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SPACELIKE CONVEX SURFACES WITH PRESCRIBED CURVATURE IN (2+1)-MINKOWSKI SPACE

FRANCESCO BONSANTE AND ANDREA SEPPI

Abstract. We prove existence and uniqueness of solutions to the Minkowski problem in any domain of dependence D in $(2+1)$-dimensional Minkowski space, provided D is contained in the future cone over a point. Namely, it is possible to find a smooth convex Cauchy surface with prescribed curvature function on the image of the Gauss map. This is related to solutions of the Monge-Ampère equation $\det D^2u(z) = (1/\psi(z))(1-|z|^2)^{-2}$ on the unit disc, with the boundary condition $u|_{\partial\mathbb{D}} = \varphi$, for ψ a smooth positive function and φ a bounded lower semicontinuous function.

We then prove that a domain of dependence D contains a convex Cauchy surface with principal curvatures bounded from below by a positive constant if and only if the corresponding function φ is in the Zygmund class. Moreover in this case the surface of constant curvature K contained in D has bounded principal curvatures, for every $K < 0$. In this way we get a full classification of isometric immersions of the hyperbolic plane in Minkowski space with bounded shape operator in terms of Zygmund functions of $\partial\mathbb{D}$.

Finally, we prove that every domain of dependence as in the hypothesis of the Minkowski problem is foliated by the surfaces of constant curvature K, as K varies in $(-\infty, 0)$.

1. Introduction

A classical theorem of Riemannian geometry states that if $\sigma : \mathbb{S}^2 \to \mathbb{R}^3$ is an isometric immersion of the round sphere into Euclidean space, then it is the standard inclusion up to an isometry of $\mathbb{R}^3$. On the other hand, Hano and Nomizu ([19]) first proved that the analogous statement does not hold in Minkowski space $\mathbb{R}^{2,1}$. Namely there are isometric embeddings of the hyperbolic plane in $\mathbb{R}^{2,1}$ which are not equivalent to the standard inclusion of the hyperboloid model of $\mathbb{H}^2$ into $\mathbb{R}^{2,1}$.

A possible way to understand the rigidity in the Euclidean case makes use of the so-called support function: basically if $\sigma : \mathbb{S}^2 \to \mathbb{R}^3$ is an isometric immersion, the image of σ must be a locally convex surface by the Gauss equation. In particular it turns out that the Gauss map $G : \mathbb{S}^2 \to \mathbb{S}^2$ is bijective, and the support function is defined as $\bar{u} : \mathbb{S}^2 \to \mathbb{R}$, $\bar{u}(x) = \langle x, G^{-1}(x)\rangle$. A simple computation shows that $\det(\mathrm{Hess}^{\mathbb{S}^2}\bar{u} + \bar{u}\mathrm{I}) = 1$, where $\mathrm{Hess}^{\mathbb{S}^2}$ is the covariant Hessian on the sphere. By a wise use of the comparison principle it turns out that the difference of any two solutions must be the restriction on $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$. This allows to conclude that every solution is of the form $\bar{u}(x) = 1 + \langle x, \xi\rangle$ for some $\xi \in \mathbb{R}^3$. Therefore the surface $\sigma(\mathbb{S}^2)$ is the round sphere of radius 1 centered at ξ.

The *support function* $\bar{u}$ can be analogously defined for a spacelike convex immersion $\sigma : \mathbb{H}^2 \to \mathbb{R}^{2,1}$. If σ is an isometric immersion, then $\bar{u}$ satisfies the equation $\det(\mathrm{Hess}^{\mathbb{H}^2}\bar{u} - \bar{u}\mathrm{I}) = 1$. However the maximum principle cannot be directly used in this context by the non-compactness of $\mathbb{H}^2$. This is a general indication that some boundary condition must be taken into account to determine the solution and the immersion σ.

The classical Minkowski problem in Euclidean space can be also formulated for Minkowski space. Given a smooth spacelike strictly convex surface S in $\mathbb{R}^{2,1}$, the curvature function is defined as $\psi : G(S) \to \mathbb{R}$, $\psi(x) = -K_S(G^{-1}(x))$, where $G : S \to \mathbb{H}^2$ is the Gauss map and

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K_S is the scalar intrinsic curvature on S. Minkowski problem consists in finding a convex surface in Minkowski space whose curvature function is a prescribed positive function ψ. Using the support function technology, the problem turns out to be equivalent to solving the equation

$$\det(\mathrm{Hess}^{\mathbb{H}^2}\bar{u} - \bar{u}\mathrm{I}) = \frac{1}{\psi}\,. \tag{1}$$

Also in this case for the well-posedness of the problem some boundary conditions must be imposed.

Using the Klein model of $\mathbb{H}^2$, Equation (1) can be reduced to a standard Monge-Ampère equation over the unit disc $\mathbb{D}$. In particular solutions of (1) explicitly correspond to solutions $u : \mathbb{D} \to \mathbb{R}$ of the equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2}\,. \tag{2}$$

It should be remarked that the correspondence can degenerate in some sense. Indeed given a convex surface S, the support function u is defined only on a convex subset of $\mathbb{D}$. On the other hand, any convex function over $\mathbb{D}$ corresponds to some convex surface S in Minkowski space, but in general S might contain lightlike rays. We say that a convex surface is a spacelike entire graph if $S = \{(p, f(p)) \,|\, p \in \mathbb{R}^2\}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function on the horizontal plane such that $||Df(p)|| < 1$ for all $p \in \mathbb{R}^2$.

In [20], Li studied the Minkowski problem in Minkowski space in any dimension showing the existence and uniqueness of the solution of (2) imposing $u|_{\partial\mathbb{D}} = \varphi$, for a given smooth φ. The result was improved in dimension $2+1$ by Guan, Jian and Schoen in [17], where the existence of the solution is proved assuming that the boundary data is only Lipschitz. The solutions obtained in both cases correspond to spacelike entire graphs.

A remarkable result in [20] is that under the assumption that the boundary data is smooth, the corresponding convex surface S has principal curvatures bounded from below by a positive constant. As a partial converse statement, if S has principal curvatures bounded from below by a positive constant, then the corresponding function $u : \mathbb{D} \to \mathbb{R}$ extends to a continuous function of the boundary of $\mathbb{D}$.

In a different direction Barbot, Béguin and Zeghib ([2]) solved the Minkowski problem for surfaces invariant by an affine deformation of a cocompact Fuchsian group. Let G be a cocompact Fuchsian group, and Γ an affine deformation of G (Γ is a group of affine transformations whose elements are obtained by adding a translation part to elements of G). If S is a Γ-invariant surface, its curvature function ψ is G-invariant. Barbot, Béguin and Zeghib proved that, given a positive G-invariant function ψ, there exists a unique solution of Minkowski problem which is Γ-invariant.

If $u : \mathbb{D} \to \mathbb{R}$ is the support function corresponding to some Γ-invariant surface S, combining the result by Li and the cocompactness of Γ, it turns out that u extends to the boundary of $\mathbb{D}$. It is not difficult to see that the extension on the boundary only depends on Γ and in particular it is independent of the curvature function. However the result in [2] is not a consequence of the results in [20, 17], as it is not likely that $u|_{\partial\mathbb{D}}$ is Lipschitz continuous. This gives an indication that in dimension $2+1$ results in [20, 17] are not sharp.

One of the goals of the paper is to determine the exact regularity class of the extension on $\partial\mathbb{D}$ of functions $u : \mathbb{D} \to \mathbb{R}$ corresponding to surfaces with principal curvatures bounded from below.

Theorem 1.1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. Then there exists a spacelike entire graph in $\mathbb{R}^{2,1}$ whose principal curvatures are bounded from below by a positive constant and whose support function u extends φ if and only if φ is in the Zygmund class.*

Recall that a function $\varphi : S^1 \to \mathbb{R}$ is in the *Zygmund class* if there is a constant C such that, for every $\theta, h \in \mathbb{R}$,

$$|\varphi(e^{i(\theta+h)}) + \varphi(e^{i(\theta-h)}) - 2\varphi(e^{i\theta})| < C|h| \,.$$

Functions in the Zygmund class are α-Hölder for every $\alpha \in (0,1)$, but in general they are not Lipschitz.

Theorem 1.1 implies that spacelike entire graphs of constant curvature -1 and with a uniform bound on the principal curvatures correspond to functions u whose extension to $\partial\mathbb{D}$ is Zygmund. We prove that also the converse holds. This gives a complete classification of such surfaces in terms of Zygmund functions.

Theorem 1.2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a function in the Zygmund class. For every $K < 0$ there is a unique spacelike entire graph S in $\mathbb{R}^{2,1}$ of constant curvature K and with bounded principal curvatures whose corresponding function u extends φ.*

The proof of Theorem 1.2 relies on a general statement we prove about solvability of Minkowski problem. We precisely prove the following Theorem.

Theorem 1.3. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a lower semicontinuous and bounded function and $\psi : \mathbb{D} \to [a,b]$ for some $0 < a < b < +\infty$. Then there exists a unique spacelike graph S in $\mathbb{R}^{2,1}$ whose support function u extends φ and whose curvature function is ψ.*

In general we say that a convex function u extends φ if $\varphi(z_0) = \liminf_{z\to z_0} u(z)$ for every $z_0 \in \partial\mathbb{D}$. By convexity, if φ is continuous this condition is equivalent to requiring that u is continuous up to the boundary and its boundary value coincides with φ.

Let us explain the geometric meaning of the boundary value of the support function of S. As $\partial\mathbb{D}$, regarded as the set of lightlike directions, parameterizes lightlike linear planes, the restriction of the support function on $\partial\mathbb{D}$ gives the height function of lightlike support planes of S, where $u(\eta) = +\infty$ means that there is no lightlike support plane orthogonal to η. It can be checked that, when S is the graph of a convex function $f : \mathbb{R}^2 \to \mathbb{R}$, the condition $u|_{\partial\mathbb{D}} = \varphi$ is also equivalent to requiring that

$$\lim_{r\to+\infty} (r - f(rz)) = \varphi(z)$$

for every $z \in \partial\mathbb{D}$. The asymptotic condition is stated in the latter fashion for instance in [29] and [9], where the existence problem for constant mean curvature surfaces is treated.

In this paper we will consider future convex surfaces with bounded support function on $\mathbb{D}$. Geometrically this means that S is contained in the future cone of some point $p \in \mathbb{R}^{2,1}$.

It is also useful to consider convex objects more general than spacelike surfaces. A *future-convex domain* is defined as an open domain in $\mathbb{R}^{2,1}$ which is the intersection of a family of future half-spaces with spacelike boundary planes. Given a future convex set, D_0 we consider a bigger domain D obtained as the intersection of the future of the lightlike support planes of D_0. From the Lorentzian point of view D is the Cauchy development of the boundary of D_0. A domain D obtained in this way is called a *domain of dependence*. An immediate consequence of Theorem 1.3 is that for any domain of dependence and any $K < 0$ there exists a unique convex surface S in $\mathbb{R}^{2,1}$ of constant curvature K whose Cauchy development coincides with D. More precisely we prove the following Theorem.

Theorem 1.4. *If D is a domain of dependence contained in the future cone of a point, then D is foliated by surfaces of constant curvature $K \in (-\infty, 0)$.*

A problem that remains open is to characterize spacelike entire graphs with bounded curvature which are complete for the induced metric. If a surface has bounded principal curvatures, the Gauss map turns to be bi-Lipschitz, hence the surface is automatically complete. On the opposite side we construct an example of non complete entire graphs which constant curvature; in our example the boundary extension φ of the support function

is bounded but not continuous. This result underlines a remarkable difference with respect to CMC surfaces. Indeed it was proved in [7, 29] that an entire CMC spacelike graph is automatically complete.

The surfaces we construct are invariant under a one-parameter parabolic group of isometries of $\mathbb{R}^{2,1}$ fixing the origin, and are isometric to a half-plane in $\mathbb{H}^2$. This strategy goes back to Hano and Nomizu, who first exhibited non-standard immersions of the hyperbolic plane in $\mathbb{R}^{2,1}$ as *surfaces of revolutions*, namely surfaces invariant under a one-parameter hyperbolic group fixing the origin.

Ingredients in the proofs. We will use a description of domains of dependence due to Mess. The key fact is that it is possible to associate to a domain of dependence D a *dual* measured geodesic lamination of $\mathbb{H}^2$. It turns out that if D is a domain of dependence, its support function u is the convex envelope of its boundary value on $\partial\mathbb{D}$. Heuristically, the graph of u is a piece-wise linear convex pleated surface. The bending lines provide a geodesic lamination over $\mathbb{H}^2$, whereas a transverse measure encodes the amount of bending.

This correspondence will be crucial in the present work. Solving the Minkowski problem with a given boundary value φ is equivalent to finding a convex entire graph S in $\mathbb{R}^{2,1}$ whose curvature function is prescribed and whose Cauchy development is the domain of dependence D determined by φ. If μ is the dual measured geodesic lamination of D, we construct a sequence of measured geodesic laminations μ_n and Fuchsian groups G_n such that μ_n is G_n-invariant and μ_n converges to μ on compact subsets of $\mathbb{H}^2$ in an appropriate sense. By a result of [2] for every n it is possible to solve the Minkowski problem. The proof of Theorem 1.3 is obtained by taking Γ_n-invariant curvature functions ψ_n converging to the prescribed curvature function, and showing that solutions S_n of Minkowski problem for the domain D_n dual to μ_n with curvature function ψ_n converge to a solution of the original Minkowski problem for D.

The convergence of solutions is obtained first by showing the convergence of the support functions and then by proving that the surface dual to the limit support function is an entire graph.

For the first step a simple application of the maximum principle implies some a priori bounds of the support functions of S_n in terms of the support function of D_n. This allows to conclude that the support functions of S_n converge to a convex function $u : \mathbb{D} \to \mathbb{R}$ extending φ. Applying standard theory of Monge-Ampère equation we have that u is a generalized solution of our problem. On the other hand, in dimension 2, Alexander-Heinz theorem implies that u is strictly convex and by standard regularity theorem we conclude that u is a classical solution.

The second step is more geometric. The key idea is to use - as *barriers* - the already mentioned constant curvature surfaces which are invariant under a 1-parameter parabolic group fixing a point $\eta_0 \in \partial\mathbb{D}$. A similar approach (also in higher dimension) using surfaces invariant for a hyperbolic group is taken in [17]. Since the parabolic-invariant surfaces are entire graphs and their support function is constant on $\partial\mathbb{D} \setminus \{\eta_0\}$, they are very appropriate to show (by applying the comparison principe) that the boundary of the domain dual to a solution of (2) cannot contain lightlike rays. The argument works well under the hypothesis that the boundary value of the support function is bounded, leading to the proof of Theorem 1.3.

The proof of Theorem 1.1 is based on a relation we point out between convex geometry in Minkowski space and the theory of infinitesimal earthquakes introduced and studied in [16, 14, 24, 27]. In particular given $\varphi : \partial\mathbb{D} \to \mathbb{R}$ we prove that the convex envelope of φ is explicitly related to the infinitesimal earthquake extending the field $\varphi\frac{\partial}{\partial\theta}$. From this correspondence we see that the dual lamination associated by Mess to the domain defined by φ is equal to the earthquake lamination. Using a result of Gardiner, Hu and Lakic ([14]) we deduce that φ is Zygmund if and only if the Thurston norm of the dual lamination is finite.

Given a convex entire graph with principal curvatures bounded from below by a positive constant, we point out by a direct geometric construction in Minkowski space an explicit estimate on the Thurston norm of the dual lamination. By the above correspondence, this proves one direction of the statement of Theorem 1.1.

Conversely, we show that a spacelike entire graph of constant curvature with bounded dual lamination has bounded principal curvatures. This proves Theorem 1.2 and shows the other implication of Theorem 1.1. The proof is obtained by contradiction. The key fact is the following: if D is a domain of dependence with bounded dual lamination and α_n is any sequence of isometries of Minkowski space such that $\alpha_n(D)$ contains a fixed point p with horizontal support plane at p, then $\alpha_n(D)$ converges to a domain of dependence with bounded dual lamination. Now let S be an entire graph of constant curvature whose domain of dependence has bounded dual lamination. If the principal curvatures of S were not bounded, we could construct a sequence of isometries α_n bringing back to a fixed point a sequence $p_n \in S$ where the principal curvature are degenerating. The statement above implies that the domain of dependence of $\alpha_n(S)$ is converging to a domain with bounded dual lamination. Applying standard regularity theory of Monge-Ampère equations to the support function of $\alpha_n(S)$ one obtains an a priori bound on the second derivatives of the support function of $\alpha_n(S)$, thus leading to a contradiction.

Discussions and possible developments. Recently several connections between Teichmüller theory and the geometry of spacelike surfaces in the Anti-de Sitter space have been exploited [21, 1, 5]. The key idea is that graphs of orientation-preserving homeomorphisms of $\partial\mathbb{D}$ are naturally curves in the asymptotic boundary of the Anti-de Sitter space spanned by spacelike surfaces. A natural construction allows to associate, to any convex surface Σ spanning the graph of a homeomorphism $f : \partial\mathbb{D} \to \partial\mathbb{D}$, a homeomorphism $F : \mathbb{D} \to \mathbb{D}$ extending f. It turns out that F is quasi-conformal if and only if the principal curvatures of Σ are bounded, showing that in this case f is quasi-symmetric.

Theorem 1.1 can be regarded as an infinitesimal version of this property. More generally we believe that the relation between the theory of convex surfaces in Minkowski space and the infinitesimal Teichmüller theory is an infinitesimal evidence of the above connection for Anti-de Sitter space. The correct geometric setting to understand this idea goes through the geometric transition introduced recently in [10, 12, 11]. In fact Danciger introduced the half-pipe geometry, which is a projective blow-up of a a spacelike geodesic plane in Anti-de Sitter space. This model turns out to be a natural parameter space for spacelike affine planes in Minkowski space. Regarding the Minkowski space as the blow-up of a point in Anti-de Sitter space, the correspondence between spacelike affine planes in Minkowski space and points in half-pipe geometry is the infinitesimal version of the projective duality between points and spacelike planes in the Anti-de Sitter space. Using this connection the graph of support functions $u : \mathbb{D} \to \mathbb{R}$ are naturally convex surfaces in half-pipe geometry.

Let Σ_t be a smooth family of convex surfaces, for $t \in [0, \epsilon)$, such that Σ_0 is a totally geodesic plane and the principal curvatures of Σ_t are $O(t)$. Using the correspondence above one can prove that the boundary of Σ_t is the graph of a differentiable family f_t of quasi-symmetric homeomorphisms of $\partial\mathbb{D}$ such that $f_0 = \mathrm{id}$.

The rescaled limit of Σ_t in the half-pipe model is the graph of the support function of a convex surface S in Minkowski space. It turns out that the principal curvatures of S are the inverse of the derivatives of the principal curvatures of Σ_t. In particular they are bounded from below by a positive constant is the principal curvatures of Σ_t are $O(t)$. Moreover the support function at infinity of S corresponds to the vector field $\dot{f}$ on $\partial\mathbb{D}$.

This gives an evidence of the fact that Theorem 1.1 is a rescaled version of the correspondence between quasi-symmetric homeomorphisms and convex surfaces in Anti de Sitter space with bounded principal curvatures.

We believe that even Theorem 1.2 is a rescaled version in this sense of the existence of a K-surface (actually, a foliation by K-surfaces, compare Theorem 1.4) with bounded principal curvature spanning a prescribed quasi-symmetric homeomorphism (for $K \in (-\infty, -1)$). However we leave this problem for a further investigation.

Finally we mention some questions still open in the Minkowski case, which are left for future developments.

- As already remarked, an interesting problem is to characterize complete spacelike entire graphs in Minkowski space in terms of the regularity of the boundary value of the support function. From Theorem 1.2 we know that the surface is complete if this regularity is Zygmund. On the other hand it is not difficult to construct an example of isometric embedding of $\mathbb{H}^2$ in Minkowski space with unbounded principal curvatures, for which the regularity of the support function will necessarily be weaker than Zygmund.
- Another interesting question is to solve Minkowski problem for domains of dependence which are not contained in the future of a point. This would include the case of domains of dependence whose support function is finite only on some subset of $\overline{\mathbb{D}}$ which is obtained as the convex hull of a subset E of $\partial\mathbb{D}$. It is simple to check that there is no solution for the constant curvature problem when E contains 0, 1 or 2 points (the corresponding domains are the whole space, the future of a lightlike plane, or the future of a spacelike line). An existence result in case where E is an interval is given in [17] with some assumption on the smoothness of the support function on E. In our opinion the construction of the support function in this setting is not difficult to generalize, but the barriers we use to prove that the corresponding surfaces are entire graph seem to be ineffective.

Organization of the paper. In Section 2 we give a short review of the different techniques used in the paper. We first recall the main definitions of the theory of future convex sets in Minkowski space, introducing the support function and the generalized Gauss map. We relate those notions to Mess' description of domains of dependence and dual laminations. Minkowski problem is then formulated in terms of Monge-Ampère equations and we collect the main result used in the paper. Finally we discuss the notion of infinitesimal earthquakes and how it is related to the infinitesimal theory of Teichmüller spaces.

In Section 3 we solve Minkowski problem for domains of dependence contained in the future cone of a point, proving Theorem 1.3. In the following section we prove the existence of the foliation by constant curvature surfaces of the same domains, as in Theorem 1.4. In Section 5 we finally prove Theorems 1.1 and 1.2. Those theorems are collected in a unique statement, see Theorem 5.1.

Finally in Appendix A we construct constant curvature surfaces invariant by a parabolic group. The discussion leads to consider two classes of surfaces: the first class (used in the proof of Theorem 1.3) includes surfaces which are entire graphs, whereas the second class gives surfaces which develop a lightlike ray. Although we do not need surfaces of the latter type in this paper, we include a short description for completeness.

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2. Preliminaries and motivation

2.1. **Convex surfaces in Minkowski space.** We denote by $\mathbb{R}^{2,1}$ the $(2+1)$-dimensional Minkowski space, namely $\mathbb{R}^3$ endowed with the bilinear quadratic form

$$\langle x, y\rangle = x^1y^1 + x^2y^2 - x^3y^3\,.$$

We denote by $\mathrm{Isom}(\mathbb{R}^{2,1})$ the group of orientation preserving and time preserving isometries of $\mathbb{R}^{2,1}$. It is know that there is an isomorphism

$$\mathrm{Isom}(\mathbb{R}^{2,1}) \cong \mathrm{SO}_0(2,1) \rtimes \mathbb{R}^{2,1}$$

where $\mathrm{SO}(2,1)$ is the group of linear isometries of Minkowski product, $\mathrm{SO}_0(2,1)$ is the connected component of the identity, and $\mathbb{R}^{2,1}$ acts on itself by translations.

A vector $v \in T_x\mathbb{R}^{2,1} \cong \mathbb{R}^{2,1}$ is called *timelike* (resp. *lightlike*, *spacelike*) if $\langle v, v\rangle < 0$ (resp. $\langle v, v\rangle = 0$, $\langle v, v\rangle > 0$). A plane P is *spacelike* (resp. *lightlike*, *timelike*) if its orthogonal vectors are timelike (resp. lightlike, spacelike). An immersed differentiable surface S in $\mathbb{R}^{2,1}$ is *spacelike* if its tangent plane T_xS is spacelike for every point $x \in S$. In this case, the symmetric 2-tensor induced on S by the Minkowski product is a Riemannian metric. For instance, the hyperboloid

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2,1} : \langle x, x\rangle = -1, x^3 > 0\}$$

is a spacelike embedded surface, and is indeed an isometric copy of hyperbolic space embedded in $\mathbb{R}^{2,1}$. An example of an embedded surface which is not spacelike - but that will still be important in the following - is the de Sitter space

$$\mathrm{d}\mathbb{S}^2 = \{x \in \mathbb{R}^{2,1} : \langle x, x\rangle = +1\}\,.$$

It is easy to see that $\mathrm{d}\mathbb{S}^2$ parametrizes oriented geodesics of $\mathbb{H}^2$. Indeed, for every point x in $\mathrm{d}\mathbb{S}^2$, the orthogonal complement $x^\perp$ is a timelike plane in $\mathbb{R}^{2,1}$, which intersects $\mathbb{H}^2$ in a complete geodesic, and coincides as a set with $(-x)^\perp$.

Given a point $x_0 \in \mathbb{R}^{2,1}$, $\mathrm{I}^+(x_0)$ denotes the *future cone* over x_0, namely the set of points x of $\mathbb{R}^{2,1}$ which are connected to x_0 by a timelike differentiable path (namely, a path with timelike tangent vector at every point) along which the x^3 coordinate is increasing. It is easy to see that $\mathrm{I}^+(x_0)$ is a translate of

$$\mathrm{I}^+(0) = \{x \in \mathbb{R}^{2,1} : (x^1)^2 + (x^2)^2 < (x^3)^2, x^3 > 0\}\,.$$

It is clear from the definition that $\mathbb{H}^2$ parametrizes spacelike linear planes in $\mathbb{R}^{2,1}$. Hence for every spacelike surface S, there is a well-defined *Gauss map*

$$G : S \to \mathbb{H}^2$$

which maps $x \in S$ to the normal of S at x, i.e. the future unit timelike vector orthogonal to T_xS.

As we stated in the introduction, one of the aims of this paper is to classify isometric (or more generally homothetic) immersions of the hyperbolic plane into $\mathbb{R}^{2,1}$ which are contained in the future cone over some point. Mess proved [21] that if the first fundamental form of a spacelike immersion is complete, then the image of the immersion is a spacelike entire graph. So we will deal with convex surfaces in $\mathbb{R}^{2,1}$ which are of the form $\{(x^1, x^2, x^3) \,|\, x^3 = f(x^1, x^2)\}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a convex function satisfying the spacelike condition $||Df||^2 < 1$, where Df is the Euclidean gradient of f. Notice however that if S is a spacelike entire graph in general it might not be complete.

It is convenient to extend the theory to the case of convex entire graphs which are not smooth and possibly contain lightlike rays. Those correspond to convex functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $||Df|| \leq 1$ almost everywhere. We will extend the notion of Gauss map to this more general class.

A *future-convex domain* in $\mathbb{R}^{2,1}$ is a closed convex set which is obtained as the intersection of future half-spaces bounded by spacelike planes. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a convex function

satisfying the condition $||Df|| \leq 1$, then the epigraph of f is a future-convex domain, and conversely the boundary of any future-convex domain is the graph of a convex function as above.

A *support plane* for a future-convex domain D is a plane $P = y + x^\perp$ such that $P \cap \mathrm{int}(D) = \emptyset$ and every translate $P' = P + v$, for v in the future of $x^\perp$, intersects $\mathrm{int}(D)$. A future-convex domain can admit spacelike and lightlike support planes. We define the *spacelike boundary* of D as the subset

$$\partial_s D = \{p \in \partial D : p \text{ belongs to a spacelike support plane of } D\}\,.$$

It can be easily seen that $\partial D \setminus \partial_s D$ is a union of lightlike geodesic rays. So ∂D is a spacelike entire graph if and only if it does not contain lightlike rays.

We can now define the Gauss map for the spacelike boundary of a future-convex set. We allow the Gauss map to be set-valued, namely

$$G(p) = \{x \in \mathbb{H}^2 : p + x^\perp \text{ is a support plane of } D\}\,.$$

By an abuse of notation, we will treat the Gauss map as a usual map with values in $\mathbb{H}^2$. The following proposition (see [22, 23]) has to be interpreted in this sense.

Proposition 2.1. *Given a future-convex domain D in $\mathbb{R}^{2,1}$, the Gauss map of $\partial_s D$ has image a convex subset of $\mathbb{H}^2$. If S is a strictly convex embedded spacelike surface, then its Gauss map is a homeomorphism onto its image.*

2.2. **The support function.** We now introduce another important tool for this work, which is the Lorentzian analogue of the support function of Euclidean convex bodies. Roughly speaking, the support function encodes the information about the support planes of a future-convex domain. This is essentially done by associating to a unit future timelike (or lightlike) vector $x \in \mathbb{H}^2$ the height of the support plane with normal unit vector x.

Definition 2.2. Given a future-convex domain D in $\mathbb{R}^{2,1}$, the *support function* of D is the function $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R} \cup \{\infty\}$ defined by

$$U(x) = \sup_{p \in D} \langle p, x \rangle\,.$$

The following is an immediate property of support functions.

Lemma 2.3. *Let D_1 and D_2 be future-convex domains with support functions U_1 and U_2. Then $U_1 \geq U_2$ if and only if $D_1 \subseteq D_2$.*

It is clear from the definition that U is 1-homogeneous, namely $U(\lambda x) = \lambda U(x)$ for every $\lambda > 0$. Moreover, U is lower semicontinuous, since it is defined as the supremum of continuous functions. It is straightforward to check that, given an isometry in $\mathrm{Isom}(\mathbb{R}^{2,1})$ with linear part A and translation part t, the support function of $D' = AD + t$ is

$$U'(x) = U(A^{-1}x) + \langle x, t \rangle \tag{3}$$

We will mostly consider the restriction of U to the Klein projective model of hyperbolic space, which is the disc

$$\mathbb{D} = \{(z, 1) \in \mathbb{R}^{2,1} : |z| < 1\}\,.$$

This restriction will be denoted by lower case letters, $u = U|_{\mathbb{D}}$, and uniquely determines the 1-homogeneous extension U. We will generally write $u(z)$ instead of $u(z, 1)$. Analogously, also the restriction of U to the hyperboloid, denoted $\bar{u} = U|_{\mathbb{H}^2}$, can be uniquely extended to a 1-homogeneous function, and will be often used in the following.

Remark 2.4. It is easy to relate the restrictions u and $\bar{u}$ of the support function to $\mathbb{D}$ and $\mathbb{H}^2$ respectively. Let us consider the radial projection $\pi : \mathbb{H}^2 \to \mathbb{D}$ defined by

$$\pi(x^1, x^2, x^3) = (x^1/x^3, x^2/x^3, 1)\,.$$

Its inverse is given by

$$\pi^{-1}(z,1) = \left(\frac{z}{\sqrt{1-|z|^2}}, \frac{1}{\sqrt{1-|z|^2}} \right) .$$

Since U is 1-homogeneous, we obtain

$$u(z) = \sqrt{1-|z|^2}\, \bar{u}(\pi^{-1}(z)) .$$

A 1-homogeneous convex function is called *sublinear*.

Lemma 2.5 ([13, Lemma 2.21])**.** *Given a future-convex domain D in $\mathbb{R}^{2,1}$, the support function $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ is sublinear and lower semicontinuous. Conversely, given a sublinear function $\hat{U}$ on $\mathrm{I}^+(0)$ (or equivalently every convex function u on $\mathbb{D}$), consider the lower semicontinuous extension $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R} \cup \{+\infty\}$, which is defined on $\partial \mathrm{I}^+(0)$ as*

$$U(x) = \liminf_{y \to x} \hat{U}(y) .$$

Then U is the support function of a future-convex domain, defined by

$$D = \{p \in \mathbb{R}^{2,1} : \langle p, x \rangle \leq U(x) \text{ for every } x \in \mathrm{I}^+(0)\} .$$

The support function of a future-convex domain D is finite on the image of the Gauss map of $\partial_s D$, since for every point x in $G(\partial_s D)$ there exists a support plane with normal vector x. Observe that $\bar{u}(x) = +\infty$ if $x \in \mathbb{H}^2 \setminus \overline{G(\partial_s D)}$. We will call *support function at infinity* the restriction of U to $\partial\mathbb{D} = \{(z,1) : |z| = 1\}$. Given $z \in \partial\mathbb{D}$, $u(z) < +\infty$ if and only if there exists a lightlike support plane P orthogonal to the lightlike vector $(z,1)$. In this case $-u(z)$ is the intercept of P on the x^3-axis.

Example 2.6. The support function of the hyperboloid $\mathbb{H}^2$ is $U(x) = -\sqrt{|\langle x, x \rangle|}$. Hence its restriction $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ is constant, $\bar{u} \equiv -1$. The support function at infinity is finite, $u|_{\partial\mathbb{D}} \equiv 0$, by an easy computation. For some less elementary examples, see Remark 3.12 and Appendix A.

In this paper, we are mostly concerned with domains of dependence for which the support function at infinity is finite, and is actually bounded. Geometrically, this means that the domain is contained in the future cone over some point.

The following lemma will be useful to compute the value of support functions at infinity.

Lemma 2.7 ([26, Theorem 7.4,7.5])**.** *Let $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ be a sublinear and lower semincontinuous function. Let $c : [0,1] \to \overline{\mathrm{I}^+(0)}$ be a spacelike line such that $x = c(1) \in \partial \mathrm{I}^+(0)$. Then $U(x) = \lim_{t \to 1} U(c(t))$.*

Some of the geometric invariants of $\partial_s D$ can be directly recovered from the support function. This is the content of next lemma. Recall that, given a C^2 embedded spacelike surface S in $\mathbb{R}^{2,1}$, its *shape operator* is a $(1,1)$-tensor which can be defined as

$$B(v) = \nabla_v N ,$$

where ∇ is the Levi-Civita connection of the first fundamental form of $\mathbb{R}^{2,1}$, N is the future unit normal vector field, and v is any tangent vector in $T_p S$. The Gauss Theorem in this Lorentzian setting gives the following relation between the intrinsic curvature K of the first fundamental form and its shape operator, which holds for every point $p \in S$:

$$K = -\det B .$$

Finally, we define the *hyperbolic Hessian* of a function $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ as the $(1,1)$-tensor

$$\mathrm{Hess}\bar{u}(v) = \nabla_v^{\mathbb{H}^2} \operatorname{grad} \bar{u} ,$$

where $\nabla^{\mathbb{H}^2}$ is the Levi-Civita connection of $\mathbb{H}^2$. We denote by $D^2 u$ the Euclidean Hessian of a function u defined on an open subset of $\mathbb{R}^2$. In the following, the identity operator is denoted by I.

Lemma 2.8 ([13, §2.10, 2.13]). *Let D be a future-convex domain in $\mathbb{R}^{2,1}$ and let $G : \partial_s D \to \mathbb{H}^2$ be its Gauss map.*

- *If the support function is C^1, then the intersection of $\partial_s D$ with any spacelike support plane consists of exactly one point. The inverse of the Gauss map G is well defined and is related to the support function of D by the formula*
$$G^{-1}(x) = \operatorname{grad} \bar{u}(x) - \bar{u}(x)x\,, \tag{4}$$
where $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ is the support function restricted to $\mathbb{H}^2$.
- *If the support function is C^2 and the operator $\mathrm{Hess}\bar{u} - \bar{u}\mathrm{I}$ is positive definite, then $\partial_s D$ is a convex C^2-surface. The inverse of its shape operator and its curvature are*
$$B^{-1} = \mathrm{Hess}\bar{u} - \bar{u}\mathrm{I}\,, \tag{5}$$
$$-\frac{1}{K(G^{-1}(x))} = \det(\mathrm{Hess}\bar{u} - \bar{u}\mathrm{I})(x) = (1-|z|^2)^2 \det D^2 u(z)\,, \tag{6}$$
where $z = \pi(x)$ is the point of $\mathbb{D}$ obtained from x by radial projection.

We will often abuse notation and write $G^{-1}(z)$ in place of $G^{-1}(\pi^{-1}(z))$ for $z \in \mathbb{D}$.

2.3. **The boundary value of the support function of an entire graph.** Let S be the boundary of a future-convex domain in $\mathbb{R}^{2,1}$. Denote by $f : \mathbb{R}^2 \to \mathbb{R}$ the function defining S as a graph and $u : \mathbb{D} \to \mathbb{R}$ the support function. We want to show that
$$\lim_{r\to\infty} (f(rz) - r) = -u(z)$$
for every unitary vector $z \in \partial\mathbb{D}$. This clarifies that the asymptotic conditions defined for instance in [29, 9] coincide with those considered here and in [17, 20].

First consider the case where $D = \mathrm{I}^+(p) = graph(f_p)$ where $p = (w_0, a_0)$ and $f_p(z) = |z - w_0| + a_0$. In this case the support function is $u_p(z) = \langle z, w_0\rangle - a_0$. A simple computation shows
$$\begin{aligned} f_p(rz) - r =& \sqrt{|w_0|^2 - 2r\langle z, w_0\rangle + r^2} - r + a_0 \\ =& \frac{-2r\langle z, w_0\rangle + |w_0|^2}{\sqrt{|w_0|^2 - 2r\langle z, w_0\rangle + r^2} + r} + a_0 \longrightarrow -\langle z, w_0\rangle + a_0 = -u_p(z)\,. \end{aligned}$$

Now consider the general case. Imposing that the point $(rz, f(rz))$ lies in the future of the support plane $\{q \in \mathbb{R}^{2,1} \mid \langle q, (z,1)\rangle = u(z)\}$ we get $f(rz) - r \geq -u(z)$. So it is sufficient to prove that $\limsup(f(rz) - r) \leq -u(z)$.

Fix $\epsilon > 0$ and consider the lightlike plane $P = \{q \in \mathbb{R}^{2,1} \mid \langle q, (z,1)\rangle = u(z) - \epsilon\}$. This plane must intersect the future of S. Let $p = (w_0, a_0)$ be a point in this intersection. The cone $\mathrm{I}^+(p)$ is contained in the future of S, hence $f_p \geq f$, where f_p is the graph function for $\mathrm{I}^+(p)$ as above.

In particular, using the computation above for $\mathrm{I}^+(p)$,
$$\limsup_{r\to+\infty}(f(rz) - r) \leq \lim_{r\to+\infty}(f_p(rz) - r) = -\langle z, w_0\rangle + a_0\,.$$
Imposing that p lies on the plane P,
$$-\langle z, w_0\rangle + a_0 = -\langle (z,1), p\rangle = -u(z) + \epsilon\,.$$
Therefore, for any $\epsilon > 0$,
$$\limsup_{r\to+\infty}(f(rz) - r) \leq -u(z) + \epsilon$$
and this concludes our claim, since ϵ is arbitrary.

2.4. **Cauchy surfaces and domains of dependence.** Given a future-convex domain D in $\mathbb{R}^{2,1}$, a *Cauchy surface* for D is a spacelike embedded surface $S \subseteq D$ such that every differentiable inextensible causal path in D (namely, such that its tangent vector is either timelike or lightlike at every point) intersects S in exactly one point. Given an embedded surface S in $\mathbb{R}^{2,1}$, the maximal future-convex domain $D(S)$ such that S is a Cauchy surface for $D(S)$ is the *domain of dependence* of S. It turns out that $D(S)$ is obtained as intersection of future half-spaces bounded by lightlike planes which do not disconnect S.

It is easy to prove the following lemma.

Lemma 2.9. *Let $h : \overline{\mathbb{D}} \to \mathbb{R}$ be the support function of a future-convex domain D, with $h|_{\partial\mathbb{D}} < \infty$. Let $S \subseteq D$ be a convex embedded surface and let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be the support function of S. Then S is a Cauchy surface for D if and only if $h|_{\partial\mathbb{D}} = u|_{\partial\mathbb{D}}$.*

Domains of dependence can be characterized in terms of the support function, see [4, Proposition 2.21].

Lemma 2.10. *Let D be a domain of dependence in $\mathbb{R}^{2,1}$, whose lightlike support planes are determined by the function $\varphi : \partial\mathbb{D} \to \mathbb{R} \cup \{\infty\}$. Then the support function $h : \overline{\mathbb{D}} \to \mathbb{R}$ of D is the convex envelope $h = \mathrm{co}\,(\varphi)$, namely:*

$$h(z) = \sup\{f(z) : f \text{ is an affine function on } \mathbb{D}, f|_{\partial\mathbb{D}} \leq \varphi\}\,.$$

A useful example of support functions of Cauchy surfaces can be obtained by looking at the leaves of the cosmological time of the domain of dependence. Observe that a *timelike distance* can be defined for two points x_1 and $x_2 \in \mathrm{I}^+(x)$ in $\mathbb{R}^{2,1}$, by means of the definition

$$d(x_1, x_2) = \sup_\gamma \int_\gamma \sqrt{|\langle\gamma'(t), \gamma'(t)\rangle|}dt\,,$$

where the supremum is taken over all causal paths γ from x_1 to x_2. This is not a distance though, because it satisfies a reverse triangle inequality; however, $d(x_1, x_2)$ is achieved along the geodesic from x_1 to x_2. Given an embedded spacelike surface S, consider the equidistant surface

$$S_d = \{x \in \mathbb{R}^{2,1} : x \in \mathrm{I}^+(S), d(x, S) = d\}\,,$$

where of course $d(x, S) = \sup_{x' \in S} d(x, x')$. If the support function of S restricted to $\mathbb{H}^2$ is $\bar{u}$, then S_d has support function (see for instance [13])

$$\bar{u}_d(x) = \bar{u}(x) - d\,.$$

This can be applied also for $\partial_s D$, instead of an embedded surface. In this way, we obtain the level sets of the *cosmological time*, namely the function $T : D \to \mathbb{R}$ defined by

$$T(x) = \sup_\gamma \int_\gamma \sqrt{|\langle\gamma'(t), \gamma'(t)\rangle|}dt\,,$$

where the supremum is taken over all causal paths γ in D with future endpoint x. If $\bar{h} : \mathbb{H}^2 \to \mathbb{R}$ is the support function of D, the level sets $L_d = \{T = d\}$ of the cosmological time have support function on the disc $\bar{h}_d(x) = \bar{h}(x) - d$. It can be easily seen that all leaves of the cosmological time of D are Cauchy surfaces for D (although only $C^{1,1}$). Indeed, the support functions $h_d : \mathbb{D} \to \mathbb{R}$ can be computed:

$$h_d(z) = h(z) - d\sqrt{1 - |z|^2}\,.$$

Therefore they all agree with h on $\partial\mathbb{D}$.

2.5. **Dual lamination.** In this paper, we will adopt the following definition of measured geodesic lamination. The equivalence with the most common definition is discussed for instance in [24]. Let $\mathcal{G}$ be the set of (unoriented) geodesics of $\mathbb{H}^2$. The space $\mathcal{G}$ is identified to $((S^1 \times S^1) \setminus diag)/\sim$ where the equivalence relation is defined by $(a,b) \sim (b,a)$. Note that $\mathcal{G}$ has the topology of an open Möbius strip. Given a subset $B \subset \mathbb{H}^2$, we denote by $\mathcal{G}_B$ the set of geodesics of $\mathbb{H}^2$ which intersect B.

Definition 2.11. A geodesic lamination on $\mathbb{H}^2$ is a closed subset of $\mathcal{G}$ such that its elements are pairwise disjoint geodesics of $\mathbb{H}^2$. A measured geodesic lamination is a locally finite Borel measure on $\mathcal{G}$ such that its support is a geodesic lamination.

A measured geodesic lamination is called *discrete* if its support is a discrete set of geodesics. A measured geodesic lamination μ is *bounded* if

$$\sup_I \mu(\mathcal{G}_I) < +\infty\,,$$

where the supremum is taken over all geodesic segments I of lenght at most 1 transverse to the support of the lamination. The *Thurston norm* of a bounded measured geodesic lamination is

$$||\mu||_{Th} = \sup_I \mu(\mathcal{G}_I)\,.$$

Elements of a geodesic lamination are called *leaves*. *Strata* of the geodesic lamination are either leaves or connected components of the complement of the geodesic lamination in $\mathbb{H}^2$.

In his groundbreaking work, Mess associated a domain of dependence D to every measured geodesic lamination μ, in such a way that the support function $h : \mathbb{D} \to \mathbb{R}$ of D is linear on every stratum of μ. Although we do not enter into details here, the measure of μ determines the bending of h. (Recall h is the convex envelope of some lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.) The domain D is determined up to translation in $\mathbb{R}^{2,1}$, and μ is called *dual lamination* of D.

Given $y_0 \in \mathbb{R}^{2,1}$ and $x_0 \in \mathbb{H}^2$, we will denote by $D(\mu, x_0, y_0)$ the domain of dependence having μ as dual laminations and $P = y_0 + x_0^\perp$ as a support plane tangent to the boundary at y_0.

We sketch here the explicit construction of $D(\mu, x_0, y_0)$. In the following, given the oriented geodesic interval $[x_0, x]$ in $\mathbb{H}^2$, $\boldsymbol{\sigma} : \mathcal{G}[x_0, x] \to \mathbb{R}^{2,1}$ is the function which assigns to a geodesic l (intersecting $[x_0, x]$) the corresponding point in $\mathrm{d}\mathbb{S}^2$, namely, the spacelike unit vector in $\mathbb{R}^{2,1}$ orthogonal to l for the Minkowski product, pointing outward with respect to the direction from x_0 to x. Then

$$y(x) = y_0 + \int_{\mathcal{G}[x_0,x]} \boldsymbol{\sigma} d\mu \tag{7}$$

is a point of the regular boundary of $D(\mu, x_0, y_0)$ such that $y(x) + x^\perp$ is a support plane for $D(\mu, x_0, y_0)$, for every $x \in \mathbb{H}^2$ such that the expression in Equation (7) is integrable. The image of the Gauss map of the regular boundary of $D(\mu, x_0, y_0)$ is composed precisely of those $x \in \mathbb{H}^2$ which satisfy this integrability condition.

In the following proposition, we give an explicit expression for the support function of the domain of dependence $D(\mu, x_0, y_0)$ we constructed. By an abuse of notation, given two points $x_0, x \in \mathrm{I}^+(0)$, we will denote by $[x_0, x]$ the geodesic interval of $\mathbb{H}^2$ obtained by projecting to the hyperboloid $\mathbb{H}^2 \subset \mathbb{R}^{2,1}$ the line segment from x_0 to x.

Proposition 2.12. *Suppose D is a domain of dependence in $\mathbb{R}^{2,1}$ with dual lamination μ and such that the plane $P = y_0 + (x_0)^\perp$ is a support plane for D, for $y_0 \in \partial_s D$ and $x_0 \in \mathbb{H}^2$. Then the support function $H : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ of D is:*

$$H(x) = \langle x, y_0 \rangle + \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu\,. \tag{8}$$

Indeed, the expression in Equation (8) holds for $x \in \mathbb{H}^2$ by Equation (7). Since the expression is 1-homogeneous, it is clear that it holds for every $x \in \mathrm{I}^+(0)$. Using Lemma 2.7, the formula holds also for the lower semicontinuous extension to $\partial \mathrm{I}^+(0)$.

It is easily seen from Equation (8) that the support function $h : \mathbb{D} \to \mathbb{R}$ (which is the restriction of H to $\mathbb{D}$) is affine on each stratum of μ. In [21, 3] it was proved that every domain of dependence can be obtained by the above construction. Hence a dual lamination is uniquely associated to every domain of dependence.

The work of Mess ([21]) mostly dealt with domains of dependence which are invariant under a discrete group of isometries $\Gamma < \mathrm{Isom}(\mathbb{R}^{2,1})$, whose linear part is a cocompact Fuchsian group. We resume here some results.

Proposition 2.13. *Let D be a domain of dependence in $\mathbb{R}^{2,1}$ with dual lamination μ. The measured geodesic lamination μ is invariant under a cocompact Fuchsian group G if and only if D is invariant under a discrete group $\Gamma < \mathrm{Isom}(\mathbb{R}^{2,1})$ such that the projection of Γ to $\mathrm{SO}(2,1)$ is an isomorphism onto G. In this case, assuming $P = y_0 + (x_0)^\perp$ is a support plane for D, for $y_0 \in \partial_s D$ and $x_0 \in \mathbb{H}^2$, the translation part of an element $g \in G$ is:*

$$t_g = \int_{\mathcal{G}[x_0, g(x_0)]} \boldsymbol{\sigma} d\mu \,.$$

2.6. **Monge-Ampère equations.** Given a smooth strictly convex spacelike surface S in $\mathbb{R}^{2,1}$, let $U : \mathrm{I}^+(0) \to \mathbb{R}$ be the support function of S and let u be its restriction to $\mathbb{D} = \mathrm{I}^+(0) \cap \{x^3 = 1\}$. Given a point $z \in D$, let $x = \pi^{-1}(z) \in \mathbb{H}^2$. The curvature of S is given by (see Lemma 2.8)

$$-\frac{1}{K(G^{-1}(x))} = (1 - |z|^2)^2 \det D^2 u(z) \,,$$

where $G : S \to \mathbb{H}^2$ is the Gauss map, which is a diffeomorphism. For K-surfaces, namely surfaces with constant curvature equal to $K \in (-\infty, 0)$, the support function satisfies the Monge-Ampère equation

$$\det D^2 u(z) = \frac{1}{|K|}(1 - |z|^2)^{-2} \,. \tag{9}$$

More generally, the Minkowski problem consists of finding a convex surface with prescribed curvature function on the image of the Gauss map. Given a smooth function $\psi : \mathbb{D} \to \mathbb{R}$, the support function of a surface with curvature $K(G^{-1}(z)) = -\psi(z)$ solves the Monge-Ampère equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2} \tag{MA}$$

We review here some key facts of Monge-Ampère theory. Given a convex function $u : \Omega \to \mathbb{R}$ for Ω a convex domain in $\mathbb{R}^2$, we define the normal mapping of u as the set-valued function N_u whose value at a point $\bar{w} \in \Omega$ is:

$$N_u(\bar{w}) = \{Df : f \text{ affine; } graph(f) \text{ is a support plane for } graph(u), (\bar{w}, u(\bar{w})) \in graph(f)\} \,.$$

In general $N_u(\bar{w})$ is a convex set; if u is differentiable at $\bar{w}$, then $N_u(\bar{w}) = \{Du(\bar{w})\}$. We define the Monge-Ampère measure on the collection of Borel subsets ω of $\mathbb{R}^2$:

$$MA_u(\omega) = \mathcal{L}(N_u(\omega))$$

where $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}^2$.

Lemma 2.14 ([30, Lemma 2.3])**.** *If u is a C^2 function, then*

$$MA_u(\omega) = \mathcal{L}(Du(\omega)) = \int_\omega (\det D^2 u) d\mathcal{L} \,.$$

In general, $\int_\omega (\partial^2 u) d\mathcal{L}$ is the regular part of the Lebesgue decomposition of $MA_u(\omega)$, where we set

$$\partial^2 u(\bar{w}) = \begin{cases} \det D^2_{\bar{w}} u & \text{if } u \text{ is twice-differentiable at } \bar{w} \\ 0 & \text{otherwise} \end{cases} .$$

Definition 2.15. Given a nonnegative measure ν on Ω, we say a convex function $u : \Omega \to \mathbb{R}$ is a generalized solution to the Monge-Ampère equation

$$\det D^2 u = \nu \tag{10}$$

if $MA_u(\omega) = \nu(\omega)$ for all Borel subsets ω. In particular, given an integrable function $f : \Omega \to \mathbb{R}$, u is a generalized solution to the equation $\det D^2 u = f$ if and only if, for all ω,

$$MA_u(\omega) = \int_\omega f d\mathcal{L} .$$

We collect here, without proofs, some facts which will be used in the following. Unless explicitly stated, the results hold in $\mathbb{R}^n$, although we are only interested in $n = 2$. Recall that, by Aleksandrov Theorem, a convex function u on Ω is twice-differentiable almost everywhere.

Lemma 2.16 ([30, Lemma 2.2])**.** *Given a sequence of convex functions u_n which converges uniformly on compact sets to u, the Monge-Ampère measure MA_{u_n} converges weakly to MA_u.*

Theorem 2.17 (Comparison principle, [30, 18])**.** *Given a bounded convex domain Ω and two convex functions u, v defined on $\overline{\Omega}$, if $MA_u(\omega) \leq MA_v(\omega)$ for every Borel subset ω, then*

$$\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v) .$$

Corollary 2.18. *Given two generalized solutions $u_1, u_2 \in C^0(\overline{\Omega})$ to the Monge-Ampère equation $\det D^2(u_i) = \nu$ on a bounded convex domain Ω, if $u_1 \equiv u_2$ on $\partial\Omega$, then $u_1 \equiv u_2$ on Ω.*

Theorem 2.19 ([8, Lemma 3], [25])**.** *Given a bounded convex domain Ω, let u be a C^4 solution to $\det D^2 u = f$ defined on $\overline{\Omega}$ which is constant on $\partial\Omega$. There is an estimate on the second derivatives of u at $x \in \Omega$ which depends only on*

$$\max_{\Omega} \left\{ |u|, ||Du||^2, ||D \log(f)||^2, \sum_{i,j} \partial_{ij} (\log(f))^2 \right\}$$

and on the distance of x to $\partial\Omega$.

The following property will be used repeatedly in the paper, and is a peculiar property of dimension $n = 2$.

Theorem 2.20 (Aleksandrov-Heinz)**.** *A generalized solution to $\det D^2 u = f$ on a domain $\Omega \subset \mathbb{R}^2$ with $f > 0$ must be strictly convex.*

We will use this theorem to prove an a priori estimate of the C^2-norm of support functions of surfaces of constant curvature in Minkowski space in terms of the C^0-norm. Although this result is well-known we sketch a proof for the convenience of the reader.

Lemma 2.21. *Let $u_n : \mathbb{D} \to \mathbb{R}$ be a sequence of smooth solutions of the Monge-Ampère equation*

$$\det D^2(u_n) = \frac{1}{|K|}(1 - |z|^2)^{-2}$$

uniformly bounded on $\mathbb{D}$. Then $||u_n||_{C^2(\Omega)}$ is uniformly bounded on any compact domain Ω contained in $\mathbb{D}$.

Proof. Assume that the conclusion is false and that there is a subsequence (which we still denote by u_n by a slight abuse of notation) for which the C^2-norm goes to infinity. Hence it suffices to show that there exists a further subsequence u_{n_k} for which $||u_{n_k}||_{C^2(\Omega)}$ is bounded. Take Ω' such that $\Omega \subset\subset \Omega' \subset\subset \mathbb{D}$. Using the uniform bound on $||u_n||_{C^0(\mathbb{D})}$ and the convexity, one can derive that the C^1-norms $||u_n||_{C^1(\overline{\Omega}')}$ are uniformly bounded by a constant C. By Ascoli-Arzelà theorem, we can extract a subsequence which converges uniformly on compact subsets of Ω'. Let u_∞ be the limit function. By Lemma 2.16, u_∞ is a generalized solution to

$$\det D^2(u_\infty) = \frac{1}{|K|}(1-|z|^2)^{-2}$$

and u_∞ is strictly convex, by Theorem 2.20.

For any $z \in \Omega$ and $n \geq 0$ we can fix an affine function $f_{n,z}$ such that $v_{n,z} = u_n + f_{n,z}$ takes its minimum at z and $v_{n,z}(z) = 0$. We claim that there are $\epsilon_0 > 0$ and $r_0 > 0$ such that

- $\min_{\partial\Omega'} v_{n,z} \geq 2\epsilon_0$ for any $n \geq 0$ and $z \in \Omega$.
- $\max_{B(z,r_0)} v_{n,z} \leq \epsilon_0$ for any $n \geq 0$ and $z \in \Omega$.

First let us show how the claim implies the statement. Indeed for any z and n consider the domain $U_{n,z} = \{z \in \Omega' \,|\, v_{n,p}(z) \leq \epsilon_0\}$. We have that $U_{n,z} \subset\subset \Omega'$ by the first point of the claim. In particular, $v_{n,z}$ is constant equal to ϵ_0 along the boundary of $U_{n,z}$. On the other hand the second point of the claim implies that the distance of z from $\partial U_{n,z}$ is at least r_0. So by Theorem 2.19 there is a constant C' depending on C and r_0 such that $||D^2u_n(z)|| = ||D^2v_{n,z}(z)|| < C'$ for all $z \in \Omega'$ and $n \geq 0$.

To prove the claim we argue by contradiction. Suppose there exist sequences $z_n, z_n' \in \Omega$ such that, defining $2\epsilon_n = \min_{\partial\Omega'} v_{n,z_n}$,

- $||z_n - z_n'|| \to 0$;
- $v_{n,z_n}(z_n') > \epsilon_n$.

Up to passing to a subsequence we may suppose that $z_n \to z_\infty$, so that $z_n' \to z_\infty$ as well. As the C^1-norm of u_n is bounded, the C^1-norm of $f_{n,z}$ is uniformly bounded for any $z \in \Omega$ and $n \geq 0$, so we may suppose that f_{n,z_n} converges to an affine function f_∞. Therefore v_{n,z_n} converges to $v_\infty = u_\infty + f_\infty$. Since, as already observed, u_∞ is strictly convex, v_∞ is strictly convex as well.

As $\lim v_{n,z_n}(z_n') = v_\infty(z_\infty) = \lim v_{n,z_n}(z_n) = 0$ we conclude that $\epsilon_n \to 0$, so that $\min_{\partial\Omega'} v_\infty = 0$. This gives a contradiction with the strict convexity of v_∞, and thus concludes the proof. □

A refinement of the above arguments leads to the proof of the regularity of strictly convex solutions of Monge-Ampère equation.

Theorem 2.22 ([30, Theorem 3.1])**.** *Let u be a strictly convex generalized solution to $\det D^2u = f$ on a bounded convex domain Ω with smooth boundary. If $f > 0$ and f is smooth, then u is smooth.*

2.7. **Universal Teichmüller space and Zygmund fields.** In this section, we want to introduce the notion of Zygmund class, which will be the relevant boundary regularity for the support functions of convex surfaces with bounded principal curvatures. Since support functions are defined as 1-homogeneous functions, we first show that a vector field on S^1 defines a 1-homogeneous function on the boundary of the null-cone in a natural way.

We consider the boundary at infinity $\partial_\infty\mathbb{H}^2$ of $\mathbb{H}^2$, as $\mathbb{P}(N) \cong S^1$, where $N = \partial\mathrm{I}^+(0)\setminus\{0\}$. In particular, we will use vector fields on S^1 to define 1-homogeneous functions on $\partial\mathrm{I}^+(0)$. We want to show that this is well-defined, i.e. does not depend on the choice of a section $S^1 \to N$.

Lemma 2.23. *There is a 1-to-1 correspondence between vector fields X on S^1 and 1-homogeneous functions $H : N \to \mathbb{R}$ satisfying the following property: if $\gamma : S^1 \to N$ is*

any C^1 spacelike section of the projection $N \to S^1$ and v is the unit tangent vector field to γ, then

$$\gamma_*(X(\xi)) = H(\gamma(\xi))v(\gamma(\xi))\,. \tag{11}$$

Proof. Consider coordinates (x^1, x^2, x^3) on $\mathbb{R}^{2,1}$ and z, θ on N given by

$$\phi : (r, \theta) \to (r\cos\theta, r\sin\theta, r) \in N.$$

In these coordinates, the restriction of the Minkowski metric to N takes the (degenerate) form

$$g = r^2 d\theta^2 \tag{12}$$

We take γ_1 to be the section $\gamma_1(\theta) = (1, \theta)$. Namely, the image of γ_1 is $N \cap \{x^3 = 1\}$. Any other section γ_2 is of the form $\gamma_2(\theta) = (r(\theta), \theta)$ and is obtained as $\gamma_2 = f \circ \gamma_1$ by a radial map $f(1, \theta) = (r(\theta), \theta)$. Let X be a vector field on S^1. We define a 1-homogeneous function H such that $(\gamma_1)_*(X(\theta)) = H(1, \theta)v_1$ and compute

$$(\gamma_2)_*(X(\theta)) = f_*(H(1, \theta)v_1) = H(1, \theta)f_*(v_1).$$

Now $f_*(v_1)$ is a tangent vector to $\gamma_2(S^1)$ whose norm (recall the form (12) of the metric) is $r(\theta)$. Therefore

$$(\gamma_2)_*(X(\theta)) = H(1, \theta)r(\theta)v_2 = H(r(\theta), \theta)v_2$$

where v_2 is the unit tangent vector. Conversely, given any 1-homogeneous function, (11) defines a vector field on S^1 which does not depend on the choice of γ. □

We now introduce a class of vector fields on S^1 whose regularity is specially interesting for this article, namely the Zygmund fields. To do so, we first introduce the notion of earthquake of $\mathbb{H}^2$ and quasi-symmetric homeomorphisms of the circle. A Zygmund field is a vector field which corresponds to an infinitesimal deformation of quasi-symmetric homeomorphisms, in a suitable sense.

Definition 2.24. A surjective map $E : \mathbb{H}^2 \to \mathbb{H}^2$ is a left earthquake if it is an isometry on the strata of a geodesic lamination of $\mathbb{H}^2$ and, for every pair of strata S and S', the composition

$$(E|_S)^{-1} \circ (E|_{S'})$$

is a hyperbolic translation whose axis weakly separates S and S' and such that S' is translated on the left as seen from S.

Let us observe that E is in general not continuous. Given an earthquake E, there is a measured geodesic lamination associated to E, called the *earthquake measure*. See [28]. The earthquake measure μ determines E up to post-composition with an hyperbolic isometry (in other words, up to the choice of the image of one stratum), and up to the ambiguity on the weighted leaves of the lamination. Hence an earthquake whose earthquake measure is μ will be denoted by E^μ.

The following theorem was proved by Thurston [28].

Theorem 2.25. *Any earthquake $E : \mathbb{H}^2 \to \mathbb{H}^2$ extends to an orientation-preserving homeomorphism of $S^1 = \partial_\infty \mathbb{H}^2$. Conversely, every orientation-preserving homeomorphism of S^1 is induced by a unique earthquake of $\mathbb{H}^2$.*

We are now going to present a characterization of earthquakes whose measured geodesic lamination is bounded in terms of their extension to the boundary at infinity, due to Gardiner-Hu-Lakic in [14]. Given an orientation preserving homeomorphism $T : S^1 \to S^1$, we consider a lifting of $\tilde{T} : \mathbb{R} \to \mathbb{R}$ to the universal covering so that

$$T(e^{i\theta}) = e^{i\tilde{T}(\theta)}$$

Definition 2.26. An orientation-preserving homeomorphism $T : S^1 \to S^1$ is quasi-symmetric if there is a constant C such that

$$\frac{1}{C} < \left| \frac{\tilde{T}(\theta + h) - \tilde{T}(\theta)}{\tilde{T}(\theta) - \tilde{T}(\theta - h)} \right| < C \,, \tag{13}$$

for all $\theta, h \in \mathbb{R}$.

The space of quasi-symmetric homeomorphisms of S^1, up to the action of Möbius transformation by post-compositions, is called *Universal Teichmüller space*. It is endowed with a smooth (actually, complex) structure, see for instance [15, §16]. We are now going to discuss briefly the tangent space to Universal Teichmüller space. We will use the notation $\hat{\varphi}$ for a vector field on S^1 and φ for the function from S^1 to $\mathbb{R}$ which corresponds to $\hat{\varphi}$ under the standard trivialization of TS^1. In the following definition, we regard tangent vectors to S^1 as elements of $\mathbb{C}$. Hence, $\hat{\varphi}(z) = iz\varphi(z)$ for every $z \in \partial\mathbb{D}$.

Definition 2.27. A function $\varphi : S^1 \to \mathbb{R}$ is in the Zygmund class if there is a constant C such that

$$|\varphi(e^{i(\theta+h)}) + \varphi(e^{i(\theta-h)}) - 2\varphi(e^{i\theta})| < C|h| \tag{14}$$

for all $\theta, h \in \mathbb{R}$. A vector field $\hat{\varphi}$ on S^1 is a Zygmund field if the associated function φ is in the Zygmund class.

Functions in the Zygmund class are α-Hölder for any $\alpha \in (0, 1)$, but in general they are not Lipschitz. The vector space of Zygmund fields, quotiented by the subspace of vector fields which are extensions on S^1 of Killing vector fields on $\mathbb{H}^2$, is precisely the tangent space at the identity of Universal Teichmüller space ([15]).

Example 2.28. Let μ be the measured geodesic lamination whose support consists of a single geodesic l, with weight 1. Then, once a point $x_0 \in \mathbb{H}^2 \setminus l$ is fixed, it is easy to describe the earthquake along μ:

$$E_l([\eta]) = \begin{cases} [\eta] & \text{if } x_0 \text{ and } [\eta] \text{ are in the same component of } (\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \setminus \bar{l} \\ [A^l(1)(\eta)] & \text{otherwise} \end{cases} \tag{15}$$

for any $\eta \in N$, where $A^l(t) \in \mathrm{SO}(2, 1)$ induces the hyperbolic isometry of $\mathbb{H}^2$ which translates on the left (as seen from x_0) along the geodesic l by lenght t.

Hence the 1-homogeneous function H associated to the Zygmund field $\dot{E}_l = \frac{d}{dt}\big|_{t=0} E_{tl}$ (as in Lemma 2.23) has the following expression, for any section $\gamma : S^1 \to N$:

$$H(\gamma(\xi)) = \begin{cases} 0 & \text{if } x_0 \text{ and } [\gamma(\xi)] \text{ are in the same component of } (\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \setminus \bar{l} \\ \langle \dot{A}^l(\gamma(\xi)), v(\gamma(\xi)) \rangle & \text{otherwise} \end{cases} \tag{16}$$

where v is any unit spacelike tangent vector field to N, in the counterclockwise orientation. Under the standard identification of S^1 with $\partial\mathbb{D}$, we obtain, for $\eta \in \partial\mathbb{D}$

$$\dot{E}_l(\eta) = \langle \dot{A}^l(\eta), v \rangle v \,.$$

where v is now the unit tangent vector to $\partial\mathbb{D}$.

We will use the following result by Gardiner-Hu-Lakic, see [14] or [24, Appendix].

Theorem 2.29. *Given a bounded measured geodesic lamination μ and a fixed point x_0 which does not lie on any weighted leaf of μ, the integral*

$$\dot{E}^\mu(\eta) = \int_{\mathcal{G}} \dot{E}_l(\eta) d\mu(l) \tag{17}$$

converges for every $\eta \in \partial\mathbb{D}$ and defines a Zygmund field $\hat{\varphi}$ on S^1, which corresponds to the infinitesimal earthquake

$$\dot{E}^\mu = \left.\frac{d}{dt}\right|_{t=0} E^{t\mu}\,.$$

Conversely, for every Zygmund field $\hat{\varphi}$ on $S^1 = \partial\mathbb{D}$, there exists a bounded measured geodesic lamination μ such that $\hat{\varphi}$ is the infinitesimal earthquake along μ, namely

$$\hat{\varphi} = \left.\frac{d}{dt}\right|_{t=0} E^{t\mu}\,,$$

up to an infinitesimal Möbius transformation.

Analogously to the case of earthquakes, although the infinitesimal earthquake $\dot{E}^\mu$ is not continuous in $\mathbb{H}^2$, its boundary value is a continuous field. Theorem 2.29 can be regarded as the infinitesimal version of the Theorem ([14], anticipated by Thurston) which states that every quasi-symmetric homeomorphism is the extension on the boundary of an earthquake with bounded measured geodesic lamination.

3. The Minkowski problem in Minkowski space

The aim of this section is to prove that, for every domain of dependence in $\mathbb{R}^{2,1}$ contained in the cone over a point, there exists a unique smooth Cauchy surface with prescribed (à la Minkowski) negative curvature, which is an entire graph. Equivalently, the main statement is the following.

Theorem 3.1. *Given a bounded lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and a smooth function $\psi : \mathbb{D} \to [a, b]$ for some $0 < a < b < +\infty$, there exists a unique smooth spacelike surface S in $\mathbb{R}^{2,1}$ with support function at infinity φ and curvature $K(G^{-1}(x)) = -\psi(x)$. Moreover, S is an entire graph and is contained in the past of the $(1/\sqrt{\inf\psi})$-level surface of the cosmological time of the domain of dependence with support function $h = \mathrm{co}\,(\varphi)$.*

The proof will be split in several steps. In Subsection 3.1 we construct a solution to the Monge-Ampère equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2} \tag{MA}$$

with the prescribed boundary condition at infinity

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

In Subsection 3.2 we study the behavior of Cauchy surfaces in terms of the support functions, and we use this condition to prove uniqueness by applying the theory of Monge-Ampère equations. Finally, in Subsection 3.3 we prove that the surface is not tangent to the boundary of the domain of dependence, and hence is a spacelike entire graph.

3.1. Existence of solutions.

The surface S will be obtained as a limit of surfaces S_Γ invariant under the action of discrete groups $\Gamma < \mathrm{Isom}(\mathbb{R}^{2,1})$, isomorphic to the fundamental group of a closed surface, acting freely and properly discontinuously on some future-convex domain in $\mathbb{R}^{2,1}$ for which S_Γ is a Cauchy surface. Indeed, such a surface S_Γ can be obtained as the lift to the universal cover of a closed Cauchy surface S_Γ/Γ in a maximal globally hyperbolic spacetime $D(S_\Gamma)/\Gamma$, and the existence of surfaces with prescribed curvature in such spacetimes is guaranteed by results of Barbot-Béguin-Zeghib in [2].

In this subsection we prove the following existence result for the Monge-Ampère equation (MA).

Theorem 3.2. *Given a bounded lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and a smooth function $\psi : \mathbb{D} \to [a, b]$ for some $0 < a < b < +\infty$, there exists a smooth solution $u : \mathbb{D} \to \mathbb{R}$ to the equation*

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2} \tag{MA}$$

such that u extends to a lower semicontinuous function on $\overline{\mathbb{D}}$ with

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

Moreover, u satisfies the inequality

$$h(z) - C\sqrt{1 - |z|^2} \leq u(z) \leq h(z)\,, \tag{CT}$$

where h is the convex envelope of φ and $C = 1/\sqrt{\inf \psi}$.

There are several notions of convergence of measured geodesic laminations, as discussed for instance in [24]. Recall in Definition 2.11 we defined a measured geodesic lamination as a locally finite Borel measure on the set of (unoriented) geodesics $\mathcal{G}$ of $\mathbb{H}^2$, with support a closed set of pairwise disjoint geodesics.

Definition 3.3. A sequence $\{\mu_n\}_n$ of measured geodesics laminations converges in the weak* topology to a measured geodesic lamination, $\mu_n \rightharpoonup \mu$, if

$$\lim_{n\to\infty} \int_{\mathcal{G}} f d\mu_n = \int_{\mathcal{G}} f d\mu$$

for every $f \in C^0_0(\mathcal{G})$.

A stronger notion of convergence is given by the Fréchet topology on the space of measured geodesic laminations. This is defined by interpreting a measured geodesic lamination as a linear functional on the space of α-Hölder functions (for every $\alpha \in (0, 1)$) with compact support. The Fréchet topology hence comes from a family of α-seminorms obtained by considering the supremum of the linear functional on α-Hölder functions supported in a box of geodesics $Q = [a, b] \times [c, d]$, where (a, b, c, d) is a symmetric quadruple of points in $\partial\mathbb{D}$. See [24] for more details.

We are going to approximate a measured geodesic lamination in the weak* topology by measured geodesic laminations which are invariant under the action of a cocompact Fuchsian group.

Lemma 3.4. *Given a measured geodesic lamination μ, there exists a sequence of measured geodesic laminations μ_n such that μ_n is invariant under a torsion-free cocompact Fuchsian group $G_n < \mathrm{Isom}(\mathbb{H}^2)$ and $\mu_n \rightharpoonup \mu$.*

Proof. Recall that $\mathcal{G}_B$ denotes the set of geodesics of $\mathbb{H}^2$ which intersect the subset $B \subseteq \mathbb{H}^2$. We construct the approximating sequence in several steps.

Step 1. We show there is a sequence of discrete measured geodesic laminations μ_n which converge to μ in the weak* topology. In [24, §7] it was proved that, if μ is bounded, there exists a sequence of discrete measured geodesic laminations which converge to μ in the Fréchet topology, which implies weak* convergence. So, assume μ is not bounded. We define ν_n by $\nu_n(A) = \mu(A \cap \mathcal{G}_{B(0,n)})$, i.e. the support of ν_n consists of the geodesics of μ which intersect $B(0, n)$. By the results in [24, §7], for every n, there exists a sequence $(\nu_{n,m})_m$ which converges to ν_n in the Fréchet sense. As a consequence of Fréchet convergence, for every n we can find $m = m(n)$ such that

$$\sup_{f \in C^0_0(\mathcal{G}_{B(0,n)})} \left| \int_{\mathcal{G}} f d\nu_n - \int_{\mathcal{G}} f d\nu_{n,m(n)} \right| \leq \frac{1}{n}\,.$$

It follows that, for every f compactly supported in $\mathcal{G}$, if $supp(f) \subset \mathcal{G}_{B(0,n_0)}$, then for $n \geq n_0$

$$\left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\nu_{n,m(n)} \right| = \left| \int_{\mathcal{G}} f d\nu_n - \int_{\mathcal{G}} f d\nu_{n,m(n)} \right| \xrightarrow{n\to\infty} 0\,.$$

Hence $\mu_n := \nu_{n,m(n)}$ gives the required approximation.

Step 2. We now modify the sequence μ_n to obtain a sequence $\mu'_n \rightharpoonup \mu$ of finite measured laminations with ultraparallel geodesics. This step is necessary because we would like to construct a fundamental polygon P_n for a cocompact Fuchsian group, so that the edges of P_n intersect the geodesics of μ_n orthogonally. We will choose edges of P_n so as to "separate" the endpoints of the geodesics of μ_n. However, for this purpose one needs that the geodesics of μ_n are ultraparallel, and thus we first need to modify μ_n to a new sequence μ'_n. We can assume the discrete laminations μ_n constructed in Step 1 are finite (namely they consist of a finite number of weighted geodesics), by taking the intersection with $\mathcal{G}_{B(0,n)}$. Let $d_{\mathcal{G}}$ be the distance induced by a Riemannian metric on $\mathcal{G}$. Suppose the leaves of μ_n are $l^n_1, \dots, l^n_{p(n)}$ with weights $a^n_1, \dots, a^n_{p(n)}$. Then we construct a finite lamination μ'_n by replacing $l^n_1, \dots, l^n_{p(n)}$ by leaves $k^n_1, \dots, k^n_{p(n)}$ so that

- k^n_i and k^n_j are ultraparallel for every $i \neq j$;
- $d_{\mathcal{G}}(l^n_i, k^n_i) \leq 1/n$;
- The weight of k^n_i is a^n_i.

Let us show that $(\mu'_n)_n$ converges weak* to μ. For this purpose, fix a function f with $supp(f) \subset \mathcal{G}_{B(0,n_0)}$ for some n_0. Fix $\epsilon > 0$. Since f is uniformly continuous, there exists n_1 such that if $d_{\mathcal{G}}(l,k) < 1/n_1$, then $|f(l) - f(k)| < \epsilon$. We have

$$\left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\mu'_n \right| \leq \left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\mu_n \right| + \left| \int_{\mathcal{G}} f d\mu_n - \int_{\mathcal{G}} f d\nu'_{n,n} \right|\,.$$

By construction, there exists n_2 such that the first term in the RHS is smaller than ϵ provided $n \geq n_2$. Now for every n, if $m \geq \max\{n_0, n_1\}$,

$$\left| \int_{\mathcal{G}} f d\mu_n - \int_{\mathcal{G}} f d\nu'_{n,m} \right| = \sum_{i=1}^{p(n)} \left(f(l^n_i) - f(k^n_i) \right) a^n_i \leq \epsilon \mu_n(\mathcal{G}_{(B(0,n_0))})\,.$$

Since $\mu_n \rightharpoonup \mu$, there exists a constant C such that $\mu_n(\mathcal{G}_{(B(0,n_0))}) \leq C$ for $n \geq n_3$. In conclusion, if $n \geq \max\{n_0, n_1, n_2, n_3\}$, then

$$\left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\mu'_n \right| \leq (1 + C)\epsilon\,.$$

Step 3. We claim it is possible to find a polygon P_n with the following properties:

- P_n contains the ball $B(0,n)$;
- The angles of P_n are $\pi/2$;
- P_n intersects the leaves of μ'_n orthogonally.

We construct the polygon P_n in the following way. For every point $z \in \partial\mathbb{D}$ which is limit of a leaf k^n_i of μ'_n, we pick a geodesic orthogonal to k^n_i which separates z from $B(0,n)$ and from all the other limit points of μ'_n. Let $\{g_1, \dots, g_p\}$ be the geodesics obtained in this way. Replacing the g_i by other geodesics further from $B(0,n)$, we can assume the geodesics $g_1, \dots, g_p$ are pairwise ultraparallel. See Figures 1 and 2.

We can now extend the family of geodesics $\{g_1, \dots, g_p\}$ to a larger family $\{g'_1, \dots, g'_{p'}\}$ satisfying:

- $\{g_1, \dots, g_p\} \subset \{g'_1, \dots, g'_{p'}\}$;
- $g'_1, \dots, g'_{p'}$ are contained in $\mathbb{H}^2 \setminus B(0,n)$
- $g'_1, \dots, g'_{p'}$ are pairwise ultraparallel;
- No geodesic g'_i separates two geodesics g'_j and g'_k in the family (so we can assume the indices in $\{g'_1, \dots, g'_{p'}\}$ are ordered counterclockwise);

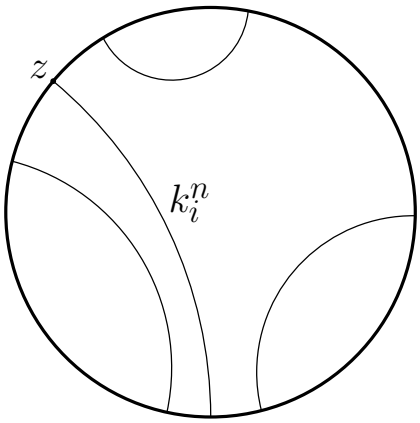

FIGURE 1. Start from a finite geodesic laminations with leaves k_i^n.

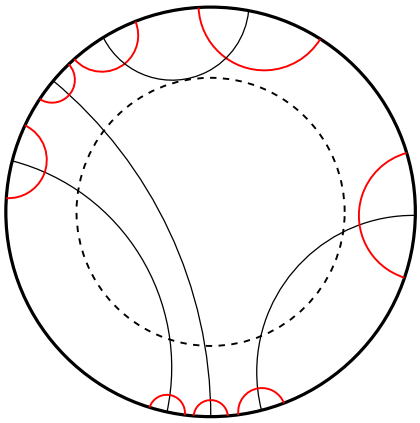

FIGURE 2. The construction of the geodesics $g_1, \dots, g_p$.

- The geodesics h_i orthogonal to g'_i and g'_{i+1} (if the indices i are considered $\bmod p'$) are contained in $\mathbb{H}^2 \setminus B(0,n)$.

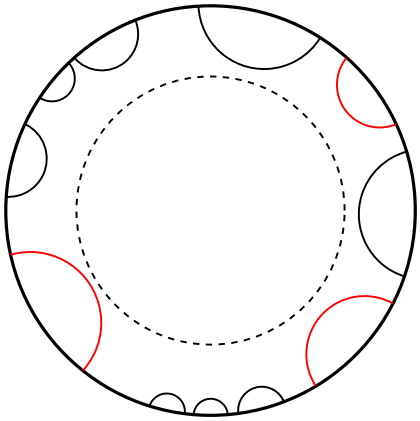

FIGURE 3. The geodesics in $\{g'_1, \dots, g'_{p'}\} \setminus \{g_1, \dots, g_p\}$.

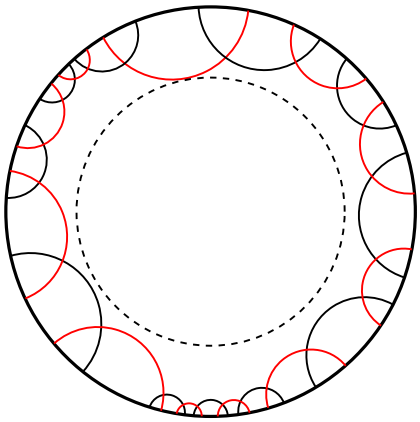

FIGURE 4. The geodesics $h_1, \dots, h_{p'}$ orthogonal to $g'_1, \dots, g'_{p'}$.

The reader can compare with Figures 3 and 4. It is clear that the polygon P_n (Figure 5) given by the connected component of $\mathbb{H}^2 \setminus \{g'_1, \dots, g'_{p'}, h_1, \dots, h_{p'}\}$ containing $B(0,n)$ satisfies the given properties.

Step 4. We finally construct a sequence $\mu''_n \rightharpoonup \mu$ with μ''_n invariant under the action of a torsion-free cocompact Fuchsian group. We consider the discrete group of isometries of $\mathbb{H}^2$ generated by reflections in the sides of the polygon P_n constructed in Step 3. The index 2 subgroup $\overline{G}_n$ of orientation-preserving isometries is a discrete cocompact group and it is well-known that $\overline{G}_n$ contains a finite index torsion-free cocompact Fuchsian group G_n. We define μ''_n as the G_n-orbit of μ'_n. It turns out that μ''_n is a measured geodesic lamination since the leaves of μ'_n intersect the sides of P_n orthogonally. Note that μ''_n is obtained by modifying μ'_n only in the complement of $B(0,n)$, since by construction $B(0,n) \subset P_n$. Hence

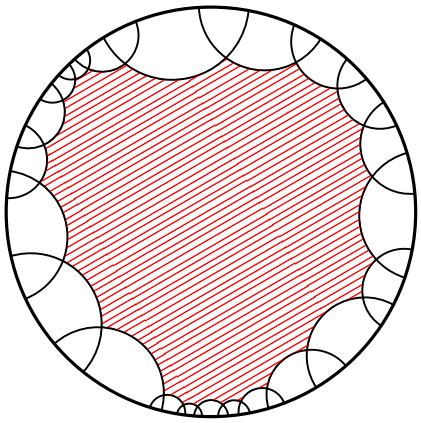

FIGURE 5. The polygon P_n.

it is clear that, if n_0 is such that $supp(f) \subset \mathcal{G}_{B(0,n_0)}$, then $\int_{\mathcal{G}} f d\mu'_n = \int_{\mathcal{G}} f d\mu''_n$ for $n > n_0$, and thus $\mu''_n \rightharpoonup \mu$. □

Lemma 3.5. *Given a sequence μ_n of measured geodesic laminations converging to μ in the weak* sense, let $D_n = D(\mu_n, x_0, y_0)$ and $D = D(\mu, x_0, y_0)$ be the domains of dependence having μ_n and μ respectively as dual laminations, and $P = y_0 + x_0^{\perp}$ as a support plane tangent to the boundary at y_0, where x_0 does not belong to a weighted leaf of μ. Let h_n and h be the support functions of D_n and D. Then h_n converges uniformly on compact sets of $\mathbb{D}$ to h.*

Proof. It suffices to prove that the convergence is pointwise, since the functions h_n and h are convex on $\mathbb{D}$, and therefore pointwise convergence implies uniform convergence on compact sets. In fact, it suffices to prove pointwise convergence for almost every point.

We will actually prove that the support functions $\bar{h}_n$ restricted to $\mathbb{H}^2$ converge pointwise to $\bar{h}$ almost everywhere, which is clearly equivalent to the claim. Let us assume x_0 and x are points which do not lie on weighted leaves of the lamination μ. Let $\mathcal{G}[x_0, x]$ be the set of geodesics which intersect the closed geodesic segment $[x_0, x]$ and $\boldsymbol{\sigma} : \mathcal{G}[x_0, x] \to \mathbb{R}^{2,1}$ be the function which assigns to a geodesic l the corresponding point in $\mathrm{d}\mathbb{S}^2$, namely, the spacelike unit vector in $\mathbb{R}^{2,1}$ orthogonal to l with respect to Minkowski product, pointing outward with respect to the direction from x_0 to x. The support function of D can be written as (compare expression (8) in Proposition 2.12):

$$\bar{h}_n(x) = \langle x, y_0 \rangle + \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_n \,.$$

We thus want to show that

$$\int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_n \xrightarrow{n\to\infty} \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu \,.$$

Note that $\mathcal{G}[x_0, x]$ is compact in $\mathcal{G}$; define φ_i a smooth function such that $\varphi_i(l) = 1$ for every $l \in \mathcal{G}[x_0, x]$ and $supp(\varphi_i) \subset \mathcal{G}(x_0 - \frac{1}{i}, x + \frac{1}{i})$. Here $(x_0 - \frac{1}{i}, x + \frac{1}{i})$ denotes the open geodesic

interval which extends $[x_0, x]$ of a lenght $1/i$ on both sides. Hence we have:

$$\left|\int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \chi_{\mathcal{G}[x_0,x]} d\mu_n - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \chi_{\mathcal{G}[x_0,x]} d\mu\right| \leq \left|\int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \chi_{\mathcal{G}[x_0,x]} d\mu_n - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \varphi_i d\mu_n\right| \quad (\star)$$

$$+ \left|\int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \varphi_i d\mu_n - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \varphi_i d\mu\right| \quad (\star\star)$$

$$+ \left|\int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \varphi_i d\mu - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma}\rangle \chi_{\mathcal{G}[x_0,x]} d\mu\right| . \quad (\star\star\star)$$

Let $\mathcal{F} = \overline{\mathcal{G}(x_0 - \frac{1}{i}, x + \frac{1}{i}) \setminus \mathcal{G}[x_0, x]}$. Since we have assumed x and x_0 are not on weighted leaves of μ, we have $(\star\star\star) \leq K\mu(\mathcal{F}) \leq K\epsilon$ if $i \geq i_0$, for some fixed i_0. By definition of weak* convergence, the term numbered $(\star\star)$ converges to zero as $n \to \infty$ for $i = i_0$ fixed, so $(\star\star) \leq \epsilon$ for $n \geq n_0$. Finally, $\limsup_{n\to\infty} \mu_n(\mathcal{F}) \leq \mu(\mathcal{F})$ by Portmanteau Theorem (which in this case can be easily proved again by an argument of enlarging the interval and approximating by bump functions). Hence there exists n_0' such that $(\star) \leq K\mu_n(\mathcal{F}) \leq 2K\epsilon$ if $n \geq n_0'$ and $i \geq i_0$. Choosing $n \geq \max\{n_0, n_0'\}$, the proof is concluded. □

Let us now consider an arbitrary measured geodesic lamination μ and take the sequence $\mu_n \rightharpoonup \mu$ constructed as in Lemma 3.4. Let $D(\mu_n, x_0, y_0)$ be the domain of dependence having dual lamination μ_n and $P = y_0 + x_0^{\perp}$ as a support plane tangent at y_0. Since μ_n is invariant under the action of a Fuchsian cocompact group G_n, $D(\mu_n, x_0, y_0)$ is a domain of dependence invariant under a discrete group Γ_n. The linear part of Γ_n is G_n and the translation part is determined (up to conjugacy) by μ_n (see Proposition 2.13). This means that $D(\mu_n)/\Gamma_n$ is a maximal globally hyperbolic flat spacetime.

Theorem 3.6 ([2])**.** *Let G_0 be a Fuchsian cocompact group and μ_0 be a G_0-invariant measured geodesic lamination. Let $\psi_0 : \mathbb{H}^2 \to (0, \infty)$ be a G_0-invariant smooth function. Let Γ_0 be a subgroup of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *whose linear part is G_0 and whose translation part is determined by μ_0. Then there exists a unique smooth Cauchy surface S_0 of curvature $K(G^{-1}(x)) = -\psi_0(x)$ in the maximal future-convex domain of dependence D_0 invariant under the action of the group Γ_0.*

We will construct a Cauchy surface S for $D = D(\mu, x_0, y_0)$ with prescribed curvature as a limit of Cauchy surfaces S_n in $D_n = D(\mu_n, x_0, y_0)$. Let h_n be the support function of D_n. By Lemma 3.5, h_n converges uniformly on compact sets of $\mathbb{D}$ to h, the support function of D. Let u_n be the support function of S_n.

Lemma 3.7. *Let S_0 be a smooth strictly convex Cauchy surface in a domain of dependence D_0 invariant under the action of a discrete group $\Gamma_0 <$* $\mathrm{Isom}(\mathbb{R}^{2,1})$*, such that D_0/Γ_0 is a maximal globally hyperbolic flat spacetime. Let $K : S_0 \to (-\infty, 0)$ be the curvature function of S_0. Then the support functions u_0 of S_0 and h_0 of $\partial_s D_0$ satisfy*

$$h_0(z) - C\sqrt{1 - |z|^2} \leq u_0(z) \leq h_0(z) \tag{18}$$

for every $z \in \mathbb{D}$. Moreover, one can take $C = 1/\sqrt{\inf |K|}$.

Proof. Since $S_0 \subset \mathrm{I}^+(\partial_s D_0)$, it is clear that $u_0 \leq h_0$. For the converse inequality, recall that we denote by $\bar{u}_0$ the restriction to $\mathbb{H}^2$ of the 1-homogeneous extension U_0 of u_0, and analogously for $\bar{h}_0$. Let us consider $x \in \mathbb{H}^2$ and show that $\bar{u}_0(x) \geq \bar{h}_0(x) - C$, for $C = 1/\sqrt{\inf |K|}$. The inequality (18) then follows, since $u_0(z) = U_0(z, 1) = \sqrt{1 - |z|^2}\bar{u}_0(x)$ and $h_0(z) = H_0(z, 1) = \sqrt{1 - |z|^2}\bar{h}_0(x)$, for $x \in \mathbb{H}^2$ which projects to $(z, 1)$. Let us consider the foliation of D_0 by leaves of the cosmological time, namely the surfaces L_T whose support function is $\bar{h}_T(x) = \bar{h}_0(x) - T$ if $x \in \mathbb{H}^2$, with $T \in (0, \infty)$. The surface S_0 descends to a compact surface in D_0/Γ_0 and therefore the time function T on S_0 achieves a maximum $T_{\max}$ at a point p (actually, a full discrete Γ_0-orbit) on S_0. It follows that the level surface $L_{T_{\max}}$ is entirely contained in the future of S_0 and tangent to S_0 at p.

By the construction of the cosmological time, for any point $p \in L_T$ there exists a hyperboloid of curvature $-1/T^2$ which is tangent to L_T and contained in the future of L_T. Hence, the surface S_0 is contained in the past of a hyperboloid of curvature $-1/T^2$, and tangent to such hyperboloid at some point, which implies that $\inf |K| \leq 1/T^2$. Therefore $T \leq C$ for $C = 1/\sqrt{\inf |K|}$. This shows that the surface S_0 is contained in the past of the level surface L_C and its support function on $\mathbb{H}^2$ satisfies $\bar{h}_0(x) - C \leq \bar{u}_0(x)$. □

We are now ready to conclude the proof.

Proof of Theorem 3.2. Given the lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, let us consider the dual lamination μ of the domain of dependence D defined by φ. Hence $D = D(\mu, x_0, y_0)$ for some x_0, y_0 and the support function h of D is the convex envelope of φ.

By Lemma 3.4, there exist measured geodesic laminations μ_n, invariant under the action of torsion-free cocompact Fuchsian groups G_n, which converge weakly to μ. Recall from the proof of Lemma 3.4 that the Fuchsian group G_n has a fundamental domain P'_n which contains the ball $B(0, n)$ for the hyperbolic metric. Let us define a G_n-invariant function $\psi_n : \mathbb{H}^2 \to \mathbb{R}$, which approximates ψ. We take a partition of unity $\{\rho_n, \varrho_n\}$ subordinate to the covering $\{B(0, n), P'_n \setminus B(0, n/2)\}$ of P'_n. We define

$$\psi_n(x) = \rho_n(x)\psi(x) + \varrho_n(x)(\inf \psi)$$

and we extend ψ_n to $\mathbb{H}^2$ by invariance under the isometries in G_n. It is clear that the sequence ψ_n converges to ψ uniformly on compact sets of $\mathbb{H}^2$, since ψ_n agrees with ψ on $B(0, n/2)$. Since ψ_n is constant on $P'_n \setminus B(0, n/2)$, the G_n-invariant extension is smooth. Finally, $\inf \psi_n = \inf \psi$. Applying Theorem 3.6, we obtain a solution u_n to the equation

$$\det D^2 u_n(z) = \frac{1}{\psi_n(z)}(1 - |z|^2)^{-2}$$

for every n.

Let $D_n = D(\mu_n, x_0, y_0)$ be the domain of dependence associated with μ_n, so that $y_0 + x_0^\perp$ is a support plane of the domain D_n. We can assume x_0 is a point of $\mathbb{H}^2$ which does not belong to a weighted leaf of any μ_n. Let H_n be the extended support function of D_n and let h_n be its usual restriction to $\mathbb{D}$. By Lemma 3.5, h_n converges uniformly on compact sets of $\mathbb{D}$ to h. Moreover by inequality (18) of Lemma 3.7, the convex functions u_n are uniformly bounded on every compact set of $\mathbb{D}$. Hence, by convexity, the u_n are equicontinuous on compact sets of $\mathbb{D}$ and therefore, by the Ascoli-Arzelà Theorem, there exists a subsequence converging uniformly on compact sets to a convex function u. The limit function u is a generalized solution of the equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2}\,. \tag{MA}$$

By Theorem 2.20, u is strictly convex and therefore is smooth by Theorem 2.22. Moreover, the functions u_n satisfy the inequality in (18) for every $z \in \mathbb{D}$:

$$h_n(z) - C\sqrt{1 - |z|^2} \leq u_n(z) \leq h_n(z)\,,$$

hence for the limit function u we have

$$h(z) - C\sqrt{1 - |z|^2} \leq u(z) \leq h(z)\,, \tag{CT}$$

where h is the limit of the support functions h_n of D_n and is the support function of D by Lemma 3.5. Since both u and h, extended to $\overline{\mathbb{D}}$, are lower semicontinuous and convex functions, and the value on a point $z \in \partial\mathbb{D}$ coincides with the limit along a radial geodesic (see Lemma 2.7), we have the boundary condition $u|_{\partial\mathbb{D}} = h|_{\partial\mathbb{D}} = \varphi$. This shows that the condition (BC) holds, and concludes the proof. □

3.2. **Uniqueness of solutions.** In this subsection, we discuss the uniqueness of the solution of Equation (MA), for which the existence was proved in Theorem 3.2. More precisely, we prove the following:

Proposition 3.8. *Given a bounded lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and a smooth function $\psi : \mathbb{H}^2 \to [a, b]$ for some $0 < a < b < +\infty$, the smooth solution $u : \mathbb{D} \to \mathbb{R}$ to the equation*

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2} \tag{MA}$$

satisfying

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

is unique.

The claim in Proposition 3.8 holds if φ is continuous, by a direct application of Theorem 2.17. The rest of this subsection will be devoted to the proof of the claim when φ is only assumed to be lower semicontinuous. The key property is that every solution of (MA) with boundary value φ, for $\psi > a > 0$, satisfies the condition (CT). Geometrically, this means that every Cauchy surface with curvature bounded away from zero has bounded cosmological time.

Proposition 3.9. *Given a smooth function $\psi : \mathbb{D} \to [a, b]$ for some $0 < a < b < +\infty$, any smooth solution $u : \mathbb{D} \to \mathbb{R}$ to the equation*

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2} \tag{MA}$$

with

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

satisfies

$$h(z) - C\sqrt{1 - |z|^2} \leq u(z) \leq h(z)\,, \tag{CT}$$

for some constant $C > 0$, where $h = \operatorname{co}\varphi$.

Proof. The statement is true if φ is continuous, as we have already observed that in that case the solution is unique, and in Theorem 3.2 we have proved the existence of a solution satisfying the required condition. Let us now consider the general case. Since u is convex, it is clear that $u \leq h$. We show the other inequality. Let $r \in (0, 1]$ and $u_r : \mathbb{D} \to \mathbb{R}$ be defined as

$$u_r(z) = u(rz)\,.$$

Since u is continuous (actually, smooth) on $\mathbb{D}$, u_r converges uniformly on compact sets of $\mathbb{D}$ to u as $r \to 1$. Let ψ_r be such that

$$\det D^2 u_r(z) = \frac{1}{\psi_r(z)}(1 - |z|^2)^{-2}\,.$$

We have

$$\det D^2 u_r(z) = r^4 \det D^2 u(rz) = \frac{r^4}{\psi(rz)}(1 - r^2|z|^2)^{-2} \leq \frac{1}{\inf\psi}(1 - |z|^2)^{-2}$$

and therefore $\psi_r(z) \geq \inf\psi$. Since u_r is continuous on $\overline{\mathbb{D}}$, by the continuous case and the above inequality we obtain

$$h_r(z) - \frac{1}{\sqrt{\inf\psi}}\sqrt{1 - |z|^2} \leq u_r(z)\,, \tag{19}$$

where $h_r = \operatorname{co}(u_r|_{\partial\mathbb{D}})$.

Fix a point $z_0 \in \mathbb{D}$. We claim that $h(z_0) \leq \liminf_r h_r(z_0)$. Indeed, given an arbitrary affine function $f : \mathbb{D} \to \mathbb{R}$ such that $f|_{\partial\mathbb{D}} < \varphi$, the set $\{z : u(z) \leq f(z)\}$ is compact in $\mathbb{D}$. Suppose it is contained in a ball of radius r_0. Hence if $r > r_0$, for every $z_1 \in \partial\mathbb{D}$

$$u_r(z_1) = u(rz_1) > f(rz_1) = f_r(z_1)\,,$$

where of course we defined $f_r(z) = f(rz)$. This shows that $u_r|_{\partial\mathbb{D}} > f_r|_{\partial\mathbb{D}}$, and therefore $h_r \geq f_r$. As $r \to 1$, we obtain $f(z_0) \leq \liminf_r h_r(z_0)$, and the claim follows since f is arbitrary.

Now taking limits in Equation (19), we conclude that $h(z_0) - C\sqrt{1-|z_0|^2} \leq u(z_0)$ for $C = 1/\sqrt{\inf \psi}$. As the point z_0 is arbitrary, we conclude the proof. □

Proof of Proposition 3.8. Let u_1, u_2 be two solutions with $u_1|_{\partial\mathbb{D}} = u_2|_{\partial\mathbb{D}} = \varphi$. By Proposition 3.9, there exists a constant C such that

$$-C\sqrt{1-|z|^2} \leq u_1(z) - u_2(z) \leq C\sqrt{1-|z|^2}\,.$$

Hence the function $u_1 - u_2$ extends continuously to zero at the boundary $\partial\mathbb{D}$. Therefore $u_1 - u_2$ has a minimum on $\overline{\mathbb{D}}$. By Theorem 2.17, the minimum cannot be achieved at an interior point. Therefore the minimum is achieved on $\partial\mathbb{D}$, which means that $u_1 \geq u_2$. By exchanging the roles of u_1 and u_2, one can conclude that $u_1 \equiv u_2$. □

3.3. **The solution is an entire graph.** In this subsection we prove that the solutions constructed in Theorem 3.2 are the support functions of spacelike entire graphs in $\mathbb{R}^{2,1}$. We will make use of barriers which are constant curvature surfaces invariant under a parabolic group. The proof of the following proposition is given in Appendix A.

Proposition 3.10. *For every $K < 0$, $C > 0$ and every null vector $v_0 \in \mathbb{R}^{2,1}$ there exists an entire graph $S_{C,v_0}(K)$ of constant curvature K which is a Cauchy surface in the domain of dependence with support function at infinity*

$$\varphi_{C,v_0}(z) = \begin{cases} -\sqrt{C} & [z] = [v_0] \\ 0 & [z] \neq [v_0] \end{cases}.$$

In Appendix A the surfaces of Proposition 3.10 are described in more detail. For instance, we remark that the surface $S_{C,v_0}(K)$ is not complete. The domain of dependence with support function equal to $\mathrm{co}\,(\varphi_{C,v_0})$ is the future of a parabola, as pictured in Figure 6, see also the discussion in the proof of of Theorem 3.1 below. In order to use the surface $S_{C,v_0}(K)$ as a barrier, we need to prove a technical lemma.

Lemma 3.11. *Let $\varphi_1, \varphi_2 : \partial\mathbb{D} \to \mathbb{R}$ be two bounded lower semicontinuous functions and let $\psi_1, \psi_2 : \mathbb{H}^2 \to [a, b]$ be two smooth functions, for some $0 < a < b$. If $\varphi_1 \leq \varphi_2$ and $\psi_1 \leq \psi_2$, then the smooth solutions $u_i : \mathbb{D} \to \mathbb{R}$ (for $i = 1, 2$) to the equation*

$$\det D^2 u_i(z) = \frac{1}{\psi_i(z)}(1-|z|^2)^{-2}$$

with

$$u_i|_{\partial\mathbb{D}} = \varphi_i$$

satisfy $u_1 \leq u_2$ on $\mathbb{D}$.

Proof. Suppose first φ_1 is continuous. Therefore also the solution u_1 is continuous on $\overline{\mathbb{D}}$, since it satisfies the condition

$$h_1(z) - C\sqrt{1-|z|^2} \leq u_1(z) \leq h_1(z)\,,$$

where h_1 is the convex envelope of φ_1. Then the function $u_2 - u_1$ is lower semicontinuous and is positive on the boundary, therefore it achieves a minimum. By Theorem 2.17, the minimum has to be on the boundary, hence $u_2 \geq u_1$ on $\overline{\mathbb{D}}$.

Now for the general case, let φ_1 be lower semicontinuous and let φ_n, $n \geq 3$, be a sequence of continuous functions which converge to φ_1 monotonically from below, namely $\varphi_n \leq \varphi_{n+1}$ and $\varphi_n \leq \varphi_1$ for every $n \geq 3$. Let u_n be the solution of the equation

$$\det D^2 u_n(z) = \frac{1}{\psi_1(z)}(1-|z|^2)^{-2}$$

with

$$u_n|_{\partial\mathbb{D}} = \varphi_n \,.$$

By the previous case, we know (if $n \geq 3$) that $u_n \leq u_{n+1}$ and $u_n \leq u_1$. Hence the u_n are uniformly bounded and convex, thus by convexity the sequence u_n converges uniformly on compact sets (up to a subsequence) to a generalized solution u_∞ of the same equation:

$$\det D^2 u_\infty(z) = \frac{1}{\psi_1(z)}(1-|z|^2)^{-2} \,.$$

It is clear that $u_\infty \leq u_1$. Let $z \in \partial\mathbb{D}$. Recall the value of u_∞ on z coincides with the limit on radial geodesics. Hence we have $u_\infty(z) = \lim_{r\to 1} u_\infty(rz) \geq \lim_{r\to 1} u_n(rz) = \varphi_n(z)$ for every n. Therefore $u_\infty|_{\partial\mathbb{D}} \equiv \varphi_1$. By the uniqueness proved in Proposition 3.8, $u_\infty \equiv u_1$. Since $u_n(z) \leq u_2(z)$ for $n \geq 3$ and for every $z \in \mathbb{D}$, we conclude that $u_1 \leq u_2$. □

We are finally ready to conclude the proof of Theorem 3.1.

Proof of Theorem 3.1. We have showed in Theorem 3.1 and Proposition 3.8 that there exists a unique solution u to Equation (MA), hence having the required curvature function. Moreover, the solution satisfies Equation (CT), which ensures that the surface S with support function u is a Cauchy surface and satisfies the estimate on the cosmological time.

It only remains to show that S is a spacelike entire graph. Suppose it is not. Therefore S is tangent to the boundary of the domain of dependence (recall Subsection 2.1) and develops a lightlike ray R at the tangency point. Suppose the lightlike ray is parallel to the null vector v_0 of $\mathbb{R}^{2,1}$. Let α be such that $\varphi(z) \leq \alpha$ for every $z \in \partial\mathbb{D}$.

We consider the function

$$\varphi_0(z) = \begin{cases} \varphi(z) & [z] = [v_0] \\ \alpha & [z] \neq [v_0] \end{cases} .$$

From Proposition 3.10, there exists a K_0-surface S_0 with support function at infinity φ_0, for $K_0 = \inf K = -\sup|K|$, which is obtained by translating vertically in $\mathbb{R}^{2,1}$ a suitably chosen surface $S_{C,v_0}(K_0)$. Geometrically, this is equivalent to choosing the domain of dependence whose boundary is the future of a parabola (see Figure 6). The parabola is obtained by intersecting the plane containing the lightlike ray R with a cone $\mathrm{I}^+(p)$ over a point p on the z-axis, sufficiently in the past, so as to contain the original surface S.

Applying Lemma 3.11, we see that S is in the future of S_0. However, S_0 is an entire graph (see Appendix A) and both S and S_0 have the same lightlike support plane with normal vector v_0. This gives a contradiction and concludes the claim that S is an entire graph. □

Remark 3.12. In the above proof of Theorem 3.1, we have actually showed that every smooth convex bounded solution $u : \mathbb{D} \to \mathbb{R}$ of

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1-|z|^2)^{-2} \tag{MA}$$

with $\psi : \mathbb{D} \to (0,\infty)$, $\sup\psi < \infty$, corresponds to a spacelike entire graph in $\mathbb{R}^{2,1}$ with curvature $K(G^{-1}(z)) = -\psi(z)$. On the other hand, the hypothesis that $\sup\psi < \infty$ is essential. We give an explicit counterexample. Let us consider the function $u : \mathbb{D} \to \mathbb{R}$ defined by $u(z) = |z|^2/2$. It is easily checked that the dual surface S defined by u is the graph of u itself, and is only defined over $\mathbb{D}$. One can see that S is tangent to the lightcone

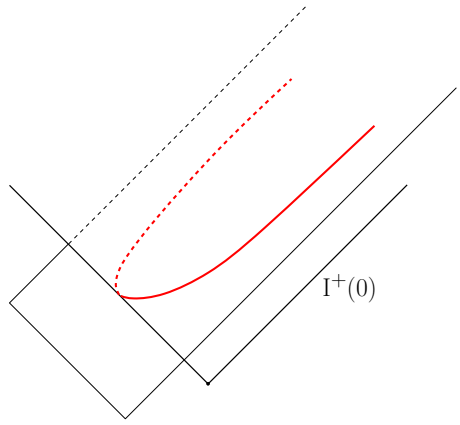

FIGURE 6. The future of a parabola is a domain of dependence invariant for the parabolic group. Its support function at infinity is lower-semicontinuous and is affine on the complement of a point of $\partial\mathbb{D}$.

centered at $(0, 0, -1/2)$ and that its curvature function is $K(G^{-1}(z)) = -\psi(z) = -\frac{1}{(1-|z|^2)^2}$, hence is unbounded. See also Figure 7.

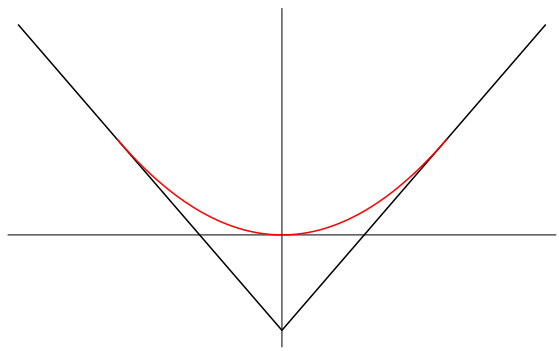

FIGURE 7. A Cauchy surface in the cone over a point, of unbounded negative curvature, which is not a spacelike entire graph. The surface (red) is obtained by revolution around the vertical axis.

4. FOLIATIONS BY CONSTANT CURVATURE SURFACES

In this section we prove that every domain of dependence defined by a bounded support function at infinity is foliated by K-surfaces, as K varies in $(-\infty, 0)$. The main statement is the following.

Theorem 4.1. *Every domain of dependence D, with bounded support function at infinity $\varphi : \partial\mathbb{D} \to \mathbb{R}$, is foliated by smooth spacelike entire graphs of constant curvature $K \in (-\infty, 0)$.*

The existence of such K-surfaces follows from Theorem 3.1, by choosing the constant function $\psi \equiv |K|$. We now show that the K-surfaces foliate the domain of dependence D.

Lemma 4.2. *Let $S_n = \mathrm{graph}(f_n)$ and $S_\infty = \mathrm{graph}(f_\infty)$ be spacelike entire graphs in $\mathbb{R}^{2,1}$ with C^1 support functions $u_n : \mathbb{D} \to \mathbb{R}$ and $u_\infty : \mathbb{D} \to \mathbb{R}$. If u_n converges to u_∞ uniformly on compact sets, then f_n converges uniformly on compact sets to f_∞.*

Proof. By a slight abuse of notation, we consider here the Gauss map $G_n : S_n \to \mathbb{D}$ using the canonical identification $\pi : \mathbb{H}^2 \to \mathbb{D}$. As by convexity we have that $Du_n(z) \to Du_\infty(z)$ for any $z \in \mathbb{D}$, Formula (4) implies that

$$G_n^{-1}(z) \to G_\infty^{-1}(z)$$

for all $z \in \mathbb{D}$. Let us set $G_n^{-1}(z) = (p_n(z), f_n(p_n(z))$, where $p_n(z)$ is the vertical projection to $\mathbb{R}^2$. We have that

$$p_n(z) \to p_\infty(z) \quad \text{and} \quad f_n(p_n(z)) \to f_\infty(p_\infty(z))\,. \tag{20}$$

Using that f_n's are 1-Lipschitz, by a standard use of Ascoli-Arzelà Theorem, we get that up to a subsequence, f_n converges uniformly on compact subset of $\mathbb{R}^2$ to some function g.

In order to prove that $g = f_\infty$, let us use again (20). We get that $f_\infty(p_\infty(z)) = \lim f_n(p_n(z)) = g(p_\infty(z))$. So f and g coincide on the image of p_∞. As we are assuming that S_∞ is a spacelike entire graph we conclude that they coincide everywhere. □

Recall that a K_0-surface $S(K_0)$ in D is constructed as limit of K_0-surfaces $S_n(K_0)$ invariant under the action of a surface group. By the work of [2], the K-surfaces $S_n(K)$ foliate the domain of dependence D_n of $S_n(K_0)$, as K varies in $(-\infty, 0)$.

Theorem 4.3 ([2])**.** *Let G_0 be a Fuchsian cocompact group and let Γ_0 be a discrete subgroup of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *whose linear part is G_0. Then the Cauchy surfaces $S_0(K)$ of constant curvature $K \in (-\infty, 0)$ foliate the maximal domain of dependence D_0 invariant under the action of the group Γ_0, in such a way that if $K_1 < K_2$, then $S_0(K_2)$ is contained in the future of $S_0(K_1)$.*

Proof of Theorem 4.1. The proof is split in several steps. First we prove the constant curvature surfaces are pairwise disjoint, then that the portion contained between two constant curvature surfaces is filled by other constant curvature surfaces, and finally that one can find a constant curvature surface arbitrarily close to the boundary of the domain of dependence and to infinity.

Step 1. Let us show that, if $K_1 < K_2$, then the constant curvature surfaces $S(K_1)$ and $S(K_2)$ are disjoint, and $S(K_2)$ is in the future of $S(K_1)$. Let $S_n(K_1)$ and $S_n(K_2)$ be approximating sequences as in the proof of Theorem 3.2, and let $u_n(K_1)$ and $u_n(K_2)$ be the corresponding support functions. From Theorem 4.3 of [2], we know that $u_n(K_2) < u_n(K_1)$. Hence in the limit $u(K_2) \leq u(K_1)$, where $u(K_i)$ is the support function of $S(K_i)$. Hence $S(K_1)$ and $S(K_2)$ do not intersect transversely. Moreover $S(K_1)$ is in the closure of the past of $S(K_2)$. Finally $S(K_1)$ and $S(K_2)$ cannot be tangent at a point, since $|K_1| > |K_2|$ and thus at least one of the eigenvalues of the shape operator of $S(K_1)$ is larger than the largest eigenvalue of $S(K_2)$.

Step 2. We show that, given two Cauchy surfaces $S(K_1)$, $S(K_2)$ in D of constant curvature $K_1 < K_2$, every point between $S(K_1)$ and $S(K_2)$ lies on a Cauchy surface of constant curvature. Let x be a point in $\mathbb{R}^{2,1}$ contained in the past of $S(K_2)$ and in the future of $S(K_1)$. For n large, x is in the past of $S_n(K_2)$ and in the future of $S_n(K_1)$. Therefore there exists a surface $S_n(K_n)$ through x, with $K_1 < K_n < K_2$. Up to a subsequence, let us assume $K_n \to K_\infty$. Using the same argument we gave in the proof of Theorem 3.2, the support functions $u_n(K_n)$ converge (up to a subsequence) uniformly of compact sets to $u_\infty(K_\infty)$, which is the support function of the K_∞-surface $S(K_\infty)$ in D. Since $x \in S_n(K_n)$ for every n, Lemma 4.2 implies that $x \in S(K_\infty)$.

Step 3. We show that for every point x nearby the boundary of the domain of dependence there is a constant curvature Cauchy surface $S(K_0)$ such that $x \in S(K_0)$. This follows from Equation (CT), which states that the K-surface $S(K)$ in D is contained in the past of the $(1/\sqrt{|K|})$-level surface of the cosmological time. Since the level surfaces $L_C = \{T = C\}$ of the cosmological time get arbitrarily close to the boundary of the domain of dependence as

C gets close to 0, it is clear that x is in the future of $S(K_0)$ if $|K_0|$ is large enough. The claim follows by Step 2.

Step 4. Finally, we show that every point far off at infinity lies on some constant curvature Cauchy surface. By contradiction, suppose there is a point x which is in the future of every K-surface $S(K)$. Let $S(K)$ be the graph of $f_K : \mathbb{R}^2 \to \mathbb{R}$. Then the f_K are uniformly bounded (they are all smaller than the function which defines $\mathrm{I}^+(x)$) and convex. Up to a subsequence $f_K \to f_\infty$ uniformly on compact sets. The function f_∞ defines a surface S_∞ contained in the domain of dependence D.

Now, f_K satisfies (see for instance [20])

$$\det D^2 f_K = |K|(1 - ||Df||^2)^2 \,.$$

Therefore, taking the limit as $K \to 0$, $\det D^2 f_\infty = 0$ in the generalized sense. The following Lemma states that f_∞ is affine along a whole line of $\mathbb{R}^2$, and this gives a contradiction, since S_∞ would contain an entire line and thus could not be contained in the domain of dependence D. □

Lemma 4.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex function which satisfies the equation* $\det D^2 f = 0$ *in the generalized sense. Then there exist a point $x_0 \in \mathbb{R}^2$, a vector $v \in \mathbb{R}^2$, and $\alpha \in \mathbb{R}$ such that $f(x_0 + tv) = f(x_0) + \alpha t$ for every $t \in \mathbb{R}$.*

Proof. By [18, Theorem 1.5.2] for any bounded convex domain $\Omega \subset \mathbb{R}^2$, $f|_\Omega$ coincides with the convex envelope of $f|_{\partial\Omega}$. It follows that for any $x \in \mathbb{R}^2$ there is $v = v(x) \in \mathbb{R}^2$ and $\alpha = \alpha(x) \in \mathbb{R}$ such that $f(x + tv) = f(x) + \alpha t$ for $t \in (-\epsilon, \epsilon)$, for some $\epsilon = \epsilon(x) > 0$.

Fix a point x_1, and set $v_1 = v(x_1)$. Up to adding an affine function we may assume that $\alpha(x_1) = 0$ and that $f(x) \geq f(x_1)$ for any $x \in \mathbb{R}^2$. If f is affine along the whole line $x_1 + \mathbb{R}v_1$, we have done. Otherwise take the maximal t_1 such that $f(x_1 + t_1 v_1) = f(x_1)$ and put $x_0 = x_1 + t_1 v_1$. Let $v_0 = v(x_0)$, clearly $v_0 \neq v_1$. As $f(x) \geq f(x_0) = f(x_1)$ for every $x \in \mathbb{R}^2$, necessarily $\alpha(x_0) = 0$. See Figure 8.

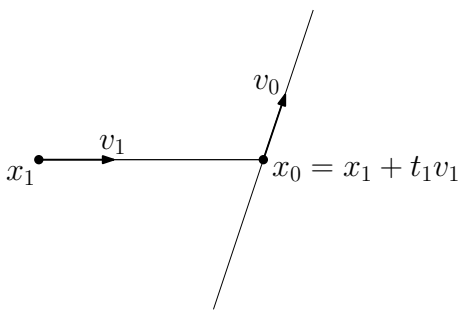

FIGURE 8. The setting of the proof of Lemma 4.4. Composing with an affine map, we can assume f is constant on the drawn segments.

We claim that $f(x) > f(x_0)$ on the half-plane P_0 bounded by $x_0 + \mathbb{R}v_0$ which does not contain x_1. Otherwise we should have that $f \equiv f(x_0)$ on the triangle with vertices $x, x_0 + \epsilon v_0, x_0 - \epsilon v_0$, but then f would be constant equal to $f(x_1)$ on some segment $[x_1, x_0 + \eta v_1]$, violating he maximality of t_1. See Figure 9.

Now suppose that $f(x_0 + tv_0) > f(x_0)$ for some t and take the maximal t_0 for which $f(x_0 + t_0 v_0) = f(x_0)$. Define $x_2 = x_0 + t_0 v_0$. As before we have that v_2 is different form v_0 and that $f \equiv f(x_0)$ on some segment of the form $[x_2 - \epsilon v_2, x_2 + \epsilon v_2]$ (see Figure 10). As this segment contains points of P_0 we get a contradiction. □

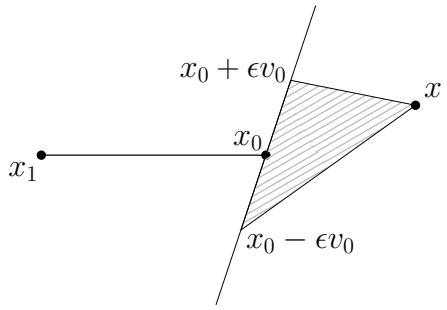

FIGURE 9. We have $f(x) > f(x_1)$ on the right halfplane bounded by the line $x_0 + \mathbb{R}v_0$, for otherwise f would be constant on the ruled triangle, contradicting the maximality of t_1.

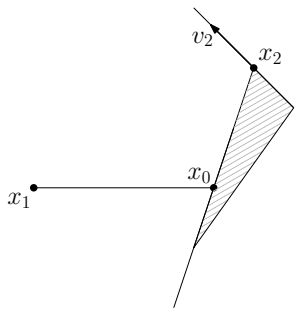

FIGURE 10. By a similar argument, f has to be constant on the entire line $x_0 + \mathbb{R}v_0$.

5. Constant curvature surfaces

In this section we give a characterization of K-surfaces with bounded principal curvatures.

Theorem 5.1. *Let D be a domain of dependence in $\mathbb{R}^{2,1}$. The following are equivalent:*

i) The measured geodesic lamination μ dual to $\partial_s D$ is bounded, i.e. $||\mu||_{Th} < +\infty$.
ii) The support function at infinity $h = H|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \mathbb{R}$ of D is in the Zygmund class.
iii) The domain of dependence D contains a convex Cauchy surface with principal curvatures bounded from below by some constant $d > 0$.
iv) The domain of dependence D is foliated by complete convex Cauchy surfaces of constant curvature K with principal curvatures bounded from below by some constant $d = d(K) > 0$, where $K \in (-\infty, 0)$.

We will give the proof in several steps. It is obvious that $iv) \Rightarrow iii)$. In Subsection 5.1 we prove that $i) \Leftrightarrow ii)$. The existence part of $ii) \Rightarrow iv)$ follows by Theorem 4.1; in Subsection 5.2 we complete the proof by showing that, if the dual lamination has finite Thurston norm, then the principal curvatures are bounded and the surface is complete. Finally, Subsection 5.3 proves $iii) \Rightarrow i)$, by giving an explicit estimate of Thurston norm of the dual lamination in terms of the supremum of the principal curvatures, which holds for any convex Cauchy surface.

5.1. Zygmund fields and bounded measured geodesic laminations.

In this part, we discuss the equivalence between $i)$ and $ii)$. We prove here the key fact for this equivalence. Given a function $\varphi : S^1 \to \mathbb{R}$, we denote by $\hat{\varphi}$ the vector field on S^1 associated to φ by means of the standard trivialization.

Proposition 5.2. *Given an infinitesimal earthquake $\hat{\varphi} = \frac{d}{dt}\big|_{t=0} E^{t\mu}$, the function $\varphi : S^1 \to \mathbb{R}$ is the support function at infinity on $\partial\mathbb{D} = S^1$ of a domain of dependence D with dual lamination μ.*

Proof. By composing $\hat{\varphi}$ with an infinitesimal Möbius tranformation, (compare Section 2.7) we can suppose the point $x_0 \in \mathbb{H}^2$ lies in a stratum of μ which is fixed by the earthquakes $E^{t\mu}$, for $t \in \mathbb{R}$. By Proposition 2.12, the support function at infinity of the domain of

dependence $D = D(\mu, x_0, 0)$ which has dual lamination μ and $x_0^\perp$ as a support plane is

$$H(\eta) = \int_{\mathcal{G}[x_0,\eta)} \langle \eta, \boldsymbol{\sigma} \rangle d\mu \,, \tag{21}$$

for every η in $\overline{\partial \mathrm{I}^+(0)}$. Here $[x_0, \eta)$ denotes the geodesic ray obtained by projecting to $\mathbb{H}^2$ the line segment connecting x_0 and η (recall the convention introduced before Proposition 2.12).

By Lemma 2.23, the vector field $\hat{\varphi}$ on S^1 defines a 1-homogeneous function Φ on $\partial \mathrm{I}^+(0)$. Since H is 1-homogeneous, it suffices to check that H and Φ agree on $\partial \mathbb{D} = \partial \mathrm{I}^+(0) \cap \{x_3 = 1\}$. Let $\eta \in \partial \mathbb{D}$ and let v be the unit vector tangent to $\partial \mathbb{D}$ in the counterclockwise orientation. By Lemma 2.23, under the standard identification of $\partial \mathbb{D}$ with S^1, we have

$$\varphi(\eta) = \langle \hat{\varphi}(\eta), v \rangle = \Phi(\eta) \,.$$

We now compute the infinitesimal earthquake φ at a point η. If l is a leaf of μ, the infinitesimal earthquake along the lamination composed of the only leaf l (as in Example 2.28) is

$$\dot{E}_l(\eta) = \langle \dot{A}^l(\eta), v \rangle v = \langle \eta \boxtimes \boldsymbol{\sigma}(l), v \rangle v \,,$$

where $\dot{A}^l = \frac{d}{dt}\big|_{t=0} A^l(t) \in \mathfrak{so}(2,1)$ is the infinitesimal generator of the 1-parameter subgroup of hyperbolic isometries $A^l(t)$ which translate on the left (as seen from x_0) along the geodesic l by lenght t. Here $\boxtimes$ denotes the Minkowski cross product. In the second equality we have used the fact that $\dot{A}^l(\eta) = \eta \boxtimes \boldsymbol{\sigma}(l)$ (see for instance [6, Appendix B]).

Using Equation (17) in Theorem 2.29, we obtain

$$\langle \hat{\varphi}(\eta), v \rangle = \int_{\mathcal{G}[x_0,\eta)} \dot{E}_l(\eta) d\mu = \int_{\mathcal{G}[x_0,\eta)} \langle \eta \boxtimes \boldsymbol{\sigma}, v \rangle d\mu \,. \tag{22}$$

We will show that $\langle \eta, \boldsymbol{\sigma}(l) \rangle = \langle \eta \boxtimes \boldsymbol{\sigma}(l), v \rangle$, from which the claim follows, by comparing Equations (21) and (22). For this purpose, let $p = (0,0,1)$ and $\eta = p + w$, so that (p, w, v) gives an orthonormal oriented triple. Suppose $\boldsymbol{\sigma}(l) = ap + bw + cv$. Then

$$\langle \eta, \boldsymbol{\sigma}(l) \rangle = a\langle p, p \rangle + b\langle w, w \rangle = b - a$$

whereas

$$\eta \boxtimes \boldsymbol{\sigma}(l) = b(p \boxtimes w) + a(w \boxtimes p) + c(p \boxtimes v + w \boxtimes v)$$

and in conclusion $\langle \eta \boxtimes \boldsymbol{\sigma}(l), v \rangle = b - a = \langle \eta, \boldsymbol{\sigma}(l) \rangle$. □

Remark 5.3. We observe that if a different base point x_0' (out of any weighted leaf of μ) is chosen, we obtain the 1-homogeneous function H' such that

$$H'(\eta) = \int_{\mathcal{G}[x_0',\eta)} \langle \eta, \boldsymbol{\sigma} \rangle d\mu = \int_{\mathcal{G}[x_0',x_0]} \langle \eta, \boldsymbol{\sigma} \rangle d\mu + \int_{\mathcal{G}[x_0',\eta)} \langle \eta, \boldsymbol{\sigma} \rangle d\mu$$

where this equality follows from the fact that μ is a measured geodesic lamination, and hence in Equation (21) the interval $[x_0, x]$ can be replaced by any path from x_0 to x transverse to the support of the lamination (see also [21]). Clearly, if a suitable normalization is chosen, also the infinitesimal earthquake $\hat{\varphi}'$ changes in the same way. Therefore H' agrees with the function Φ' obtained from $\hat{\varphi}'$.

Finally, recall that a tangent element to Universal Teichmüller space is defined as a vector field on S^1 satisfying the Zygmund condition (14) up to a first-order deformation by Möbius transformations. Therefore a different representative $\hat{\varphi}'$ in the same equivalence class of $\hat{\varphi}$ differs by

$$\hat{\varphi}'(\eta) = \hat{\varphi}(\eta) + \frac{d}{dt}\bigg|_{t=0} A(t)(\eta) = \hat{\varphi}(\eta) + \dot{A}(\eta) = \hat{\varphi}(\eta) + \eta \boxtimes y_0$$

where $A(t) \in \mathrm{Isom}(\mathbb{R}^{2,1})$, $A(0) = \mathrm{I}$ and $\dot{A} = \frac{d}{dt}\big|_{t=0} A(t) \in \mathfrak{so}(2,1)$. By the same computation as above, $\langle \hat{\varphi}'(\eta), v \rangle = H'(\eta)$ where H' is the support function of the domain of dependence $D(\mu, x_0, y_0)$.

By applying Proposition 5.2 and results presented in [14] or [24] on the convergence of the integral in Equation (17), one deduces that, if D is a domain of dependence whose dual lamination has finite Thurston norm, then the support function at infinity of D is finite. We will give a quantitative version of this fact in Proposition 5.6 - which will be useful because it gives a uniform bound on the support function in terms of the Thurston norm. Here we draw another consequence of Proposition 5.2, namely the equivalence of conditions $i)$ and $ii)$.

Corollary 5.4. *Given a domain of dependence D, the dual lamination μ has finite Thurston norm if and only if the support function h of D extends to a Zygmund field on $\partial\mathbb{D}$.*

Proof. If $||\mu||_{Th} < +\infty$, from Proposition 5.2 we know that $h|_{\partial\mathbb{D}}$ coincides with the infinitesimal earthquake along μ, hence is a Zygmund field. Viceversa, if $h|_{\partial\mathbb{D}}$ is a Zygmund field, by Theorem 2.29 there exists a bounded lamination μ such that $h|_{\partial\mathbb{D}}$ is the infinitesimal earthquake along μ, and we conclude again by Proposition 5.2. □

5.2. **Boundedness of curvature.** To prove the implication $i) \Rightarrow iv)$, we have showed in Theorem 3.1 the existence of constant curvature Cauchy surfaces $S(K)$, while in Theorem 4.1 we proved that the surfaces $S(K)$ foliate the domain of dependence as $K \in (-\infty, 0)$. It remains to show that the principal curvatures of the Cauchy K-surfaces $S(K)$ we constructed are bounded provided the dual measured geodesic lamination has finite Thurston norm. This will also imply that $S(K)$ is complete, since (by boundedness of the curvature and of the principal curvatures) the Gauss map is bi-Lipschitz with respect to the induced metric on S and the hyperbolic metric of $\mathbb{H}^2$.

Proposition 5.5. *Given a K-surface S, if the lamination μ dual to the domain of dependence $D(S)$ has finite Thurston norm, then the principal curvatures of S are uniformly bounded.*

We will prove the proposition by contradiction. If the statement did not hold, there would exist a sequence of points $x_n \in S$ such that the principal curvatures diverge (since the product of the principal curvatures is constant, necessarily one principal curvature will tend to zero and the other to infinity). Roughly speaking, we will choose isometries A_n so that the points x_n are sent to a compact region of $\mathbb{R}^{2,1}$, and consider the surfaces $S_n = A_n(S)$. Essentially, a contradiction will be obtained by showing that the sequence S_n contains a subsequence converging to a constant curvature smooth surface S_∞ - using the boundedness of the dual lamination - and that this gives bounds on the principal curvatures at x_n. Hence it is not possible that principal curvatures diverge.

In order to apply the above argument, we need to prove a uniform bound on the support functions, depending only on the Thurston norm of the dual lamination.

Proposition 5.6. *Let $D_0 = D(\mu_0, x_0, y_0)$ be a domain of dependence whose dual lamination μ_0 has Thurston norm $||\mu_0||_{Th} < M$, such that $P = y_0 + x_0^\perp$ is a support plane tangent to $\partial_s D_0$ at y_0. Let h_0 be the support function of D_0. Then $h_0 \leq C$ on $\overline{\mathbb{D}}$ for a constant C which only depends on M, x_0, y_0.*

Proof. The support function of D_0 restricted to $\mathbb{H}^2$, under the hypothesis, is given by (see Proposition 2.12):

$$\bar{h}_0(x) = \langle x, y_0 \rangle + \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_0 \,.$$

It is harmless to assume that $x_0 = (0,0,1)$ and $y_0 = 0$; indeed, composing with an isometry of $\mathbb{R}^{2,1}$, the support function h changes by an affine map on $\overline{\mathbb{D}}$. Hence we give an estimate of the integral term in $\bar{h}_0$. Let γ be a unit speed parametrization of the geodesic segment $[x_0, x]$. Note that if $\gamma(s)$ is on a geodesic l, for every $x \in \mathbb{H}^2$,

$$|\langle x, \boldsymbol{\sigma}(l) \rangle| = \sinh d_{\mathbb{H}^2}(x, l) \leq \sinh d_{\mathbb{H}^2}(x, \gamma(s)).$$

Hence, consider the partition $\gamma(0) = x_0, \gamma(1), \dots, \gamma(N), \gamma(d_{\mathbb{H}^2}(x_0, x)) = x$, for N the integer part of $d_{\mathbb{H}^2}(x_0, x)$. We have

$$\left| \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_0 \right| \leq \sum_{i=1}^{N+1} \sinh(i) \mu_0([\gamma(i-1), \gamma(i)])$$
$$\leq \sum_{i=1}^{N+1} \frac{M}{2} e^i = \frac{M}{2} e \frac{e^{N+1} - 1}{e - 1} \leq \frac{M}{2} \frac{e^2}{e-1} e^{d_{\mathbb{H}^2}(x_0,x)} .$$

We can finally give a bound for the support function h_0 on $\mathbb{D}$. If $\pi(x) = z \in \mathbb{D}$,

$$h_0(z) = \frac{\bar{h}_0(x)}{\cosh d_{\mathbb{H}^2}(x_0, x)} \leq \frac{M}{2} \frac{e^2}{e-1} \frac{e^{d_{\mathbb{H}^2}(x_0,x)}}{\cosh d_{\mathbb{H}^2}(x_0, x)} \leq M \frac{e^2}{e-1} .$$

This shows that the function h_0 is bounded by a constant which only depends on M and on x_0, y_0. By Lemma 2.7, the bound also holds on $\partial\mathbb{D}$. □

Proof of Proposition 5.5. Let $A_n \in \mathrm{SO}_0(2,1)$ be a linear isometry such that $A_n(G(x_n)) = (0,0,1)$, where $G : S \to \mathbb{H}^2$ is the Gauss map of S. Let D'_n be the domain of dependence of the surface $A_n(S)$. Now let $t_n \in \mathbb{R}^{2,1}$ be such that the t_n-translate of $\partial_s D'_n$ has a support plane with normal vector $(0,0,1)$ with tangency point the origin.

Let $S_n = A_n(S) + t_n$. Let u_n be the support function on $\mathbb{D}$ of S_n and h_n be the support function of its domain of dependence $D_n = D'_n + t_n$.

The support functions u_n are uniformly bounded on any Ω with compact closure in $\mathbb{D}$, since we have $h_n - (1/\sqrt{|K|})\sqrt{1-|z|^2} \leq u_n \leq h_n$ and by Proposition 5.6 the support functions h_n of D_n are uniformly bounded on Ω. Hence $||u_n||_{C^0(\mathbb{D})} < C$ for some constant C. Let B_n be the shape operator of S_n. Equation (5) in Lemma 2.8, for $z = 0$, gives $B_n^{-1} = \mathrm{Hess}(u_n)$. Applying Lemma 2.21, the inverse of the principal curvatures of S_n, which are the eigenvalues of B_n^{-1}, cannot go to infinity at the origin. This concludes the proof that principal curvatures of S_n cannot become arbitrarily small. □

5.3. **Cauchy surfaces with bounded principal curvatures have finite dual lamination.** In this part, we will show that a Cauchy surface S with principal curvatures bounded below, $\lambda_i \geq d$ for some $d > 0$, are such that the measured geodesic lamination μ dual to $D(S)$ has finite Thurston norm. This shows the implication *iii)* $\Rightarrow$ *i)*. More precisely, we prove:

Proposition 5.7. *Let B be the shape operator of a convex spacelike surface S in $\mathbb{R}^{2,1}$ such that the Gauss map is a homeomorphism. Let μ be the measured geodesic lamination dual to $D(S)$, where $D(S)$ is the domain of dependence of S. Then*

$$||\mu||_{Th} \leq 2\sqrt{2(1+\cosh(1))}||B^{-1}||_{op} \tag{23}$$

where $||B^{-1}||_{op} = \sup\{||B^{-1}(v)||/||v|| : v \in TS\}$ is the operator norm.

Note that $||B^{-1}||_{op}$ is the supremum of the inverse of the principal curvatures of S; alternatively, it is the inverse of the infimum of the principal curvatures of S. To prove Proposition 5.7, we consider $x_1, x_2 \in \mathbb{H}^2$ with $d_{\mathbb{H}^2}(x_1, x_2) \leq 1$ and take points y_1, y_2 on $\partial_s D(S)$ such that $P_1 = y_1 + x_1^\perp$ and $P_2 = y_2 + x_2^\perp$ are support planes of D. Recall $\mu(\mathcal{G}[x_1, x_2])$ denotes the value taken by the dual lamination μ on the geodesic segment $[x_1, x_2]$ which joins x_1 and x_2. We will also denote $||v||_- = \sqrt{\langle v, v \rangle}$ if $v \in \mathbb{R}^{2,1}$ is spacelike.

Lemma 5.8. *Let $y_1, y_2 \in \partial_s D(S)$ and $P_1 = y_1 + x_1^\perp$ and $P_2 = y_2 + x_2^\perp$ be support planes for $\partial_s D(S)$ tangent to $\partial_s D(S)$ at y_1 and y_2. If x_1 and x_2 do not lie on any weighted leaf of the dual lamination μ of $D(S)$, then*

$$\mu(\mathcal{G}[x_1, x_2]) \leq ||y_1 - y_2||_- .$$

Proof. Assume first $\mathrm{supp}\mu \cap \mathcal{G}[x_1, x_2]$ determines a finite lamination, i.e. μ restricted to $\mathcal{G}[x_1, x_2]$ is composed of a finite number of weighted leaves $g_1, \dots, g_p$ with weights $a_1, \dots, a_p$. Then we have (compare Proposition 2.12)

$$y_2 - y_1 = \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu = \sum_{i=1}^{p} a_i \boldsymbol{\sigma}(g_i).$$

Since the geodesics $g_1, \dots, g_p$ are pairwise disjoint, the unit normal vectors $\boldsymbol{\sigma}(g_1), \dots, \boldsymbol{\sigma}(g_p)$ are such that $\langle \boldsymbol{\sigma}(g_i), \boldsymbol{\sigma}(g_j)\rangle \geq 1$. Hence

$$\langle y_2 - y_1, y_2 - y_1\rangle = \sum_{i=1}^{p} a_i^2 + 2\sum_{i<j} a_i a_j \langle \boldsymbol{\sigma}(g_i), \boldsymbol{\sigma}(g_j)\rangle \geq \left(\sum_{i=1}^{p} a_i\right)^2 = \mu(\mathcal{G}[x_1, x_2])^2.$$

This shows that the following inequality holds for a finite lamination μ:

$$\left(\int_{\mathcal{G}[x_1,x_2]} d\mu\right)^2 \leq \langle \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu, \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu\rangle \,. \tag{24}$$

In general, if μ restricted to $\mathcal{G}[x_1, x_2]$ is not a finite lamination, we can approximate in the weak* topology the lamination μ by finite laminations μ_n (compare Lemma 3.4). As in Lemma 3.5, one can show

$$\int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu_n \xrightarrow{n\to\infty} \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu = y_2 - y_1 \tag{25}$$

and

$$\int_{\mathcal{G}[x_1,x_2]} d\mu_n \xrightarrow{n\to\infty} \int_{\mathcal{G}[x_1,x_2]} d\mu = \mu(\mathcal{G}[x_1, x_2]) \,. \tag{26}$$

Since (24) holds for the finite laminations in the LHS of Equations (25) and (26), the proof is complete. □

Lemma 5.9. *Let S be a convex surface in $\mathbb{R}^{2,1}$ such that the principal curvatures λ_i of S are bounded below, $\lambda_i \geq d > 0$. For every point $p \in S$, the surface S is contained in the future of the hyperboloid through p, tangent to T_pS and with curvature $-d^2$.*

Proof. Let $u : \mathbb{D} \to \mathbb{R}$ and $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$, as usual, denote the support function of S restricted to $\mathbb{D}$ and to $\mathbb{H}^2$. Analogously, let $v : \mathbb{D} \to \mathbb{R}$ and $\bar{v} : \mathbb{H}^2 \to \mathbb{R}$ be the support function of the hyperboloid $p + (1/d)\mathbb{H}^2$, as in the hypothesis. Composing with an isometry, we can assume $p = (0, 0, 1/d)$ and T_pS is the horizontal plane $x_3 = 1/d$. Hence $\bar{v} \equiv -1/d$, while from Equation (4) in Lemma 2.8 for the inverse of the Gauss map of S

$$G^{-1}(x) = \mathrm{grad}\,\bar{u}(x) - \bar{u}(x)x \,,$$

one can deduce $\bar{u}((0, 0, 1)) = -1/d$ and $\mathrm{grad}\,\bar{u}((0, 0, 1)) = 0$. From the hypothesis, the eigenvalues of the shape operator of S are larger than d at every point. On the other hand the shape operator of the hyperboloid of curvature $-d^2$ is $d\mathrm{I}$. Therefore $\mathrm{Hess}\bar{u} - \bar{u}\mathrm{I} < \mathrm{Hess}\bar{v} - \bar{v}\mathrm{I}$ and it follows (recalling the technology from Subsection 2.2 of the support function restricted on $\mathbb{H}^2$ and $\mathbb{D}$) that $v - u$ is a convex function on $\mathbb{D}$ with a minimum at $0 \in \mathbb{D}$, where $(v - u)(0) = 0$. Therefore $v - u$ is positive on $\mathbb{D}$, which shows that $v \geq u$ and thus proves the statement by Lemma 2.3. □

The following Lemma is a direct consequence.

Lemma 5.10. *Let S be a convex spacelike surface in $\mathbb{R}^{2,1}$ such that the principal curvatures of S are bounded below, $\lambda_i \geq d > 0$, and that the Gauss map G is a homeomorphism. Then for every $p \in S$, $D(S) \subset \mathrm{I}^+(r_d(p))$, where*

$$r_d(p) = p - \frac{1}{d}G(p).$$

Proof. By Lemma 5.9, $S \subset \mathrm{I}^+(r_d(p))$ for every $p \in S$. It follows that the entire domain of dependence of S is contained in $\mathrm{I}^+(r_d(p))$. □

Proof of Proposition 5.7. Assume the principal curvatures of S are bounded below by $d > 0$, and d is the infimum of the principal curvatures of S. Hence $||B^{-1}||_{op} = 1/d$.

Let us take $x_1, x_2 \in \mathbb{H}^2$, which do not lie on any weighted leaf of the dual lamination μ of $D(S)$, with $d_{\mathbb{H}^2}(x_1, x_2) \leq 1$. Suppose $y_1, y_2 \in \partial_s D(S)$ are such that $P_1 = y_1 + x_1^\perp$ and $P_2 = y_2 + x_2^\perp$ are tangent planes for $\partial_s D(S)$. We will show that $||y_1 - y_2||_- \leq 2\sqrt{2(1+\cosh(1))}/d$ and thus the estimate (23) will follow by Lemma 5.8.

Suppose moreover $p_1, p_2 \in S$ are such that $p_1 + x_1^\perp$ and $p_2 + x_2^\perp$ are tangent planes for S. Let us denote $U_i = \mathrm{I}^+(r_d(p_i)) \cap \overline{\mathrm{I}^-(T_{p_i}S)}$ and $V_i = \mathrm{I}^+(r_d(p_i)) \cap \overline{\mathrm{I}^-(P_i)}$, for $i = 1, 2$. See Figure 11 and 12. Note that $y_1, y_2 \in \mathrm{I}^+(r_d(p_i))$, for $i = 1, 2$, by Lemma 5.10.

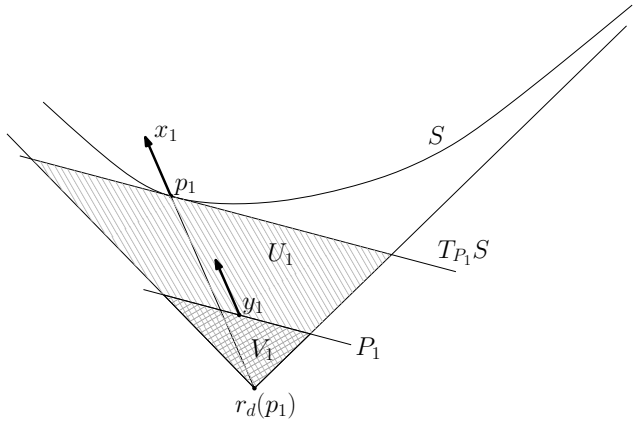

FIGURE 11. The setting of the proof and the definitions of the sets U_1 and V_1.

By construction, $y_1 \in V_1 \subset U_1$ and $y_2 \in V_2 \subset U_2$. Let us consider separately three cases:

Case 1: $y_2 \in V_1$. Then both y_1 and y_2 are contained in U_1. Since U_1 can be mapped isometrically to the region $\left\{(x_1, x_2, x_3) : x_3^2 \geq x_1^2 + x_2^2,\ x_3 \leq 1/d\right\}$ (see Figure 13), it is easy to see that a spacelike segment contained in U_1 can have lenght at most $2/d$, which gives the statement in this particular case.

Case 2: $y_1 \in V_2$. The estimate $||y_1 - y_2||_- \leq 2/d$ is obtained in a completely analogous way.

Case 3: $y_1 \notin V_2$ and $y_2 \notin V_1$. We claim that in this case $P_1 \cap P_2$ contains a point y_3 in $\mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$. Indeed, if $P_1 \cap P_2$ did not contain such a point, then the line $P_1 \cap P_2$ would be disjoint from $\mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$ and there would be two possibilities. Either $P_2 \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$ is contained in $\mathrm{I}^+(P_1) \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$, or $P_1 \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$ is contained in $\mathrm{I}^+(P_2) \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$. The former case implies that V_2 contains $P_1 \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$, which is not possible since by assumption $y_1 \in P_1$ is not in V_2. The latter case analogously contradicts $y_2 \notin V_1$.

Consider now the geodesic segments $y_3 - y_1$ and $y_2 - y_3$. Since y_3 and y_1 are both contained in V_1, we have $||y_3 - y_1||_- \leq 2/d$. Analogously $||y_2 - y_3||_- \leq 2/d$. If the plane Q containing y_1, y_2, y_3 is spacelike or lightlike, then

$$||y_2 - y_1||_- \leq ||y_2 - y_3||_- + ||y_3 - y_1||_- \leq \frac{4}{d}\,.$$

If Q is timelike (meaning that the induced metric on Q is a Lorentzian metric), then

$$\langle y_2 - y_1, y_2 - y_1\rangle = ||y_2 - y_3||_-^2 + ||y_3 - y_1||_-^2 + 2\langle y_2 - y_3, y_3 - y_1\rangle\,.$$

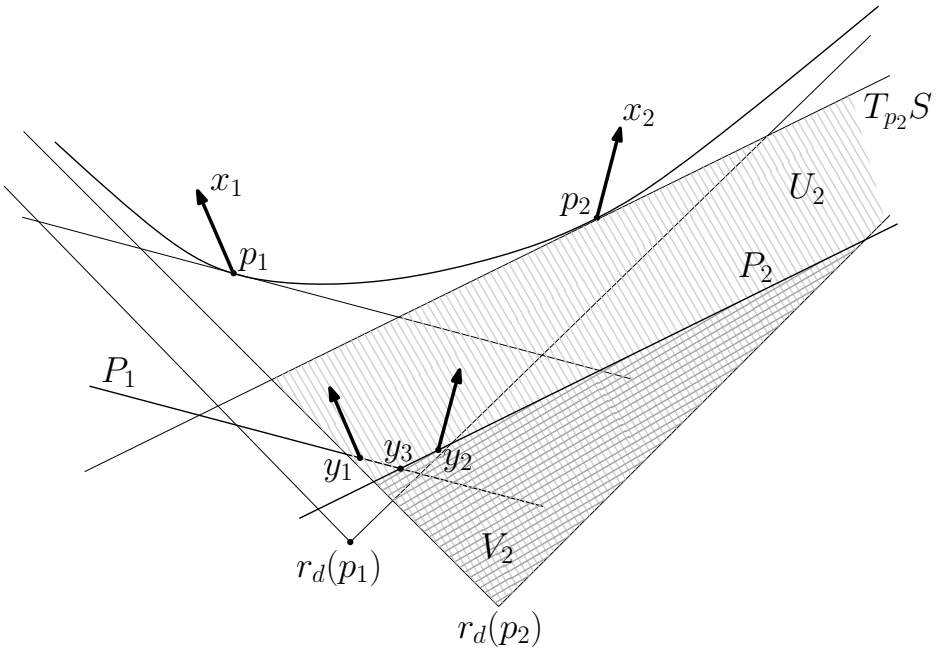

FIGURE 12. Analogously, the definitions of U_2 and V_2. The case pictured is for $y_1 \notin V_2$ and $y_2 \notin V_1$, hence there is a point y_3 in $P_1 \cap P_2$, which lies in $\mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$.

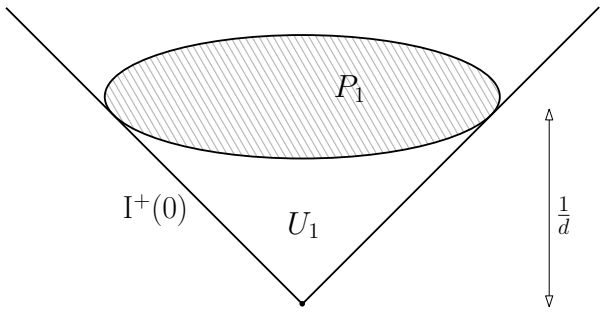

FIGURE 13. An isometric image of U_1.

Let v_1 and v_2 be the future unit vectors in Q orthogonal to $y_3 - y_1$ and $y_2 - y_3$. It is easy to check that

$$|\langle y_2 - y_3, y_3 - y_1\rangle| = ||y_2 - y_3||_- ||y_3 - y_1||_- |\langle v_1, v_2\rangle| = ||y_2 - y_3||_- ||y_3 - y_1||_- \cosh d_{\mathbb{H}^2}(v_1, v_2)\,.$$

We claim that v_i is the orthogonal projection in $\mathbb{H}^2$ of x_i to the geodesic determined by Q (namely, the geodesic through v_1 and v_2). Composing with an isometry, we can assume $y_3 = 0$, the direction spanned by $y_3 - y_1$ is the line $x_2 = x_3 = 0$ and $Q = \{x_2 = 0\}$. Then $v_1 = (0, 0, 1)$ and $x_1 = (0, \sinh t, \cosh t)$, where $t = d_{\mathbb{H}^2}(v_1, x_1)$, and thus the claim holds. Of course the proof for x_2 and v_2 is analogous. This concludes the proof by Lemma 5.8, since

$$||y_2 - y_1||_-^2 \leq \frac{4}{d^2}(2 + 2\cosh(1))\,,$$

where we have used that y_3 and y_2 are contained in $P_2 \subset U_2$, y_1 and y_3 are contained in $P_1 \subset U_1$ and so $||y_2 - y_3||_-, ||y_3 - y_1||_- \leq 2/d$ as above, and (since the projection to a line in $\mathbb{H}^2$ is distance-contracting) $d_{\mathbb{H}^2}(v_1, v_2) \leq d_{\mathbb{H}^2}(x_1, x_2) \leq 1$. $\square$

5.4. **Proof of Theorem 5.1.** All the elements to conclude the proof of Theorem 5.1 have been obtained. Let us summarize the necessary steps. We must show the equivalence of the four points *i*)-*iv*). The equivalence of *i*) and *ii*) is the content of Corollary 5.4.

To show that *ii*) implies *iv*), the existence of a foliation by surfaces of constant curvature was proved in Theorem 4.1, while the fact that the principal curvatures are bounded was proved in Proposition 5.5. As observed, this also implies completeness of the induced metric.

The condition of point *iv*) is obviously stronger than the condition of *iii*). Hence to conclude, it suffices to show that *iii*) implies either *i*) or *ii*). In fact, Proposition 5.7 proves that if *iii*) holds, namely if the principal curvatures are bounded from below by a positive constant, then *i*) holds, i.e. the Thurston norm of the dual lamination is bounded (and actually Proposition 5.7 provides a quantitative version of this fact).

Appendix A. Constant curvature surfaces invariant under a parabolic group

In the appendix we construct some explicit solutions to the Monge-Ampère equation associated to surfaces with constant curvature $K < 0$, namely

$$\det D^2 u(z) = \frac{1}{|K|}(1 - |z|^2)^{-2} \,. \tag{27}$$

We study constant curvature surfaces invariant under a one-parameter parabolic subgroup of isometries. In order to have such a surface, the subgroup must necessarily fix the origin.

Hence, let us denote by $A_\bullet : \mathbb{R} \to \mathrm{Isom}(\mathbb{R}^{2,1})$ the representation associated to the linear parabolic subgroup. Let us choose a basis $\{v_0, v_1, v_2\}$ of $\mathbb{R}^{2,1}$ such that v_0 is the null vector fixed by the parabolic group, v_1 is a null vector with $\langle v_0, v_1\rangle = -1$, and v_2 is a spacelike unit vector orthogonal to both v_0 and v_1.

The parabolic group is acting by

$$\begin{aligned} A_t(v_0) =& v_0 \,; \\ A_t(v_1) =& (t^2/2)v_0 + v_1 + tv_2 \,; \\ A_t(v_2) =& v_2 + tv_0 \,. \end{aligned}$$

Let $\gamma_0(s) = \frac{\sqrt{2}}{2}(e^s v_0 + e^{-s} v_1)$ be the unit speed geodesic of $\mathbb{H}^2$ with endpoints $[v_1]$ (for $s \to -\infty$) and $[v_0]$ (for $s \to +\infty$). Let us consider the following parametrization of $\mathbb{H}^2$:

$$\sigma(t, s) = A_t(\gamma_0(s)) \,,$$

namely, the levels $\{s = c\}$ are horocycles, while the levels $\{t = c\}$ are geodesics asymptotic to $[v_0]$. In these coordinates, the metric of $\mathbb{H}^2$ takes the form $ds^2 + (e^{-2s}/2)dt^2$.

We consider support functions restricted to $\mathbb{H}^2$, which we denote as usual by $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$, corresponding to surfaces of constant curvature. Hence we want to find solutions of the equation

$$\det(\mathrm{Hess}\,\bar{u} - \bar{u}\,\mathrm{I}) = \frac{1}{|K|} \,, \tag{28}$$

where K is a negative constant. Since are imposing that the surface dual to $\bar{u}$ is invariant for the parabolic group (with no translation), recalling Equation (3) for the transformation of support functions under isometries of $\mathbb{R}^{2,1}$, we look for a solution which only depends on s, namely a solution of the form $\bar{u}(t, s) = f(s)$.

By a direct computation, one can see that the gradient and the Hessian of $\bar{u}$ for the hyperbolic metric in this coordinate frame has the form:

$$\begin{aligned} \mathrm{grad}\,\bar{u} =& f'(s)\partial_s \\ \mathrm{Hess}(\bar{u})(\partial_s) =& f''(s)\partial_s \\ \mathrm{Hess}(\bar{u})(\partial_t) =& -\, f'(s)\partial_t \,, \end{aligned}$$

therefore the constant curvature condition (28) gives

$$(f''(s) - f(s))(-f'(s) - f(s)) = 1/|K|\,. \tag{29}$$

We now solve Equation (29). By convexity, we impose that both eigenvalues $(f''(s) - f(s))$ and $(-f'(s) - f(s))$ are positive. Let us perform the change of variables

$$g(s) = -f'(s) - f(s)\,, \tag{30}$$

so that Equation (29) becomes

$$g(s)(g(s) - g'(s)) = 1/|K|\,, \tag{31}$$

whose general positive solution is, as C varies in $\mathbb{R}$,

$$g(s) = \sqrt{|K|^{-1} + Ce^{2s}}\,. \tag{32}$$

A.1. **Solutions for $C = 0$.** We observe that the case $C = 0$ gives the trivial solution, namely the hyperboloid. Indeed f can be recovered by integrating (30), hence obtaining

$$f(s) = e^{-s}\left(D - \int_0^s e^x g(x)dx\right)\,. \tag{33}$$

Observe that the term $e^{-s}D$ corresponds to a translation in the direction $-\sqrt{2}Dv_0$. Hence, as the parameter D varies over $\mathbb{R}$, the corresponding surface varies by a translation in the line spanned by v_0. If $C = 0$, we have $g \equiv 1/\sqrt{|K|}$. By choosing D suitably, we obtain the solution

$$f_{0,K}(s) = -\frac{1}{\sqrt{|K|}}\,, \tag{34}$$

which is the support function of a hyperboloid of curvature K centered at the origin.

A.2. **Solutions for $C > 0$.** If $C > 0$, from Equation (33) we obtain the solution (for a constant D which we will fix later)

$$f_{C,K}(s) = -\frac{1}{2}\sqrt{|K|^{-1} + Ce^{2s}} - \frac{1}{2|K|\sqrt{C}}e^{-s}\log\left(\sqrt{C}\sqrt{|K|^{-1} + Ce^{2s}} + Ce^{s}\right) + e^{-s}D\,. \tag{35}$$

We now describe some of the properties of the surface $S_{C,v_0}(K)$ whose support function is $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$, for any fixed curvature $K < 0$.

First, we want to determine the value on $\partial\mathbb{D}$ of the support function $u_{C,K}$, namely the restriction to $\mathbb{D}$ of the 1-homogeneous extension of $\bar{u}_{C,K}$. Let us denote by $\sigma(t,s)_z$ the vertical component of $\sigma(t,s)$. Then we have

$$u_{C,K}(t,s) = \frac{f_{C,K}(s)}{\sigma(t,s)_z}\,.$$

Using the invariance for the parabolic group in $\mathrm{Isom}(\mathbb{R}^{2,1})$, it suffices to consider the case $t = 0$. We have

$$\sigma(0,s)_z = -\langle\sigma(0,s), \frac{v_0 + v_1}{\sqrt{2}}\rangle = \cosh(s)\,.$$

Observe that the term $e^{-s}D/\cosh(s)$ tends to 0 when $s \to \infty$ and to $2D$ when $s \to -\infty$. By an explicit computation, choosing D suitably, we can obtain

$$\lim_{s\to-\infty} u_{C,K}(t,s) = \lim_{s\to-\infty}\frac{f_{C,K}(s)}{\sigma(t,s)_z} = 0\,,$$

while

$$\lim_{s\to-\infty} u_{C,K}(t,s) = \lim_{s\to+\infty}\frac{f_{C,K}(s)}{\sigma(t,s)_z} = -\sqrt{C}\,,$$

hence the support function at infinity is

$$u_{C,K}|_{\partial\mathbb{D}}(z) = \begin{cases} -\sqrt{C} & [z] = [v_0] \\ 0 & [z] \neq [v_0] \end{cases}\,.$$

Geometrically, this means that the domain of dependence of the surface is the future of a parabola, obtained as the intersection of the cone centered at the origin (whose support function is identically 0) with a lightlike plane with normal vector v_0 (whose intercept on the z-axis is $\sqrt{C}$). See Figure 6 in subsection 3.2. It is easy to see that the dual lamination is the measured geodesic lamination of $\mathbb{H}^2$ whose leaves are all geodesics asymptotic to $[v_0]$, with a measure invariant under the parabolic group (a multiple of the Lebesgue measure, in the upper half-space model, pictured in Figures 14 and 15).

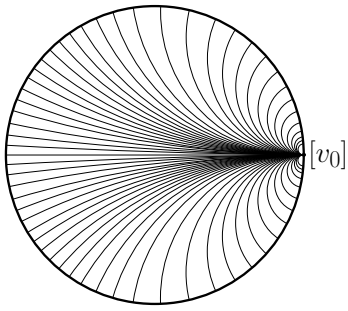

FIGURE 14. The dual lamination to the boundary of the domain of dependence, which is the future of a parabola.

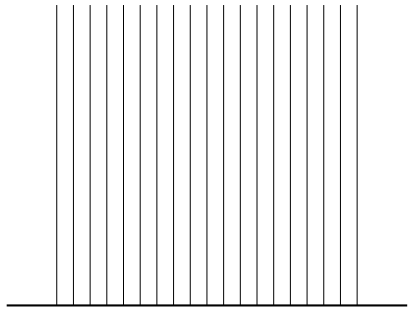

FIGURE 15. In the upper half space model, with fixed point at infinity, the measure is the Lebesgue measure.

The surface $S_{C,v_0}(K)$ dual to $u_{C,K}$ can be described by an explicit parametrization, recalling that the inverse of the Gauss map of the surface is $G^{-1}(x) = \operatorname{grad} \bar{u}_{C,K}(x) - \bar{u}_{C,K}(x)x$. In these coordinates,

$$G^{-1}(\sigma(t,s)) = \operatorname{grad} \bar{u}_{C,K}(t,s) - \bar{u}_{C,K}(t,s)\sigma(t,s) \tag{36}$$

$$= \frac{\sqrt{2}}{2} f'(s)(e^s v_0 - e^{-s} A_t(v_1)) - \frac{\sqrt{2}}{2} f(s)(e^s v_0 + e^{-s} A_t(v_1)) \tag{37}$$

$$= \frac{\sqrt{2}}{2} \big((f'(s) - f(s))e^s v_0 - (f'(s) + f(s))e^{-s} A_t(v_1)\big) \tag{38}$$

$$= \frac{\sqrt{2}}{2} \big(-(g(s) + 2f(s))e^s v_0 + g(s)e^{-s} A_t(v_1)\big) \ . \tag{39}$$

We now want to show that the surface is an entire graph. For this purpose, we will use the following criterion.

Lemma A.1. *Let $u : \mathbb{D} \to \mathbb{R}$ be a C^2 support function with positive Hessian, and suppose that the inverse of the Gauss map $G^{-1} : \mathbb{D} \to \mathbb{R}^{2,1}$ is proper. Then the boundary of the future-convex domain D defined by u is a spacelike entire graph.*

Proof. As u is C^1, we can use Equation (4) and get $G^{-1}(x) = \operatorname{grad} \bar{u}(x) - \bar{u}(x)x$ for every $x \in \mathbb{H}^2$. It can be readily shown that this implies that the vertical projection of $G^{-1}(x)$ is $Du(z)$, where $z = \pi(x) \in \mathbb{D}$, see Lemma 2.8 of [4]. As the Hessian of u is positive, the image of the gradient map of u is an open subset of $\mathbb{R}^2$, so it follows that $\partial_s D$ is open in ∂D. Since G^{-1} is proper, $\partial_s D$ is also closed in ∂D, and this concludes the proof. □

We will actually show that the height function given by

$$z(t,s) = -\langle G^{-1}(\sigma(t,s)), \frac{v_0 + v_1}{\sqrt{2}} \rangle$$

is proper. By a direct computation,

$$\begin{aligned}2z(t,s) &= -(g(s)+2f(s))e^s + g(s)e^{-s}\left(\frac{t^2}{2}+1\right)\\ &= \frac{1}{|K|\sqrt{C}}\log\left(\sqrt{C}\sqrt{|K|^{-1}+Ce^{2s}}+Ce^s\right)+e^{-s}\sqrt{|K|^{-1}+Ce^{2s}}\left(\frac{t^2}{2}+1\right)\\ &\geq \frac{1}{|K|\sqrt{C}}s + e^{-s}|K|^{-1/2}\left(\frac{t^2}{2}+1\right)-C_0\,,\end{aligned}$$

for some constant C_0. Observe that $z(t,s)$ tends to infinity for $s\to\pm\infty$. It is easily checked that on a sequence $\sigma(t_n,s_n)$ which escapes from every compact, $t_n^2+s_n^2\to\infty$, and thus $z(t_n,s_n)\to\infty$. This concludes the claim that $S_{C,v_0}(K)$ is a spacelike entire graph, by Lemma A.1.

Finally, we briefly discuss the isometry type of the induced metric. By an explicit computation using the expression in Equation (36), we find the pull-back of the induced metric via G^{-1}:

$$(G^{-1}\circ\sigma)^*(g_{\mathbb{R}^{2,1}}) = (f''(s)-f(s))^2ds^2+\frac{1}{2}e^{-2s}g(s)^2dt^2\,, \tag{40}$$

where it turns out that

$$f''(s)-f(s)=\frac{1}{|K|\sqrt{|K|^{-1}+Ce^{2s}}}$$

By an explicit change of variables

$$r(s)=\frac{1}{\sqrt{|K|}}\operatorname{arctanh}\left(\frac{1}{\sqrt{1+|K|Ce^{2s}}}\right),$$

so that $(r'(s))^2=(f''(s)-f(s))^2$, by computing

$$g(s)^2=\frac{1}{|K|}(1+|K|Ce^{2s})=\frac{1}{|K|\tanh^2(r\sqrt{|K|})}$$

and

$$e^{-2s}=|K|C\sinh^2\left(r\sqrt{|K|}\right)$$

one obtains that the induced metric is

$$dr^2+\left(\frac{C}{2}\right)\cosh^2\left(r\sqrt{|K|}\right)dt^2\,.$$

Rescaling t, one obtains

$$dr^2+\cosh^2\left(r\sqrt{|K|}\right)dt^2\,,$$

that is, the first fundamental form of $S_{C,v_0}(K)$ is isometric to a half-plane of constant curvature K, namely, to the region of a hyperboloid bounded by a geodesic l.

We resume the content of this subsection in the following proposition. Let us denote by $\mathbb{H}^2(K)$ the rescaled hyperbolic plane, of curvature $K<0$, and by $\mathbb{H}^2(K)_+$ a half-plane in $\mathbb{H}^2(K)$. Observe that $\mathbb{H}^2(K)_+$ has a one-parameter group of isometries $T(l)$ which consists of (the restriction of) hyperbolic translations along the geodesic l which bounds $\mathbb{H}^2(K)_+$.

Proposition A.2. *For every $K<0$, $C>0$ and every null vector $v_0\in\mathbb{R}^{2,1}$ there exists an isometric embedding*

$$i_{K,C,v_0}:\mathbb{H}^2(K)_+\to\mathbb{R}^{2,1}$$

with image a Cauchy surface $S_{C,v_0}(K)$ in the domain of dependence whose support function at infinity is

$$\varphi(z)=\begin{cases}-\sqrt{C} & [z]=[v_0]\\ 0 & [z]\neq[v_0]\end{cases}.$$

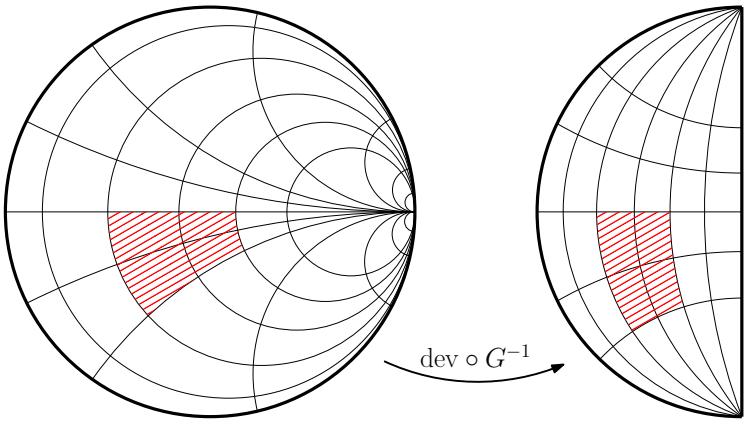

FIGURE 16. A developing map dev : $S_{C,v_0}(K) \to \mathbb{H}^2$ for the induced metric, composed with the inverse of the Gauss map $G^{-1} : \mathbb{H}^2 \to S_{C,v_0}(K)$, in the Poincaré disc model of $\mathbb{H}^2$.

The surface $S_{C,v_0}(K)$ is a spacelike entire graph and i_{K,C,v_0} is equivariant with respect to the group of isometries $T(l)$ of $\mathbb{H}^2(K)_+$ and the parabolic linear subgroup of $\mathrm{Isom}(\mathbb{R}^{2,1})$ *fixing v_0.*

A.3. **Solutions for $C < 0$.** If $C < 0$, the function $g(s) = \sqrt{|K|^{-1} + Ce^{2s}}$ is only defined for

$$s \leq \frac{1}{2} \log \left(\frac{1}{|CK|} \right) .$$

From Equation (33) we can explicitly write the solution (again D is a constant to be fixed):

$$f_{C,K}(s) = -\frac{1}{2}\sqrt{|K|^{-1} + Ce^{2s}} - \frac{1}{2|K|\sqrt{|C|}} e^{-s} \arctan \left(\frac{\sqrt{|C|}e^s}{\sqrt{|K|^{-1} + Ce^{2s}}} \right) + e^{-s} D . \quad (41)$$

Again, we study briefly the properties of the surface $S_{C,v_0}(K)$ whose support function is $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$. Observe that the solution (41) is only defined in the range $s \leq \frac{1}{2} \log \left(\frac{1}{|CK|} \right)$, namely, in the complement of a horoball. Let us notice that, in the same notation as before, the limit of the support function (which only makes sense for $s \to -\infty$) is

$$\lim_{s \to -\infty} u_{C,K}(t,s) = \lim_{s \to -\infty} \frac{f_{C,K}(s)}{\sigma(t,s)_z} = 0 ,$$

provided we choose $D = 0$. On the other hand, as $s \to \frac{1}{2} \log \left(\frac{1}{|CK|} \right)$, the function $f_{C,K}(s)$ has the finite limit $-(\pi/4)\sqrt{|K|^{-1}}$. We observe that $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$ can be extended to a convex function defined on the whole $\mathbb{H}^2$ by declaring

$$f_{C,K}(s) = -\frac{\pi}{4} \frac{1}{|K|\sqrt{|C|}} e^s$$

for $s \geq \frac{1}{2} \log \left(\frac{1}{|CK|} \right)$. We will now denote by $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$ the function extended in this way. The surface $S_{C,v_0}(K)$ is thus a constant curvature surface which develops a singular point, namely it intersects the boundary of the domain of dependence, which in this case is just $\mathrm{I}^+(0)$. The inverse of the Gauss map sends the whole horoball $\{s \geq \frac{1}{2} \log \left(\frac{1}{|CK|} \right)\}$ to the point $\frac{\sqrt{2}}{2} \frac{1}{|K|\sqrt{|C|}} \frac{\pi}{4} v_0$.

We remark that $u_{C,K}$ is a generalized solution to the Monge-Ampère equation on the disc $\det D^2 u = \nu$, where ν in this case is a measure which coincides with $(1/|K|)(1 - |z|^2)^{-2}\mathcal{L}$

on the complement of the horoball (where $\mathcal{L}$ is the Lebesgue measure), and is 0 inside the horoball.

By a computation analogous to the previous case, to compute the induced metric we manipulate Equation (40): setting

$$r(s) = \frac{1}{\sqrt{|K|}} \operatorname{arctanh}(\sqrt{1+|K|Ce^{2s}}),$$

and replacing

$$g(s)^2 = \frac{1}{|K|}(1+|K|Ce^{2s}) = \frac{1}{|K|}\tanh^2(r\sqrt{|K|})$$

and

$$e^{-2s} = |K|C\cosh^2(r\sqrt{|K|})$$

we obtain the expression for the metric

$$dr^2 + \left(\frac{C}{2}\right)\sinh^2\left(r\sqrt{|K|}\right)dt^2,$$

or, after rescaling of t,

$$dr^2 + \sinh^2\left(r\sqrt{|K|}\right)dt^2.$$

This shows that the first fundamental form of $S_{C,v_0}(K)$ is isometric to the universal cover of the complement of a point in $\mathbb{H}^2(K)$, which we will denote by $\widetilde{\mathbb{H}^2(K)\setminus p}$. Let $R(p)$ the group of rotations of $\mathbb{H}^2(K)$ fixing a point p and let $\widetilde{R(p)}$ be its universal cover. We conclude by including all the information in the following proposition.

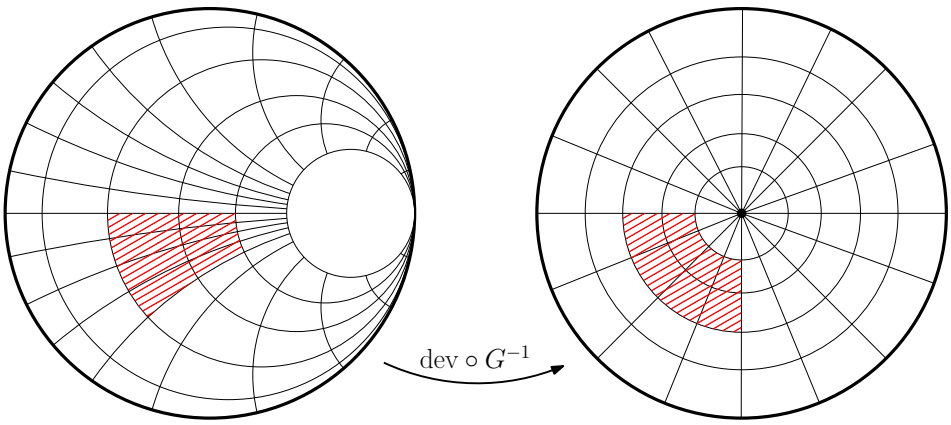

FIGURE 17. Again, the developing map in the Poincaré disc model. In the case $C<0$, the inverse of the Gauss map $G^{-1}:\mathbb{H}^2\to S_{C,v_0}(K)$ shrinks a horoball to a point.

Proposition A.3. *For every $K<0$, $C<0$ and every null vector $v_0\in\mathbb{R}^{2,1}$ there exists an isometric embedding*

$$i_{K,C,v_0}:\widetilde{\mathbb{H}^2(K)\setminus p}\to\mathbb{R}^{2,1}$$

with image a Cauchy surface $S_{C,v_0}(K)$ for $\mathrm{I}^+(0)$. The closure of the surface $S_{C,v_0}(K)$ intersects the null cone $\partial\mathrm{I}^+(0)$ in the point βv_0, where

$$\beta = \frac{\sqrt{2}}{2}\frac{1}{|K|\sqrt{|C|}}\frac{\pi}{4}.$$

The inverse of the Gauss map of the closure of $S_{C,v_0}(K)$ *maps a horoball of* $\mathbb{H}^2$ *to* βv_0. *Finally* i_{K,C,v_0} *is equivariant with respect to the group of isometries* $\widetilde{R(p)}$ *of* $\widetilde{\mathbb{H}^2(K)} \setminus p$ *and the parabolic linear subgroup of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *fixing* v_0.

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