, we lift the map from H_{k+1} to N''_k (resp. from $S_k \oplus \widetilde{G}_{k+1}$ to N''_k) to a map π_k (resp. $\widetilde{\pi}_k$) from H_{k+1} to N'_k (resp. from $S_k \oplus \widetilde{G}_k$ to H_{k+1}). If $G_{k+1} = H_{k+1} \oplus S_k \oplus \widetilde{G}_{k+1}$, there is a natural surjection from G_{k+1} to N'_k obtained by adding π_k and the map from $S_k \oplus \widetilde{G}_{k+1}$ to N'_k. Then we define F_{k+1} by $F_{k+1} = \ker\big(\widetilde{\pi}_k, \mathrm{id}\big)$.

If $k = N$, $F_{N+1} = N_{N+1}$ and $G_{N+1} = N'_{N+1}$, so that $F_\bullet$ and $G_\bullet$ are locally free resolutions of $\mathcal{F}$ and $\mathcal{G}$. □

We apply now this result in our context. If $(\mathfrak{X}, \mathcal{A})$ is a relative analytic space, then $\mathfrak{X}$ is also a differentiable orbifold, and $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}$ is a subsheaf of $\mathcal{C}^\infty_{\mathfrak{X}}$. Therefore, we can associate with every relatively coherent analytic sheaf $\mathcal{F}$ on $\mathfrak{X}$ the sheaf $\mathcal{F}^\infty$ defined by $\mathcal{F}^\infty = \mathcal{F} \otimes_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}} \mathcal{C}^\infty_{\mathfrak{X}}$, which is a sheaf of $\mathcal{C}^\infty_{\mathfrak{X}}$–modules. This sheaf admits a finite free resolution in a neighbourhood of any point of $\mathfrak{X}$, thanks to the lemma:

Lemma 3.18 *Let U be an open subset of $\mathbb{R}^n$, G be a finite group of diffeomorphisms of U and Z be a smooth analytic set. If $W = U/G$, then $\mathcal{C}^\infty_{Z\times W}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$.*

Proof As in Lemma 3.5, it suffices to prove that $\mathcal{O}^{\mathrm{rel}}_{Z\times U}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{O}_Z$. If Y is a real-analytic manifold, let $\mathcal{C}^\omega_Y$ be the sheaf of real-analytic functions on Y. Then, $\mathcal{C}^\omega_Z$ is flat over $\mathcal{O}_Z$, $\mathcal{C}^\omega_{Z\times U}$ is flat over $\mathrm{pr}_1^{-1}\,\mathcal{C}^\omega_Z$ and $\mathcal{C}^\infty_{Z\times U}$ is flat over $\mathcal{C}^\omega_{Z\times U}$ by [32, Th. 2]. □

The Grothendieck group of the category of complex topological vector bundles on a topological space Y will be called $K(Y)$. Besides, the class in $K(Y)$ of a complex vector bundle E on Y will be denoted by $[E]$. Then Proposition 3.17 and Lemma 3.18 yield:

Proposition 3.19 *If $(\mathfrak{X}, \mathcal{A})$ is a relative analytic space and if $\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}$, then the sheaf $\mathcal{F}^\infty$ admits a finite locally free resolution in a neighbourhood of any compact subset of $\mathfrak{X}$. Besides, if $\mathfrak{U}$ is a relatively compact open subset of $\mathfrak{X}$ and $\left(E_i\right)_{0\leq i\leq N}$ is a locally free resolution of $\mathcal{F}^\infty$ on $\mathfrak{U}$, then the element $\sum_{i=0}^N (-1)^i \left[E_i\right]$ of $K^{\mathrm{rel}}(\mathfrak{U})$ is independent of $E_\bullet$ and depends only on the class of $\mathcal{F}$ in $K^{\mathrm{rel}}(\mathfrak{X})$.*

In conclusion, we can associate with each relatively coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ a topological class $\left[\mathcal{F}^\infty\right]$ in $\varprojlim_{\mathfrak{U}} K(\mathfrak{U})$, where $\mathfrak{U}$ runs through all the relatively compact open subsets of $\mathfrak{X}$. Furthermore, if $\mathfrak{Z}$ is a closed subset of $\mathfrak{X}$ and $\mathcal{F}$ is supported in $\mathfrak{Z}$, then the topological class of $\mathcal{F}$ lies naturally in $\varprojlim_{\mathfrak{U}} K_{\mathfrak{U}\cap\mathfrak{Z}}(\mathfrak{U})$.

We are going to prove two fundamental properties of the topological class, namely the functoriality by pullback and the homotopy invariance. We start with the first one.

Proposition 3.20 *Let f be a weak morphism between two relative analytic spaces $\mathfrak{X}$ and $\mathfrak{X}'$, $\mathfrak{Z}$ be a relative analytic subspace of $\mathfrak{X}'$ and $\mathcal{F}$ be a relatively coherent sheaf on $\mathfrak{X}'$ supported in $\mathfrak{Z}$. Then the pullback morphism*

$$f^*\colon \varprojlim_{\mathfrak{U}'\subset\subset\mathfrak{X}'} K_{\mathfrak{U}'\cap\mathfrak{Z}}(\mathfrak{U}') \longrightarrow \varprojlim_{\mathfrak{U}\subset\subset\mathfrak{X}} K_{\mathfrak{U}\cap f^{-1}(\mathfrak{Z})}(\mathfrak{U})$$

maps $\left[\mathcal{F}^\infty\right]$ to $\left[\left(f^!\mathcal{F}\right)^\infty\right]$.

Proof We can assume that $\mathcal{F}^\infty$ admits a global locally free resolution on $\mathfrak{X}'$. Let $\left(E_i\right)_{0\leq i\leq N}$ be such a resolution. Then for every nonnegative integer k, the cohomology sheaf of $f^*E_\bullet$ in degree k is $\mathcal{T}or^k_{f^{-1}\mathcal{C}^\infty_{\mathfrak{X}'}}\left(f^{-1}\mathcal{F}^\infty, \mathcal{C}^\infty_{\mathfrak{X}}\right)$, which is isomorphic to $\mathcal{T}or^k_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}\left(f^{-1}\mathcal{F}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\right)^\infty$ by Lemma 3.5 and Lemma 3.18. We define the sheaves $\mathcal{N}_k$ and $\mathcal{I}_k$ by $\mathcal{N}_k = \ker\left(f^*E_k \to f^*E_{k-1}\right)$ and $\mathcal{I}_k = \mathrm{Im}\left(f^*E_{k+1} \to f^*E_k\right)$. Then we have exact sequences

$$0 \longrightarrow \mathcal{N}_k \longrightarrow f^*E_k \longrightarrow \mathcal{I}_{k-1} \longrightarrow 0 \tag{3–3}$$

and

$$0 \longrightarrow \mathcal{I}_k \longrightarrow \mathcal{N}_k \longrightarrow \mathcal{T}or^k_{f^{-1}\mathcal{C}^\infty_{\mathfrak{X}'}}\left(f^{-1}\mathcal{F}^\infty, \mathcal{C}^\infty_{\mathfrak{X}}\right) \longrightarrow 0. \tag{3–4}$$

If $\mathfrak{U}$ is relatively compact in $\mathfrak{X}$, then the sheaves $\mathcal{T}or^i_{f^{-1}\mathcal{C}^\infty_{\mathfrak{X}'}}\big(f^{-1}\mathcal{F}^\infty, \mathcal{C}^\infty_{\mathfrak{X}}\big)$ admit, by Proposition 3.19, a global locally free resolution on $\mathfrak{U}$. Since $\mathcal{N}_0 = E_0$, the repeated use of Proposition 3.17 (iv) with the exact sequences (3–3) and (3–4) shows that the sheaves $\mathcal{N}_i$ and $\mathcal{I}_i$ admit global locally free resolutions on $\mathfrak{U}$, and that the following identities hold in $K_{\mathfrak{U}\cap f^{-1}(\mathfrak{Z})}(\mathfrak{U})$:

$$\big[f^*E_k\big] = \big[\mathcal{N}_k\big] + \big[\mathcal{I}_{k-1}\big] \ \text{ and } \ \big[\mathcal{N}_k\big] = \big[\mathcal{I}_k\big] + \Big[\mathcal{T}or^k_{f^{-1}\mathcal{C}^\infty_{\mathfrak{X}'}}\big(f^{-1}\mathcal{F}^\infty, \mathcal{C}^\infty_{\mathfrak{X}}\big)\Big].$$

As a consequence, $\sum_{i=0}^N (-1)^i \big[f^*E_i\big] = \sum_{i=0}^N (-1)^i \left[\mathcal{T}or^i_{f^{-1}\mathcal{C}^\infty_{\mathfrak{X}'}}\big(f^{-1}\mathcal{F}^\infty, \mathcal{C}^\infty_{\mathfrak{X}}\big)\right]$. This yields the result. □

We can now come to the homotopy invariance of the topological class:

Proposition 3.21 *Let $\mathfrak{Y}$ be a relative analytic space over $B \times [0,1]$, $\mathfrak{R}$ be a relative analytic subspace of $\mathfrak{Y}$ and $\mathcal{F}$ be a relatively coherent sheaf on $\mathfrak{Y}$ supported in $\mathfrak{R}$. Assume that the pair $\big(\mathfrak{R}, \mathfrak{Y}\big)$ is topologically trivial over $[0,1]$ (ie for any t_0 in $[0,1]$, if $\mathfrak{X} = \mathfrak{Y}_{t_0}$ and $\mathfrak{Z} = \mathfrak{R}_{t_0}$, then there exists a homeomorphism between $\mathfrak{Y}$ and $\mathfrak{X} \times [0,1]$ over $B \times [0,1]$ mapping $\mathfrak{R}$ to $\mathfrak{Z} \times [0,1]$). For every t in $[0,1]$, let $i_t\colon \mathfrak{Y}_t \longrightarrow \mathfrak{Y}$ be the natural inclusion. Then, via the homeomorphism between $\mathfrak{Y}_t$ and $\mathfrak{X}$, the topological class $\big[(i_t^*\mathcal{F})^\infty\big]$ in $\varprojlim_{\mathfrak{U}\subset\subset\mathfrak{X}} K_{\mathfrak{U}\cap\mathfrak{Z}}(\mathfrak{U})$ is independent of t.*

Proof Let $\mathfrak{U}$ be an open relatively compact subset of $\mathfrak{X}$, $\varphi\colon \mathfrak{X} \times [0,1] \longrightarrow \mathfrak{Y}$ be a homeomorphism trivializing the pair $(\mathfrak{R}, \mathfrak{Y})$ and let $\mathfrak{U}' = \varphi(\mathfrak{U} \times [0,1])$. Then $\mathfrak{U}'$ is open and relatively compact in $\mathfrak{Y}$. We take a locally free resolution $\big(E_i\big)_{0\le i\le N}$ of $\mathcal{F}^\infty$ on $\mathfrak{U}'$. By the homotopy invariance property for topological K–theory, the class $\sum_{i=0}^N (-1)^i \big[i_t^*\mathcal{F}^\infty\big]$ in $K_{\mathfrak{U}\cap\mathfrak{Z}}(\mathfrak{U})$ is independent of t. By Proposition 3.16 (vi) and Proposition 3.20, $i_t^*E_\bullet$ is a locally free resolution of $i_t^*\mathcal{F}^\infty$ in $\mathfrak{U}$. This yields the desired result. □

3.5 Relative incidence sheaves

Let (X, J) be an almost-complex compact four-manifold and n be a *fixed* positive integer. If W, W' and W'' are small neighbourhoods of the incidence loci $Z_{n\times 1}$, Z_n and Z_{n+1} introduced in (2–5) and (2–6), let $J^{\mathrm{rel}}_{n\times 1}$, J^{rel}_n and J^{rel}_{n+1} be relative integrable complex structures on W, W' and W'' respectively. To simplify the notations, we put

$$X^{[n]} = X^{[n]}_{J^{\mathrm{rel}}_n},\ X^{[n+1]} = X^{[n+1]}_{J^{\mathrm{rel}}_{n+1}},\ X^{[n+1,n]} = X^{[n+1,n]}_{J^{\mathrm{rel}}_{n\times 1}},\ X^{[n]\times[1]} = \big(X^{[n]\times[1]}, J^{\mathrm{rel}}_{n\times 1}, J^{\mathrm{rel}}_{n\times 1}\big).$$

Definition 3.22 The four relative incidence sets $\mathfrak{Y}_n$, $\mathfrak{Y}_{n+1}$, $\widetilde{\mathfrak{Y}}_n$ and $\widetilde{\mathfrak{Y}}_{n+1}$ are defined by:

- $\mathfrak{Y}_n = \{(\xi, w; \underline{x})$ in $W'^{[n]}_{\text{rel}} \times_{X^{(n)}} W'$ such that $w \in \text{supp}(\xi)\}$
- $\mathfrak{Y}_{n+1} = \{(\xi, w; \underline{y})$ in $W''^{[n+1]}_{\text{rel}} \times_{X^{(n+1)}} W''$ such that $w \in \text{supp}(\xi)\}$
- $\widetilde{\mathfrak{Y}}_n = \{(\xi, w; \underline{x}, p)$ in $W^{[n]}_{\text{rel}} \times_{X^{(n)} \times X} W$ such that $w \in \text{supp}(\xi)\}$
- $\widetilde{\mathfrak{Y}}_{n+1} = \{(\xi, w; \underline{x}, p)$ in $W^{[n+1]}_{\text{rel}} \times_{X^{(n)} \times X} W$ such that $w \in \text{supp}(\xi)\}$

The relative incidence sets $\mathfrak{Y}_n$, $\mathfrak{Y}_{n+1}$, $\widetilde{\mathfrak{Y}}_n$ and $\widetilde{\mathfrak{Y}}_{n+1}$ are relative analytic subspaces of $W'^{[n]}_{\text{rel}} \times_{X^{(n)}} W'$, $W''^{[n+1]}_{\text{rel}} \times_{X^{(n+1)}} W''$, $W^{[n]}_{\text{rel}} \times_{X^{(n)} \times X} W$ and $W^{[n+1]}_{\text{rel}} \times_{X^{(n)} \times X} W$ respectively. For instance, if $\{\phi_i \colon U_i \xrightarrow{\sim} \Omega_i \times V_i\}_{i \in I}$, where V_i and Ω_i are open subsets of $X^{(n)}$ and $\mathbb{C}^2$ respectively, is a maximal relative atlas on W', the associated atlas on $W'^{[n]}_{\text{rel}} \times_{X^{(n)}} W'$ is the completed atlas of $\{\phi_i^{[n]} \times_{V_i} \phi_i \colon U_i^{[n]} \times_{V_i} U_i \xrightarrow{\sim} \Omega_i^{[n]} \times \Omega_i \times V_i\}_{i \in I}$. For any i in I, let $Y_{n,i}$ be the incidence locus in $\Omega_i^{[n]} \times \Omega_i$ defined by (2–1). Then we have $\left(\phi_i^{[n]} \times_{V_i} \phi_i\right) \left[\left(U_i^{[n]} \times_{V_i} U_i\right) \cap \mathfrak{Y}_n\right] = Y_{n,i} \times V_i$.

Definition 3.23 (i) The *relative exceptional divisor* D_{rel} is the subset of $W^{[n+1,n]}_{\text{rel}}$ defined by

$$D_{\text{rel}} = \{(\xi,\, \xi';\, \underline{x},\, p) \text{ in } W^{[n+1,n]}_{\text{rel}} \text{ such that } \text{supp}(\xi) = \text{supp}(\xi')\}.$$

(ii) The *relative residual morphism* $\overline{\rho} \colon W^{[n+1,n]}_{\text{rel}} \longrightarrow W$ is defined by

$$\overline{\rho}\left(\xi\,, \xi'\,; \underline{x}\,, p\right) = \left(\text{supp}\left(\xi/\xi'\right); \underline{x}, p\right)$$

(iii) The *relative diagonal* Δ_{rel} is the subset of $W \times_{X^{(n)} \times X} W$ defined by

$$\Delta_{\text{rel}} = \{(w_1,\, w_2\,; \underline{x},\, p) \text{ in } W \times_{X^{(n)} \times X} W \text{ such that } w_1 = w_2\}.$$

The set D_{rel} is a relative analytic subspace of $W^{[n+1,n]}_{\text{rel}}$ of relative codimension one, the fibers $D_{\text{rel},\underline{x},p}$ of D_{rel} over $X^{(n)} \times X$ being the usual exceptional divisors in $W^{[n+1,n]}_{\underline{x},p}$ defined by (2–2). The ideal sheaf $\mathcal{J}^{\text{rel}}_{D_{\text{rel}}}$ is locally free of rank one on $W^{[n+1,n]}_{\text{rel}}$, so that $\mathcal{J}^{\infty}_{D_{\text{rel}}}$ is a complex line bundle on $W^{[n+1,n]}_{\text{rel}}$.

Let us give a list of notations which are extensively used in the sequel of this section as well as in § 5.2 and § 6.

Notations 3.24 SHEAVES

– The relatively coherent sheaves $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{Y}_n}$, $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{Y}_{n+1}}$, $\mathcal{O}^{\mathrm{rel}}_{\widetilde{\mathfrak{Y}}_n}$ and $\mathcal{O}^{\mathrm{rel}}_{\widetilde{\mathfrak{Y}}_{n+1}}$ (resp. $\mathcal{J}^{\mathrm{rel}}_{\mathfrak{Y}_n}$, $\mathcal{J}^{\mathrm{rel}}_{\mathfrak{Y}_{n+1}}$, $\mathcal{J}^{\mathrm{rel}}_{\widetilde{\mathfrak{Y}}_n}$ and $\mathcal{J}^{\mathrm{rel}}_{\widetilde{\mathfrak{Y}}_{n+1}}$) on $W'^{[n]}_{\mathrm{rel}} \times_{X^{(n)}} W'$, $W''^{[n+1]}_{\mathrm{rel}} \times_{X^{(n+1)}} W''$, $W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$ and $W^{[n+1]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$ defined by Proposition 3.11 are denoted by $\mathcal{O}_n$, $\mathcal{O}_{n+1}$, $\widetilde{\mathcal{O}}_n$ and $\widetilde{\mathcal{O}}_{n+1}$ (resp. $\mathcal{J}_n$, $\mathcal{J}_{n+1}$, $\widetilde{\mathcal{J}}_n$ and $\widetilde{\mathcal{J}}_{n+1}$).
– The ideal sheaf $\mathcal{J}^{\mathrm{rel}}_{D_{\mathrm{rel}}}$ is denoted by $\mathcal{L}$.
– The relatively coherent sheaf $\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}}$ on $W \times_{X^{(n)}\times X} W$ is denoted by $\mathcal{O}_\Delta$.

COHOMOLOGY CLASSES

– The restriction to $X^{[n+1,n]}$ of the first Chern class of $\mathcal{L}^\infty$ in $H^2\big(X^{[n+1,n]},\mathbb{Q}\big)$ is denoted by l.
– If $0 \leq i \leq 2n+2$, we define a class $\mu_{i,n}$ in $H^{2i}\big(X^{[n]}\times X,\mathbb{Q}\big)$ by the formula $\mu_{i,n} = c_i\big(\big[\widetilde{\mathcal{O}}_n^\infty\big]\big)_{|X^{[n]}\times[1]}$.

MORPHISMS

– The natural morphisms from $W^{[n+1,n]}_{\mathrm{rel}}$ to $W^{[n]}_{\mathrm{rel}}$ and $W^{[n+1]}_{\mathrm{rel}}$ are denoted by ϕ and ψ respectively.
– The natural projections from $W'^{[n]}_{\mathrm{rel}} \times_{X^{(n)}} W'$ (resp. $W''^{[n+1]}_{\mathrm{rel}} \times_{X^{(n+1)}} W''$, resp. $W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$, resp. $W^{[n+1]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$, resp. $W^{[n+1,n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$) to $W'^{[n]}_{\mathrm{rel}}$ (resp. $W''^{[n+1]}_{\mathrm{rel}}$, resp. $W^{[n]}_{\mathrm{rel}}$, resp. $W^{[n+1]}_{\mathrm{rel}}$, resp. $W^{[n+1,n]}_{\mathrm{rel}}$) are denoted by $\overline{p}$ (resp. $\overline{q}$, resp. p, resp. q, resp. $\widetilde{p}$).
– The morphism $\overline{\rho}\colon W^{[n+1,n]}_{\mathrm{rel}} \longrightarrow W$ has already been defined in 3.23 (ii).
– The morphism $\overline{\sigma}\colon W^{[n+1,n]}_{\mathrm{rel}} \longrightarrow W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$ is defined by $\overline{\sigma} = (\phi, \overline{\rho})$.
– The morphism $j\colon W^{[n+1,n]}_{\mathrm{rel}} \longrightarrow W^{[n+1,n]}_{\mathrm{rel}}{}_{X^{(n)}\times X} W$ is defined by $j = (\mathrm{id}, \overline{\rho})$.
– If $f\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ is a morphism of relative analytic spaces over $X^{(n)}\times X$, we define f_W by

$$f_W = f \times_{X^{(n)}\times X} \mathrm{id}_W\colon \mathfrak{X} \times_{X^{(n)}\times X} W \longrightarrow \mathfrak{X}' \times_{X^{(n)}\times X} W.$$

The relative incidence sheaves $\mathcal{O}_n$, $\mathcal{O}_{n+1}$, $\widetilde{\mathcal{O}}_n$ and $\widetilde{\mathcal{O}}_{n+1}$ are related if the relative integrable complex structures J^{rel}_n, J^{rel}_{n+1} and $J^{\mathrm{rel}}_{n\times 1}$ satisfy some compatibility conditions.

– If the compatibility condition (A) of § 2.4 is satisfied in the case $m = n+1$, let $k : W'^{[n]}_{\mathrm{rel}} \times_{X^{(n)}} \big(X^{(n)}\times X\big) \hookrightarrow W^{[n]}_{\mathrm{rel}}$ be the associated injection. If r denotes the base change morphism from $W'^{[n]}_{\mathrm{rel}} \times_{X^{(n)}} \big(X^{[n]}\times X\big)$ to $W'^{[n]}_{\mathrm{rel}}$, then $r^*\mathcal{O}_n = k^*\widetilde{\mathcal{O}}_n$ and $r^*\mathcal{J}_n = k^*\widetilde{\mathcal{J}}_n$.

– If the condition (B) of § 2.4 is satisfied, a weak morphism s from $W^{[n+1]}_{\mathrm{rel}}$ to $W''^{\,[n+1]}_{\mathrm{rel}}$ can be obtained by composing the isomorphism $W^{[n+1]}_{\mathrm{rel}} \xrightarrow{\sim} W''^{\,[n+1]}_{\mathrm{rel}} \times_{X^{(n+1)}} \big(X^{(n)} \times X\big)$ with the base change morphism from $W''^{\,[n+1]}_{\mathrm{rel}} \times_{X^{(n+1)}} \big(X^{(n)} \times X\big)$ to $W''^{\,[n+1]}_{\mathrm{rel}}$. Then $s^*\mathcal{O}_{n+1} = \widetilde{\mathcal{O}}_{n+1}$ and $s^*\mathcal{J}_{n+1} = \widetilde{\mathcal{J}}_{n+1}$.

The sheaves $\widetilde{\mathcal{O}}_n$ and $\widetilde{\mathcal{O}}_{n+1}$ fit into an important exact sequence:

Proposition 3.25 *Let $\mathfrak{X}$ be the relative analytic space $W^{[n+1,n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$.*

(i) There is a natural exact sequence on $\mathfrak{X}$:

$$0 \longrightarrow j_*\mathcal{L} \longrightarrow \psi_W^*\,\widetilde{\mathcal{O}}_{n+1} \longrightarrow \phi_W^*\,\widetilde{\mathcal{O}}_n \longrightarrow 0.$$

(ii) The relatively coherent sheaves $j_*\mathcal{L}$ and $\widetilde{p}^{\,*}\mathcal{L} \otimes \overline{\rho}^{\,*}_W\mathcal{O}_\Delta$ are isomorphic on $\mathfrak{X}$.

(iii) The three sheaves $\mathcal{T}or^{\,i}_{\overline{\rho}_W^{-1}\mathcal{O}^{\mathrm{rel}}_W}\big(\overline{\rho}_W^{-1}\mathcal{O}_\Delta, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\big)$, $\mathcal{T}or^{\,i}_{\psi_W^{-1}\mathcal{O}^{\mathrm{rel}}}\big(\psi_W^{-1}\widetilde{\mathcal{O}}_{n+1}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\big)$ and $\mathcal{T}or^{\,i}_{\phi_W^{-1}\mathcal{O}^{\mathrm{rel}}}\big(\phi_W^{-1}\widetilde{\mathcal{O}}_n, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}\big)$ vanish for any positive integer i.

Proof Using Proposition 3.6, the proposition is an immediate consequence of the analogous results in the integrable case. For instance, the exact sequence of (i) is obtained via relative holomorphic charts on W using the exact sequence (2–3). □

We now turn to the computation of the classes $\mu_{i,n}$.

Lemma 3.26 *For every positive integer i, the classes $\mu_{i,n}$ are independent of the relative integrable complex structure J^{rel}_n.*

Proof Let $J^{\mathrm{rel}}_{0,n\times 1}$ and $J^{\mathrm{rel}}_{1,n\times 1}$ be two relative integrable complex structures on a neighbourhood W of $Z_{n\times 1}$ such that $\|J^{\mathrm{rel}}_{0,n\times 1} - J\|_{C^0,g,W}$ and $\|J^{\mathrm{rel}}_{1,n\times 1} - J\|_{C^0,g,W}$ are strictly smaller than the bound ε_0 given in Proposition 2.13. Then there exist a neighbourhood U of $Z_{n\times 1}$ included in W as well as a relative integrable complex structure $\widetilde{J}^{\mathrm{rel}}_{n\times 1}$ on $U \times [0,1]$ (considered as a relative differentiable space over the base $X^{(n)} \times X \times [0,1]$) such that $\widetilde{J}^{\mathrm{rel}}_{n\times 1\,|U\times\{0\}} = J^{\mathrm{rel}}_{0,n\times 1\,|U}$ and $\widetilde{J}^{\mathrm{rel}}_{n\times 1\,|U\times\{1\}} = J^{\mathrm{rel}}_{1,n\times 1\,|U}$. If $\mathfrak{X} = \big(U \times [0,1], \widetilde{J}^{\mathrm{rel}}_{n\times 1}\big)$, let us introduce the incidence set

$$\widehat{\mathfrak{Y}}_n = \{(\xi, w; \underline{x}, p, t) \ \text{ in } \ \mathfrak{X}^{[n]} \times_{X^{(n)}\times X\times[0,1]} \mathfrak{X} \ \text{ such that } \ w \in \operatorname{supp}\xi\}.$$

Then $\widehat{\mathfrak{Y}}_n$ is a relative analytic subset of $\mathfrak{X}^{[n]} \times_{X^{(n)}\times X\times[0,1]} \mathfrak{X}$. Furthermore, for any t in $[0,1]$, if $i_t\colon \mathfrak{X}^{[n]}_t \times_{X^{(n)}\times X\times\{t\}} \mathfrak{X}_t \longrightarrow \mathfrak{X}^{[n]} \times_{X^{(n)}\times X\times[0,1]} \mathfrak{X}$ is the natural injection,

then $i_t^*\mathcal{O}^{\mathrm{rel}}_{\widehat{\mathfrak{Y}}_n}$ is the incidence sheaf $\widetilde{\mathcal{O}}_n$ on $U^{[n]} \times_{X^{(n)}\times X} U$, where U is endowed with the relative integrable complex structure $\widetilde{J}^{\mathrm{rel}}_{n\times 1\,|U\times\{t\}}$. Since the relative product Hilbert scheme $\left(X^{[n]\times[1]}, \widetilde{J}^{\mathrm{rel}}_{n\times 1}\right)$ is a topological fibration over $[0,1]$, the required result is a direct consequence of Proposition 3.21. □

We compute now the cohomology classes $\mu_{i,n}$. Let $\rho\colon X^{[n+1,n]} \longrightarrow X$ be the restriction of $\mathrm{pr}_1 \circ \overline{\rho}$ to $X^{[n+1,n]}$. If λ is the canonical homotopy class in $\left[X^{[n+1,n]} : X^{[n]}\right]$ introduced in § 2.4, we define σ in $\left[X^{[n+1,n]} : X^{[n]} \times X\right]$ by $\sigma = (\lambda, \rho)$.

Proposition 3.27 *If $1 \leq i \leq 2n+2$, $\mu_{i,n} = (-1)^i \sigma_*(l^i)$.*

Proof Let us assume that $\left(W, J^{\mathrm{rel}}_{n\times 1}\right)$ and $\left(W', J^{\mathrm{rel}}_{n}\right)$ satisfy the compatibility condition (A) of § 2.4 for $m = n+1$. Then $X^{[n]\times[1]} = X^{[n]} \times X$. In this proof, we denote the two relative analytic spaces $W^{[n+1,n]}_{\mathrm{rel}}$ and $W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$ by $\mathfrak{X}$ and $\mathfrak{X}'$ respectively. The homotopy class of the restriction of $\overline{\sigma}\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ to $X^{[n+1,n]}$ is σ.

Let $\{\phi_i\colon U_i \xrightarrow{\sim} \Omega_i \times V_i\}_{i\in I}$, where Ω_i and V_i are open subsets of $\mathbb{C}^2$ and $X^{(n)} \times X$ respectively, be a maximal relative holomorphic atlas on W. If $\psi_i = \phi_i^{[n+1,n]}$, if $\gamma_i = \phi_i^{[n]} \times_{V_i} \phi_i$ and if $O_i = U^{[n]}_{i,\mathrm{rel}} \times_{V_i} U_i$, then the family $\{\psi_i\colon U^{[n+1,n]}_{i,\mathrm{rel}} \xrightarrow{\sim} \Omega_i^{[n+1,n]} \times V_i\}_{i\in I}$ (resp. $\{\gamma_i\colon O_i \xrightarrow{\sim} \Omega_i^{[n]} \times \Omega_i \times V_i\}_{i\in I}$) is a relative holomorphic atlas on $\mathfrak{X}$ (resp. $\mathfrak{X}'$). For any i in I, let $Y_{n,i}$ be the incidence locus in $\Omega_i^{[n]} \times \Omega_i$ defined by (2–2), and let $0 \longrightarrow \mathcal{A}_i \longrightarrow \mathcal{B}_i$ be a locally free resolution of length two of the ideal sheaf $\mathcal{J}_{Y_{n,i}}$ (see § 2.1). By the very construction of global smooth resolutions for relatively coherent sheaves (Proposition 3.19), we can assume, after shrinking W if necessary, that there exists a locally free resolution $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}$ of length 2 of $\widetilde{\mathcal{J}}_n^{\infty}$ on $\mathfrak{X}'$ such that for any i in I, $0 \longrightarrow \gamma_i^{-1}\mathcal{A}_i^{\infty} \longrightarrow \gamma_i^{-1}\mathcal{B}_i^{\infty}$ is a subresolution of $0 \longrightarrow \mathcal{A}_{|O_i} \longrightarrow \mathcal{B}_{|O_i}$ (by a slight abuse of notation, we write $\mathcal{A}_i^{\infty}$ instead of $\mathrm{pr}_1^{-1}\mathcal{A}_i \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{\Omega_i^{[n]}\times\Omega_i}} \mathcal{C}^{\infty}_{\Omega_i^{[n]}\times\Omega_i\times V_i}$ and $\mathcal{B}_i^{\infty}$ instead of $\mathrm{pr}_1^{-1}\mathcal{B}_i \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{\Omega_i^{[n]}\times\Omega_i}} \mathcal{C}^{\infty}_{\Omega_i^{[n]}\times\Omega_i\times V_i}$).

Let $\mathbb{P}(\mathcal{B})$ be the projective bundle of $\mathcal{B}$ (using Grothendieck's convention), $\mathcal{K}$ be the universal quotient line bundle on $\mathbb{P}(\mathcal{B})$, $\pi\colon \mathbb{P}(\mathcal{B}) \longrightarrow W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W$ be the projection and s be the section of $\pi^*\mathcal{A}^* \otimes \mathcal{K}$ obtained via the morphism $\pi^*\mathcal{A} \longrightarrow \pi^*\mathcal{B} \longrightarrow \mathcal{K}$.

Lemma 3.28 *(i) The vanishing locus $Z(s)$ of s is canonically isomorphic to $\mathfrak{X}$.*

(ii) After changing base from $X^{(n)} \times X$ to $X^n \times X$, the section s is transverse to the zero section.

(iii) If j is the embedding of $\mathfrak{X}$ into $\mathbb{P}(\mathcal{B})$, then $j^*\mathcal{K} \simeq \mathcal{L}^\infty$.

Proof (i) By Proposition 3.17 (i), we can split the injection of $\gamma_i^{-1}\mathcal{A}_i^\infty$ in $\mathcal{A}_{|O_i}$. If $\mathcal{A}_{|O_i} \simeq \gamma_i^{-1}\big(\mathcal{A}_i^\infty \oplus \mathcal{R}_i\big)$ is such a splitting, it induces another splitting $\mathcal{B}_{|O_i} \simeq \gamma_i^{-1}\big(\mathcal{B}_i^\infty \oplus \mathcal{R}_i\big)$. The resolution $0 \longrightarrow \mathcal{A}_{|O_i} \longrightarrow \mathcal{B}_{|O_i}$ is therefore isomorphic to $0 \longrightarrow \gamma_i^{-1}\big(\mathcal{A}_i^\infty \oplus \mathcal{R}_i\big) \xrightarrow{N_i} \gamma_i^{-1}\big(\mathcal{B}_i^\infty \oplus \mathcal{R}_i\big)$, where $N_i = \begin{pmatrix} \gamma_i^{-1}M_i & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. For any point p in $\Omega_i^{[n+1,\,n]} \times V_i$, we have an obvious bijection between the two sets $\{u$ in $\big(\mathcal{B}_{i\,|p}^\infty \oplus \mathcal{R}_{i\,|p}\big)^*$ such that u vanishes on $\big(\mathcal{A}_{i\,|p}^\infty \oplus \mathcal{R}_{i\,|p}\big)\}$ and $\{u$ in $\big(\mathcal{B}_{i\,|p}^\infty \oplus \mathcal{R}_{i\,|p}\big)^*$ such that $u \equiv 0$ on $\mathcal{R}_{i\,|p}$ and u vanishes $\mathcal{A}_{i\,|p}^\infty\}$. If we consider the embedding of $\mathbb{P}\big(\gamma_i^{-1}\mathcal{B}_i^\infty\big)$ in $\mathbb{P}\big(\mathcal{B}_{|O_i}\big)$ given by the splitting of $\mathcal{B}_{|O_i}$, this means that $Z(s) \cap \pi^{-1}(O_i)$ lies in $\mathbb{P}\big(\gamma_i^{-1}\mathcal{B}_i^\infty\big)$. Furthermore, it is easy to see that the embedding of $Z(s) \cap \pi^{-1}(O_i)$ in $\mathbb{P}\big(\gamma_i^{-1}\mathcal{B}_i^\infty\big)$ is independent of the splitting. If $\widetilde{\pi}\colon \mathbb{P}(\mathcal{B}_i) \longrightarrow \Omega_i^{[n]} \times \Omega_i$ is the projection of the projective bundle of $\mathcal{B}_i$, if $\widetilde{s}$ is the section of $\widetilde{\pi}^*\mathcal{A}_i^*(1)$ given by the morphism $\widetilde{\pi}^*\mathcal{A}_i \longrightarrow \widetilde{\pi}^*\mathcal{B}_i \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{B}_i)}(1)$ and if $\delta_i\colon \mathbb{P}\big(\gamma_i^{-1}\mathcal{B}_i^\infty\big) \xrightarrow{\sim} \mathbb{P}(\mathcal{B}_i^\infty)$ is the natural isomorphism induced by γ_i, then $Z(s) \cap \pi^{-1}(O_i) = \delta_i^{-1}\big(Z(\widetilde{s}) \times V_i\big)$. The zero locus $Z(\widetilde{s})$ being canonically isomorphic to $\Omega_i^{[n+1,\,n]}$, we get a commutative diagram

$$\begin{array}{ccc} \mathbb{P}\big(\gamma_i^{-1}\mathcal{B}_i^\infty\big) & \xrightarrow[\delta_i]{\sim} & \mathbb{P}(\mathcal{B}_i^\infty) \\ \uparrow & & \uparrow \\ U_{i,\,\mathrm{rel}}^{[n+1,\,n]} & \xrightarrow[\psi_i]{\sim} & Z(\widetilde{s}) \times V_i \end{array}$$

Thus $Z(s) \cap \pi^{-1}(O_i) \simeq U_{i,\,\mathrm{rel}}^{[n+1,\,n]}$.

(ii) Let p be in $\big(\Omega_i^{[n]} \times \Omega_i\big) \times V_i$. We choose a neighbourhood $U_p \times V_p$ of p such that $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{R}_i$ are trivial on U_p, U_p and $U_p \times V_p$, of respective ranks r, $r+1$ and m. Let q be a point in $\widetilde{\pi}^{-1}(p) \cap \big(Z(\widetilde{s}) \times V_i\big)$. We can find an affine hyperplane H_1 in $\mathbb{C}^{r+1}$ which does not contain zero and an hyperplane H_2 in $\mathbb{C}^m$ such that the open subset $(H_1 \times H_2 \times U_p) \times V_p$ of $\mathbb{P}(\mathcal{B}_i^\infty \oplus \mathcal{R}_i)$ contains q. The restriction of s to $H_1 \times \{0\} \times U_p$ is a holomorphic map from $H_1 \times U_p$ to $\big(\mathbb{C}^r\big)^*$ corresponding to the section $\widetilde{s}$, and $s\colon (H_1 \times H_2 \times U_p) \times V_p \longrightarrow \big(\mathbb{C}^r \oplus \mathbb{C}^m\big)^*$ can be expressed as $s(u_1, u_2, z; v)(\alpha, \beta) = \widetilde{s}(u_1, z)(\alpha) + u_2(\beta)$. After changing base from $X^{(n)} \times X$ to $X^n \times X$, the variable v lies in the preimage of V_i in $X^n \times X$, which is smooth. Since $\widetilde{s}$ is transverse to the zero section, the result ensues.

(iii) We write j on $W_{i,\,\mathrm{rel}}^{[n+1,\,n]}$ as the composed of the following morphisms:

$$W_{i,\,\mathrm{rel}}^{[n+1,\,n]} \xrightarrow[\psi_i]{\sim} \big(\Omega_i^{[n]} \times \Omega_i\big) \times V_i \hookrightarrow \mathbb{P}(\mathcal{B}_i) \times V_i \simeq \mathbb{P}(\mathcal{B}_i^\infty) \hookrightarrow \mathbb{P}\big(\mathcal{B}_{|O_i}\big)$$

Since the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{B}_i)}(1)$ to $\Omega_i^{[n]} \times \Omega_i$ is the ideal sheaf of the exceptional divisor in $\Omega_i^{[n]} \times \Omega_i$, the sheaves $j^*\mathcal{K}$ and $\mathcal{L}^\infty$ are isomorphic on $W_{i,\mathrm{rel}}^{[n+1,n]}$. It is then a routine verification to check that these isomorphisms patch together into a global isomorphism between $j^*\mathcal{K}$ and $\mathcal{L}^\infty$. □

We now finish the proof of Proposition 3.27. Let $r\colon \mathbb{P}(\mathcal{B}) \times_{X^{(n)}\times X} \big(X^n \times X\big) \longrightarrow \mathbb{P}(\mathcal{B})$ be the natural base change map. Point (ii) of Lemma 3.28 imply that $Z(r^*s)$ is Poincaré dual to the top Chern class of $r^*\big(\pi^*\mathcal{A}^* \otimes \mathcal{K}\big)$. Since the pull-back morphism r^* is injective in cohomology with rational coefficients, $Z(s)$ is Poincaré dual to the top Chern class of $\pi^*\mathcal{A}^* \otimes \mathcal{K}$ in $H_{8n+8}\big(\mathbb{P}(\mathcal{B}),\mathbb{Q}\big)$. Besides, if ε is the first Chern class of $\mathcal{K}$ in $H^2\big(\mathbb{P}(\mathcal{B}),\mathbb{Q}\big)$, point (iii) of Lemma 3.28 imply that $l = j^*\varepsilon_{|X^{[n+1,n]}}$. As $\overline{\sigma}$ is proper, we get for $1 \le i \le 2n+2$:

$$\begin{aligned}\overline{\sigma}_*(j^*\varepsilon^i) &= \pi_* j_*(j^*\varepsilon^i) = \pi_*\big([\mathfrak{X}]\,.\,\varepsilon^i\big) = \pi_*\big(c_r(\pi^*\mathcal{A}^* \otimes \mathcal{K})\,.\,\varepsilon^i\big) = \sum_{k=0}^{r} c_k(\mathcal{A}^*)\,\pi_*\,\varepsilon^{r+i-k}\\ &= \sum_{k=0}^{r} c_k(\mathcal{A}^*)\,s_{i-k}(\mathcal{B}^*) = c_i(\mathcal{A}^* - \mathcal{B}^*) = (-1)^i c_i\big(\widetilde{\mathcal{J}}_n^\infty\big) = (-1)^i c_i\big(\widetilde{\mathcal{O}}_n^\infty\big).\end{aligned}$$

Let u (resp. v) denote the embedding of $X^{[n+1,n]}$ (resp. $X^{[n]} \times X$) in $\mathfrak{X}$ (resp. in $\mathfrak{X}'$). Since $\overline{\sigma}_*\overline{\sigma}^* = \mathrm{id}$ and $\sigma_*\sigma^* = \mathrm{id}$, we get

$$v^*\overline{\sigma}_*(j^*\varepsilon^i) = \sigma_*\sigma^* v^*\overline{\sigma}_*(j^*\varepsilon^i) = \sigma_* u^*\overline{\sigma}^*\,\overline{\sigma}_*(j^*\varepsilon^i) = \sigma_* u^* j^*\varepsilon^i = \sigma_* l^i,$$

so that $\sigma_* l^i = (-1)^i v^* c_i\big(\widetilde{\mathcal{O}}_n^\infty\big) = (-1)^i\,\mu_{i,n}$. □

4 The boundary operator

4.1 Lehn's formula in the almost-complex case

Let (X,J) be an almost-complex compact four-manifold, and W', W be respective neighbourhoods of Z_n and $Z_{n\times 1}$ endowed with relative integrable complex structures J_n^{rel} and $J_{n\times 1}^{\mathrm{rel}}$. If $\mathcal{L}$ is the invertible sheaf defined in Notations 3.24, let F be the restriction of $\mathcal{L}^\infty$ to $X^{[n+1,n]}$. By the homotopy invariance of topological K–theory, the class of F in $K\big(X^{[n+1,n]}\big)$ is independent of $J_{n\times 1}^{\mathrm{rel}}$, and its first Chern class is l.

If $\{\phi_i\colon U_i \xrightarrow{\sim} Z_i \times V_i\}_{i\in I}$ is a maximal relative holomorphic atlas on W, the tautological sheaves $\mathrm{pr}_1^{-1}\,\mathcal{O}_{Z_i^{[n]}}^{[n]} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z_i^{[n]}}} \mathcal{O}_{Z_i^{[n]}\times V_i}^{\mathrm{rel}}$ patch together into a locally free sheaf $\mathbb{T}_{\mathrm{rel}}^{[n]}$ on $W_{\mathrm{rel}}^{[n]}$. The restriction $\mathbb{T}^{[n]}$ of $\mathbb{T}_{\mathrm{rel}}^{[n]}$ to $X^{[n]}$ satisfies the properties (see [23, Prop. 5.3, Lemma 5.4, Prop. 5.5]):

(i) The class of $\mathbb{T}^{[n]}$ in $K\big(X^{[n]}\big)$ is independent of J_n^{rel}.

(ii) The cohomology class $-2c_1\big(\mathbb{T}^{[n]}\big)$ in $H^2(X^{[n]},\mathbb{Q})$ is Poincaré dual to the fundamental cohomology class of $\partial X^{[n]}$, where

$$\partial X^{[n]} = \{(\xi;\underline{x}) \text{ in } X^{[n]} \text{ such that there exists } p \text{ in } \underline{x} \text{ with } \mathrm{length}_p(\xi) \geq 2\}$$

is the so-called *boundary of* $X^{[n]}$.

(iii) If λ and ν are the homotopy classes introduced in § 2.4 in the case $m = n+1$, then $\nu^*\mathbb{T}^{[n+1]} - \lambda^*\mathbb{T}^{[n]} = F$ in $K\big(X^{[n+1,n]}\big)$.

If m and n are positive integers with $m > n$, we define a class $I_{\mathbb{T}}^{[m,n]}$ in $K\big(X^{[m,n]}\big)$ by the formula $I_{\mathbb{T}}^{[m,n]} = \nu^*\mathbb{T}^{[m]} - \lambda^*\mathbb{T}^{[n]}$. Then (iii) implies that $I_{\mathbb{T}}^{[n+1,n]}$ and F are equal in $K\big(X^{[n+1,n]}\big)$.

Let us recall Lehn's definition of the boundary operator [26, Def. 3.8]:

Definition 4.1 Let $\mathbb{H} = \bigoplus_{n\geq 0} H^*\big(X^{[n]},\mathbb{Q}\big)$.

(i) The *boundary operator* $\mathfrak{d}\colon \mathbb{H} \longrightarrow \mathbb{H}$ is defined by

$$\mathfrak{d}\big[(\alpha_n)_{n\geq 0}\big] = \big(c_1\big(\mathbb{T}^{[n]}\big) \cup \alpha_n\big)_{n\geq 0}.$$

(ii) If A is an endomorphism of $\mathbb{H}$, *the derivative* A' of A is defined by the formula

$$A' = [\mathfrak{d}, A] = \mathfrak{d}\circ A - A\circ\mathfrak{d}.$$

We now state a partial extension of Lehn's main formula [26, Th. 3.10] for almost-complex four-manifolds.

Theorem 4.2 *Let (X,J) be an almost-complex compact four-manifold. Then there exist classes $(e_n)_{n\geq 0}$ in $H^2(X,\mathbb{Q})$ such that for all integers m, n and for all rational cohomology classes α and β on X,*

$$\big[\mathfrak{q}_n'(\alpha), \mathfrak{q}_m(\beta)\big] = -nm\,\mathfrak{q}_{n+m}(\alpha\beta) + \delta_{n+m,0}\Big(\int_X e_{|n|}\alpha\beta\Big)\,\mathrm{id}_{\mathbb{H}}\,.$$

Proof Exactly as in [26, Lemma 3.9], we start by proving that correspondences actions induced by homology classes on incidence varieties are stable under derivation. We denote by PD the Poincaré duality map between the homology groups and the cohomology groups of a compact topological manifold.

Lemma 4.3 *Let m, n be two positive integers with $m > n$, u be a rational homology class on $X^{[m,n]}$ and $u_*\colon H^*\big(X^{[n]},\mathbb{Q}\big) \longrightarrow H^*\big(X^{[m]},\mathbb{Q}\big)$ be the correspondence map given by the formula $u_*(\tau) = PD^{-1}\big[\nu_*(u\cap\lambda^*\tau)\big]$. Then $\big(u_*\big)' = \big[u\cap c_1\big(I_{\mathbb{T}}^{[m,n]}\big)\big]_*$.*

Proof For every rational cohomology class τ on $X^{[n]}$,

$$\begin{aligned}
\big(u_*\big)'\tau &= c_1\big(\mathbb{T}^{[m]}\big) \cup u_*\tau - u_*\big(c_1\big(\mathbb{T}^{[n]}\big) \cup \tau\big) \\
&= PD^{-1}\Big[\big(\nu_*(u \cap \lambda^*\tau)\big) \cap c_1\big(\mathbb{T}^{[m]}\big) - \nu_*\big(u \cap \lambda^*\big(c_1\big(\mathbb{T}^{[n]}\big) \cup \tau\big)\big)\Big] \\
&= PD^{-1}\Big[\nu_*\Big(u \cap \big[\big(\nu^* c_1\big(\mathbb{T}^{[m]}\big) - \lambda^* c_1\big(\mathbb{T}^{[n]}\big)\big) \cup \lambda^*\tau\big]\Big)\Big] \\
&= PD^{-1}\,\nu_*\Big(\big[u \cap c_1\big(I_{\mathbb{T}}^{[m,n]}\big)\big] \cap \lambda^*\tau\Big) \\
&= \big[u \cap c_1\big(I_{\mathbb{T}}^{[m,n]}\big)\big]_*(\tau).
\end{aligned}$$

□

By this Lemma, the proof of the theorem boils down to computing the commutator of two correspondences. Lehn's proof can be adapted exactly as we did in [23] for the Nakajima relations. This yields (see [22, § 4.3] for a detailed exposition):

– For all integers m and n such that $m+n$ is nonzero, there exists an excess multiplicity $\mu_{n,m}$ in $\mathbb{Z}$ such that for all rational cohomology classes α and β on X, the commutation relation $\big[\mathfrak{q}'_n(\alpha), \mathfrak{q}_m(\beta)\big] = \mu_{n,m}\,\mathfrak{q}_{n+m}(\alpha\beta)$ holds.

– For every nonnegative integer k, there exists an excess intersection class e_k in $H^2(X,\mathbb{Q})$ such that for every integer n and for all rational cohomology classes α and β on X, the identity $\big[\mathfrak{q}'_n(\alpha), \mathfrak{q}_{-n}(\beta)\big] = \big(\int_X e_{|n|}\,\alpha\beta\big)\,\mathrm{id}_{\mathbb{H}}$ holds.

The terms $\mu_{n,m}$ and e_k are the excess contributions. The multiplicity $\mu_{n,m}$ can be computed locally on X, so that Lehn's proof is valid and gives $\mu_{n,m} = -nm$. □

Unlike the multiplicities $\mu_{n,m}$, the excess classes e_k involve the *global* geometry of X. We compute these classes in § 4.3 under the additional assumption that X be symplectic.

Corollary 4.4 *When α runs through a basis of $H^*(X,\mathbb{Q})$, the operators $\mathfrak{d}$ and $\mathfrak{q}_1(\alpha)$ generate $\mathbb{H}$ from the vector 1.*

Proof The corollary is a straightforward consequence of the commutation relations $\big[q'_1(\alpha), q_m(1)\big] = -m\,q_{m+1}(\alpha),\ m > 0$. □

4.2 Holomorphic curves in symplectic four-manifolds

Until now, we have only considered integrable structures in small open sets of (X,J). To compute the excess classes e_n appearing in Theorem 4.2, we construct pseudo-holomorphic curves in X for perturbed almost-complex structures. To do so we use the following theorem of Donaldson, which is a symplectic analog of Kodaira's embedding theorem:

Theorem 4.5 [12, Th. 1] *Let (V,ω) be a symplectic manifold of dimension $2n$ such that ω is an integral class and let $\widetilde{\omega}$ be a lift of ω in $H^2(V,\mathbb{Z})$. For any sufficiently large positive integer k, the Poincaré dual of $k\widetilde{\omega}$ in $H_{2n-2}(V,\mathbb{Z})$ is the homology class of a closed symplectic submanifold of V. More precisely, if J is an almost-complex structure on V compatible with ω, there is a positive constant C such that for any large integer k, there exist an almost-complex structure J_k on X and a J_k–holomorphic submanifold S_k of codimension two in V such that $k\widetilde{\omega}$ is Poincaré dual to S_k and that $\|J_k - J\|_{C^0} \leq C/\sqrt{k}$.*

We apply this theorem to our situation:

Proposition 4.6 *Let (X,ω) be a symplectic compact four-manifold, J be an adapted almost-complex structure on X and N be the second Betti number of X. Then there exist almost-complex structures $\left(J_i\right)_{1\leq i\leq N}$ arbitrary close to J in C^0–norm and two-dimensional submanifolds $\left(C_i\right)_{1\leq i\leq N}$ such that:*

(i) For any i with $1 \leq i \leq N$, C_i is J_i–holomorphic and J_i is integrable in a neighbourhood of C_i.

(ii) The homology classes of $C_1,\dots,C_N$ span $H_2(X,\mathbb{Q})$ over $\mathbb{Q}$.

Proof Let $\alpha_1,\dots,\alpha_N$ be closed differential two-forms on X such that the $\omega+\alpha_i$'s are rational symplectic forms whose cohomology classes span $H^2(X,\mathbb{Q})$. Then there exist almost-complex structures $(\widetilde{J}_i)_{1\leq i\leq N}$ on X such that for every i, $\widetilde{J}_i$ is adapted to $\omega+\alpha_i$. Besides, if g is a Riemannian metric on X and if ε is a positive real number, we can assume by choosing the α_j's small enough that $\|\widetilde{J}_i - J\|_{C^0,g} < \varepsilon$ for $1 \leq i \leq N$. Let $m_1,\dots,m_N$ be positive integers such that $m_1(\omega+\alpha_1),\dots,m_N(\omega+\alpha_N)$ are integral classes. By Theorem 4.5, there exist a positive integer k, a family $(J_i')_{1\leq i\leq N}$ of almost-complex structures on X and a family $(C_i)_{1\leq i\leq N}$ of two-dimensional submanifolds of X such that for $1 \leq i \leq N$, C_i is J_i'–holomorphic, $\|J_i' - \widetilde{J}_i\|_{C^0,g} < \varepsilon$ and $km_i[\omega+\alpha_i]$ is Poincaré dual to $[C_i]$. Thus, for each integer i between 1 and N, J_i' defines an

almost-complex structure on C_i, which is integrable since C_i is two-dimensional. Furthermore, J_i' endows the normal bundle $N_{C_i/X}$ with the structure of a complex vector bundle over the Riemann surface (C_i, J_i'). By the Koszul-Malgrange integrability theorem [13, Th. 2.1.53], there exists a structure of holomorphic line bundle on $N_{C_i/X}$. Let U_i be a tubular neighbourhood of C_i in X, diffeomorphic to a neighbourhood of the zero section in $N_{C_i/X}$. Pulling back the integrable complex structure on $N_{C_i/X}$ by this diffeomorphism, we obtain an integrable complex structure J_i'' on U_i whose restriction to C_i is equal to J_i'. Since we are free to restrict U_i, we can assume that $\|J_i'' - J_i'\|_{C^0, g, U_i} < \varepsilon$, so that $\|J_i'' - J\|_{C^0, g, U_i} < 3\varepsilon$. If ε is small enough, this implies that there exist a relatively compact neighbourhood V_i of C_i in U_i as well as a smooth family $(J_t)_{0\leq t\leq 1}$ of almost-complex structures on V_i such that $J_0 = J_{|V_i}$, $J_1 = J_{i\,|V_i}''$ and for every t in $[0,1]$, $\|J - J_t\|_{C^0, g, V_i} < 3\varepsilon$. Let χ be a smooth real-valued function on X supported in V_i such that $\chi \equiv 1$ in a neighbourhood of C_i. We define an almost-complex structure J_i on X by $J_i(p) = J_{\chi(p)}(p)$. Then J_i is integrable on V_i, C_i is J_i–holomorphic and $\|J_i - J\|_{C^0} < 3\varepsilon$. □

4.3 Computation of the excess term in the symplectic case

Our aim in this section is to prove Lehn's formula in full generality for symplectic four-manifolds:

Theorem 4.7 *Let (X, ω) be a symplectic compact four-manifold and J be is an almost-complex structure compatible with ω. If n is a nonnegative integer, the excess contribution e_n of Theorem 4.2 is given by*

$$e_n = \frac{1}{2}\, n^2(n-1)c_1(X).$$

This means that for all integers n, m and for all rational cohomology classes α and β on X,

$$\big[\mathfrak{q}_n'(\alpha), \mathfrak{q}_m(\beta)\big] = -nm\Big\{\mathfrak{q}_{n+m}(\alpha\beta) - \frac{|n|-1}{2}\,\delta_{n+m,0}\Big(\int_X c_1(X)\alpha\beta\Big)\,\mathrm{id}_{\mathbb{H}}\Big\}.$$

In the integrable case, the statement of the theorem is [26, Prop. 3.15], with slightly different notations. We start by an outline of Lehn's original proof [26, § 3.4], then we show how to adapt it in the symplectic case. In the sequel, if Z is a triangulable cycle in a topological manifold Y, we denote by $[Z]$ the cohomological cycle class of Z.

If X is a smooth projective surface and if C is a smooth algebraic curve on X, a result of Grojnowski (see Lemma 4.8 below) describes explicitly the class $\big[C^{[n]}\big]$ in $H^{2n}\big(X^{[n]}, \mathbb{Q}\big)$

in terms of the classes $\mathfrak{q}_{i_1}([C])\dots\mathfrak{q}_{i_N}([C])\,.\,1$, where $i_1,\dots,i_N$ are positive integers of total sum n.

Let $X_0^{[n]}$ be the set of elements in $X^{[n]}$ whose support is a single point. If $\partial C^{[n]}$ denotes the intersection $C^{[n]}\cap\partial X^{[n]}$, the term $I=\int_{X^{[n]}}\big[X_0^{[n]}\big]\,.\,\big[\partial C^{[n]}\big]$ can be computed in two different ways:

(i) The integral I is equal to $\mathfrak{q}_{-n}(1)\left(\big[\partial C^{[n]}\big]\right)$. Since $C^{[n]}$ and $\partial X^{[n]}$ intersect generically transversally, $\big[\partial C^{[n]}\big]=\big[\partial X^{[n]}\big]\,.\,\big[C^{[n]}\big]=-2\,c_1(\mathbb{T}^n)\,.\,\big[C^{[n]}\big]=-2\,\mathfrak{d}\big(\big[C^{[n]}\big]\big)$. Thus, I is a linear combination of terms $\mathfrak{q}_{-n}(1)\,\mathfrak{q}_{i_1}([C])\dots\mathfrak{q}'_{i_k}([C])\dots\mathfrak{q}_{i_N}([C])\,.\,1$, where $i_1,\dots,i_N$ are positive integers of total sum n. These terms vanish except in two cases:

– $N=1$, $i_1=n$. Then $\mathfrak{q}_{-n}(1)\,\mathfrak{q}'_n([C])\,.\,1=-\int_X e_n\,.\,[C]$.

– $N=2$, $i_1+i_2=n$. Then $\mathfrak{q}_{-n}(1)\,\mathfrak{q}_k([C])\,\mathfrak{q}'_{n-k}([C])\,.\,1=0$ and

$$\mathfrak{q}_{-n}(1)\,\mathfrak{q}'_k([C])\,\mathfrak{q}_{n-k}([C])\,.\,1=-nk\,\mathfrak{q}_{k-n}([C])\,\mathfrak{q}_{n-k}([C])\,.\,1=nk(n-k)\,[C]^2.$$

This computation gives $I=\dfrac{1}{n}\int_X e_n\,.\,[C]+\dbinom{n}{2}\,[C]^2$.

(ii) The cycle $C^{[n]}$ intersects transversally $X_0^{[n]}$ in its smooth locus. Besides, the intersection $C^{[n]}\cap X_0^{[n]}$ is $C_0^{[n]}$, which is canonically isomorphic to C. Therefore $I=\int_{X^{[n]}}\big[X_0^{[n]}\big]\,.\,\big[C^{[n]}\big]\,.\,\big[\partial X^{[n]}\big]=\deg_C\big[\mathcal{O}_{X^{[n]}}\big(\partial X^{[n]}\big)\big]=\deg_C\big[\mathcal{O}_{C^{[n]}}\big(\partial C^{[n]}\big)\big]$, which is $-n(n-1)\,\deg_C K_X$ by direct computation.

The excess terms e_n lie in the Neron-Severi group of X so that it is enough to show that for every smooth algebraic curve C, $\int_X\Big[e_n-\dfrac{1}{2}\,n^2(n-1)\,c_1(X)\Big]\,.\,[C]=0$. This is proved by comparison of the two expressions obtained for I.

Proof of Theorem 4.7. If γ is a rational cohomology class on X of even degree, we define the vertex operators $\big(S_m(\gamma)\big)_{m\geq 0}$ acting on $\mathbb{H}$ by the formula

$$\sum_{m\geq 0}S_m(\gamma)\,t^m=\exp\left(\sum_{n>0}\frac{(-1)^{n-1}}{n}\,\mathfrak{q}_n(\gamma)\,t^n\right).$$

Since γ is of even degree, the operators $\big(\mathfrak{q}_i(\gamma)\big)_{i>0}$ commute in the usual sense, so that the definition of $S_m(\gamma)$ is unambiguous. The following lemma is due to Grojnowski [24] in the integrable case; we refer the reader to [36, § 9.3] for a detailed exposition.

Lemma 4.8 *Let $\widetilde{J}$ be an almost-complex structure on X in the deformation class of J, and let C be a $\widetilde{J}$–holomorphic curve on X. Assume that $\widetilde{J}$ is integrable in*

a neighbourhood of C. Then for any positive integer n, $\left[C^{[n]}\right] = S_n\big([C]\big).1$ in $H^{2n}\big(X^{[n]},\mathbb{Q}\big)$.

Proof Let V be a small neighbourhood of C in X such that $\widetilde{J}$ is integrable on V, U be a relatively compact neighbourhood of C in V, m and n be two positive integers such that $m > n$ and $J^{\mathrm{rel}}_{n\times(m-n)}$ be a relative integrable complex structure in a neighbourhood W of $Z_{n\times(m-n)}$. If $\widetilde{W} = V \times U^{(n)} \times U^{(m-n)}$, let us assume that $\widetilde{W} \subseteq W$, that $J^{\mathrm{rel}}_{n\times(m-n)|\widetilde{W}} = \widetilde{J}$ and that $J^{\mathrm{rel}}_{n\times(m-n)}$ is close to $\widetilde{J}$ in C^0–norm. Then $\widetilde{W}^{[m,n]}_{\mathrm{rel}} \cap \big(X^{[m,n]}, J^{\mathrm{rel}}_{n\times(m-n)}\big)$ is exactly the usual incidence variety $U^{[m,n]}$, where U is endowed with the integrable complex structure $\widetilde{J}$. We denote by $\widehat{q}_i(\alpha)$ the usual Nakajima operators on U, in order to distinguish them from the almost-complex Nakajima operators on X. Then we have a commutative diagram:

$$\begin{tikzcd}
H^*_c\big(U^{[n]},\mathbb{Q}\big) \arrow[r] \arrow[d, "\widehat{q}_{m-n}([C])"'] & H^*\big(X^{[n]},\mathbb{Q}\big) \arrow[d, "q_{m-n}([C])"] \\
H^*_c\big(U^{[m]},\mathbb{Q}\big) \arrow[r] & H^*\big(X^{[m]},\mathbb{Q}\big)
\end{tikzcd}$$

Besides $\widehat{q}_n([C]).1$ lies in $H^*_c\big(U^{[n]},\mathbb{Q}\big)$ and its image in $H^*\big(X^{[n]},\mathbb{Q}\big)$ is $q_n([C]).1$. Since the identity of the lemma holds in $H^{2n}_c\big(U^{[n]},\mathbb{Q}\big)$ for the classical Nakajima operators, we obtain the result. □

If $(C,\widetilde{J})$ satisfies the hypotheses of Lemma 4.8, Lehn's computations recalled above apply verbatim and give $\displaystyle\int_X \Big[e_n - \frac{1}{2}\,n^2(n-1)\,c_1(X)\Big].[C] = 0$. By Proposition 4.6, $H^2(X,\mathbb{Q})$ is spanned by cohomology classes of such holomorphic curves. Since the intersection form of X is nondegenerate, we get $e_n = \dfrac{1}{2}\,n^2(n-1)\,c_1(X)$. □

The derivative of the Nakajima operators can be explicitly expressed using the Virasoro operators $\mathfrak{L}_n(\alpha)$ defined in [26, § 3.1]:

Corollary 4.9 *If (X,ω) is a symplectic compact four-manifold and if J is a compatible almost-complex structure, then for every n in $\mathbb{Z}$,*

$$\mathfrak{q}'_n(\alpha) = n\,\mathfrak{L}_n(\alpha) - \frac{1}{2}\,n(|n|-1)\mathfrak{q}_n\big(c_1(X)\,\alpha\big).$$

For the proof, see [26, p. 180].

5 The ring structure of $H^*\big(X^{[n]},\mathbb{Q}\big)$

5.1 Geometric tautological Chern characters

Let (X,J) be an almost-complex compact four-manifold, n be a positive integer and λ, ν be the homotopy classes in $\big[X^{[n+1,\,n]}:X^{[n]}\big]$ and $\big[X^{[n+1,\,n]}:X^{[n+1]}\big]$ introduced in § 2.4.

If E is a complex vector bundle on X, it is possible to associate with E a sequence of tautological vector bundles $\big(E^{[n]}\big)_{n>0}$ on $X^{[n]}$. These tautological bundles are constructed in [23, § 5] using relative holomorphic structures on E, and their classes in complex K–theory are shown to be independent of these auxiliary structures [23, Prop. 5.3]. This construction yields tautological morphisms from $K(X)$ to $K\big(X^{[n]}\big)$.

If F is the class in $K\big(X^{[n+1,\,n]}\big)$ defined at the beginning of § 4.1, then the tautological bundles $E^{[n]}$ and $E^{[n+1]}$ are related through the identity $\nu^*E^{[n+1]}=\lambda^*E^{[n]}+\rho^*E\otimes F$ in $K\big(X^{[n+1,\,n]}\big)$, which is a K–theoretical analog of (2–4) (see [23, Prop. 5.5]). This gives in $H^{\mathrm{even}}\big(X^{[n+1,\,n]},\mathbb{Q}\big)$ the relation

$$\nu^*\big(\mathrm{ch}\big(E^{[n+1]}\big)\big)=\lambda^*\big(\mathrm{ch}\big(E^{[n]}\big)\big)+\rho^*\,\mathrm{ch}(E)\,.\,\mathrm{ch}(F).$$

Lemma 5.1 *For every class α in $H^{\mathrm{even}}(X,\mathbb{Q})$ and every positive integer n, there exists a unique class $G(\alpha,n)$ in $H^{\mathrm{even}}\big(X^{[n]},\mathbb{Q}\big)$ such that $G(\alpha,1)=\alpha$ and for every positive integer n,*

$$\nu^*G(\alpha,n+1)-\lambda^*G(\alpha,n)=\rho^*\alpha\,.\,\mathrm{ch}(F).$$

Proof The Chern character on X gives an isomorphism between $K(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $H^{\mathrm{even}}(X,\mathbb{Q})$. Therefore, we can define the classes $G(\alpha,n)$ in $H^{\mathrm{even}}\big(X^{[n]},\mathbb{Q}\big)$ as follows: if y is the unique class in $K(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ such that $\mathrm{ch}(y)=\alpha$, then $G(\alpha,n)=\mathrm{ch}\big(y^{[n]}\big)$. Furthermore, $G(\alpha,n)$ is unique since $\nu_*\nu^*=\frac{1}{n+1}\,\mathrm{id}$. □

5.2 Virtual tautological Chern characters

In this section, we extend Lemma 5.1 to odd cohomology classes. We adapt the method originally developed in the projective case by Li, Qin and Wang in [30, § 5].

Proposition 5.2 *For every class α in $H^*(X,\mathbb{Q})$ and every n in $\mathbb{N}^*$, there exists a unique class $G(\alpha,n)$ in $H^*\big(X^{[n]},\mathbb{Q}\big)$ such that $G(\alpha,1)=\alpha$ and for every positive integer n,*

$$\nu^*G(\alpha,n+1)-\lambda^*G(\alpha,n)=\rho^*\alpha\,.\,\mathrm{ch}(F).$$

Remark 5.3 If X is a projective surface, if Y_n is the incidence locus in $X^{[n]} \times X$ and if $\mathrm{td}(X)$ is the Todd class of X, then $G(\alpha, n) = \mathrm{pr}_1^* \big[\mathrm{ch}(\mathcal{O}_{Y_n}) \,.\, \mathrm{pr}_2^* \,\alpha \,.\, \mathrm{pr}_2^* \,\mathrm{td}(X)\big]$ (see [30, Lemma 5.8]).

Proof We adopt the notations of § 3.5, especially those of 3.24.

– The projection from W (resp. W', resp. W'') to X is denoted by t (resp. t', resp. t'').
– The projection from $W'^{[n]}_{\mathrm{rel}} \times_{X^{(n)}} W'$ (resp. $W''^{[n+1]}_{\mathrm{rel}} \times_{X^{(n+1)}} W''$, resp. $W^{[n]}_{\mathrm{rel}} \times_{X^{(n)} \times X} W$, resp. $W^{[n+1]}_{\mathrm{rel}} \times_{X^{(n)} \times X} W$, resp. $W^{[n+1,\,n]}_{\mathrm{rel}} \times_{X^{(n)} \times X} W$) to W' (resp. W'', resp. W, resp. W, resp. W) is denoted by π_1 (resp. π_2, resp. π_3, resp. π_4, resp. π_5).
– The first and seconde projections from $W \times_{X^{(n)} \times X} W$ to W are denoted by π_6 and π_7.

Thanks to Proposition 3.19, we can assume that $\mathcal{O}_n^\infty$, $\mathcal{O}_{n+1}^\infty$, $\widetilde{\mathcal{O}}_n^\infty$ and $\widetilde{\mathcal{O}}_{n+1}^\infty$ admit global smooth locally free resolutions. Let μ_n (resp. μ_{n+1}, resp. $\widetilde{\mu}_n$, resp. $\widetilde{\mu}_{n+1}$) be the Chern character of the relative incidence sheaf $\mathcal{O}_n$ (resp. $\mathcal{O}_{n+1}$, resp. $\widetilde{\mathcal{O}}_n$, resp. $\widetilde{\mathcal{O}}_{n+1}$) in $H^*_{\mathfrak{Y}_n}\big(W'^{[n]}_{\mathrm{rel}} \times_{X^{(n)}} W'\big)$ (resp. $H^*_{\mathfrak{Y}_{n+1}}\big(W''^{[n+1]}_{\mathrm{rel}} \times_{X^{(n+1)}} W''\big)$, resp. $H^*_{\widetilde{\mathfrak{Y}}_n}\big(W^{[n]}_{\mathrm{rel}} \times_{X^{(n)} \times X} W\big)$, resp. $H^*_{\widetilde{\mathfrak{Y}}_{n+1}}\big(W^{[n+1]}_{\mathrm{rel}} \times_{X^{(n)} \times X} W\big)$). Since $\overline{p}$ (resp. $\overline{q}$, resp. p, resp. q) is finite on $\mathfrak{Y}_n$ (resp. $\mathfrak{Y}_{n+1}$, resp. $\widetilde{\mathfrak{Y}}_n$, resp. $\widetilde{\mathfrak{Y}}_{n+1}$), we can define six cohomology classes as follows:

– $K(\alpha, n) = \overline{p}_* \big[\mu_n \,.\, \pi_1^* \, t'^{\,*}(\alpha \,.\, \mathrm{td}(X))\big]$ in $H^*\big(W'^{[n]}_{\mathrm{rel}}, \mathbb{Q}\big)$,
– $K(\alpha, n+1) = \overline{q}_* \big[\mu_{n+1} \,.\, \pi_2^* \, t''^{\,*}(\alpha \,.\, \mathrm{td}(X))\big]$ in $H^*\big(W''^{[n+1]}_{\mathrm{rel}}, \mathbb{Q}\big)$,
– $\widetilde{K}(\alpha, n) = p_* \big[\widetilde{\mu}_n \,.\, \pi_3^* \, t^*(\alpha \,.\, \mathrm{td}(X))\big]$ in $H^*\big(W^{[n]}_{\mathrm{rel}}, \mathbb{Q}\big)$,
– $\widetilde{K}(\alpha, n+1) = q_* \big[\widetilde{\mu}_{n+1} \,.\, \pi_4^* \, t^*(\alpha \,.\, \mathrm{td}(X))\big]$ in $H^*\big(W^{[n+1]}_{\mathrm{rel}}, \mathbb{Q}\big)$,
– $G(\alpha, n) = K(\alpha, n)_{|X^{[n]}}$ in $H^*\big(X^{[n]}, \mathbb{Q}\big)$,
– $G(\alpha, n+1) = K(\alpha, n+1)_{|X^{[n+1]}}$ in $H^*\big(X^{[n+1]}, \mathbb{Q}\big)$.

Then, by Proposition 3.10 (ii),

$$\begin{aligned}\psi^* \widetilde{K}(\alpha, n+1) - \phi^* \widetilde{K}(\alpha, n) &= \widetilde{p}_* \big[(\psi_W^* \, \widetilde{\mu}_{n+1} - \phi_W^* \, \widetilde{\mu}_n) \,.\, \pi_5^* \, t^* \, (\alpha \,.\, \mathrm{td}(X))\big] \\ &= \widetilde{p}_* \big[\widetilde{p}^* \, \mathrm{ch}(\mathcal{L}) \,.\, \overline{\rho}^{\,*}_W \, \mathrm{ch}\big(\big[\mathcal{O}_\Delta^\infty\big]\big) \,.\, \overline{\rho}^{\,*}_W \, \pi_7^* \, t^* \, (\alpha \,.\, \mathrm{td}(X))\big].\end{aligned}$$

Since $\mathrm{ch}\big(\big[\mathcal{O}_\Delta^\infty\big]\big)$ is supported in Δ_{rel}, for every β in $H^*\big(W \times_{X^{(n)} \times X} W, \mathbb{Q}\big)$, we obtain that $\mathrm{ch}\big(\big[\mathcal{O}_\Delta^\infty\big]\big) \,.\, \pi_6^* \beta = \mathrm{ch}\big(\big[\mathcal{O}_\Delta^\infty\big]\big) \,.\, \pi_7^* \beta$. Using the diagram

$$\begin{tikzcd} W^{[n+1,\,n]}_{\mathrm{rel}} \times_{X^{(n)} \times X} W \arrow[r, "\overline{\rho}^{\,*}_W"] \arrow[d, "\widetilde{p}"'] & W_{X^{(n)} \times X} W \arrow[d, "\pi_6"] \\ W^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "\overline{\rho}"'] & W \end{tikzcd}$$

we get

$$\begin{aligned}\psi^*\widetilde{K}(\alpha,n+1)-\phi^*\widetilde{K}(\alpha,n) &= \mathrm{ch}(\mathcal{L})\widetilde{p}_*\,\overline{\rho}^{\,*}_W\big[\mathrm{ch}\big(\big[\mathcal{O}^\infty_\Delta\big]\big)\,.\,\pi_6^*\,t^*\,(\alpha\,.\,\mathrm{td}(X))\big]\\ &= \mathrm{ch}(\mathcal{L})\,\overline{\rho}^{\,*}\big(\pi_{6*}\,\mathrm{ch}\big(\big[\mathcal{O}^\infty_\Delta\big]\big)\,.\,t^*(\alpha\,.\,\mathrm{td}(X))\big).\end{aligned}$$

Lemma 5.4 *(i) If $\widetilde{t}=t\times_{X^{(n)}\times X}t\colon W_{X^{(n)}\times X}W\longrightarrow X\times X$ and if $i\colon X\longrightarrow X\times X$ is the diagonal injection, then $\mathrm{ch}\big(\big[\mathcal{O}^\infty_\Delta\big]\big)=\widetilde{t}^*\,i_*\,\mathrm{td}(X)^{-1}$.*

(ii) $\lambda^*G(\alpha,n)=\phi^*\widetilde{K}(\alpha,n)_{|X^{[n+1,\,n]}}$.

(iii) $\nu^*G(\alpha,n+1)=\psi^*\widetilde{K}(\alpha,n+1)_{|X^{[n+1,\,n]}}$.

Proof (i) The class $\big[\mathcal{O}^\infty_\Delta\big]$ in $K_{\Delta_{\mathrm{rel}}}\big(W_{X^{(n)}\times X}W\big)$ is the pullback by the map $\widetilde{t}$ of the class $\big[\mathcal{C}^\infty_{\Delta_X}\big]$ in $K_{\Delta_X}(X\times X)$. Besides, the differentiable Grothendieck–Riemann–Roch theorem for immersions [4, Th. 3.3] yields $\mathrm{ch}\big(\big[\mathcal{C}^\infty_{\Delta_X}\big]\big)=i_*\,\mathrm{td}(X)^{-1}$.

(ii) If $\big(W,J^{\mathrm{rel}}_{n\times 1}\big)$ and $\big(W',J^{\mathrm{rel}}_n\big)$ satisfy the compatibility condition (A) of § 2.4, then we have $\lambda^*G(\alpha,n)=\phi^*\widetilde{K}(\alpha,n)_{|X^{[n+1,\,n]}}$. To conclude in the general case, we argue exactly as in Lemma 3.26.

(iii) The proof is the same as (ii), replacing condition (A) by condition (B). □

By (i), we get $\pi_{6*}\,\mathrm{ch}\big(\big[\mathcal{O}^\infty_\Delta\big]\big)=\pi_{6*}\,\widetilde{t}^*\,i_*\,\mathrm{td}(X)^{-1}=t^*\,\mathrm{pr}_{1*}\;i_*\,\mathrm{td}(X)^{-1}=t^*\,\mathrm{td}(X)^{-1}$, so that $\psi^*\widetilde{K}(\alpha,n+1)-\phi^*\widetilde{K}(\alpha,n)=\mathrm{ch}(\mathcal{L})\,\overline{\rho}^{\,*}\,t^*\alpha$. By the points (ii) and (iii), we obtain the relation $\lambda^*G(\alpha,n)-\nu^*G(\alpha,n+1)=\mathrm{ch}(F)\,.\,\rho^*\alpha$. This finishes the proof of Proposition 5.2. □

5.3 The ring structure and the crepant resolution conjecture

In this section, X is a symplectic compact four-manifold endowed with a compatible almost-complex structure.

We introduce operators acting on $\mathbb{H}=\bigoplus_{n\in\mathbb{N}}H^*\big(X^{[n]},\mathbb{Q}\big)$ by cup product with the components of the virtual tautological Chern characters constructed in § 5.2.

If α is a homogeneous rational cohomology class on X and if i, n are positive integers, we denote by $G_i(\alpha,n)$ the $(|\alpha|+2i)$–th component of $G(\alpha,n)$ and by $\mathfrak{S}_i(\alpha)$ the operator on $\mathbb{H}$ that acts by cup product with $G_i(\alpha,n)$ on $H^*(X^{[n]},\mathbb{Q})$.

We now state a result, originally proved by Lehn for geometric tautological Chern characters [26, Th. 4.2] and generalized by Li–Qin–Wang for virtual ones [30, Lemma 5.8]. We include a proof for the sake of completeness.

Proposition 5.5 *For all homogeneous rational cohomology classes α, β on X and for any positive integer k, $\big[\mathfrak{S}_k(\alpha), \mathfrak{q}_1(\beta)\big] = \frac{1}{k!}\,\mathfrak{q}_1^{(k)}(\alpha\beta)$.*

Proof Let α, β be homogeneous rational cohomology classes on X and n be a positive integer. The operator $\mathfrak{q}_1(\beta)\colon H^*\big(X^{[n]},\mathbb{Q}\big) \longrightarrow H^*\big(X^{[n+1]},\mathbb{Q}\big)$ is given by $\mathfrak{q}_1(\beta)\,.\,\tau = \nu_*(\lambda^*\tau\,.\,\rho^*\beta)$. Therefore

$$\begin{aligned}
&G(\alpha,n+1)\,.\,(\mathfrak{q}_1(\beta)\,.\,\tau) - \mathfrak{q}_1(\beta)\,.\,(G(\alpha,n)\,.\,\tau)\\
&\qquad = \nu_*\big(\lambda^*\tau\,.\,\nu^*G(\alpha,n+1)\,.\,\rho^*\beta\big) - \nu_*\big(\lambda^*(\tau\,.\,G(\alpha,n))\,.\,\rho^*\beta\big)\\
&\qquad = \nu_*\big(\mathrm{ch}(F)\,.\,\lambda^*\tau\,.\,\rho^*(\alpha\beta)\big) \qquad\qquad \text{by Proposition 5.2}\\
&\qquad = \sum_{k\geq 0}\frac{1}{k!}\,\nu_*\big(c_1(F)^k\,\lambda^*\tau\,.\,\rho^*(\alpha\beta)\big).
\end{aligned}$$

On the other hand, since $\mathfrak{q}_1(\alpha\beta)$ is given on $H^*\big(X^{[n]},\mathbb{Q}\big)$ by the action by correspondence of the Poincaré dual of $\rho^*(\alpha\beta)$, we have by Lemma 4.3

$$\begin{aligned}
\mathfrak{q}_1^{(k)}(\alpha\beta) &= \big[\big(PD\,(\rho^*(\alpha\beta))\big)_*\big]^{(k)} = \big[\big(PD\,(\rho^*(\alpha\beta))\cap c_1(F)\big)_*\big]^{(k-1)}\\
&= \cdots = \big[PD\,(\rho^*(\alpha\beta))\cap c_1(F)^k\big]_*
\end{aligned}$$

This yields the result. □

As explained in [26, Rem. 4.5], Proposition 5.5, Theorem 4.7 and Corollary 4.4 yield a complete description of the operators $\mathfrak{S}_k(\alpha)$.

The forthcoming theorems 5.6, 5.7 and 5.9 extend to Hilbert schemes of symplectic manifolds the analogous results for projective surfaces of [30, Th. 5.2], [31, Th. 4.1] and, for Theorem 5.9, of [28, Th. 1.1] and [38, Th. 5.13]. The first two results are formal consequences of the various relations between $\mathfrak{q}_n(\alpha)$, $\mathfrak{d}$, $\mathfrak{L}_n(\alpha)$ and $\mathfrak{S}_i(\alpha)$ listed in [31, Th. 2.1]. Thus, the two following theorems are formal consequences of Theorem 4.7, Corollary 4.9 and Proposition 5.5:

Theorem 5.6 *If $0 \leq i < n$ and if α runs through a fixed basis of $H^*(X,\mathbb{Q})$, the classes $G_i(\alpha,n)$ generate the ring $H^*\big(X^{[n]},\mathbb{Q}\big)$.*

Theorem 5.7 *For every integer n, the ring $H^*\big(X^{[n]},\mathbb{Q}\big)$ can be built by universal formulae from the ring $H^*(X,\mathbb{Q})$ and the first Chern class of X in $H^2(X,\mathbb{Q})$.*

In the case where $b_1(X)$ vanishes, Theorem 5.6 implies that the rings $H^*\big(X^{[n]},\mathbb{Q}\big)$ are generated by the components of the tautological Chern characters $\mathrm{ch}(E^{[n]})$, where E runs through all complex vector bundles on X.

We now turn to the study of a particular case of Ruan's crepant resolution conjecture. Orbifold cohomology provides a geometrical approach to the rings $H^*\big(X^{[n]},\mathbb{Q}\big)$. If J is an adapted almost-complex structure on X, the symmetric product $X^{(n)}$ is an almost-complex Gorenstein orbifold. The orbifold cohomology ring $H^*_{\mathrm{orb}}\big(X^{(n)},\mathbb{Q}\big)$ is $\mathbb{Z}$–graded and depends only on the deformation class of J (see [10], [1], [16]).

After works by Lehn–Sorger [27], [28], Li–Qin–Wang [29], [30], [31], Fantechi–Göttsche [16] and Uribe [43], Qin and Wang [38, § 2.6] developed a set of axioms that characterize $H^*_{\mathrm{orb}}\big(X^{(n)},\mathbb{Q}\big)$ as a ring. Here is their result, as stated in [1, Th. 5.24]:

Theorem 5.8 *Let A be a graded unitary ring, (X,J) be an almost-complex compact four-manifold and $\mathcal{H}\big(H^*(X,\mathbb{C})\big)$ be the Heisenberg super-algebra of $H^*(X,\mathbb{C})$. We assume that:*

(i) The ring A is an irreducible $\mathcal{H}\big(H^*(X,\mathbb{C})\big)$–module and 1 is a highest weight vector.

(ii) For any α in $H^*(X,\mathbb{C})$ and for any nonnegative integer i, there exist classes $O_i(\alpha,n)$ in $A^{|\alpha|+2i}$ such that if $\mathfrak{D}_i(\alpha)$ is the left multiplication by $\bigoplus_n O_i(\alpha,n)$ on A and if $\mathfrak{d}=\mathfrak{D}_1(1)$, then

– For all α, β in $H^*(X,\mathbb{C})$, for every nonnegative integer k,

$$\big[\mathfrak{D}_k(\alpha),\mathfrak{q}_1(\beta)\big]=\mathfrak{q}_1^{(k)}(\alpha\beta).$$

– If δ_X is the class in $H^*(X,\mathbb{C})^{\otimes 3}$ mapped by the Künneth isomorphism to the cycle class of the diagonal in X^3, then $\displaystyle\sum_{l_1+l_2+l_3=0} :\mathfrak{q}_{l_1}\mathfrak{q}_{l_2}\mathfrak{q}_{l_3}:(\delta_X)=-6\,\mathfrak{d}.$

Then the rings A and $H^*_{\mathrm{orb}}\big(X^{(n)},\mathbb{C}\big)$ are isomorphic.

(We use the physicists' normal ordering convention

$$:\mathfrak{q}_{l_1}\mathfrak{q}_{l_2}\mathfrak{q}_{l_3}:=\mathfrak{q}_{m_1}\mathfrak{q}_{m_2}\mathfrak{q}_{m_3},\ \text{where } \{l_1,l_2,l_3\}=\{m_1,m_2,m_3\} \text{ and } m_1\le m_2\le m_3).$$

We apply this theorem to prove Ruan's conjecture for the symmetric products of a symplectic four-manifold with torsion first Chern class.

Theorem 5.9 *Let (X,ω) be a symplectic compact four-manifold with vanishing first Chern class in $H^2(X,\mathbb{Q})$. Then, for every positive integer n, Ruan's crepant conjecture holds for $X^{(n)}$, ie the rings $H^*\big(X^{[n]},\mathbb{C}\big)$ and $H^*_{\mathrm{orb}}\big(X^{(n)},\mathbb{C}\big)$ are isomorphic.*

Proof Let $O_k(\alpha,n)=k!\,\mathfrak{S}_k(\alpha,n)$. The first condition of Theorem 5.8 (ii) is exactly Proposition 5.5. The second condition is a formal consequence of the Nakajima relations and of the formulae $\big[\mathfrak{q}'_n(\alpha),\mathfrak{q}_m(\beta)\big]=-nm\,\mathfrak{q}_{n+m}(\alpha\beta)$, $\mathfrak{q}'_n(\alpha)=n\,\mathfrak{L}_n(\alpha)$. □

6 The cobordism class of $X^{[n]}$

In this section, (X, J) is an almost-complex compact four-manifold, and no symplectic hypotheses are required. The almost-complex Hilbert schemes $X^{[n]}$ are endowed with a stable almost complex structure, hence define almost-complex cobordism classes. By classical results of Novikov [37] and Milnor [34], the almost-complex cobordism class of $X^{[n]}$ is completely determined by the Chern numbers $\int_{X^{[n]}} P\big[c_1(X^{[n]}), \dots, c_{2n}(X^{[n]})\big]$, where P runs through all polynomials P in $\mathbb{Q}\big[T_1, \dots, T_{2n}\big]$ of weighted degree $4n$, each variable T_k having degree $2k$. We intend to prove the following result:

Theorem 6.1 *The almost-complex cobordism class of $X^{[n]}$ depends only on the almost-complex cobordism class of X.*

This means that if P is a weighted polynomial in $\mathbb{Q}\big[T_1, \dots, T_{2n}\big]$ of degree $4n$, there exists a weighted polynomial $\widetilde{P}\big[T_1, T_2\big]$ of degree 4, depending only on P and n, such that

$$\int_{X^{[n]}} P\big[c_1(X^{[n]}), \dots, c_{2n}(X^{[n]})\big] = \int_X \widetilde{P}\big[c_1(X), c_2(X)\big].$$

This result has been proved by Ellinsgrud, Göttsche and Lehn [15, Th. 0.1] when X is projective. In § 6.1, 6.2 and 6.3, we adapt the authors' original proof in a relative setting. Throughout this section, we use extensively the notations of § 3.5, especially Notations 3.24.

6.1 Computation of $TX^{[n]}$ in K–theory

Let $J^{\mathrm{rel}}_{n\times 1}$ (resp. J^{rel}_n, resp. J^{rel}_{n+1}) be a relative integrable complex structure in a neighbourhood W (resp. W', resp. W'') of $Z_{n\times 1}$ (resp. Z_n, resp. Z_{n+1}). We denote the class of $T^{\mathrm{rel}}W^{[n]}_{\mathrm{rel}}$ (resp. $T^{\mathrm{rel}}W^{[n+1]}_{\mathrm{rel}}$, resp. $T^{\mathrm{rel}}W'^{[n]}_{\mathrm{rel}}$, resp. $T^{\mathrm{rel}}W''^{[n+1]}_{\mathrm{rel}}$) in $K^{\mathrm{rel}}\big(W^{[n]}_{\mathrm{rel}}\big)$ (resp. $K^{\mathrm{rel}}\big(W^{[n+1]}_{\mathrm{rel}}\big)$, resp $K^{\mathrm{rel}}\big(W'^{[n]}_{\mathrm{rel}}\big)$, resp. $K^{\mathrm{rel}}\big(W''^{[n+1]}_{\mathrm{rel}}\big)$) by $\widetilde{\kappa}_n$ (resp. $\widetilde{\kappa}_{n+1}$, resp. κ_n, resp. κ_{n+1}).

Lemma 6.2 *The restriction to $X^{[n]}$ (resp. $X^{[n+1]}$) of the topological class of κ_n (resp. κ_{n+1}) is the class of the complex vector bundle $TX^{[n]}$ (resp. $TX^{[n+1]}$) in $K(X^{[n]})$ (resp. $K(X^{[n+1]})$) given by the stable almost-complex structure on $X^{[n]}$ (resp. $X^{[n+1]}$).*

Proof If J_n^{rel} satisfies the conditions ($\mathcal{C}$) listed in [44, p. 711], then $X^{[n]}_{J_n^{\mathrm{rel}}}$ is smooth. Besides, the construction of the stable almost-complex structure of $X^{[n]}_{J_n^{\mathrm{rel}}}$ performed in [44] shows that $TX^{[n]}$ and $T^{\mathrm{rel}}W^{[n]}_{\mathrm{rel}|X^{[n]}}$ have the same class in $K(X^{[n]})$. Since relative almost-complex structures satisfying the conditions ($\mathcal{C}$) can be chosen arbitrary close to J in C^0–norm, Proposition 2.13 implies, after shrinking W if necessary, that J_n^{rel} can be joined by a smooth path $\{J^{\mathrm{rel}}_{n,t}\}_{t\in[0,1]}$ to another relative integrable structure satisfying the conditions ($\mathcal{C}$). By rigidity of the topological K–theory, the class of $T^{\mathrm{rel}}\big[W^{[n]}_{\mathrm{rel}}, J^{\mathrm{rel}}_{n,t}\big]_{|X^{[n]}}$ in $K\big(X^{[n]}\big)$ is independent of t. The result follows. □

Remark 6.3 For an arbitrary J_n^{rel}, by Theorem 2.18 (ii), $X^{[n]}_{J_n^{\mathrm{rel}}}$ is only a topological manifold. Therefore, the advantage of using $T^{\mathrm{rel}}W^{[n]}_{\mathrm{rel}}$ is that this complex vector bundle is defined for *any* relative integrable complex structure J_n^{rel}.

Proposition 6.4 *The following identities hold in $K^{\mathrm{rel}}\big(W^{[n]}_{\mathrm{rel}}\big)$ and $K^{\mathrm{rel}}\big(W^{[n+1]}_{\mathrm{rel}}\big)$ respectively:*

$$\widetilde{\kappa}_n = p_*\big(\widetilde{\mathcal{O}}_n + \widetilde{\mathcal{O}}_n^\vee - \widetilde{\mathcal{O}}_n \,.\, \widetilde{\mathcal{O}}_n^\vee\big), \qquad \widetilde{\kappa}_{n+1} = q_*\big(\widetilde{\mathcal{O}}_{n+1} + \widetilde{\mathcal{O}}_{n+1}^\vee - \widetilde{\mathcal{O}}_{n+1} \,.\, \widetilde{\mathcal{O}}_{n+1}^\vee\big).$$

Proof Let $\phi\colon U \xrightarrow{\sim} Z\times V$ be a relative holomorphic chart on W', $\mathfrak{p}$ be the first projection from $Z^{[n]}\times Z$ to $Z^{[n]}$ and Y_n be the incidence locus in $Z^{[n]}\times Z$ defined by (2–1). Since p is finite on $\widetilde{\mathfrak{Y}}_n$, we obtain by Proposition 3.13:

$$\begin{aligned} p_*\mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big)_{|U} &\simeq \phi^{-1}\big[\mathrm{pr}_1^{-1}\big[\mathfrak{p}_*\,\mathcal{H}om_{\mathcal{O}_{Z^{[n]}\times Z}}\big(\mathcal{J}_{Y_n}, \mathcal{O}_{Y_n}\big)\big]\otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z^{[n]}\times Z}} \mathcal{O}^{\mathrm{rel}}_{Z^{[n]}\times Z\times V}\big] \\ &\simeq \phi^{-1}\big[\mathrm{pr}_1^{-1}\,TZ^{[n]} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z^{[n]}\times Z}} \mathcal{O}^{\mathrm{rel}}_{Z^{[n]}\times Z\times V}\big] \simeq \big(T^{\mathrm{rel}}\,W^{[n]}_{\mathrm{rel}}\big)_{|U}. \end{aligned}$$

These local isomorphisms patch together into a global isomorphism between the two relative holomorphic bundles $p_*\,\mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big)$ and $TW^{[n]}_{\mathrm{rel}}$. If i is a nonnegative integer, Proposition 3.6 yields the isomorphism

$$\mathcal{E}xt^i_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{O}}_n, \mathcal{O}^{\mathrm{rel}}\big)_{|U} \simeq \phi^{-1}\big(\mathrm{pr}_1^{-1}\,\mathcal{E}xt^i_{\mathcal{O}_{Z^{[n]}\times Z}}(\mathcal{O}_{Y_n}, \mathcal{O}_{Z^{[n]}\times Z}) \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_{Z^{[n]}\times Z}} \mathcal{O}^{\mathrm{rel}}_{Z^{[n]}\times Z\times V}\big).$$

Since Y_n has codimension 2 in $Z^{[n]}\times Z$, $\mathcal{E}xt^i_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{O}}_n, \mathcal{O}^{\mathrm{rel}}\big) = 0$ for $i<2$ by [14, Prop. 18.4 and Th. 18.7]. Besides, $\widetilde{\mathcal{J}}_n$ locally admits a free resolution of length 2.

Using Proposition 3.16 (i), we get the following equalities in $K^{\mathrm{rel}}_{\widetilde{\mathfrak{Y}}_n}\big(W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W\big)$:

$$\begin{aligned}
\widetilde{\mathcal{J}}_n^{\vee} \,.\, \widetilde{\mathcal{O}}_n &= \mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) - \mathcal{E}xt^1_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) + \mathcal{E}xt^2_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) \\
&= \mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) - \mathcal{E}xt^2_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{O}}_n, \widetilde{\mathcal{O}}_n\big) \\
&= \mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) - \mathcal{E}xt^2_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{O}}_n, \mathcal{O}^{\mathrm{rel}}\big) \\
&= \mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) - \widetilde{\mathcal{O}}_n^{\vee}
\end{aligned}$$

so that $p_*\mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}}\big(\widetilde{\mathcal{J}}_n, \widetilde{\mathcal{O}}_n\big) = p_*\big[\big(\mathcal{O}^{\mathrm{rel}} - \widetilde{\mathcal{O}}_n^{\vee}\big) \,.\, \widetilde{\mathcal{O}}_n + \widetilde{\mathcal{O}}_n^{\vee}\big]$. The proof of the second identity is exactly the same. □

6.2 Comparison of $TX^{[n]}$ and $TX^{[n+1]}$ via the incidence variety $X^{[n+1,\,n]}$

We use the notations of the previous section. The relative canonical bundle of W is denoted by K^{rel}_W.

Proposition 6.5 *The following identity holds in* $K^{\mathrm{rel}}\big(W^{[n+1,\,n]}_{\mathrm{rel}}\big)$:

$$\begin{aligned}
\psi^{\,!}\widetilde{\kappa}_{n+1} = \phi^{\,!}\widetilde{\kappa}_n + \mathcal{L} + \mathcal{L}^{\vee} \,.\, \overline{\rho}^{\,!} K_W^{\mathrm{rel}\,\vee} - \overline{\rho}^{\,!}\big(\mathcal{O}_W^{\mathrm{rel}} - T^{\mathrm{rel}}W + K_W^{\mathrm{rel}\,\vee}\big) \\
- \mathcal{L} \,.\, \overline{\sigma}^{\,!}\widetilde{\mathcal{O}}_n^{\vee} - \mathcal{L}^{\vee} \,.\, \overline{\rho}^{\,!} K_W^{\mathrm{rel}\,\vee} \,.\, \overline{\sigma}^{\,!}\widetilde{\mathcal{O}}_n.
\end{aligned}$$

Proof By Proposition 6.4, $\psi^{\,!}\widetilde{\kappa}_{n+1} = \psi^{\,!}\widetilde{p}_*\big(\widetilde{\mathcal{O}}_{n+1} + \widetilde{\mathcal{O}}_{n+1}^{\vee} - \widetilde{\mathcal{O}}_{n+1} \,.\, \widetilde{\mathcal{O}}_{n+1}^{\vee}\big)$. Let us consider the cartesian diagrams

$$\begin{tikzcd}
W^{[n+1,\,n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W \arrow[r, "\psi_W"] \arrow[d, "\widetilde{p}"'] & W^{[n+1]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W \arrow[d, "q"] \\
W^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "\psi"] & W^{[n+1]}_{\mathrm{rel}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
W^{[n+1,\,n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W \arrow[r, "\phi_W"] \arrow[d, "\widetilde{p}"'] & W^{[n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W \arrow[d, "p"] \\
W^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "\phi"] & W^{[n]}_{\mathrm{rel}}
\end{tikzcd}$$

Since p (resp. q) is finite on $\widetilde{\mathfrak{Y}}_n$ (resp. $\widetilde{\mathfrak{Y}}_{n+1}$), Proposition 6.4 and Proposition 3.16 (iv) yield

$$\psi^!\widetilde{\kappa}_{n+1} = \widetilde{p}_*\psi_W^!\big(\widetilde{\mathcal{O}}_{n+1} + \widetilde{\mathcal{O}}_{n+1}^\vee - \widetilde{\mathcal{O}}_{n+1}\,.\,\widetilde{\mathcal{O}}_{n+1}^\vee\big) \qquad \text{and}$$
$$\phi^!\widetilde{\kappa}_n = \widetilde{p}_*\phi_W^!\big(\widetilde{\mathcal{O}}_n + \widetilde{\mathcal{O}}_n^\vee - \widetilde{\mathcal{O}}_n\,.\,\widetilde{\mathcal{O}}_n^\vee\big).$$

Thus, we obtain by Proposition 3.25 and Proposition 3.16 (ii):

$$\psi^!\widetilde{\kappa}_{n+1} = \phi^!\widetilde{\kappa}_n + \widetilde{p}_*\Big[\widetilde{p}^!\mathcal{L}\,.\,\overline{\rho}^{\,!}_W\mathcal{O}_\Delta + \widetilde{p}^!\mathcal{L}^\vee.\,\overline{\rho}^{\,!}_W\mathcal{O}_\Delta^\vee - \widetilde{p}^!\big(\mathcal{L}\,.\,\mathcal{L}^\vee\big)\,.\,\overline{\rho}^{\,!}_W\big(\mathcal{O}_\Delta.\,\mathcal{O}_\Delta^\vee\big)$$
$$- \widetilde{p}^!\mathcal{L}\,.\,\overline{\rho}^{\,!}_W\mathcal{O}_\Delta\,.\,\phi_W^!\,\widetilde{\mathcal{O}}_n^\vee - \widetilde{p}^!\mathcal{L}^\vee.\,\overline{\rho}^{\,!}_W\,\mathcal{O}_\Delta^\vee\,.\,\phi_W^!\,\widetilde{\mathcal{O}}_n\Big].$$

Note that $\mathcal{L}\,.\,\mathcal{L}^\vee = \mathcal{O}^{\mathrm{rel}}_{W^{[n+1,\,n]}_{\mathrm{rel}}}$. Let $\pi\colon W\times_{X^{(n)}\times X}W \longrightarrow W$ be the first projection. Since π is injective on Δ_{rel}, the diagram

$$\begin{tikzcd}
W^{[n+1,\,n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W \arrow[r, "\overline{\rho}_W"] \arrow[d, "\widetilde{p}"'] & W\times_{X^{(n)}\times X} W \arrow[d, "\pi"] \\
W^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "\overline{\rho}"] & W
\end{tikzcd}$$

and Propositions 3.16 (iv) and 3.25 (ii) give

$$\psi^!\widetilde{\kappa}_{n+1} = \phi^!\widetilde{\kappa}_n + \mathcal{L}\,.\,\overline{\rho}^{\,!}\,\pi_*\mathcal{O}_\Delta + \mathcal{L}^\vee.\,\overline{\rho}^{\,!}\,\pi_*\mathcal{O}_\Delta^\vee - \overline{\rho}^{\,!}\,\pi_*\big(\mathcal{O}_\Delta\,.\,\mathcal{O}_\Delta^\vee\big)$$
$$- \widetilde{p}_*\big(j_*\mathcal{L}\,.\,\phi_W^!\,\widetilde{\mathcal{O}}_n^\vee\big) - \mathcal{L}^\vee.\,\widetilde{p}_*\Big[\overline{\rho}^{\,!}_W\,\mathcal{O}_\Delta^\vee\,.\,\phi_W^!\,\widetilde{\mathcal{O}}_n\Big].$$

Now $\pi_*\mathcal{O}_\Delta = \mathcal{O}_W^{\mathrm{rel}}$, $\pi_*\mathcal{O}_\Delta^\vee = K_W^{\mathrm{rel}\,\vee}$, $\mathcal{O}_\Delta\,.\,\mathcal{O}_\Delta^\vee = \mathcal{O}_W^{\mathrm{rel}} - T_W^{\mathrm{rel}} + K_W^{\mathrm{rel}\,\vee}$ and if $\delta\colon W\longrightarrow W\times_{X^{(n)}\times X}W$ is the diagonal injection, $\overline{\rho}^{\,!}_W\,\mathcal{O}_\Delta^\vee = \overline{\rho}^{\,!}_W\,\delta_*\,K_W^{\mathrm{rel}\,\vee} = j_*\overline{\rho}^{\,!}\,K_W^{\mathrm{rel}\,\vee}$ thanks to Proposition 3.16 (v) and to the diagram

$$\begin{tikzcd}
W^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "j"] \arrow[d, "\overline{\rho}"] & W^{[n+1,\,n]}_{\mathrm{rel}} \times_{X^{(n)}\times X} W \arrow[d, "\overline{\rho}_W"] \\
W \arrow[r, "\delta"] & W\times_{X^{(n)}\times X} W
\end{tikzcd}$$

By Proposition 3.16 (ii) and (iii), we get:

$$\psi^!\widetilde{\kappa}_{n+1} = \phi^!\widetilde{\kappa}_n + \mathcal{L} + \mathcal{L}^\vee\,.\,\overline{\rho}^{\,!}\,K_W^{\mathrm{rel}\,\vee} - \overline{\rho}^{\,!}\big(\mathcal{O}_W^{\mathrm{rel}} - T_W^{\mathrm{rel}} + K_W^{\mathrm{rel}\,\vee}\big)$$
$$- (\widetilde{p}\circ j)_*\,\big[\mathcal{L}\,.\,\big(\phi_W\circ j\big)^!\,\widetilde{\mathcal{O}}_n^\vee\big] - \mathcal{L}^\vee.\,(\widetilde{p}\circ j)_*\,\big(\phi_W\circ j\big)^!\,\widetilde{\mathcal{O}}_n.$$

Since $\widetilde{p}\circ j = \mathrm{id}$ and $\phi_W\circ j = \overline{\sigma}$, we obtain the result. $\square$

6.3 Cohomological computations

Lemma 6.6 *If i is a positive integer, the following identities hold in $H^{2i}\big(X^{[n+1,\,n]},\mathbb{Q}\big)$:*

(i) $c_i\big(\phi^{\,!}\widetilde{\kappa}_n\big)_{|\,X^{[n+1,\,n]}} = \lambda^* c_i\big(X^{[n]}\big)$ *and* $c_i\big(\psi^{\,!}\widetilde{\kappa}_{n+1}\big)_{|\,X^{[n+1,\,n]}} = \nu^* c_i\big(X^{[n+1]}\big)$.

(ii) $c_i\big[\big(\overline{\sigma}^{\,!}\,\widetilde{\mathcal{O}}_n\big)^{\infty}\big]_{|\,X^{[n+1,\,n]}} = \sigma^*\mu_{i,\,n}$.

(iii) $c_i\big(\overline{\rho}^{\,*}\,T^{\mathrm{rel}}W\big)_{|\,X^{[n+1,\,n]}} = \rho^* c_i(X)$.

Proof (i) By the homotopy invariance of topological K–theory, the cohomology class $c_i\big(\phi^{\,!}\widetilde{\kappa}_n\big)_{|\,X^{[n+1,\,n]}}$ in $H^{2i}\big(X^{[n+1,\,n]},\mathbb{Q}\big)$ is independent of $J^{\mathrm{rel}}_{n\times 1}$. Thus, we can assume that W and W' satisfy the compatibility condition (A) of § 2.4. It follows from Lemma 6.2 that $c_i\big(\phi^{\,!}\widetilde{\kappa}_n\big)_{|\,X^{[n+1,\,n]}} = \lambda^* c_i\big(\kappa_n\big)_{|\,X^{[n]}} = \lambda^* c_i\big(X^{[n]}\big)$. Using the compatibility condition (B) instead of (A) we obtain the second identity.

(ii) By Proposition 3.20, $\big[\big(\overline{\sigma}^{\,!}\widetilde{\mathcal{O}}_n\big)^{\infty}\big] = \overline{\sigma}^*\big[\widetilde{\mathcal{O}}_n^{\infty}\big]$. This gives the result.

(iii) Let $\overline{W} = W \times_{X^{(n)}} X$, where the base change morphism is given by the diagonal injection of X in $X^{(n)}$. We consider the diagram:

$$\begin{tikzcd}
W^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "\overline{\rho}"] & W & \overline{W} \arrow[l, hook'] \\
X^{[n+1,\,n]} \arrow[u, hook] \arrow[r, "\rho"'] & X \arrow[u, hook] \arrow[r, equal] & X \arrow[u, hook]
\end{tikzcd}$$

Then, $c_i\big(\overline{\rho}^{\,*}\,T^{\mathrm{rel}}W\big)_{|\,X^{[n+1,\,n]}} = \rho^* c_i\big(T^{\mathrm{rel}}\,\overline{W}\big)_{|X}$. Since $\overline{W}$ is a neighbourhood of Δ_X in $X\times X$, $T^{\mathrm{rel}}\,\overline{W}_{|X} \simeq TX$, so that $c_i\big(T^{\mathrm{rel}}\overline{W}\big)_{|X} = c_i(X)$. □

For any nonnegative integer k, let d_k be the k–th Chern class of $\big[\mathcal{C}^{\infty}_{\Delta_X}\big]$ in $H^{2k}(X\times X,\mathbb{Q})$. If $i\colon X \longrightarrow X\times X$ is the diagonal injection, the differentiable Grothendieck–Riemann–Roch theorem for immersions of [4, Th. 3.3] gives $d_0 = 1$, $d_1 = 0$, $d_2 = -i_*(1)$, $d_3 = -i_*\,[c_1(X)]$ and $d_k = 0$ for $k \geq 4$. Thus $i^*d_0 = 1$, $i^*d_1 = 0$, $i^*d_2 = -c_2(X)$, $i^*d_3 = -c_1(X)\,c_2(X)$ and $i^*d_k = 0$ for $k \geq 4$.

Proposition 6.7 *Let i be a positive integer. Then*

(i) $(\nu,\mathrm{id})^*\mu_{i,\,n+1} - (\lambda,\mathrm{id})^*\mu_{i,\,n} = \displaystyle\sum_{k=0}^{i} \mathrm{pr}_1^*\; l^k\,.\,(\rho,\mathrm{id})^*\,d_{i-k}$.

(ii) $\nu^* c_i\big(X^{[n+1]}\big) - \lambda^* c_i\big(X^{[n]}\big)$ *is a universal polynomial in the classes l, $\rho^* c_j(X)$ and $\sigma^*\mu_{j,\,n}$.*

Proof (i) Let $Z_{n\times 1\times 1}$ be the incidence locus in $X\times\big(X^{(n)}\times X\times X\big)$ defined by

$$Z_{n\times 1\times 1}=\{(p;\,\underline{x},\,q,\,r)\ \text{ in }\ X\times\big(X^{(n)}\times X\times X\big)\ \text{ such that }\ p\in\underline{x}\cup q\cup r\}$$

and let $J^{\mathrm{rel}}_{n\times 1\times 1}$ be a relative integrable complex structure in a neighbourhood $\widehat{W}$ of $Z_{n\times 1\times 1}$. We can assume that $\big(W,J^{\mathrm{rel}}_{n\times 1}\big)$ and $\big(\widehat{W},J^{\mathrm{rel}}_{n\times 1\times 1}\big)$ satisfy as relative analytic spaces the compatibility condition $W\times_{X^{(n)}\times X}\big(X^{(n)}\times X\times X\big)\subseteq\widehat{W}$, the base change map being given by the diagonal injection of X into $X\times X$.

Exactly as in § 3.5, we can construct relative incidence sheaves $\widehat{\mathcal{O}}_n$ and $\widehat{\mathcal{O}}_{n+1}$ on $\widehat{W}^{[n]}_{\mathrm{rel}}$ and $\widehat{W}^{[n+1]}_{\mathrm{rel}}$ as well as a relative exceptional divisor $\widehat{D}_{\mathrm{rel}}$ in $\widehat{W}^{[n+1,\,n]}_{\mathrm{rel}}$. Let $\widehat{\phi}_{\widehat{W}}$, $\widehat{\psi}_{\widehat{W}}$, $\widehat{\rho}_{\widehat{W}}$, $\widehat{j}$, $\widehat{p}$, $\widehat{\mathcal{L}}$, $\widehat{\Delta}_{\mathrm{rel}}$ and $\mathcal{O}_{\widehat{\Delta}}$ be the analogs of ϕ_W, ψ_W, $\overline{\rho}_W$, j, $\widetilde{p}$, $\mathcal{L}$, Δ_{rel} and $\mathcal{O}_{\Delta}$. In this context, Proposition 3.25 takes the following form:

– There is a natural exact sequence on $\widehat{W}^{[n+1,\,n]}_{\mathrm{rel}}\times_{X^{(n)}\times X\times X}\widehat{W}$ relating $\widehat{\mathcal{O}}_n$ and $\widehat{\mathcal{O}}_{n+1}$, namely:

$$0\longrightarrow\widehat{j}_*\widehat{\mathcal{L}}\longrightarrow\widehat{\psi}^*_{\widehat{W}}\,\widehat{\mathcal{O}}_{n+1}\longrightarrow\widehat{\phi}^*_{\widehat{W}}\,\widehat{\mathcal{O}}_n\longrightarrow 0.$$

– $\widehat{j}_*\widehat{\mathcal{L}}=\widehat{p}^{\,*}\widehat{\mathcal{L}}\otimes\widehat{\rho}^*_{\widehat{W}}\mathcal{O}_{\widehat{\Delta}}$.

– $\widehat{\rho}^*_{\widehat{W}}\mathcal{O}_{\widehat{\Delta}}=\widehat{\rho}^{\,!}_{\widehat{W}}\mathcal{O}_{\widehat{\Delta}}$, $\quad\widehat{\psi}^*_{\widehat{W}}\,\widehat{\mathcal{O}}_{n+1}=\widehat{\psi}^{\,!}_{\widehat{W}}\,\widehat{\mathcal{O}}_{n+1}$ and $\widehat{\phi}^*_{\widehat{W}}\,\widehat{\mathcal{O}}_n=\widehat{\phi}^{\,!}_{\widehat{W}}\,\widehat{\mathcal{O}}_n$.

Arguing as in Lemma 6.3 and using Proposition 3.20, we get $c_1\big(\widehat{p}^{\,*}\widehat{\mathcal{L}}\big)_{|X^{[n+1,\,n]}\times X}=\mathrm{pr}_1^*\,l$ and for any nonnegative integer k,

– $c_k\big[\big(\widehat{\rho}^*_{\widehat{W}}\mathcal{O}_{\widehat{\Delta}}\big)^{\infty}\big]_{|X^{[n+1,\,n]}\times X}=(\rho,\mathrm{id})^*d_k$,

– $c_k\big[\big(\widehat{\psi}^*_{\widehat{W}}\,\widehat{\mathcal{O}}_{n+1}\big)^{\infty}\big]_{|X^{[n+1,\,n]}\times X}=(\nu,\mathrm{id})^*\mu_{k,\,n+1}$,

– $c_k\big[\big(\widehat{\phi}^*_{\widehat{W}}\,\widehat{\mathcal{O}}_n\big)^{\infty}\big]_{|X^{[n+1,\,n]}\times X}=(\lambda,\mathrm{id})^*\mu_{k,\,n}$.

This yields the result.

(ii) This is a straightforward consequence of Proposition 6.5 and Lemma 6.6. □

We are now going to perform in our context the induction step of [15]. For any subset I of $\{0,\dots,m\}$, we denote by pr_I the projection from $X^{[n+1]}\times X^m$ to the product of the factors indexed by I.

Proposition 6.8 *If m is a positive integer, let P be a polynomial in the cohomology classes $\mathrm{pr}_0^*\,c_i\big(X^{[n+1]}\big)$, $\mathrm{pr}_{0k}^*\,\mu_{i,\,n+1}$, $\mathrm{pr}_{kl}^*\,d_i$, $\mathrm{pr}_k^*\,c_i(X)$ $(1\le k,l\le m)$ on $X^{[n+1]}\times X^m$. Then there exists a polynomial $\widetilde{P}$ depending only on P, in the analogous classes on $X^{[n]}\times X^{m+1}$, such that $\displaystyle\int_{X^{[n+1]}\times X^m}P=\int_{X^{[n]}\times X^{m+1}}\widetilde{P}$.*

Proof We consider the incidence diagram

$$\begin{tikzcd} & X^{[n+1,\,n]} \times X^m \arrow[dl, "(\nu,\,\mathrm{id})"'] \arrow[dr, "(\sigma,\,\mathrm{id})"] & \\ X^{[n+1]} \times X^m & & \big(X^{[n]} \times X\big) \times X^m \end{tikzcd}$$

Since (ν, id) and (σ, id) are generically finite of degrees $n+1$ and 1 respectively,

$$\int_{X^{[n+1]} \times X^m} P = \frac{1}{n+1} \int_{X^{[n]} \times X^{m+1}} (\sigma, \mathrm{id})_* \big[(\nu, \mathrm{id})^* P\big].$$

Let i be a positive integer. The class $(\nu, \mathrm{id})^* \,\mathrm{pr}_0^*\, c_i\big(X^{[n+1]}\big) - (\sigma, \mathrm{id})^* \,\mathrm{pr}_0^*\, c_i\big(X^{[n]}\big)$ is, by Proposition 6.7 (ii), a polynomial in the classes $\mathrm{pr}_0^*\, l$, $(\sigma, \mathrm{id})^* \,\mathrm{pr}_1^*\, c_j(X)$ and $(\sigma, \mathrm{id})^* \,\mathrm{pr}_{01}^*\, \mu_{j,\,n}$; and Proposition 3.27 gives $(\sigma, \mathrm{id})_* \,\mathrm{pr}_0^*\, l^j = (-1)^j \,\mathrm{pr}_{01}^*\, \mu_{j,\,n}$. Thus, $(\sigma, \mathrm{id})_* (\nu, \mathrm{id})^* \,\mathrm{pr}_0^*\, c_i\big(X^{[n+1]}\big) - \mathrm{pr}_0^*\, c_i\big(X^{[n]}\big)$ is a polynomial in the classes $\mathrm{pr}_{01}^*\, \mu_{j,\,n}$ and $\mathrm{pr}_1^*\, c_j(X)$.

By Proposition 6.7 (i), $(\nu, \mathrm{id})^* \,\mathrm{pr}_{0k}^*\, \mu_{i,\,n+1} - (\sigma, \mathrm{id})^* \,\mathrm{pr}_{0,k+1}^*\, \mu_{i,\,n}$ is a polynomial in the classes $\mathrm{pr}_0^*\, l$ and $(\sigma, \mathrm{id})^* \,\mathrm{pr}_{1,\,k+1}^*\, d_j$. Applying proposition 3.27 again, we obtain that $(\sigma, \mathrm{id})_* (\nu, \mathrm{id})^* \,\mathrm{pr}_{0k}^*\, \mu_{i,\,n+1} - \mathrm{pr}_{0,k+1}^*\, \mu_{i,\,n}$ is a polynomial in the classes $\mathrm{pr}_{01}^*\, \mu_{j,\,n}$ and $\mathrm{pr}_{1k}^*\, d_j$.

To conclude, we use the relations

$$(\nu, \mathrm{id})^* \,\mathrm{pr}_{kl}^*\, d_i = (\sigma, \mathrm{id})^* \,\mathrm{pr}_{k+1,\,l+1}^*\, d_i$$

$$\text{and} \qquad (\nu, \mathrm{id})^* \,\mathrm{pr}_k^*\, c_i(X) = (\sigma, \mathrm{id})^* \,\mathrm{pr}_{k+1}^*\, c_i(X).$$

□

We can now finish the proof of Theorem 6.1. We write

$$\int_{X^{[n]}} P\Big(c_1\big(X^{[n]}\big), \ldots, c_{2n}\big(X^{[n]}\big)\Big) = \int_{X^{[n-1]} \times X} \widetilde{P}_1 = \int_{X^{[n-2]} \times X^2} \widetilde{P}_2 = \cdots = \int_{X^n} \widetilde{P}$$

where $\widetilde{P}$ is a universal polynomial in the classes $\mathrm{pr}_k^*\, c_i(X)$ and $\mathrm{pr}_{kl}^*\, d_i$. By the explicit expression of the classes d_i, we obtain that $\int_{X^n} \widetilde{P}$ is a universal polynomial in the Chern numbers $c_1(X)^2$ and $c_2(X)$. □

7 Appendix: existence of relative integrable complex structures

This appendix is devoted to the proof of Proposition 2.13. This proof is carried out in several steps. We introduce at first some notations and terminology:

– If Y and A are two subsets of $X \times B$ and B respectively, we put $Y_{|A} = Y \cap (X \times A)$.
– If Y is a subset of $X \times B$, we say that Y is *adapted to* Z if Y contains $\mathrm{pr}_2^{-1}[\mathrm{pr}_2(Y) \cap Z]$.
– If W and W' are open subsets of $X \times B$, we write $W \lesssim W'$ if $W \cap W'$ is nonempty and if $W_{|\,\mathrm{pr}_2(W) \cap \mathrm{pr}_2(W')}$ is included in $W'_{|\,\mathrm{pr}_2(W) \cap \mathrm{pr}_2(W')}$.

Then we have:

Lemma 7.1 *Let K be a compact subset of B. Then for any neighbourhood W of $Z_{|K}$ in $X \times B$, there exists a neighbourhood U of K in B such that $W_{|U}$ is adapted to Z.*

Proof Let $\pi\colon Z \longrightarrow B$ be the restriction of the first projection to Z. Since π is finite, π is closed. Thus, if $U = B \setminus \pi\big[\big((X \times B) \setminus W\big) \cap Z\big]$, U is an open neighbourhood of K in B, and $W_{|U}$ is adapted to Z. □

We are now going to study in detail some very special open subsets of $X \times B$, which are essentially tubular neighbourhoods of Z. The construction of these open sets relies on the (probably well-known) lemma:

Lemma 7.2 *Let Y be a differentiable orbifold endowed with its natural stratification, S be a stratum of T and K be a compact subset of Y. Then there exist a neighbourhood U of K in Y and a smooth retraction $R\colon U \longrightarrow U \cap S$.*

Proof We start by a local construction. Let x be a point in S and U_x be a neighbourhood of x in Y such that U_x is isomorphic as an orbifold to V/G, where V is an open subset of $\mathbb{R}^n$ and G is a finite group of diffeomorphisms of V. Let $\pi\colon V \longrightarrow U_x$ be the quotient map and y be an element of $\pi^{-1}(x)$. We can assume that G is equal to the stabilizer G_y of y. In this case, $\pi^{-1}(S \cap U_x)$ is exactly the subset of the points of V fixed by all the elements of G; we denote it by V^G.

Let us construct an embedding of U_x into an Euclidean space. By Bochner's linearization theorem, we can assume that the finite group G acts linearly on V. This means that the action of G is induced by a linear representation of G in $\mathrm{GL}(n, \mathbb{R})$ if we choose y as the origin of $\mathbb{R}^n$. Then G also acts on the algebra $\mathbb{R}[X_1, \dots, X_n]$ of polynomial functions on $\mathbb{R}^n$ and the algebra $\mathbb{R}[X_1, \dots, X_n]^G$ of G–invariant polynomials is finitely generated [41, Appendix 4, Prop. 1]. Let $\phi_1, \dots \phi_d$ be a set of generators and $\phi = (\phi_1, \dots \phi_d)$. The map $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^d$ induces a smooth map $\widetilde{\phi}$ from $\mathbb{R}^n/G$ to $\mathbb{R}^d$ which is easily checked to be injective. Since U_x is open in $\mathbb{R}^n/G$, $\widetilde{\phi}$ gives an embedding of U_x in $\mathbb{R}^d$.

Let us now prove that the restriction of $\widetilde{\phi}$ to $S \cap U_x$ is an immersion. Since $\pi_{|V^G}\colon V^G \longrightarrow S \cap U_x$ is a diffeomorphism, this is equivalent to show that the restriction of ϕ to V^G is an immersion. Let p be a point in V^G, h be a tangent vector in T_pV^G, and assume that $\phi_*(h) = 0$. If we put $\psi_i = \phi_i - \phi_i(p)$ for $1 \leq i \leq d$, then every G–invariant polynomial P on $\mathbb{R}^n$ can be decomposed as $P_1\psi_1 + \cdots + P_d\psi_d + P(p)$, where the P_i's are G–invariant. This proves that $P_*(h) = 0$. Let A be the algebra of polynomial functions on the vector space $(\mathbb{R}^n)^G$. Then A is a quotient of $\mathbb{R}[X_1, \ldots, X_n]$ and G acts trivially on A, so that A is a quotient of $\mathbb{R}[X_1, \ldots, X_n]^G$. Let us choose a linear form u on $(\mathbb{R}^n)^G$ such that, via the identification between T_pV^G and $(\mathbb{R}^n)^G$, $u(h)$ is nonzero. If $\widetilde{u}$ is a lift of u in $\mathbb{R}[X_1, \ldots, X_n]^G$, then $\widetilde{u}_*(h)$ is nonzero, which is a contradiction.

We can now argue as in the proof of the Whitney embedding theorem in the compact case: using a partition of unity, we obtain an embedding Γ of a neighbourhood U of K into some Euclidean space $\mathbb{R}^N$ such that $\Gamma_{|U\cap S}$ is an immersion. Since $\Gamma(U \cap S)$ is a submanifold of $\mathbb{R}^N$, after shrinking U if necessary, we can consider a tubular neighbourhood Ω of $\Gamma(U \cap S)$ in $\mathbb{R}^N$ such that $\Gamma(U)$ is included in Ω. If $\widetilde{R}\colon \Omega \longrightarrow \Gamma(U \cap S)$ is the retraction associated with this tubular neighbourhood, we define $R\colon U \longrightarrow U \cap S$ by composing $\widetilde{R} \circ \Gamma$ with the inverse of the diffeomorphism $\Gamma_{|U\cap S}\colon U \cap S \xrightarrow{\sim} \Gamma(U \cap S)$. $\square$

Let r be the injectivity radius of X. Since X is compact, r is positive. For any x in X and ε satisfying $0 < \varepsilon < r$, the exponential map $\exp_g(x)$ at x associated with the Riemannian metric g is a diffeomorphism between the Euclidean ball of T_xX of radius ε centered at the origin and the geodesic ball $B_g(x, \varepsilon)$. The former being endowed with the integrable complex structure J_x, we get a canonical integrable complex structure on the latter.

Let us fix a stratum B_λ of B and a compact subset K_λ of B_λ. We denote by $d(\lambda)$ the number of sheets of the covering map $\mathrm{pr}_{2\,|Z_\lambda}\colon Z_\lambda \longrightarrow B_\lambda$. If η is a sufficiently small number in $]0, r[$, there exists a small neighbourhood V_λ of K_λ in B_λ such that for any b in V_λ, if $Z_{|b} = \{x_1, \ldots, x_{d(\lambda)}\}$, then the geodesic balls $\big(B_g(x_i, \eta)\big)_{1\leq i\leq d(\lambda)}$ are pairwise disjoint in X. By Lemma 7.2, we can assume that there exist a neighbourhood U_λ of K_λ in B such that $U_\lambda \cap B_\lambda = V_\lambda$ and a smooth retraction $R\colon U_\lambda \longrightarrow V_\lambda$. Then we put $\Omega = \coprod_{b\in V_\lambda} \coprod_{x\in Z_{|R_\lambda(b)}} B_g(x, \eta)$. Since Ω is a neighbourhood of $Z_{|K_\lambda}$, Lemma 7.1 shows that Ω is adapted to Z if U_λ is a sufficiently small neighbourhood of K_λ. We call such an open set Ω an *η–neighbourhood* of Z above K_λ. It is easy to prove that if W is any neighbourhood of $Z_{|K_\lambda}$, then there exists an η–neighbourhood of Z above K_λ contained in W if η is a sufficiently small positive real number.

We now explain how to cover Z in a compatible way by a finite number of η–neighbourhoods.

Lemma 7.3 *Let W be an open subset of $X \times B$ adapted to Z, B_λ be a stratum of B such that ∂B_λ is contained in $\mathrm{pr}_2(W)$ and η_0 be a positive real number. Then, for every relatively compact open subset V of $\mathrm{pr}_2(W)$ containing ∂B_λ, there exist a compact subset K_λ of B_λ, a positive real number η smaller than η_0 and a η–neighbourhood Ω of Z above K_λ such that $\overline{B}_\lambda \subseteq V \cup K_\lambda$ and $\Omega \lesssim W_{|V}$.*

Proof Let U be a relatively compact neighbourhood of ∂B_λ in $\overline{B}_\lambda \cap V$ and put $K_\lambda = \overline{B}_\lambda \setminus U$. Then K_λ is compact in B_λ and $\overline{B}_\lambda \subseteq V \cap K_\lambda$. If $K' = (\overline{V} \cap \overline{B}_\lambda) \setminus U$, then K' is a compact subset of B_λ included in K_λ and W is a neighbourhood of $Z_{|K'}$. Thus, for η small enough, there exists an η–neighbourhood Ω' of Z above K' contained in W. We can even assume that $\Omega' = \Omega_{|O}$, where Ω is an η–neighbourhood of Z above K_λ and O is a small neighbourhood of K' in B. Since $\Omega_{|O} \subseteq W$, $\Omega_{|O \cap V} \subseteq W_{|O \cap V}$, so that $\Omega \lesssim W_{|V}$. □

Then we get:

Lemma 7.4 *For every positive real number η_0, we can construct a finite covering $\left(W_i\right)_{1 \leq i \leq N}$ of Z by open subsets of $X \times B$ such that:*

– For each i, there exists η_i in $]0, \eta_0]$ such that W_i is a disjoint finite union of η_i–neighbourhoods of Z above compact subsets of the strata of B.
– For every i such that $2 \leq i \leq N$, $W_i \lesssim W_{i-1} \cup \cdots \cup W_1$.
– The integer N is smaller than the number of strata of B.

Proof Let Λ be the finite set of strata of B and $\left(\Lambda_i\right)_{1 \leq i \leq N}$ be the partition of Λ defined as follows: Λ_1 is the set of elements of Λ corresponding to closed (ie minimal) strata in B, and the other Λ_i's are defined inductively for $i \geq 2$ by the expression

$$\Lambda_i = \{\lambda \text{ in } \Lambda \setminus (\Lambda_1 \cup \cdots \cup \Lambda_{i-1}) \text{ such that } \partial B_\lambda \subseteq \bigcup_{j=1}^{i-1} \bigcup_{\mu \in \Lambda_j} B_\mu\}.$$

For $1 \leq i \leq N$, we put $S_i = \bigcup_{j=1}^{i} \bigcup_{\lambda \in \Lambda_j} B_\lambda$. The S_i's are closed subsets of B. Let $\Lambda_1 = \{\lambda_1, \ldots, \lambda_k\}$. For η_1 sufficiently small, we can pick pairwise η_1–neighbourhoods of Z above the B_{λ_i}'s ($1 \leq i \leq k$). We denote their union by W_1. Then $\mathrm{pr}_2(W_1)$ is a neighbourhood of S_1 in B. Let V be a relatively compact neighbourhood of S_1 in

$\mathrm{pr}_2(W_1)$. If $\Lambda_2 = \{\mu_1, \dots, \mu_l\}$, we can find by Lemma 7.3 some compact subsets $K_1, \dots, K_l$ of $B_{\mu_1}, \dots, B_{\mu_l}$ and η_2–neighbourhoods $\Omega_1, \dots, \Omega_l$ of Z above $K_1, \dots, K_l$ (where $0 \leq \eta_2 \leq \eta_1$) such that for $1 \leq i \leq l$, $\overline{B}_{\mu_i} \subseteq V \cup K_i$ and $\Omega_i \lesssim W_{1\,|V}$. If η_2 is small enough, we can assume that the Ω_i's are pairwise disjoint. If we replace W_1 by $W_{1\,|V}$ and if we put $W_2 = \coprod_{i=1}^{l} \Omega_i$, then $\mathrm{pr}_2(W_1)$ is still a neighbourhood of S_1 in B and $\mathrm{pr}_2(W_1 \cup W_2)$ is a neighbourhood of S_2 in B. Besides, $W_2 \lesssim W_1$. To construct W_3, we add the strata in Λ_3 and so on. □

We now turn to the construction of relative integrable complex structures. If we take a covering $\big(W_i\big)_{1 \leq i \leq N}$ of Z given by Lemma 7.4, each W_i (more generally each η–neighbourhood of $\overline{Z}$ above a compact subset of a stratum of B) is endowed with a canonical relative integrable complex structure, but these various structures do not match on the intersections $W_i \cap W_j$. This is why it is necessary to use a gluing argument. We start by some preliminaries.

Let E be any finite-dimensional real vector space of even dimension $2k$. If $\mathcal{J}(E)$ is the set of complex structures on E, $\mathcal{J}(E)$ is a homogeneous space isomorphic to $\mathrm{GL}(2k, \mathbb{R})/\mathrm{GL}(k, \mathbb{C})$, so it is a submanifold of $\mathrm{End}(E)$. We define a subset $\mathfrak{J}$ of the vector bundle $\mathrm{End}(TX)$ by $\mathfrak{J} = \coprod_{x \in X} \mathcal{J}(T_x X)$. Then $\mathfrak{J}$ is a differentiable manifold and the projection $p\colon \mathfrak{J} \longrightarrow X$ is a smooth fibration. Besides, the almost-complex structure J of X is a smooth section of this fibration. For $\varepsilon > 0$ and x in X, we put $\mathcal{J}(T_x X) = \{M \text{ in } \mathcal{J}(T_x X) \text{ such that } \|M - J(x)\|_g < \varepsilon\}$ and we define $\mathfrak{J}_\varepsilon = \coprod_{x \in X} \mathcal{J}_\varepsilon(T_x X)$. For any $\varepsilon > 0$, $\mathfrak{J}_\varepsilon$ is a neighbourhood of the image of J in $\mathfrak{J}$. Furthermore, since X is compact, there exists $\varepsilon_0 > 0$ such that for every ε smaller than ε_0, $\mathfrak{J}_\varepsilon$ is a smooth fibration whose fiber is diffeomorphic to an Euclidean ball.

Let η be a positive real number strictly smaller than r. For x in X and t in $[0,1]$, we define $u_t\colon B_g(x,\eta) \longrightarrow B_g(x, t\eta)$ by $u_t(x)(p) = \exp_g(x)\big[t \exp_g(x)^{-1}(p)\big]$. For any y in $B_g(x,\eta)$, there is a canonical isomorphism $\phi_t(x,y)$ between $T_y X$ and $T_{u_t(x)(y)} X$ obtained via the differential of $\exp_g(x)$ from the canonical isomorphism between $T_{\exp_g(x)^{-1}(y)}(T_x X)$ and $T_{t \exp_g(x)^{-1}(y)}(T_x X)$. Since X is compact, there is a positive constant A_η depending only on η such that $\lim_{\eta \to 0} A_\eta = 1$, and for every x in X, every y in $B_g(x,\eta)$ and every t in $[0,1]$, $\|\phi_t(x,y)\|_g \leq A_\eta$ and $\|\phi_t(x,y)^{-1}\|_g \leq A_\eta$. If x is in X and t in $[0,1]$, let $\widetilde{J}$ be an almost-complex structure on $B_g(x, t\eta)$; if $t = 0$, we take the convention that $\widetilde{J}$ is a complex structure on $T_x X$. Then we can define a rescaled almost-complex structure $R_t(\widetilde{J})$ on $B_g(x,\eta)$ by the formula

$$\forall y \in B_g(x,\eta), \;\; \forall h \in T_y X, \;\; R_t(\widetilde{J})(y)(h) = \phi_t(x,y)^{-1}\big[\widetilde{J}\big(u_t(x)(y)\big)\big(\phi_t(x,y)(h)\big)\big].$$

Then, for every t in $[0,1]$ and every almost-complex structures $\widetilde{J}_0$ and $\widetilde{J}_1$ on $B_g(x,t\eta)$, we have

$$\|R_t(\widetilde{J}_1) - R_t(\widetilde{J}_0)\|_{C^0,\,g,\,B_g(x,\,\eta)} \leq A_\eta^2\ \|\widetilde{J}_1 - \widetilde{J}_0\|_{C^0,\,g,\,B_g(x,\,t\eta)}.$$

Remark that if $\widetilde{J}$ is an almost-complex structure on $B_g(x,t\eta)$, then $R_t(\widetilde{J}) = u_t(x)^*\widetilde{J}$ if $t>0$, and $R_t(\widetilde{J}) = \exp_g(x)_*\big(\widetilde{J}(x)\big)$ if $t=0$. Thus, if $t>0$ and $\widetilde{J}$ is integrable or if $t=0$, then $R_t(\widetilde{J})$ is integrable.

We end with another estimate. For any x in X and any t in $[0,1]$, $R_t(J)(x) = J(x)$. Since X is compact, there exists a positive constant B_η depending only on η such that $\lim_{\eta\to 0} B_\eta = 0$, and for every x in X and every t in $[0,1]$, $\|J - R_t(J)\|_{C^0,\,g,\,B_g(x,\,\eta)} \leq B_\eta$.

We are ready to prove the essential gluing lemma:

Lemma 7.5 *Let ε, ε' and η be positive real numbers satisfying the conditions $\eta < r$, $0<\varepsilon'<\varepsilon\leq\varepsilon_0$ and $\varepsilon' A_\eta^4 + B_\eta(A_\eta^2+1) < \varepsilon$, and let W be a η–neighbourhood of Z above a compact subset K_λ of a stratum B_λ of B. If J_0^{rel} and J_1^{rel} are two relative integrable complex structures in $\mathcal{B}_{g,\,\varepsilon'}(W)$, then there exists a smooth family $\big(J_s^{\mathrm{rel}}\big)_{0\leq s\leq 1}$ of relative integrable complex structures in $\mathcal{B}_{g,\,\varepsilon}(W)$ joining J_0^{rel} and J_1^{rel}.*

Proof Let $\widetilde{J}_0^{\mathrm{rel}} = R_0\big(J_0^{\mathrm{rel}}\big)$ and $\widetilde{J}_1^{\mathrm{rel}} = R_0\big(J_1^{\mathrm{rel}}\big)$. Then the families $\big(R_t(J_0^{\mathrm{rel}})\big)_{0\leq t\leq 1}$ and $\big(R_t(J_1^{\mathrm{rel}})\big)_{0\leq t\leq 1}$ are two smooth families of relative integrable complex structures on W joining J_0^{rel} and J_1^{rel} to $\widetilde{J}_0^{\mathrm{rel}}$ and $\widetilde{J}_1^{\mathrm{rel}}$ respectively. Besides, for every t in $[0,1]$ and i in $\{0,1\}$,

$$\|R_t(J_i^{\mathrm{rel}}) - J\|_{C^0,\,g,\,W} \leq \|R_t(J_i^{\mathrm{rel}}) - R_t(J)\|_{C^0,\,g,\,W} + \|R_t(J) - J\|_{C^0,\,g,\,W} < A_\eta^2\,\varepsilon' + B_\eta.$$

Let $\widetilde{\varepsilon} = A_\eta^2\,\varepsilon' + B_\eta$, $V = \mathrm{pr}_2(W)\cap B_\lambda$ and $R_\lambda\colon\ \mathrm{pr}_2(W)\longrightarrow V$ be the smooth retraction associated with W. Since $\widetilde{\varepsilon}$ is smaller than ε_0, $\mathfrak{J}_{\widetilde{\varepsilon}}\times_X Z_{|V}$ is a smooth fibration over $Z_{|V}$ whose fiber is diffeomorphic to an Euclidean ball (hence contractible).

The relative integrable complex structures $\widetilde{J}_0^{\mathrm{rel}}$ and $\widetilde{J}_1^{\mathrm{rel}}$ define two sections J_0' and J_1' of the fibration $\mathfrak{J}_{\widetilde{\varepsilon}}\times_X Z_{|V}$; they are given for (x,v) in $Z_{|V}$ by $J_i'(x,v) = \widetilde{J}_{i,\,v}^{\mathrm{rel}}(x)$. Then we can find a smooth family $\big(J_t'\big)_{0\leq t\leq 1}$ of sections of $\mathfrak{J}_{\widetilde{\varepsilon}}\times_X Z_{|V}$ joining J_0' and J_1'. Each J_t' defines a relative integrable complex structure ${J_t'}^{\mathrm{rel}}$ on W as follows: for b in $\mathrm{pr}_2(W)$, let $v = R_\lambda(b)$ and $Z_{|v} = \{x_1,\dots,x_d\}$. Then $W_b = \coprod_{i=1}^d B_g(x_i,\eta)$ and ${J_t'}^{\mathrm{rel}}$ is equal to $R_0\big(J_t'(x_i,v)\big)$ on $B_g(x_i,\eta)$. Note that $\widetilde{J}_i^{\mathrm{rel}} = {J_i'}^{\mathrm{rel}}$ for $i=0,1$. Besides,

$$\|{J_t'}^{\mathrm{rel}} - J\|_{C^0,\,g,\,W} \leq \|{J_t'}^{\mathrm{rel}} - R_0(J)\|_{C^0,\,g,\,W} + \|R_0(J) - J\|_{C^0,\,g,\,W} < A_\eta^2\,\widetilde{\varepsilon} + B_\eta < \varepsilon.$$

Thus we have found a piecewise smooth family of relative integrable complex structures in $\mathcal{B}_{g,\varepsilon}(W)$ joining J_0^{rel} and J_1^{rel}. This family can be made smooth by reparametrization. $\square$

We can now finish the proof of Proposition 2.13. Fix ε and ε' which satisfy the conditions $0 < \varepsilon' < \varepsilon \leq \varepsilon_0$. If d is the number of strata of B and η is a positive real number, we define a sequence $\left(\varepsilon_i\right)_{1\leq i\leq d}$ by $\varepsilon_1 = \varepsilon'$ and, for $1 \leq i \leq d-1$, $\varepsilon_{i+1} = A_\eta^4\,\varepsilon_i + B_\eta(A_\eta^2+1)$. Then we can find $\eta_0 > 0$ such that for every η smaller than η_0, the ε_i's are strictly smaller than ε for $1 \leq i \leq d$ and B_η is strictly smaller than ε'. We pick such an η_0 and take a covering $\left(W_i\right)_{1\leq i\leq N}$ of Z satisfying the conditions of Lemma 7.3. Each open subset W_i is endowed with a relative integrable complex structure J_i^{rel} such that $\|J_i^{\mathrm{rel}} - J\|_{C^0,g,W_i} < \varepsilon'$.

Since $W_2 \lesssim W_1$, $W_1 \cap W_2$ is a finite disjoint union of η_2–neighbourhoods of Z (where $\eta_2 \leq \eta_0$) above compact subsets of the strata of B. The relative integrable complex structures J_1^{rel} and J_2^{rel} being in $\mathcal{B}_{g,\varepsilon_1}(W_1 \cap W_2)$, there exists by Lemma 7.5 a smooth family $\left(J_t^{\mathrm{rel}}\right)_{1\leq t\leq 2}$ in $\mathcal{B}_{g,\varepsilon_2}(W_1 \cap W_2)$ joining J_1^{rel} and J_2^{rel}.

Let V be a relatively compact open subset of $\mathrm{pr}_2(W_1)$. If V is large enough, the open sets $\left(W_{1|V}, W_2, \ldots, W_N\right)$ still satisfy the conditions of Lemma 7.3. Let χ be a smooth function from $\mathrm{pr}_2(W_1 \cup W_2)$ to $[1,2]$ such that $\chi \equiv 1$ on $\overline{V}$ and $\mathrm{supp}(\chi - 2)$ is relatively compact in $\mathrm{pr}_2(W_1)$. Then we define a relative integrable complex structure $\widetilde{J}^{\mathrm{rel}}$ in $\mathcal{B}_{g,\varepsilon_2}(W_{1|V} \cup W_2)$ by the formula $\widetilde{J}_b^{\mathrm{rel}} = J_{\chi(b),b}^{\mathrm{rel}}$ for b in $V \cup \mathrm{pr}_2(W_2)$. The same argument can be used to glue $\widetilde{J}^{\mathrm{rel}}$ and J_3^{rel} together, and so on. Finally, we get a relative integrable complex structure in $\mathcal{B}_{g,\varepsilon_N}(W)$, where W is a neighbourhood of Z in $X \times B$. Thus $\mathcal{B}_{g,\varepsilon}$ is nonempty.

We now prove the connectedness of $\mathcal{B}_{g,\varepsilon}$ for $0 < \varepsilon \leq \varepsilon_0$. Let J_0^{rel} and J_1^{rel} be two relative integrable structures in $\mathcal{B}_{g,\varepsilon}(W)$, where W is a neighbourhood of the incidence set Z in $X \times B$. We put $\varepsilon' = \|J_0^{\mathrm{rel}} - J_1^{\mathrm{rel}}\|_{C^0,g,W}$. As we did previously, we can find $\eta_0 > 0$ such that for every positive η smaller than η_0, the ε_i's are strictly smaller than ε for $1 \leq i \leq d+1$. For this η_0, we take a covering $\left(W_i\right)_{1\leq i\leq N}$ of Z satisfying the conditions of Lemma 7.3 such that each W_i is contained in W. Then for each i, ${J_0^{\mathrm{rel}}}_{|W_i}$ and ${J_1^{\mathrm{rel}}}_{|W_i}$ are elements of $\mathcal{B}_{g,\varepsilon_1}(W_i)$, so that, by Lemma 7.5, there exists a smooth family $\left(J_{i,t}^{\mathrm{rel}}\right)_{0\leq t\leq 1}$ in $\mathcal{B}_{g,\varepsilon_2}(W_i)$ joining ${J_0^{\mathrm{rel}}}_{|W_i}$ and ${J_1^{\mathrm{rel}}}_{|W_i}$. Using exactly the same methods as before, we can glue the families $\{\left(J_{i,t}^{\mathrm{rel}}\right)_{0\leq t\leq 1}\}_{1\leq i\leq N}$ together and get a

smooth family in $\mathcal{B}_{g,\varepsilon_{N+1}}(\widetilde{W})$ joining J_0^{rel} to J_1^{rel}, where $\widetilde{W}$ is a neighbourhood of Z in $X \times B$.

The weak contractibility follows from the connectedness of $\mathcal{B}_{g,\varepsilon}$ over the bases $B \times \mathbb{S}^p$.

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