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Geometric optics for surface waves in nonlinear elasticity

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Abstract

This work is devoted to the analysis of high frequency solutions to the equations of nonlinear elasticity in a half-space. We consider surface waves (or more precisely, Rayleigh waves) arising in the general class of isotropic hyperelastic models, which includes in particular the Saint Venant-Kirchhoff system. Work has been done by a number of authors since the 1980s on the formulation and well-posedness of a nonlinear evolution equation whose (exact) solution gives the leading term of an *approximate* Rayleigh wave solution to the underlying elasticity equations. This evolution equation, which we refer to as “the amplitude equation”, is an integrodifferential equation of nonlocal Burgers type. We begin by reviewing and providing some extensions of the theory of the amplitude equation. The remainder of the paper is devoted to a rigorous proof in 2D that exact, highly oscillatory, Rayleigh wave solutions u^ε to the nonlinear elasticity equations exist on a fixed time interval independent of the wavelength ε , and that the approximate Rayleigh wave solution provided by the analysis of the amplitude equation is indeed close in a precise sense to u^ε on a time interval independent of ε . The paper focuses mainly on the case of Rayleigh waves that are *pulses*, which have profiles with continuous Fourier spectrum, but our method applies equally well to the case of wavetrains, whose Fourier spectrum is discrete.

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Chapter 1

General introduction

This work is devoted to the rigorous justification of weakly nonlinear geometric optics expansions for a class of second order hyperbolic initial boundary value problems. These are evolution problems for which we ask the two main questions of geometric optics:

1. Does an exact solution, say u_ε , exist for $\varepsilon \in (0, 1]$ on a fixed time interval $[0, T_0]$ independent of ε , where ε represents the small wavelength of the data?
2. Suppose the answer to the first question is yes. If we let u_ε^{app} denote an approximate solution on $[0, T_0]$ constructed by the methods of nonlinear geometric optics (that is, solving eikonal equations for phases and suitable transport equations for profiles), how well does u_ε^{app} approximate u_ε for ε small? For example, is it true that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-\alpha} \|u_\varepsilon - u_\varepsilon^{app}\|_{L^\infty} \rightarrow 0?$$

where α is a scaling parameter such that both u_ε and u_ε^{app} have $O(\varepsilon^\alpha)$ L^∞ norm (in that case $u_\varepsilon - u_\varepsilon^{app}$ is meant to be a remainder of higher order in ε than ε^α).

Pursuing a long-term program, we deal here with the above two issues in the context of elastodynamics in a fixed domain, with boundary conditions giving rise to Rayleigh waves. These are hyperbolic initial boundary value problems for which the so-called uniform Lopatinskiĭ condition¹ fails in a manner that is manifested by the presence of surface waves.² At the linear level such waves were identified long ago, and they display an exponential decay with respect to the normal variable to the boundary of the space domain. Our goal here is to understand the behavior of small amplitude, highly oscillating surface waves in the so-called weakly nonlinear regime, where nonlinearity is visible at the leading order of the solution, meaning that both u_ε and u_ε^{app} are constructed by solving nonlinear evolution equations. More precisely, the phrase “weakly nonlinear” indicates that the L^∞ norm of u_ε^{app} , say ε^α , is taken to be as small as it can be while still satisfying the requirement that the evolution equation for the leading term of u_ε^{app} be nonlinear. This particular choice of amplitude ε^α (for waves oscillating with wavelength ε) is what we mean by the “weakly nonlinear scaling”.

Before reviewing earlier works on weakly nonlinear surface waves, let us recall several facts on hyperbolic initial boundary value problems and geometric optics expansions. Well-posedness of linear, and

¹This is often referred to as the uniform Kreiss-Lopatinskiĭ condition; problems satisfying this condition are said to be “strongly” or “uniformly” stable.

²We use the phrase “surface waves” as a general term that includes both surface wavetrains and surface pulses.

subsequently nonlinear, hyperbolic initial boundary value problems depends on whether or not the uniform Lopatinskii condition is satisfied, and if not, *how* it fails; see, e.g., [Kre70, Sak82, CP82] and [BGS07, Chapter 4]. In the most favorable case where this condition is satisfied, the corresponding initial boundary value problems satisfy *maximal* energy estimates with no loss of derivatives from the data to the solution, including for the trace of the solution at the boundary. Nonlinear problems may then be solved by fairly standard Picard iteration techniques. For uniform Lopatinskii problems the regime of weakly nonlinear geometric optics corresponds to the same scaling as for the Cauchy problem in the whole space. The main features of geometric optics in that case include: propagation at some given group velocities inside the space domain, and reflection of (outgoing) oscillating wave packets at the boundary. The reflection coefficients which determine incoming oscillating wave packets in terms of outgoing ones are *finite* as a byproduct of the uniform Lopatinskii condition. A complete justification of the high frequency WKB expansions for quasilinear problems is provided in the references [Wil02, CGW11, Her15] for wavetrains, and in [CW13] for pulses. The difference between the two problems is that for wavetrains, the data, exact and approximate solutions depend in a periodic way on the fast variables Φ/ε (here Φ represents the phases of the oscillations arising in the solution), while for pulses the fast variables all lie in \mathbb{R} with ‘some’ decay property at infinity. In problems with several interacting phases the pulse setting is slightly more favorable because pulses associated with different phases do not interact at leading order, while wavetrains may interact at leading order when *resonances* occur between several phases. A consequence is that, for pulses, one can usually derive a rate of convergence between exact and approximate solutions [CW13, CW14], while this is usually out of reach for interacting wavetrains [JMR93, JMR95]. An offsetting difficulty in the pulse setting is that it is usually much harder to construct correctors, and almost never possible to construct many correctors.

In this work we study a situation where the uniform Lopatinskii condition fails, but in a controlled way. The degeneracy of this condition may occur in different ways, and we consider here the case where surface waves of finite energy occur, see [BGS07, Chapter 7]. Let us mention right away the work of Marcou [Mar10] for first order nonlinear systems in the surface wavetrain case. In [Mar10] Marcou provided a complete justification of weakly nonlinear geometric optics expansions for surface wavetrains arising in first order conservation laws with linear, homogeneous boundary conditions. In contrast our main focus will be on the second order hyperbolic systems with fully nonlinear, nonhomogeneous boundary conditions arising in elasticity theory. We shall study surface pulses (or more precisely, Rayleigh pulses) for the most part, and these require very different methods. Moreover, as we explain below, it seems that the rigorous justification even of wavetrain solutions in nonlinear elasticity requires methods different from those of [Mar10].

As pointed out by Serre [Ser06], surface waves arise in a rather systematic way in second order hyperbolic problems with a variational formulation, and we shall therefore consider this framework, especially in sections 2.1 and 2.2, before returning to elasticity in section 2.3 and for the rest of the paper. Since the uniform Lopatinskii condition fails, well-posedness of the initial boundary value problem is an issue because maximal energy estimates are known to fail. For a class of isotropic, hyperelastic, Neumann type problems in two space dimensions, a class that includes the Saint-Venant Kirchhoff system (3.1) as a basic representative, Sablé-Tougeron [ST88] has proved precise *microlocal* energy estimates which take advantage of the fact that the uniform Lopatinskii condition degenerates in the *elliptic* region (Definition 3.4) of the frequency domain.³ The main feature of the energy estimates proved in [ST88] is that there is no loss of regularity from the initial data and interior source term to the solution. The only loss with respect to the strongly stable case previously mentioned is a loss of one derivative from the boundary data

³As we explain in section 6.3, there is still a serious obstacle to deriving such estimates in dimensions $d \geq 3$.

to the trace of the solution, but one recovers an optimal control in the interior of the space domain due to the exponential decay of the surface waves. This is consistent with one of the main results in [Ser06], which shows that a related class of linear initial boundary value problems with *homogeneous* boundary conditions may be solved, at least in cases of time-independent coefficients, by the Hille-Yosida Theorem.

The class treated in [Ser06] consists of linear problems that arise as in (2.1), (2.2) from a stored energy function $W(\nabla u)$ that is quadratic in ∇u , where u is the displacement; but the Saint Venant-Kirchhoff energy function is fourth order in ∇u , and it leads to more complicated linearized problems. Only the quadratic part of $W(\nabla u)$ contributes to the linearization at $\nabla u = 0$, but to solve the nonlinear problem studied here, we like [ST88] must consider the linearizations at nearby nonzero states, where the higher order terms of W do affect the linearized problem. In the literature the phrase “linearized elasticity” or “classical linearized elasticity” is often used to refer to the problem arising from a quadratic $W(\nabla u)$, or from one of the form $W(x, \nabla u)$, where W is quadratic in ∇u [HM78, Cia88, CC05, Ser06]. These notions of linearized problem are *different* from the one we must work with here; this is clear from a quick inspection of the principal symbol (6.10) of the Saint Venant-Kirchhoff system linearized about a state $\nabla u \neq 0$.

By combining precise estimates for the linearized problem with a clever iteration scheme based on considering the original nonlinear system together with two auxiliary systems obtained from it by differentiation, [ST88] was able to prove a short-time existence theorem for the so-called “traction” boundary problem in nonlinear elasticity (3.1) where Rayleigh waves appear. In spite of the loss of derivatives, an essentially classical iteration scheme did the job. Our work builds on this approach, rather than on alternative approaches for nonlinear elasticity, see, e.g., [Kat85, SN89]. If one applies the result of [ST88] directly to the nonlinear problem with highly oscillatory boundary data considered here, one obtains a time of existence T_ε converging to zero with the wavelength ε . The main new challenge addressed in this paper is to obtain estimates that are *uniform* with respect to small ε , and which hold on a time interval independent of ε .

We recall now some of the earlier works devoted to constructing approximate, oscillatory surface waves. The appropriate scaling and leading order amplitude equation were identified in pioneering works going back, at least, to Lardner, Parker and others, see for instance [Lar83, Lar86, PT85, Par88, HIZ95]. The analogous theory in the case of first order hyperbolic systems was initiated by Hunter [Hun89]. In both first and second order problems, the leading amplitude equation takes the form of a nonlinear integro-differential evolution equation of nonlocal Burgers type. The solution of this equation determines the trace of the leading part of the approximate solution u_ε^{app} . Once the trace is known, the leading profile of u_ε^{app} is obtained by a simple lifting procedure used for example in [Les07]. Thus, the main step in constructing the approximate solution is to prove a well-posedness result for such nonlocal versions of the more standard Burgers equation. This well-posedness problem has been solved in [Hun06, BG09], so constructing the leading profile of u_ε^{app} will not be our main concern here. However, in order to have a functional framework in which we can show that approximate solutions are close to exact solutions, we shall need to extend somewhat the constructions in [Lar83, Lar86] and recent generalizations in [BGC12, BGC15].

Unlike most earlier works we consider a WKB ansatz which incorporates all possible slow and fast variables. One needs at least the tangential slow spatial variables and the slow time variable in order to specify exactly along which space-time curve on the boundary a Rayleigh pulse propagates. In addition we must also include the slow variable normal to the boundary to have profiles that can be used in the rigorous error analysis. The analysis of section 2.2, which works in all space dimensions $d \geq 2$, computes the exact group velocity of Rayleigh waves and shows that the propagation curves are characteristics of the Lopatinski determinant. In addition we try to unify previous works and clarify the arguments that are used to derive the leading order amplitude equation. In particular, we uncover a cancellation property

(2.25), (2.26) that was overlooked by [Lar83, Lar86], and which greatly simplifies the derivation of the amplitude equation. This is achieved in Chapter 2 below. Chapter 2 is the most formal one, in which we keep the discussion in a rather broad variational context. Consistently with earlier references from the physics literature, in section 2.3 we identify the nonlocal amplitude equation for the class of isotropic hyperelastic models, which includes the Saint Venant-Kirchhoff model. We analyze the WKB ansatz in the case of pulses, that is, when the fast variables lie in \mathbb{R} . Similar arguments can be applied in the case of wavetrains, and we leave these rather small modifications to the interested reader.

Hyperbolic boundary problems in which the uniform Lopatinski condition fails typically exhibit some kind of amplification of solutions [CGW14a, CW14, CW16] at the boundary. In the second order problem with first order boundary condition (3.4)-(3.6) considered here, interior forcing of size ε (in L^∞) with zero boundary and zero initial data gives rise to a surface wave solution of size $O(\varepsilon^2)$; here the response is the same size as for the forward problem in the whole space, or for a boundary problem of the same order that satisfies the uniform Lopatinski condition. This is compatible with the absence of any loss of derivatives from interior forcing to solution in the linearized estimates. On the other hand, boundary forcing of size $O(\varepsilon^2)$ with zero interior forcing and zero initial data gives rise to a surface wave solution of size $O(\varepsilon^2)$, while for a boundary problem of the same order that satisfies the uniform Lopatinski condition, the solution (which would not be a surface wave!) is of size $O(\varepsilon^3)$. This discrepancy is compatible with the loss of one derivative from boundary forcing to solution in the linearized estimates. Thus, there is a sense, perhaps slightly contorted, in which “amplification” does occur here.

At this stage, it remains unclear whether or not the leading order amplitude, which we know to exist on a time interval that is independent of the small wavelength ε , is a reasonable approximation of the exact solution. It is also unclear whether or not the exact solution exists on a time interval that is independent of the small wavelength ε . This part of the analysis is achieved for wavetrains in first order systems in [Mar10] by using high order approximate solutions to construct nearby exact solutions. This type of argument dates back to [Guè93]. In this case the Fourier transform of profiles with respect to the periodic fast variable θ is a function of $k \in \mathbb{Z}$ rather than $k \in \mathbb{R}$. Marcou [Mar10] took advantage of this discrete Fourier spectrum to construct arbitrarily many correctors yielding an arbitrarily high order approximate solution, and then added a small remainder to this to obtain a nearby exact solution on a fixed time interval independent of ε . In particular, her approximate solutions involved profiles that decayed at a rate $e^{-\delta z}$ in the fast variable $z = x_2/\varepsilon$ normal to the boundary for a *fixed* δ independent of $k \in \mathbb{Z} \setminus \{0\}$. In the pulse setting the presence of Fourier spectrum arbitrarily close to $k = 0$ makes it impossible to construct such “strongly evanescent” profiles; indeed, now $\delta = \delta(k) \rightarrow 0$ as $k \rightarrow 0$. In the case of surface waves given by pulses, as in other problems involving pulses [CW13, CW14], there is no hope of constructing high order approximate solutions. Indeed, in the pulse setting it is usually impossible to construct high order correctors even in problems where the uniform Lopatinski condition is satisfied [CW13]. In the Rayleigh pulse problem studied here, we are actually able to construct only a *single* corrector.

The first-order systems in [Mar10] were assumed to be symmetric with maximal dissipative boundary conditions; moreover, the boundary conditions were linear and *homogeneous*, so she was able to use the well-posedness estimates without loss of derivatives which hold for such systems to rigorously justify the high-order expansions. We see no way to use estimates without loss in a similar way in the traction problem of nonlinear elasticity, even if one is trying to justify high order approximate *wavetrain* solutions. To sum up, for several reasons a quite different approach is needed to both of the open questions described above. Our alternative approach to both questions depends on the study of singular systems. This is an idea that goes back to [JMR95] for problems in free space, and to [Wil02] for problems on domains with boundary.

As we explain in more detail in the introduction to Chapter 3, if one looks for an *exact* solution $U^\varepsilon(t, x)$ to the system of 2D nonlinear elasticity (3.4) in the form

$$(1.1) \quad U^\varepsilon(t, x) = u^\varepsilon(t, x, \theta)|_{\theta = \frac{\beta \cdot (t, x_1)}{\varepsilon}}, \text{ where } \beta = (\beta_0, \beta_1) \text{ and } x = (x_1, x_2),$$

then by plugging the ansatz (1.1) into (3.4), one obtains a similar system for the function $u^\varepsilon(t, x, \theta)$, except that the derivatives $\partial_t, \partial_{x_1}$ are now replaced by

$$(1.2) \quad \partial_{t,\varepsilon} := \partial_t + \beta_0 \frac{\partial \theta}{\varepsilon}, \quad \partial_{x_1,\varepsilon} := \partial_{x_1} + \beta_1 \frac{\partial \theta}{\varepsilon}$$

wherever they occur. We refer to this new problem (3.8) as the *singular system* associated with (3.4). One gain is immediately apparent: whereas high Sobolev norms of the boundary data $\varepsilon^2 G(t, x_1, \frac{\beta \cdot (t, x_1)}{\varepsilon})$ in (3.4) clearly blow up as $\varepsilon \rightarrow 0$, the Sobolev norms of the boundary data $\varepsilon^2 G(t, x_1, \theta)$ in (3.8) *go to zero* as $\varepsilon \rightarrow 0$. A significant price is also apparent. The problem (3.8) is now singular in two different senses; first because of the factors $1/\varepsilon$ that appear in the derivatives, and second because derivatives now occur in the linear combinations (1.2). Even if ε is fixed, the second sense still makes the singular system hard to study. The first sense of singularity implies that when the boundary, $x_2 = 0$, is noncharacteristic, one cannot hope to control norms of normal (∂_{x_2}) derivatives of solutions, uniformly with respect to ε , simply by employing the classical device of first controlling tangential derivatives and then using the equation. Our decision to build on the approach of [ST88] to the traction problem, rather than that of [Kat85] or [SN89], was based on the fact that we saw no way to implement either of the latter approaches in the associated *singular* traction problem.

In Chapter 3 we prove the existence of exact solutions to the original nonlinear Saint Venant-Kirchhoff system (3.4) on a fixed time interval independent of ε as a consequence of such an existence theorem for the associated singular system (3.8). In this approach the exact solution is obtained without any reliance on the approximate solution. Employing an idea used by [ST88] in the nonsingular setting, we consider not just (3.8) but the trio of coupled singular systems (3.11)-(3.13), where (3.11) and (3.12) are obtained from the original system by differentiating it, and (3.13) is essentially the same as (3.8). In contrast to [ST88], it turns out that we are not able to use an iteration scheme modeled on the one employed in that paper, or indeed any iteration scheme at all, to prove estimates uniform with respect to ε . The uniform estimates depend on knowing the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$, which holds (for short times) between the solution v^ε of (3.11)-(3.12) and u^ε of (3.13), and this relation holds only for exact solutions *not* for iterates.⁴ Instead, we use a continuous induction argument based on local existence and continuation theorems for singular systems with ε fixed (Propositions 3.83 and 3.87) and the uniform a priori estimate of Proposition 3.20 for the coupled systems. The proof of the latter proposition, based on *simultaneous* estimation of the trio of modified singular systems (3.86)-(3.88), is the core of the rigorous analysis of this paper. The continuous induction argument is summarized in more detail in the introduction to Chapter 3, and our uniform existence result for exact solutions is stated in Theorem 3.7.

The analysis of singular systems in this paper, for example the proof of the basic estimates for the linearized systems corresponding to (3.11)-(3.13) given in Proposition 3.11, uses two main tools. The first is the calculus of singular pseudodifferential operators for pulses constructed in [CGW14b], and the second is the collection of estimates proved in section 3.4 of singular norms of nonlinear functions of u .

The calculus of [CGW14b] is a calculus for symbols of finite (t, x, θ) regularity which allows us to compose and take adjoints of operators that are, roughly speaking, pseudodifferential with respect to the

⁴Here $\nabla_\varepsilon = (\partial_{x_1,\varepsilon}, \partial_{x_2})$.

vector fields $\partial_{t,\varepsilon}, \partial_{x_1,\varepsilon}$ (1.2). The operators are defined in (A.2).⁵ This is a “first-order” calculus in the sense that only the principal symbols of compositions or adjoints are given; but there are explicit formulas for the error terms which give a clear picture of how big they are as $\varepsilon \rightarrow 0$. Using this calculus, one can for example construct a Kreiss symmetrizer for a singular problem simply by taking the classical symbol of the Kreiss symmetrizer for the corresponding nonsingular problem and quantizing it in the singular calculus by the process described in (A.2). The main results of this calculus are recalled in Appendix A. The calculus was originally created with applications to first order systems in mind. Here we deal with second order singular systems, so commutators arise, sometimes involving operators of fractional order, which can not be treated using the results of [CGW14b]. Thus we have had to extend the calculus of [CGW14b] in several ways, and these extensions are given in section A.5.

A quick inspection of the estimates of Proposition 3.11 for the linearized singular problems shows that a variety of “singular norms” occur there. These are norms of the form

$$|\Lambda_D^r u|_{H^m(t,x,\theta)},$$

where Λ_D^r is the singular operator associated to the symbol $\langle \xi' + \beta \frac{k}{\varepsilon}, \gamma \rangle^r$, and r and m are (usually) nonnegative constants. Clearly, to apply these estimates to nonlinear problems we need to be able to estimate singular norms

$$|\Lambda_D^r f(u)|_{H^m}$$

of nonlinear functions of u in terms of singular norms of u . Estimates of this kind, which are new, are proved in section 3.4 for analytic functions $f(u)$; both tame and simpler non-tame estimates are given.⁶ In addition, to take advantage of the extra microlocal precision in the estimates provided, for example, by the terms involving the singular pseudodifferential cutoffs $\phi_{j,D}$ in (3.37) and (3.39), we need to show that in some cases extra singular *microlocal* regularity of u is preserved under nonlinear functions $f(u)$. A result of this type, which can be viewed as a singular version of the classical Rauch’s lemma [Rau79], is given in Proposition 3.28.

Once we have the leading order approximate solution and the exact solution, the error analysis leading to the full justification of geometric optics relies on the construction of an appropriate *corrector*. More precisely, we need to add a small corrector to the leading order approximate solution in order to be able to control its difference with the exact solution. In Chapter 2 equations are derived (2.14) that one “would like” the corrector to satisfy. The amplitude equation (Proposition 2.2), which determines the trace of the leading order term of the approximate solution, is a solvability condition for the Fourier transform with respect to θ of these corrector equations (2.16). In Chapter 4 we solve the transformed equations (2.16) for each $k \neq 0$ (here k is the Fourier transform variable dual to θ) and discover that the corrector is $O(1/k^2)$ near $k = 0$. Roughly, these two factors of $1/k$ reflect the two integrations in θ , each on the unbounded domain \mathbb{R} , that are needed to construct a corrector in this second order problem; on the θ side, each integration in θ introduces growth with respect to θ in the corrector. This illustrates the difficulty of constructing correctors in pulse problems. Our solution of the transformed equations (2.16) behaves too badly at $k = 0$ to be inverse transformed, so we now regard it as a preliminary corrector. Using an idea we learned from [AR03], we modify this object by multiplying it by a low frequency cutoff, $\chi(k/\varepsilon^b)$, where $\chi(s)$ is a smooth cutoff supported away from 0 and equal to one on $|s| \geq 1$, and $b > 0$ is a constant to be chosen. This modification introduces new errors of course, but we show in the error analysis of

⁵In [CGW14b] a singular calculus for wavetrains was also constructed that was an improvement over the one first constructed in [Wil02].

⁶The tame estimates are needed mainly for the continuation result, Proposition 3.87.

Chapter 5 that the new errors are offset by the presence of the factor ε^3 on the corrector, provided b is chosen correctly.

It is natural to wonder if one could construct correctors with better decay properties in θ if one started with boundary data given by a function $G(t, x_1, \theta)$ with stronger than H^s type decay in θ . It turns out that even if one assumes $G(t, x_1, \theta)$ decays *exponentially* as $|\theta| \rightarrow \infty$, the leading term of the approximate solution generally exhibits no better than H^s type decay in θ ; see Remark 4.1. This loss of θ -decay from data to solution is typical of evanescent pulses. It is a linear phenomenon and occurs even in problems where the uniform Lopatinskii condition is satisfied [Wil15]. Evanescent pulses are generally not even L^1 in θ , so unlike evanescent wavetrains they have no well-defined mean.

With the exact and approximate solutions in hand we are ready in Chapter 5 for the error analysis. As in Chapter 3 the estimates, which involve some norms that cannot be localized in time, must be done on the full half-space Ω . In particular, we need extensions of the approximate solutions, which at this point are only defined on a short time interval, to the full half space. This extension process has to be done carefully and, in fact, it turns out to require most of theory of Chapter 3. Parallel to the trio of modified singular systems (3.86)-(3.88) that were estimated in the study of the exact solution in Chapter 3, we define a trio of approximate solution systems (5.7)-(5.9) whose solutions provide the needed extensions; a causality argument (Remark 3.85) shows that the solutions agree with the original approximate solutions for small times. The exact solution trio (3.86)-(3.88) had solutions $(v^\varepsilon, u^\varepsilon)$ on Ω , while the approximate solution trio (5.7)-(5.9) has solutions $(v_a^\varepsilon, u_a^\varepsilon)$. Naturally, then, we consider the trio of error equations (5.11)-(5.13) satisfied by the differences $(w^\varepsilon, z^\varepsilon) := (v^\varepsilon - v_a^\varepsilon, u^\varepsilon - u_a^\varepsilon)$. As in the earlier cases, this trio must be estimated simultaneously in order to take advantage of the relation $w^\varepsilon = \nabla_\varepsilon z^\varepsilon$ that holds for short times. These arguments are summarized in the introduction to Chapter 5. Ultimately, we are able to derive a rate of convergence to zero for the $E_{m,\gamma}$ norm (3.77) of the difference between the exact and approximate solutions; this result is stated in Theorem 3.8 and Corollary 3.9.⁷

In Chapter 6 we explain how the main theorems, Theorems 3.7 and 3.8, extend to general isotropic hyperelastic materials governed by an analytic stored energy function, and also how those theorems extend readily to the wavetrain case. In section 6.3 we discuss the only obstruction that remains to extending these theorems to dimensions $d \geq 3$. We show that in $d \geq 3$ the linearized problem has characteristics of variable multiplicity that fail to be algebraically regular in the sense of [MZ05], and which are at the same time glancing. Thus, the problem falls outside the scope of existing Kreiss symmetrizer technology.

We have avoided any energy dissipation argument in the construction of exact solutions, for this gives us hope to extend our work to free boundary problems for first order hyperbolic problems that also give rise to surface waves. Such situations arise indeed in the modeling of liquid-vapor phase transitions or in magnetohydrodynamics, see [BG98, AH03]. We believe that the error analysis in Chapter 5 is flexible enough so that, provided one has an exact solution in such a problem, the construction of a corrector and the estimates of the error terms involved in Chapter 5 could be adapted in order to yield a full justification of the high frequency asymptotics.

Chapters 2, 3, 4 and 5 are meant to be as independent as possible. Chapter 5 is really the only chapter that depends substantially on others, those being mainly Chapters 3 and 4. We have included detailed and mostly non-technical introductions to these chapters in an effort to make the reader familiar with the

⁷Several papers, for example [JMR95, CGW11, CW13, Her15], in which singular systems were used to rigorously justify approximate solutions for quasilinear problems, used the method of simultaneous Picard iteration, which estimates the difference between exact solution iterates and approximate solution iterates. That method was not an option here, since we were unable to prove the uniform existence of exact solutions by any kind of iteration scheme. A variant of the method in section 5.5, based on direct estimation of the difference between exact and approximate solutions, can be used to avoid the use of simultaneous Picard iteration in the earlier works.

main ideas before having to plunge into the estimates. We hope the reader will forgive a certain amount of repetition arising from this.

Notation Throughout this work, we let $\mathcal{M}_{n,N}(\mathbb{K})$ denote the set of $n \times N$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and we use the notation $\mathcal{M}_N(\mathbb{K})$ when $n = N$. The trace of a matrix $M \in \mathcal{M}_N(\mathbb{K})$ is denoted $\text{tr } M$. The transpose of a matrix (or vector) M is denoted M^T . We let I denote the identity matrix, without mentioning the dimension. The norm of a (column) vector $X \in \mathbb{C}^N$ is $|X| := (X^* X)^{1/2}$, where the row vector X^* denotes the conjugate transpose of X . If X, Y are two vectors in \mathbb{C}^N , we let $X \cdot Y$ denote the (bilinear) quantity $\sum_j X_j Y_j$, which coincides with the usual scalar product in \mathbb{R}^N when X and Y are real. We often use Einstein's summation convention in order to make some expressions easier to read.

The Fourier transform of a function f from \mathbb{R} to \mathbb{C} is defined as

$$\forall k \in \mathbb{R}, \quad \widehat{f}(k) := \int_{\mathbb{R}} e^{-ik\theta} f(\theta) d\theta.$$

The Hilbert transform $\mathcal{H}f$ of f is then defined by

$$\widehat{\mathcal{H}f}(k) := -i \text{sgn}(k) \widehat{f}(k),$$

where sgn denotes the sign function.

The letter C always denotes a positive constant that may vary from line to line or within the same line. Dependence of the constant C on various parameters is made precise throughout the text. The sign \lesssim means \leq up to a multiplicative constant.

To avoid having expressions like $v_{1,T}^{\varepsilon,s}$ or $v_T^{\varepsilon,s}$ appear repeatedly in Chapter 3 and later, we shall often suppress the ε and T indices and write simply v_1^s or v^s instead. Here v_1^ε is the first component of $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$. The s and T on $v_{1,T}^{\varepsilon,s}$ indicate that we are taking a Seeley extension (Proposition 3.15) to all time of $v_1^\varepsilon|_{t < T}$. Almost every function that occurs in the study of singular systems has ε dependence, so suppressing ε should cause no trouble. The presence of the superscript s should *always* be taken to imply the presence of a suppressed subscript T . The positive number T will always be small, and sometimes will lie in a range that depends on ε , $0 < T \leq T_\varepsilon$. The coefficients in the linearized systems we study on the whole half-space, for example those appearing in (3.86)-(3.88), will usually be functions of terms like v^s . Later reminders of this will be provided. Further notation is made precise in the body of the text.

Chapter 2

Derivation of the weakly nonlinear amplitude equation

In this Chapter, we revisit the weakly nonlinear asymptotic analysis for second order hyperbolic initial boundary value problems that come from a variational principle. Our main goal is to derive an amplitude equation that governs the evolution of small amplitude high frequency solutions in the case where the so-called WKB ansatz incorporates all possible slow and fast variables. As expected from pioneering works devoted to nonlinear elasticity, the amplitude equation we derive takes the form of a scalar integro-differential equation taking place on the boundary of the space domain. This equation displays two main features: propagation at an appropriate *group velocity* according to the slow spatial variables along the boundary of the space domain, and nonlinearity due to the presence of a *bilinear Fourier multiplier* that governs the evolution with respect to the fast variable. We unify previous works arising either from the ‘applied’ or more ‘theoretical’ literature and clarify which solvability condition is actually needed in order to derive our main amplitude equation.

Taking slow variables into account is crucial in the upcoming Chapters 3, 4, and 5 for providing a functional framework in which we are able to construct and analyze *exact* pulse solutions. In the present Chapter, the analysis is mostly *formal* and we aim at identifying the leading order term in the presumably valid asymptotic expansion of exact solutions. Showing that exact solutions are indeed well approximated by this leading order term is the purpose of Chapter 5.

Though our main concern in this work is the system of nonlinear elasticity, we aim at keeping the discussion in a rather general framework whenever possible, which might accelerate the adaptation of the present work to related problems with surface waves arising for instance in the modeling of liquid vapor phase transitions, magnetohydrodynamics and/or liquid crystals, see, e.g., [BG98, AH03, Sax89, AH13] and further references therein.

2.1 The variational setting: assumptions

In this Section, we revisit the analysis of [BGC15] and derive the amplitude equation that governs the evolution of weakly nonlinear surface (or Rayleigh) waves. Keeping the discussion in a rather general framework, we start from a Lagrangian of the form:

$$\mathcal{L}[\mathbf{u}] := \int_0^T \int_{\Omega} \left(\frac{1}{2} |\mathbf{u}_t|^2 - W(\nabla \mathbf{u}) \right) dx dt.$$

Here the space domain $\Omega \subset \mathbb{R}^d$ is a half-space, $\mathbf{u} \in \mathbb{R}^N$ is the (possibly vector-valued) unknown function, \mathbf{u}_t denotes its time derivative and $\nabla \mathbf{u}$ denotes its spatial Jacobian matrix (which we sometimes call its gradient). The function W plays the role of a ‘stored elastic energy’, and the total amount of energy over the space domain Ω is then denoted

$$\mathscr{W}[\mathbf{u}] := \int_{\Omega} W(\nabla \mathbf{u}) \, dx .$$

Opposite to the case considered in [BGC15] we assume here for simplicity that W only depends on the unknown \mathbf{u} through its (spatial) gradient. Hence Assumptions (H1) and (H2) in [BGC15] are trivially satisfied. In the case of hyperelastic materials, which is our main concern here, there holds $N = d$, with either $d = 2$ or $d = 3$.

In the following calculations, Greek letters α, β, γ usually correspond to indices for the coordinates of the vector \mathbf{u} and therefore run through the set $\{1, \dots, N\}$. Roman letters j, ℓ, m refer to the coordinates of the space variable x and therefore run through $\{1, \dots, d\}$. For instance, the (α, j) coordinate of the matrix $\nabla \mathbf{u}$ is denoted $u_{\alpha, j}$, where the subscript ‘ j ’ is a short cut for denoting partial differentiation with respect to x_j . We thus consider W as a function from $\mathcal{M}_{N, d}(\mathbb{R})$ into \mathbb{R} and tacitly assume that it is as smooth as we want (at least \mathcal{C}^3 as far as the derivation of the amplitude equation is concerned). We use from now on Einstein’s summation convention over repeated indices, unless otherwise stated.

Let us write the space domain as $\Omega = \{x \cdot \boldsymbol{\nu} > 0\}$, with $\boldsymbol{\nu}$ the normal vector to $\partial\Omega$ pointing inwards. We are then interested in *critical points* of the above Lagrangian \mathcal{L} . These correspond to functions \mathbf{u} that satisfy the interior equations:

$$(2.1) \quad \forall \alpha = 1, \dots, N, \quad \partial_t^2 u_{\alpha} - \left(\frac{\partial W}{\partial u_{\alpha, j}}(\nabla \mathbf{u}) \right)_{,j} = 0,$$

with boundary conditions:

$$(2.2) \quad \forall \alpha = 1, \dots, N, \quad \nu_j \frac{\partial W}{\partial u_{\alpha, j}}(\nabla \mathbf{u}) \Big|_{\partial\Omega} = 0,$$

In what follows, we assume that all constant states $\mathbf{u} \in \mathbb{R}^N$ are critical points of \mathcal{L} , independently of the choice of the space domain Ω . We also normalize the stored energy so that it vanishes at the origin. Equivalently, we make the following assumption:

$$(H1) \quad W(0) = 0 \text{ and } \frac{\partial W}{\partial u_{\alpha, j}}(0) = 0 \text{ for all } \alpha, j.$$

Assumption (H1) holds when the stored energy W depends quadratically on ∇u , and in that case the equations (2.1)-(2.2) are linear with respect to \mathbf{u} . Here we shall be interested in solutions \mathbf{u} that are small perturbations of a constant state, say 0 (up to translating in \mathbf{u}). In the context of nonlinear elasticity, $\mathbf{u}(t, x)$ refers to the displacement with respect to an ‘equilibrium configuration’. The deformation of the equilibrium configuration is given by the mapping $(x \mapsto x + \mathbf{u}(t, x))$.

For future use, we introduce the coefficients involved in the Taylor expansion of W up to the third order at the origin:

$$(2.3) \quad c_{\alpha j \beta \ell} := \frac{\partial^2 W}{\partial u_{\alpha, j} \partial u_{\beta, \ell}}(0), \quad d_{\alpha j \beta \ell \gamma m} := \frac{\partial^3 W}{\partial u_{\alpha, j} \partial u_{\beta, \ell} \partial u_{\gamma, m}}(0).$$

The linearization of (2.1)-(2.2) at the constant solution $\mathbf{u} \equiv 0$ reads

$$(2.4) \quad \begin{cases} \partial_t^2 v_\alpha - c_{\alpha j \beta \ell} v_{\beta, j \ell} = 0, & \forall \alpha = 1, \dots, N, \quad x \in \Omega, \\ \nu_j c_{\alpha j \beta \ell} v_{\beta, \ell} \Big|_{\partial \Omega} = 0, & \forall \alpha = 1, \dots, N, \end{cases}$$

and the determination of formal and/or rigorous high frequency weakly nonlinear solutions to (2.1)-(2.2) heavily depends on the stability properties of (2.4). In what follows, we shall assume that the linearized problem (2.4) admits a one-dimensional space of ‘surface waves’ which, in the context of linearized elasticity, correspond to Rayleigh waves. Let us be a little bit more specific. We first assume that the linearization of the stored energy W at 0 is strictly rank one convex (a strong form of the Legendre-Hadamard condition), that is:

(H2) There exists a constant $c > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $v \in \mathbb{R}^N$, there holds

$$c_{\alpha j \beta \ell} v_\alpha \xi_j v_\beta \xi_\ell \geq c |v|^2 |\boldsymbol{\xi}|^2.$$

Assumption (H2) ensures that the Cauchy problem

$$\partial_t^2 v_\alpha - c_{\alpha j \beta \ell} v_{\beta, j \ell} = 0, \quad \alpha = 1, \dots, N, \quad x \in \mathbb{R}^d,$$

is well-posed in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d; \mathbb{R}^N)$, see [Ser06]. Hence (2.4) is a linear hyperbolic boundary value problem which should be supplemented with some initial data for v , which we do not write for the moment. The analysis of (2.4) follows the general theory of [Kre70, Sak82] and relies on the so-called normal mode analysis. We therefore look for solutions to (2.4) of the form

$$\mathbf{v}(t, x) = e^{i(\tau - i\gamma)t + i\boldsymbol{\eta} \cdot x} V(\boldsymbol{\nu} \cdot x),$$

with $\gamma > 0$, $\boldsymbol{\eta}$ in the cotangent space to $\partial \Omega$ (which reduces here to assuming that $\boldsymbol{\eta}$ is orthogonal to $\boldsymbol{\nu}$ because Ω is a half-space) and a profile V vanishing at $+\infty$. Plugging the previous ansatz for $\mathbf{v}(t, x)$ in (2.4), we are led to determining the set of functions V , from \mathbb{R}^+ into \mathbb{C}^N , that vanish at $+\infty$ and satisfy the second-order differential problem:

$$(2.5) \quad \begin{cases} (\tau - i\gamma)^2 V_\alpha + c_{\alpha j \beta \ell} (i\eta_j + \nu_j \partial_z) (i\eta_\ell + \nu_\ell \partial_z) V_\beta = 0, & \forall \alpha = 1, \dots, N, \quad z > 0, \\ \nu_j c_{\alpha j \beta \ell} (i\eta_\ell + \nu_\ell \partial_z) V_\beta \Big|_{z=0} = 0, & \forall \alpha = 1, \dots, N. \end{cases}$$

It is convenient to rewrite (2.5) in a more compact form, and we therefore introduce the following $N \times N$ matrices for all vector $\boldsymbol{\xi} \in \mathbb{R}^d$:

$$(2.6) \quad \Sigma(\boldsymbol{\xi}) := \left(c_{\alpha j \beta \ell} \xi_j \xi_\ell \right)_{\alpha, \beta=1, \dots, N}, \quad A_j(\boldsymbol{\xi}) := \left(c_{\alpha j \beta \ell} \xi_\ell \right)_{\alpha, \beta=1, \dots, N}, \quad \forall j = 1, \dots, d.$$

With the previous notation, (2.5) reads

$$(2.7) \quad \begin{cases} ((\tau - i\gamma)^2 I_N - \Sigma(\boldsymbol{\eta})) V + i(\nu_j A_j(\boldsymbol{\eta}) + \nu_j A_j(\boldsymbol{\eta})^T) \partial_z V + \Sigma(\boldsymbol{\nu}) \partial_z^2 V = 0 & z > 0, \\ i \nu_j A_j(\boldsymbol{\eta}) V + \Sigma(\boldsymbol{\nu}) \partial_z V \Big|_{z=0} = 0, \end{cases}$$

where we have used the Schwarz relation which translates into symmetry properties for the coefficients $c_{\alpha j \beta \ell}$ (namely, $c_{\alpha j \beta \ell} = c_{\beta \ell \alpha j}$). The strict rank one convexity assumption (H2) shows that $\Sigma(\boldsymbol{\nu})$ is symmetric positive definite, and therefore the second-order boundary value problem (2.7) can be equivalently

recast as a first-order augmented system

$$(2.8) \quad \begin{cases} \partial_z \begin{pmatrix} V \\ W \end{pmatrix} = \mathbf{J} H(\tau - i\gamma, \boldsymbol{\eta}) \begin{pmatrix} V \\ W \end{pmatrix}, & z > 0, \\ W(0) = 0, \end{cases}$$

where, using standard block matrix notation, we have set

$$(2.9) \quad \mathbf{J} := \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \quad H(\zeta, \boldsymbol{\eta}) := \begin{pmatrix} K_0(\zeta, \boldsymbol{\eta}) & i\nu_j A_j(\boldsymbol{\eta})^T \Sigma(\boldsymbol{\nu})^{-1} \\ -i\Sigma(\boldsymbol{\nu})^{-1} \nu_j A_j(\boldsymbol{\eta}) & \Sigma(\boldsymbol{\nu})^{-1} \end{pmatrix},$$

and the upper-left block K_0 of H in (2.9) is defined by:

$$(2.10) \quad K_0(\zeta, \boldsymbol{\eta}) := \zeta^2 I_N - \Sigma(\boldsymbol{\eta}) + (\nu_j A_j(\boldsymbol{\eta})^T) \Sigma(\boldsymbol{\nu})^{-1} (\nu_\ell A_\ell(\boldsymbol{\eta})).$$

The first-order ‘Hamiltonian’ formulation (2.8) was already introduced in [Ser06] and it was used in [BGC12] in order to show structural properties for the amplitude equation which we shall derive here in a slightly more general context. The second-order formulation (2.7) was adopted in [BGC15] and we mainly follow this approach here. However, the equivalent formulation (2.8) will be used at some point, which is why we recall it here with notation similar to those in [BGC12].

The aim of the normal mode analysis is to determine the frequency pairs $(\tau - i\gamma, \boldsymbol{\eta})$ with $\gamma \geq 0$ for which (2.7) admits a nontrivial solution. The subtlety lies in the definition of which solutions are admissible when γ equals zero. However, the strict rank one convexity assumption (H2) above yields the classical result that the matrix $\mathbf{J} H(\tau - i\gamma, \boldsymbol{\eta})$ is *hyperbolic* (in the sense of dynamical systems, meaning that it has no purely imaginary eigenvalue) as long as $\gamma \neq 0$ or¹

$$\gamma = 0 \quad \text{and} \quad \tau^2 < \min_{\omega \in \mathbb{R}} \lambda_{\min}(\Sigma(\boldsymbol{\eta} + \omega \boldsymbol{\nu})).$$

The latter case corresponds to ‘elliptic frequencies’ $(\tau, \boldsymbol{\eta})$ of the cotangent bundle of $\mathbb{R}_t \times \partial\Omega$. Observe that by the strict rank one convexity assumption (H2), this set contains a cone of the form $\{|\tau| < c|\boldsymbol{\eta}|\}$ for some $c > 0$. For such elliptic frequencies, $H(\tau, \boldsymbol{\eta})$ in (2.9) is Hermitian, and eigenvalues of $\mathbf{J} H(\tau, \boldsymbol{\eta})$ come in pairs $(-\omega, \bar{\omega})$ because the adjoint matrix $(\mathbf{J} H(\tau, \boldsymbol{\eta}))^*$ is conjugated to $-\mathbf{J} H(\tau, \boldsymbol{\eta})$. Here, and from now on, we adopt the convention $\text{Re } \omega > 0$. The stable subspace $\mathbb{E}^s(\tau - i\gamma, \boldsymbol{\eta})$ of $\mathbf{J} H(\tau - i\gamma, \boldsymbol{\eta})$ is therefore well-defined, has dimension N and depends analytically on $(\tau - i\gamma, \boldsymbol{\eta})$ on the connected set that is the union of $\{\gamma > 0\}$ and of the set of elliptic frequencies. Furthermore, it is known from the general theory in [Kre70, Sak82, Mét00] that, at least in the case where the hyperbolic operator in (2.4) has constant multiplicity, the stable subspace \mathbb{E}^s admits a continuous extension up to $\gamma = 0$ for all $(\tau, \boldsymbol{\eta}) \neq (0, 0)$. In what follows, $\mathbb{E}^s(\tau, \boldsymbol{\eta})$ denotes this continuous extension for all $(\tau, \boldsymbol{\eta})$, which coincides with the ‘true’ stable subspace of $\mathbf{J} H(\tau, \boldsymbol{\eta})$ for elliptic frequencies².

Well-posedness of the linear initial boundary value problem (2.4), with prescribed initial data $v|_{t=0}$, was investigated thoroughly by Serre in [Ser06] where he obtained the striking result that strong well-posedness holds *if and only if* the energy

$$\mathscr{W}_2[\mathbf{u}] := \int_{\Omega} W_2(\nabla \mathbf{u}) \, dx, \quad W_2(F) := c_{\alpha j \beta \ell} F_{\alpha, j} F_{\beta, \ell}, \quad \forall F \in \mathscr{M}_{N, d}(\mathbb{R}),$$

¹Here $\lambda_{\min}(S)$ denotes the smallest eigenvalue of a real symmetric matrix S .

²The analysis in this Chapter is only concerned with elliptic frequencies, which is the reason why we do not emphasize too much the assumptions ensuring that the continuous extension of \mathbb{E}^s up to $\gamma = 0$ is well-defined. Such assumptions will be enforced in Chapter 3 where continuous extension of \mathbb{E}^s will play a major role.

is coercive (and thereby strictly convex) over $\dot{H}^1(\Omega)$. The subscript 2 in \mathscr{W}_2, W_2 refers to the Taylor expansion at order 2 of the original stored energy W at the origin. When \mathscr{W}_2 is coercive, the analysis in [Ser06] also shows that there exist *surface wave* solutions to (2.4), which we shall assume here to be of *finite energy*. More precisely, in agreement with the results in [Ser06], we make the following assumption:

(H3) For all $\boldsymbol{\eta} \neq 0$ in the cotangent space to $\partial\Omega$, there exists a real $\tau_r(\boldsymbol{\eta})$ satisfying

$$0 < \tau_r(\boldsymbol{\eta}) < \left(\min_{\omega \in \mathbb{R}} \lambda_{\min}(\Sigma(\boldsymbol{\eta} + \omega \boldsymbol{\nu})) \right)^{1/2},$$

such that for all $(\gamma, \tau, \boldsymbol{\eta}) \neq (0, 0, 0)$, there holds:

$$\left\{ \begin{pmatrix} V \\ W \end{pmatrix} \in \mathbb{E}^s(\tau - i\gamma, \boldsymbol{\eta}) \mid W = 0 \right\} \neq \{0\}$$

if and only if $\gamma = 0$ and $\tau = \pm\tau_r(\boldsymbol{\eta})$. Furthermore, for all $\underline{\boldsymbol{\eta}} \neq 0$, the matrix $\mathbf{J}H(\tau, \boldsymbol{\eta})$ is geometrically regular near $(\pm\tau_r(\underline{\boldsymbol{\eta}}), \underline{\boldsymbol{\eta}})$, meaning that it admits a basis of eigenvectors $(\mathbf{R}_1, \dots, \mathbf{R}_{2N})(\tau, \boldsymbol{\eta})$ that depends analytically on $(\tau, \boldsymbol{\eta})$ near $(\pm\tau_r(\underline{\boldsymbol{\eta}}), \underline{\boldsymbol{\eta}})$ with the convention that $\mathbf{R}_1, \dots, \mathbf{R}_N$ span the stable subspace \mathbb{E}^s , and the $N \times N$ determinant

$$\Delta(\tau, \boldsymbol{\eta}) := \det(S_1(\tau, \boldsymbol{\eta}) \cdots S_N(\tau, \boldsymbol{\eta})), \quad \mathbf{R}_\alpha := \begin{pmatrix} R_\alpha \\ S_\alpha \end{pmatrix},$$

has a simple root (with respect to τ) at $(\pm\tau_r(\underline{\boldsymbol{\eta}}), \underline{\boldsymbol{\eta}})$.

Let us recall that due to the fact that the so-called Lopatinskii determinant Δ has a simple root at $(\tau_r(\underline{\boldsymbol{\eta}}), \underline{\boldsymbol{\eta}})$, the subspace

$$\left\{ \begin{pmatrix} V \\ W \end{pmatrix} \in \mathbb{E}^s(\tau_r(\underline{\boldsymbol{\eta}}), \underline{\boldsymbol{\eta}}) \mid W = 0 \right\}$$

is one-dimensional. (Observe also that $\mathbb{E}^s(\tau, \boldsymbol{\eta})$ does not depend on the sign of τ , which is why we restrict here to positive values of τ .) Consequently, for all tangential wave vector $\boldsymbol{\eta} \neq 0$, there exists a one-dimensional family of surface waves

$$e^{\pm i \tau_r(\boldsymbol{\eta}) t + i \boldsymbol{\eta} \cdot \boldsymbol{x}} V(\boldsymbol{\nu} \cdot \boldsymbol{x}),$$

solution to (2.4) with $V(+\infty) = 0$ (the decay is actually exponential).

By rescaling, it is readily observed from (2.7) that for all $k > 0$, $\tau_r(k \boldsymbol{\eta}) = k \tau_r(\boldsymbol{\eta})$ and if

$$e^{\pm i \tau_r(\boldsymbol{\eta}) t + i \boldsymbol{\eta} \cdot \boldsymbol{x}} V(\boldsymbol{\nu} \cdot \boldsymbol{x}),$$

is a (nontrivial) surface wave solution to (2.4), then

$$e^{\pm i k \tau_r(\boldsymbol{\eta}) t + i k \boldsymbol{\eta} \cdot \boldsymbol{x}} V(k \boldsymbol{\nu} \cdot \boldsymbol{x}),$$

is a (nontrivial) surface wave solution to (2.4) for the rescaled frequencies $k(\tau_r(\boldsymbol{\eta}), \boldsymbol{\eta})$.

We have now made all our assumptions on the linearized problem (2.4) and turn to the derivation of the amplitude equation governing weakly nonlinear asymptotic solutions to the nonlinear problem (2.1), (2.2).

2.2 Weakly nonlinear asymptotics

In this Section, we show formally that high frequency weakly nonlinear solutions to the nonlinear equations (2.1), (2.2) are governed by a nonlocal Burgers type equation that is similar to the ones derived in [BGC12] or [BGC15] for second-order equations, or in [Hun89, Mar10] for first-order hyperbolic systems. In the case of elastodynamics, the derivation of such amplitude equations dates back at least to [Lar83] for two-dimensional elasticity. Part of the analysis in [Lar83] was later simplified in [PT85, Par88] though those last two references did not include any dependence of the weakly nonlinear solution on what we call below ‘slow spatial variables’. Those slow variables were considered in [Lar83], leading to rather complicated calculations and a somehow mysterious relation (Equation (A17) in [Lar83]) stating that weakly nonlinear Rayleigh waves actually propagate at the Rayleigh speed with respect to the slow tangential variables along the boundary of the elastic material. The analogous relation in the case of anisotropic materials (Equation (51) in [Lar86]) is explained in a more convincing way, though it seems to rely heavily on the fact that one considers two-dimensional materials. Independently of one’s ability to verify the accuracy of the expressions given in [Lar83, Lar86], there is a puzzling fact arising, which is that the propagation of the weakly nonlinear Rayleigh wave in the slow tangential space variables along the boundary seems to be governed by the Rayleigh speed, which rather arises as a *phase velocity*, while one might reasonably expect to see a *group velocity* arise. As we show below, the conclusions in [Lar83, Lar86] are indeed correct but they are linked to the fact that the boundary has only one spatial dimension so the dispersion relation between τ_r and $\boldsymbol{\eta}$ becomes linear. We also simplify below some of the calculations in [Lar83, Lar86] by showing that the substitution of normal derivatives in term of tangential ones made in those two references is actually unnecessary, since one can directly derive the amplitude equation with slow spatial derivatives taking place only along the boundary of Ω . This ‘cancellation’ property, which seems to have been unnoticed in all above mentioned references, is our main improvement with respect to [BGC15] where only fast spatial variables were taken into account.

From now on, we fix a nonzero wave vector $\boldsymbol{\eta}$ in the cotangent space to $\partial\Omega$, and we fix, as in [Lar83], the frequency $\tau := -\tau_r(\boldsymbol{\eta}) > 0$ (the case $\tau = \tau_r(\boldsymbol{\eta})$ is entirely similar). We also fix a nonzero (exponentially decaying at $+\infty$) profile V such that

$$e^{i\tau t + i\boldsymbol{\eta} \cdot \boldsymbol{x}} V(\boldsymbol{\nu} \cdot \boldsymbol{x}),$$

is a (surface wave) solution to (2.4).

We look for asymptotic solutions \mathbf{u}^ε to the nonlinear equations (2.1) satisfying an inhomogeneous version of the boundary conditions (2.2), namely

$$(2.11) \quad \forall \alpha = 1, \dots, N, \quad \nu_j \frac{\partial W}{\partial u_{\alpha,j}} (\nabla \mathbf{u}^\varepsilon) \Big|_{x \in \partial\Omega} = \varepsilon^2 G \left(t, x, \frac{\tau t + \boldsymbol{\eta} \cdot \boldsymbol{x}}{\varepsilon} \right),$$

where G is defined on $(-\infty, T] \times \partial\Omega \times \mathbb{R}$ for some given time $T > 0$, and decays at infinity with respect to its last argument, which we denote θ from now on. We do not give a precise meaning to the ‘decay at infinity’ for G , but we tacitly assume at least that G is L^2 with respect to θ so that we can apply the Fourier transform³. We expect the exact solution \mathbf{u}^ε to (2.1), (2.11) to have an asymptotic expansion of the form

$$(2.12) \quad \mathbf{u}^\varepsilon \sim \varepsilon^2 \mathbf{u}^{(1)} \left(t, x, \frac{\tau t + \boldsymbol{\eta} \cdot \boldsymbol{x}}{\varepsilon}, \frac{\boldsymbol{\nu} \cdot \boldsymbol{x}}{\varepsilon} \right) + \varepsilon^3 \mathbf{u}^{(2)} \left(t, x, \frac{\tau t + \boldsymbol{\eta} \cdot \boldsymbol{x}}{\varepsilon}, \frac{\boldsymbol{\nu} \cdot \boldsymbol{x}}{\varepsilon} \right) + \dots,$$

³Unlike several previous works on pulse-like solutions, we shall not consider here any polynomial decay with respect to the fast variable θ , though such technical considerations can be skipped in the framework of this Chapter and postponed to Chapter 3.

and we wish to determine the ‘amplitude equation’ governing the evolution of the leading profile $\mathbf{u}^{(1)}$. For convenience, we have considered here the regime of weakly nonlinear high frequency waves, which means that the ‘slow’ variables are (t, x) and the ‘fast’ variables are $(\theta, z) := ((\tau t + \boldsymbol{\eta} \cdot x)/\varepsilon, \boldsymbol{\nu} \cdot x/\varepsilon)$, while in [Lar83, Lar86, Hun89, BGC12, BGC15] the authors considered the regime of weakly nonlinear modulation on large times where the ‘slow’ variables are $(\varepsilon t, \varepsilon x)$ and the analogue of the ‘fast’ variables are $(\tau t + \boldsymbol{\eta} \cdot x, \boldsymbol{\nu} \cdot x)$. The link between the two regimes comes from the scale invariance properties of (2.1), (2.2). Namely, since the half-space Ω is invariant by dilation, if $\mathbf{u}(t, x)$ is a solution to (2.1), (2.2), then $\alpha \mathbf{u}(t/\alpha, x/\alpha)$ is also a solution for any $\alpha > 0$. This explains why the scaling in (2.12) involves solutions of amplitude $O(\varepsilon^2)$ while the references [Lar83, Lar86, Hun89, BGC12, BGC15] consider solutions of amplitude $O(\varepsilon)$. One advantage of our scaling is to deal with approximate and/or exact solutions to (2.1), (2.11) that are defined on a *fixed* time interval independent of the wavelength ε .

We follow the calculations in [BGC15], with the novelty here that the profiles $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ also depend on the slow spatial variables x while only a slow time variable was considered in [BGC15]. Recalling the definition (2.6), we introduce the so-called ‘fast-fast’/‘fast-slow’ operators in the interior and the ‘fast’/‘slow’ operators on the boundary:

$$\begin{aligned}\mathcal{L}_{\text{ff}} &:= (\tau^2 I - \Sigma(\boldsymbol{\eta})) \partial_{\theta\theta}^2 - (\nu_j A_j(\boldsymbol{\eta}) + \nu_j A_j(\boldsymbol{\eta})^T) \partial_{\theta z}^2 - \Sigma(\boldsymbol{\nu}) \partial_{zz}^2, \\ \mathcal{L}_{\text{fs}} &:= 2\tau \partial_{t\theta}^2 - (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \partial_{j\theta}^2 - (A_j(\boldsymbol{\nu}) + A_j(\boldsymbol{\nu})^T) \partial_{jz}^2, \\ \ell_{\text{f}} &:= \nu_j A_j(\boldsymbol{\eta}) \partial_{\theta} + \Sigma(\boldsymbol{\nu}) \partial_z, \\ \ell_{\text{s}} &:= A_j(\boldsymbol{\nu})^T \partial_j,\end{aligned}$$

as well as the ‘fast’ quadratic operators:

$$\begin{aligned}\mathcal{Q}[\mathbf{u}]_{\alpha} &:= d_{\alpha j \beta \ell \gamma m} (\eta_j \partial_{\theta} + \nu_j \partial_z) \left\{ ((\eta_{\ell} \partial_{\theta} + \nu_{\ell} \partial_z) u_{\beta}) ((\eta_m \partial_{\theta} + \nu_m \partial_z) u_{\gamma}) \right\}, \\ \mathcal{M}[\mathbf{u}]_{\alpha} &:= \nu_j d_{\alpha j \beta \ell \gamma m} \left\{ ((\eta_{\ell} \partial_{\theta} + \nu_{\ell} \partial_z) u_{\beta}) ((\eta_m \partial_{\theta} + \nu_m \partial_z) u_{\gamma}) \right\},\end{aligned}$$

where α runs through $\{1, \dots, N\}$. The operator $\mathcal{Q}[\cdot]$, resp. $\mathcal{M}[\cdot]$, is then defined as the vector valued operator in \mathbb{R}^N whose α -coordinate is $\mathcal{Q}[\cdot]_{\alpha}$, resp. $\mathcal{M}[\cdot]_{\alpha}$.

Plugging the ansatz (2.12) in (2.1), (2.11) and collecting the powers of ε , we are led to constructing profiles $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ solutions to the equations

$$(2.13) \quad \begin{cases} \mathcal{L}_{\text{ff}} \mathbf{u}^{(1)} = 0, & x \in \Omega, \quad z > 0, \\ \ell_{\text{f}} \mathbf{u}^{(1)} \Big|_{x \in \partial\Omega, z=0} = 0, \end{cases}$$

$$(2.14) \quad \begin{cases} \mathcal{L}_{\text{ff}} \mathbf{u}^{(2)} = -\mathcal{L}_{\text{fs}} \mathbf{u}^{(1)} + \frac{1}{2} \mathcal{Q}[\mathbf{u}^{(1)}], & x \in \Omega, \quad z > 0, \\ \ell_{\text{f}} \mathbf{u}^{(2)} \Big|_{x \in \partial\Omega, z=0} = G - \left(\ell_{\text{s}} \mathbf{u}^{(1)} + \frac{1}{2} \mathcal{M}[\mathbf{u}^{(1)}] \right) \Big|_{x \in \partial\Omega, z=0}. \end{cases}$$

One major difference here with respect to [BGC15] is that the interior equations on $\mathcal{L}_{\text{ff}} \mathbf{u}^{(1,2)}$ hold for any $x \in \Omega$ and $z > 0$ while the boundary conditions in (2.13)-(2.14) correspond to a ‘double trace’ $x \in \partial\Omega$ and $z = 0$. However, x enters as a parameter in the interior equations since the fast-fast operator \mathcal{L}_{ff} only involves differentiation with respect to (θ, z) . We can therefore take the trace of the interior equation on

$\partial\Omega$. In particular, the leading profile $\mathbf{u}^{(1)}$ should satisfy

$$\begin{cases} \mathcal{L}_{\text{ff}} \left(\mathbf{u}^{(1)}|_{x \in \partial\Omega} \right) = 0, & z > 0, \\ \ell_{\text{f}} \left(\mathbf{u}^{(1)}|_{x \in \partial\Omega} \right) = 0, & z = 0, \end{cases}$$

which is the fast problem considered in [BGC15] (Equation (P1) there, though with slightly different notation for the fast variables). The Fourier transform (with respect to the fast variable θ) of the trace $\mathbf{u}^{(1)}|_{\partial\Omega}$ therefore satisfies

$$(2.15) \quad \mathbf{v}^{(1)}(t, x, k, z) := \mathcal{F}_{\theta} \mathbf{u}^{(1)}|_{x \in \partial\Omega} = \widehat{w}(t, x, k) \widehat{\mathbf{r}}(k, z), \quad \widehat{\mathbf{r}}(k, z) := \begin{cases} V(kz), & k > 0, \\ V(-kz), & k < 0. \end{cases}$$

We emphasize that, though we let x denote one of the arguments of $\mathbf{v}^{(1)}$, it should be kept in mind that x here is restricted to the boundary of Ω . The first and actually main task is to determine the unknown scalar function w (or equivalently its Fourier transform with respect to θ), which will determine the leading profile $\mathbf{u}^{(1)}$ at the boundary of Ω . In Chapter 4 we shall go further in the construction of the asymptotic expansion of \mathbf{u}^{ε} .

Before going on, let us introduce two operators similar to the ones defined in [BGC15], and that arise after performing the Fourier transform with respect to θ on \mathcal{L}_{ff} and ℓ_{f} :

$$\begin{aligned} \mathbf{L}^k &:= -(k\tau)^2 I + \Sigma(k\boldsymbol{\eta}) - i(\nu_j A_j(k\boldsymbol{\eta}) + \nu_j A_j(k\boldsymbol{\eta})^T) \partial_z - \Sigma(\boldsymbol{\nu}) \partial_{zz}^2, \\ \mathbf{C}^k &:= i\nu_j A_j(k\boldsymbol{\eta}) + \Sigma(\boldsymbol{\nu}) \partial_z, \end{aligned}$$

The first corrector $\mathbf{u}^{(2)}$ to the leading profile $\mathbf{u}^{(1)}$ should satisfy the system (2.14). In particular, the Fourier transform $\mathbf{v}^{(2)}$ with respect to θ of the trace $\mathbf{u}^{(2)}|_{x \in \partial\Omega}$ should satisfy

$$(2.16) \quad \forall k \neq 0, \quad \begin{cases} \mathbf{L}^k \mathbf{v}^{(2)} = \mathbb{F}(k, z), & z > 0, \\ \mathbf{C}^k \mathbf{v}^{(2)} = \mathbb{G}(k), & z = 0, \end{cases}$$

where the source terms \mathbb{F}, \mathbb{G} are defined by

$$(2.17) \quad \mathbb{F} := \mathcal{F}_{\theta} \left(-\mathcal{L}_{\text{fs}} \mathbf{u}^{(1)} + \frac{1}{2} \mathcal{Q}[\mathbf{u}^{(1)}] \right) \Big|_{x \in \partial\Omega}, \quad \mathbb{G} := \widehat{G} - \mathcal{F}_{\theta} \left(\ell_{\text{s}} \mathbf{u}^{(1)} + \frac{1}{2} \mathcal{M}[\mathbf{u}^{(1)}] \right) \Big|_{x \in \partial\Omega, z=0}.$$

The main problem at this stage is that both operators $\mathcal{L}_{\text{fs}}, \ell_{\text{s}}$ involve a normal derivative with respect to $\partial\Omega$, which led Lardner [Lar83, Lar86] to perform substitutions of normal derivatives in terms of tangential ones (enforcing a ‘compatibility’ condition between the source terms \mathbb{F}, \mathbb{G} which precludes a secular growth phenomenon in z) and thereby deriving an amplitude equation for w (called γ in [Lar83, Lar86]) along the boundary $\partial\Omega$. It turns out that these manipulations in [Lar83, Lar86] are *unnecessary* here. The only compatibility condition we need between the source terms \mathbb{F} and \mathbb{G} aims at providing with the existence of a corrector $\mathbf{v}^{(2)}$ solution to (2.16) that should be at least bounded in z for all $k \neq 0$. Deriving accurate bounds for such a corrector will be one of the main issues in the error analysis of Chapter 4.

Let us now recall the following duality relation which was exhibited in [BGC15] (an analogous duality relation was proved in [BGC12] for the first order Hamiltonian formulation (2.8)):

$$\int_0^{+\infty} \mathbf{v} \cdot \mathbf{L}^k \mathbf{w} \, dz - \left(\mathbf{v} \cdot \mathbf{C}^k \mathbf{w} \right) \Big|_{z=0} = \int_0^{+\infty} \mathbf{L}^{-k} \mathbf{v} \cdot \mathbf{w} \, dz - \left(\mathbf{C}^{-k} \mathbf{v} \cdot \mathbf{w} \right) \Big|_{z=0},$$

where we use the notation $\mathbf{v} \cdot \mathbf{v}'$ for the quantity $v_\alpha v'_\alpha$, and vectors have indifferently real or complex coordinates. Actually, by translating in z , it is not hard to see that the same duality relation holds on any interval $[z, +\infty)$, namely:

$$(2.18) \quad \int_z^{+\infty} \mathbf{v} \cdot \mathbf{L}^k \mathbf{w} \, dZ - \left(\mathbf{v} \cdot \mathbf{C}^k \mathbf{w} \right) \Big|_{Z=z} = \int_z^{+\infty} \mathbf{L}^{-k} \mathbf{v} \cdot \mathbf{w} \, dZ - \left(\mathbf{C}^{-k} \mathbf{v} \cdot \mathbf{w} \right) \Big|_{Z=z}.$$

The surface wave profile $\widehat{\mathbf{r}}(k, \cdot)$ in (2.15) satisfies

$$\forall k \neq 0, \quad \begin{cases} \mathbf{L}^k \widehat{\mathbf{r}}(k, \cdot) = 0, & z > 0, \\ \mathbf{C}^k \widehat{\mathbf{r}}(k, \cdot) = 0, & z = 0, \end{cases}$$

and $\widehat{\mathbf{r}}(-k, z) = \overline{\widehat{\mathbf{r}}(k, z)}$. Applying the duality relation (2.18) with $z = 0$, we find that for a ‘reasonable’ solution $\mathbf{v}^{(2)}$ to the problem (2.16) to exist, the source terms \mathbb{F}, \mathbb{G} in (2.16) should satisfy the Fredholm type condition

$$(2.19) \quad \forall k \neq 0, \quad \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathbb{F}(k, z) \, dz - \overline{\widehat{\mathbf{r}}(k, 0)} \cdot \mathbb{G}(k) = 0.$$

We do not make precise for the moment the meaning of ‘reasonable solution’ for the corrector problem (2.16). Let us merely say that for all fixed $k \neq 0$, the function \mathbf{v} to which we apply the duality relation (2.18) is $\widehat{\mathbf{r}}(-k, z)$ and it thus has exponential decay in z (with an exponential decay rate that depends on k). For the integrals in (2.18) to make sense, the corrector $\mathbf{v}^{(2)}$ should be for instance bounded in z (with first two z -derivatives also bounded). The main problem to avoid is to have a corrector $\mathbf{v}^{(2)}$ that displays exponential growth in z .

In what remains of this Section, we make Equation (2.19) more explicit in terms of the scalar function w that defines the trace of the leading profile $\mathbf{u}^{(1)}$ at the boundary $\partial\Omega$. Namely, our goal is to prove the following result.

Proposition 2.1. *Under Assumptions (H1), (H2), (H3), consider the source terms \mathbb{F} and \mathbb{G} defined by (2.17), where the profile $\mathbf{u}^{(1)}$ has the form (2.15) on the boundary $\partial\Omega$, and satisfies*

$$\forall x \in \Omega, \quad \begin{cases} \mathcal{L}_{\mathbb{F}} \mathbf{u}^{(1)} = 0, & z > 0, \\ \lim_{z \rightarrow +\infty} \mathbf{u}^{(1)}(t, x, \theta, z) = 0. \end{cases}$$

Then the compatibility condition (2.19) is a closed evolution equation for the scalar function w . For concreteness, let us assume furthermore that Ω is the half-space $\{x_d > 0\}$. Then (2.19) equivalently reads⁴

$$(2.20) \quad \partial_t w + \sum_{j=1}^{d-1} \partial_{\eta_j} \tau_r(\boldsymbol{\eta}) \partial_j w + \mathcal{H}(\mathcal{B}(w, w)) = g,$$

where \mathcal{H} denotes the Hilbert transform with respect to the variable θ , \mathcal{B} is the bilinear Fourier multiplier

$$(2.21) \quad \widehat{\mathcal{B}(w, w)}(k) := -\frac{1}{4\pi c_0} \int_{\mathbb{R}} b(-k, k - \xi, \xi) \widehat{w}(k - \xi) \widehat{w}(\xi) \, d\xi,$$

where the constant c_0 is defined in Equation (2.22) below and the kernel b is defined in (2.29) (the slow variables (t, x) enter as parameters in the definition of \mathcal{B}). Eventually the source term g in (2.20) depends linearly on G . Its expression is given in (2.31).

⁴Here the sign convention for τ plays a role. If we had chosen $\tau = \tau_r$ rather than $\tau = -\tau_r$, then the group velocity $\nabla_{\boldsymbol{\eta}} \tau_r$ in (2.20) would have been changed into its opposite.

Proof. The first main point is to clarify whether (2.19) reads as a closed equation on the function w . This is not obvious at first sight because both source terms \mathbb{F} and \mathbb{G} involve the trace of the normal derivative of $\mathbf{u}^{(1)}$ at the boundary. It is only after taking the scalar product with $\widehat{\mathbf{r}}$ in (2.19) that these normal derivatives will actually drop out. We thus first focus on the ‘linear’ terms in (2.19), meaning on all those terms in \mathbb{F} and \mathbb{G} where the leading profile $\mathbf{u}^{(1)}$ appears linearly.

• The slow time derivative. This term coincides with that obtained in [BGC15], and we find that the only term on the left hand side of (2.19) where a t -derivative of $\mathbf{u}^{(1)}$ appears is

$$\begin{aligned} \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathcal{F}_\theta \left(-2\tau \partial_{t\theta}^2 \mathbf{u}^{(1)} \right) \Big|_{x \in \partial\Omega} dz &= -2i\tau \operatorname{sgn}(k) \int_0^{+\infty} |\widehat{\mathbf{r}}(1, z)|^2 dz \partial_t \widehat{w} \\ &=: i c_0 \operatorname{sgn}(k) \partial_t \widehat{w}, \end{aligned}$$

with (recall that τ is nonzero due to Assumption (H3)):

$$(2.22) \quad c_0 := -2\tau \int_0^{+\infty} |\widehat{\mathbf{r}}(1, z)|^2 dz \neq 0.$$

• Slow spatial derivatives. We now examine those terms in (2.19) that involve a partial derivative with respect to x_j . (Here j is fixed and there is temporarily no summation over j even though this index might be repeated.) One contribution comes from \mathbb{F} and another contribution comes from \mathbb{G} , giving the term

$$\begin{aligned} \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathcal{F}_\theta \left((A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \partial_{j\theta}^2 \mathbf{u}^{(1)} + (A_j(\boldsymbol{\nu}) + A_j(\boldsymbol{\nu})^T) \partial_{jz}^2 \mathbf{u}^{(1)} \right) \Big|_{x \in \partial\Omega} dz \\ + \overline{\widehat{\mathbf{r}}(k, 0)} \cdot \mathcal{F}_\theta \left(A_j(\boldsymbol{\nu})^T \partial_j \mathbf{u}^{(1)} \right) \Big|_{x \in \partial\Omega, z=0}. \end{aligned}$$

Letting \mathbf{v}_j denote the Fourier transform of the trace $\partial_j \mathbf{u}^{(1)}|_{\partial\Omega}$, the latter term reduces to

$$\begin{aligned} (2.23) \quad \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(k, z)} \cdot (ik (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \mathbf{v}_j + (A_j(\boldsymbol{\nu}) + A_j(\boldsymbol{\nu})^T) \partial_z \mathbf{v}_j) dz + \overline{\widehat{\mathbf{r}}(k, 0)} \cdot A_j(\boldsymbol{\nu})^T \mathbf{v}_j|_{z=0} \\ = \int_0^{+\infty} \left(\overline{\widehat{\mathbf{r}}(k, z)} \cdot ik (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \mathbf{v}_j + \overline{\widehat{\mathbf{r}}(k, z)} \cdot A_j(\boldsymbol{\nu}) \partial_z \mathbf{v}_j - A_j(\boldsymbol{\nu}) \partial_z \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathbf{v}_j \right) dz. \end{aligned}$$

The problem of course is that the x_j -derivative and the trace operator on $\partial\Omega$ do not necessarily commute, so the latter term is not obviously computable in terms of the scalar function w . More precisely, we can decompose the function $\partial_j \mathbf{u}^{(1)}|_{\partial\Omega}$ as

$$(2.24) \quad \partial_j \mathbf{u}^{(1)}|_{\partial\Omega} = \nu_j \partial_{\mathbf{n}} \mathbf{u}^{(1)}|_{\partial\Omega} + \partial_{\tan, j} \mathbf{u}^{(1)}|_{\partial\Omega},$$

with $\partial_{\mathbf{n}} \mathbf{u}^{(1)}$ the normal derivative of $\mathbf{u}^{(1)}$ with respect to $\partial\Omega$ and $\partial_{\tan, j}$ a tangential vector field along $\partial\Omega$. It remains to verify that all the contributions from the terms (2.23) that involve the normal derivative of $\mathbf{u}^{(1)}$ at $\partial\Omega$ sum to zero.

Given any point $x \in \Omega$, the profile $\mathbf{u}^{(1)}$ should satisfy

$$\mathcal{L}_{\mathbb{F}} \mathbf{u}^{(1)} = 0, \quad z > 0,$$

and $\mathbf{u}^{(1)}(z = +\infty) = 0$. This means that for all $x \in \Omega$ and all $k \neq 0$, the Fourier transform $\widehat{\mathbf{u}}^{(1)}(k, z)$ has exponential decay in z and satisfies

$$\mathbf{L}^k \widehat{\mathbf{u}}^{(1)}(k, z) = 0, \quad z > 0.$$

Taking the normal derivative on $\partial\Omega$, we find that the Fourier transform $\mathbf{v}_\mathbf{n}$ of $\partial_\mathbf{n}\mathbf{u}^{(1)}|_{\partial\Omega}$ satisfies

$$\mathbf{L}^k \mathbf{v}_\mathbf{n}(k, z) = 0, \quad z > 0,$$

and $\mathbf{v}_\mathbf{n}(z = +\infty) = 0$. Let us now apply the duality condition (2.18) on an interval $[z, +\infty)$ with $\mathbf{v} := \overline{\widehat{\mathbf{r}}(k, z)}$ and $\mathbf{w} := \mathbf{v}_\mathbf{n}(k, z)$. We get the relation

$$\forall z > 0, \quad \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathbf{C}^k \mathbf{v}_\mathbf{n}(k, z) = \mathbf{C}^{-k} \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathbf{v}_\mathbf{n}(k, z),$$

that is

$$(2.25) \quad \overline{\widehat{\mathbf{r}}(k, z)} \cdot \left(ik (\nu_\ell A_\ell(\boldsymbol{\eta}) + \nu_\ell A_\ell(\boldsymbol{\eta})^T) + \Sigma(\boldsymbol{\nu}) \partial_z \right) \mathbf{v}_\mathbf{n} = \partial_z \overline{\widehat{\mathbf{r}}(k, z)} \cdot \Sigma(\boldsymbol{\nu}) \mathbf{v}_\mathbf{n}.$$

We now use the decomposition (2.24) in (2.23), and sum with respect to $j = 1, \dots, d$ the contributions involving the normal derivative $\mathbf{v}_\mathbf{n}$, which yields (here the summation convention with respect to j is used again):

$$(2.26) \quad \int_0^{+\infty} \left(\overline{\widehat{\mathbf{r}}(k, z)} \cdot ik (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \nu_j \mathbf{v}_\mathbf{n} + \overline{\widehat{\mathbf{r}}(k, z)} \cdot \nu_j A_j(\boldsymbol{\nu}) \partial_z \mathbf{v}_\mathbf{n} - \nu_j A_j(\boldsymbol{\nu}) \partial_z \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathbf{v}_\mathbf{n} \right) dz \\ = \int_0^{+\infty} \left(\overline{\widehat{\mathbf{r}}(k, z)} \cdot ik (\nu_j A_j(\boldsymbol{\eta}) + \nu_j A_j(\boldsymbol{\eta})^T) \mathbf{v}_\mathbf{n} + \overline{\widehat{\mathbf{r}}(k, z)} \cdot \Sigma(\boldsymbol{\nu}) \partial_z \mathbf{v}_\mathbf{n} - \Sigma(\boldsymbol{\nu}) \partial_z \overline{\widehat{\mathbf{r}}(k, z)} \cdot \mathbf{v}_\mathbf{n} \right) dz = 0,$$

where we have used the relation (see (2.6)):

$$\forall \boldsymbol{\xi}, \quad \xi_j A_j(\boldsymbol{\xi}) = \Sigma(\boldsymbol{\xi}),$$

and the cancellation property (2.25). In other words, we have proved that the sum with respect to j of the slow spatial derivative terms in (2.19) reads

$$(2.27) \quad \partial_{\tan, j} \widehat{w} \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(k, z)} \cdot ik (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \widehat{\mathbf{r}}(k, z) + \overline{\widehat{\mathbf{r}}(k, z)} \cdot A_j(\boldsymbol{\nu}) \partial_z \widehat{\mathbf{r}}(k, z) - A_j(\boldsymbol{\nu}) \partial_z \overline{\widehat{\mathbf{r}}(k, z)} \cdot \widehat{\mathbf{r}}(k, z) dz.$$

For concreteness, let us assume from now on that the half-space Ω is $\{x_d > 0\}$ so that for $j = d$, one has $\partial_{\tan, j} = 0$, and for $j = 1, \dots, d-1$, one has $\partial_{\tan, j} = \partial_j$. (There also holds that $\boldsymbol{\nu}$ is the last vector in the canonical basis of \mathbb{R}^d though we do not simplify the expressions that depend on $\boldsymbol{\nu}$ accordingly.) Then the slow spatial derivative terms arising in (2.19) read

$$i \operatorname{sgn}(k) \sum_{j=1}^{d-1} c_j \partial_j \widehat{w},$$

with

$$(2.28) \quad c_j := \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(1, z)} \cdot (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \widehat{\mathbf{r}}(1, z) dz + 2 \operatorname{Im} \int_0^{+\infty} \overline{\widehat{\mathbf{r}}(1, z)} \cdot A_j(\boldsymbol{\nu}) \partial_z \widehat{\mathbf{r}}(1, z) dz.$$

• **Quadratic terms.** All remaining terms in (2.19) correspond to the contributions from the quadratic operators \mathcal{Q} in $\{z > 0\}$ and \mathcal{M} at $z = 0$. These terms are exactly identical to the ones considered in [BGC15], and we shall therefore not reproduce the (lengthy) calculations leading to the final expression

$$-\frac{1}{4\pi} \int_{\mathbb{R}} b(-k, k - \xi, \xi) \widehat{w}(t, x, k - \xi) \widehat{w}(t, x, \xi) d\xi,$$

with

$$(2.29) \quad b(\xi_1, \xi_2, \xi_3) := \int_0^{+\infty} d_{\alpha j \beta \ell \gamma m} (\nu_j \rho_{\alpha, z} + i \xi_1 \eta_j \rho_\alpha) (\nu_\ell \rho_{\beta, z} + i \xi_2 \eta_\ell \rho_\beta) (\nu_m \rho_{\gamma, z} + i \xi_3 \eta_m \rho_\gamma) dz, \\ \rho_\alpha := \widehat{r}_\alpha(\xi_1, z), \quad \rho_\beta := \widehat{r}_\beta(\xi_2, z), \quad \rho_\gamma := \widehat{r}_\gamma(\xi_3, z),$$

and the subscript ‘, z ’ denotes differentiation with respect to z .

At this stage, we have found that the amplitude equation (2.19) reads

$$i c_0 \operatorname{sgn}(k) \partial_t \widehat{w} + i \operatorname{sgn}(k) \sum_{j=1}^{d-1} c_j \partial_j \widehat{w} - \frac{1}{4\pi} \int_{\mathbb{R}} b(-k, k - \xi, \xi) \widehat{w}(t, x, k - \xi) \widehat{w}(t, x, \xi) d\xi = \overline{\widehat{\mathbf{r}}(k, 0)} \cdot \widehat{G}(k),$$

where the constants c_0, \dots, c_{d-1} are defined in (2.22), (2.28), and the kernel b is given in (2.29). Multiplying by $-i \operatorname{sgn}(k) c_0^{-1}$, and taking the inverse Fourier transform in θ , we find that weakly nonlinear high frequency solutions to (2.1), (2.11) are governed, at the leading order, by the amplitude equation

$$(2.30) \quad \partial_t w + \sum_{j=1}^{d-1} \frac{c_j}{c_0} \partial_j w + \mathcal{H}(\mathcal{B}(w, w)) = g,$$

where, as announced in the statement of Proposition 2.1, \mathcal{H} denotes the Hilbert transform with respect to the fast variable θ (namely, $\widehat{\mathcal{H}f}(k) := -i \operatorname{sgn}(k) \widehat{f}(k)$), and \mathcal{B} is the bilinear Fourier multiplier defined in (2.21). The source term g in (2.30) is obtained by setting

$$(2.31) \quad \widehat{g}(k) := -\frac{i \operatorname{sgn}(k)}{c_0} \overline{\widehat{\mathbf{r}}(k, 0)} \cdot \widehat{G}(k),$$

which defines a real valued function g provided that G is real valued, which was tacitly assumed in (2.11).

• Simplifying the linear terms. We shall have therefore proved Proposition 2.1 provided that we get the relations

$$(2.32) \quad \forall j = 1, \dots, d-1, \quad \frac{c_j}{c_0} = \partial_{\eta_j} \tau_r(\boldsymbol{\eta}),$$

which express that the propagation of the amplitude w along the boundary $\partial\Omega$ is governed by the *group velocity* of the variety along which the Lopatinskii determinant Δ vanishes. The proof of the relations (2.32) follows some arguments that already appeared in [BGC12] for the first-order Hamiltonian formulation (2.8), and which we adapt⁵ here to the second-order formulation (2.7). From Assumption (H3), we know that the matrix $\mathbf{J}H(\tau, \boldsymbol{\eta})$ is diagonalizable and hyperbolic, the diagonalization being locally analytic in the frequencies $(\tau, \boldsymbol{\eta})$. For clarity, we denote from now on the reference frequency in (2.12) $(\underline{\tau}, \underline{\boldsymbol{\eta}})$ and keep the notation $(\tau, \boldsymbol{\eta})$ for real frequencies that are close to $(\underline{\tau}, \underline{\boldsymbol{\eta}})$, though not necessarily linked by the dispersion relation $\tau + \tau_r(\boldsymbol{\eta}) = 0$. We can choose a smooth basis $(\mathbf{R}_1, \dots, \mathbf{R}_{2N})(\tau, \boldsymbol{\eta})$ of \mathbb{C}^{2N} such that

$$(2.33) \quad \forall \alpha = 1, \dots, N, \quad \begin{cases} \mathbf{J}H(\tau, \boldsymbol{\eta}) \mathbf{R}_\alpha(\tau, \boldsymbol{\eta}) = -\omega_\alpha(\tau, \boldsymbol{\eta}) \mathbf{R}_\alpha(\tau, \boldsymbol{\eta}), \\ \mathbf{J}H(\tau, \boldsymbol{\eta}) \mathbf{R}_{N+\alpha}(\tau, \boldsymbol{\eta}) = \overline{\omega_\alpha(\tau, \boldsymbol{\eta})} \mathbf{R}_{N+\alpha}(\tau, \boldsymbol{\eta}), \end{cases}$$

with the normalization convention

$$(\mathbf{R}_1 \quad \dots \quad \mathbf{R}_{2N})^* \mathbf{J} (\mathbf{R}_1 \quad \dots \quad \mathbf{R}_{2N}) = -\mathbf{J}.$$

⁵It is also the opportunity to correct some (harmless) normalizing mistakes that appeared in [BGC12].

Defining some vectors \mathbf{L}_α by

$$\forall \alpha = 1, \dots, 2N, \quad \mathbf{L}_\alpha(\tau, \boldsymbol{\eta}) := \begin{cases} \mathbf{R}_{N+\alpha}, & \text{if } \alpha \leq N, \\ -\mathbf{R}_{\alpha-N}, & \text{if } \alpha > N, \end{cases}$$

we get the orthogonality relations

$$(2.34) \quad \forall \alpha, \beta = 1, \dots, 2N, \quad \mathbf{L}_\alpha^* \mathbf{J} \mathbf{R}_\beta = \delta_{\alpha\beta},$$

which hold for all $(\tau, \boldsymbol{\eta})$ close to $(\underline{\tau}, \underline{\boldsymbol{\eta}})$, though we shall only use them at the reference frequency $(\underline{\tau}, \underline{\boldsymbol{\eta}})$. The Lopatinskii determinant is then defined⁶ as

$$\Delta(\tau, \boldsymbol{\eta}) := \det(S_1(\tau, \boldsymbol{\eta}) \cdots S_N(\tau, \boldsymbol{\eta})), \quad \mathbf{R}_\alpha := \begin{pmatrix} R_\alpha \\ S_\alpha \end{pmatrix}.$$

In what follows, we decompose all vectors $\mathbf{R}_\alpha \in \mathbb{C}^{2N}$ as above, and not only those corresponding to $\alpha = 1, \dots, N$. Underlined quantities refer to evaluation at the frequency $(\underline{\tau}, \underline{\boldsymbol{\eta}})$. Since the vectors $\underline{S}_1, \dots, \underline{S}_N$ span an $N - 1$ -dimensional subspace in \mathbb{C}^N , we can fix a nonzero vector $(\lambda_1, \dots, \lambda_N)$ in \mathbb{C}^N such that (here we use the convention $\lambda_\gamma = 0$ if $\gamma \geq N + 1$):

$$(2.35) \quad \lambda_\gamma \underline{S}_\gamma = 0.$$

There is no loss of generality in assuming $\lambda_1 \neq 0$ and even in normalizing the λ_γ 's by assuming $\lambda_1 = 1$, which means that $\underline{S}_2, \dots, \underline{S}_N$ are linearly independent. In that case, the surface wave V which appears in the definition (2.15) of $\hat{\mathbf{r}}$ reads⁷

$$(2.36) \quad V(z) = \lambda_\gamma e^{-\omega_\gamma z} \underline{R}_\gamma.$$

We now observe that the two linear forms

$$X \in \mathbb{C}^N \mapsto \det(X \underline{S}_2 \cdots \underline{S}_N), \quad X \in \mathbb{C}^N \mapsto (\lambda_\gamma \underline{R}_\gamma)^* X,$$

are nonzero and vanish on the hyperplane spanned by $\underline{S}_1, \dots, \underline{S}_N$ (use (2.35) and the orthogonality relations (2.34) for the latter). Hence there exists a nonzero constant $\kappa \in \mathbb{C}$ such that

$$\forall X \in \mathbb{C}^N, \quad \det(X \underline{S}_2 \cdots \underline{S}_N) = \kappa (\lambda_\gamma \underline{R}_\gamma)^* X.$$

Let us assume for simplicity that the half-space Ω is $\{x_d > 0\}$, which means that the tangential wave vectors $\boldsymbol{\eta}$ are parametrized by $\eta_1, \dots, \eta_{d-1}$. Then the partial derivative $\partial\Delta$ of Δ with respect to any of the variables $\tau, \eta_1, \dots, \eta_{d-1}$ is given by

$$\partial\Delta(\underline{\tau}, \underline{\boldsymbol{\eta}}) = \det(\lambda_\alpha \partial S_\alpha(\underline{\tau}, \underline{\boldsymbol{\eta}}) \underline{S}_2 \cdots \underline{S}_N) = \kappa \lambda_\alpha (\lambda_\gamma \underline{R}_\gamma)^* \partial S_\alpha(\underline{\tau}, \underline{\boldsymbol{\eta}}).$$

For each partial derivative ∂ , with respect to either of the variables $\tau, \eta_1, \dots, \eta_{d-1}$, we follow [BGC12] and decompose

$$(2.37) \quad \forall \alpha = 1, \dots, N, \quad \partial \mathbf{R}_\alpha(\underline{\tau}, \underline{\boldsymbol{\eta}}) = \mu_{\alpha\beta} \underline{\mathbf{R}}_\beta,$$

⁶Observe that the value of Δ depends on the basis of \mathbb{E}^s with which it is defined but the location of the roots and their multiplicity does not, which is why we can equivalently define Δ with our basis (\mathbf{R}_α) .

⁷Or at least we can fix it this way, since V is defined up to a nonzero multiplicative constant.

where summation with respect to β includes all indices $\beta = 1, \dots, 2N$. We thus get the expression

$$\partial\Delta(\underline{\tau}, \underline{\eta}) = \kappa \lambda_\alpha \mu_{\alpha\beta} (\lambda_\gamma \underline{\mathbf{R}}_\gamma)^* \underline{\mathbf{S}}_\beta,$$

where summation runs over $\alpha, \gamma = 1, \dots, N$ (recall $\lambda_\alpha = 0$ for $\alpha \geq N+1$) and $\beta = N+1, \dots, 2N$ (because of the orthogonality relation $(\lambda_\gamma \underline{\mathbf{R}}_\gamma)^* \underline{\mathbf{S}}_\beta = 0$ for all $\beta = 1, \dots, N$). We thus need to determine the coefficients $\mu_{\alpha\beta}$ in (2.37) for $\alpha \leq N$ and $\beta \geq N+1$. These coefficients are obtained by differentiating (2.33) and using the orthogonality relations (2.34). We get

$$\forall \alpha = 1, \dots, N, \quad \forall \beta = N+1, \dots, 2N, \quad \mu_{\alpha\beta} = \frac{\underline{\mathbf{L}}_\beta^* \partial H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha}{\underline{\omega}_\alpha + \underline{\omega}_{\beta-N}} = -\frac{\underline{\mathbf{R}}_{\beta-N}^* \partial H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha}{\underline{\omega}_\alpha + \underline{\omega}_{\beta-N}},$$

and therefore

$$\partial\Delta(\underline{\tau}, \underline{\eta}) = -\kappa \lambda_\alpha \frac{\underline{\mathbf{R}}_\beta^* \partial H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha}{\underline{\omega}_\alpha + \underline{\omega}_\beta} (\lambda_\gamma \underline{\mathbf{R}}_\gamma)^* \underline{\mathbf{S}}_{\beta+N},$$

where summation now runs over $\alpha, \beta, \gamma = 1, \dots, N$. It remains to observe that we have the relation (use (2.35)):

$$\forall \beta = 1, \dots, N, \quad (\lambda_\gamma \underline{\mathbf{R}}_\gamma)^* \underline{\mathbf{S}}_{\beta+N} = (\lambda_\gamma \mathbf{R}_\gamma)^* \mathbf{J} \mathbf{R}_{\beta+N} = -(\lambda_\gamma \mathbf{L}_{\gamma+N})^* \mathbf{J} \mathbf{R}_{\beta+N} = -\overline{\lambda}_\beta,$$

and the partial derivative of the Lopatinskii determinant thus reduces to its final expression

$$(2.38) \quad \partial\Delta(\underline{\tau}, \underline{\eta}) = \kappa \lambda_\alpha \overline{\lambda}_\beta \frac{\underline{\mathbf{R}}_\beta^* \partial H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha}{\underline{\omega}_\alpha + \underline{\omega}_\beta}.$$

We first use the relation (2.38) for the τ -partial derivative, and recall the expression (2.9) of H :

$$\partial_\tau \Delta(\underline{\tau}, \underline{\eta}) = 2\tau \kappa \lambda_\alpha \overline{\lambda}_\beta \frac{\underline{\mathbf{R}}_\beta^* \underline{\mathbf{R}}_\alpha}{\underline{\omega}_\alpha + \underline{\omega}_\beta} = 2\tau \kappa \int_0^{+\infty} |\widehat{\mathbf{r}}(1, z)|^2 dz = -\kappa c_0.$$

The computation of $\partial_{\eta_j} \Delta$ is slightly more complicated but not too troublesome either. We differentiate H in (2.9) with respect to η_j and use the definitions of A_j, Σ in (2.6) to get

$$\begin{aligned} \underline{\mathbf{R}}_\beta^* \partial_{\eta_j} H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha &= -\underline{\mathbf{R}}_\beta^* (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \underline{\mathbf{R}}_\alpha \\ &\quad + \underline{\mathbf{R}}_\beta^* A_j(\boldsymbol{\nu}) \Sigma(\boldsymbol{\nu})^{-1} (\nu_\ell A_\ell(\boldsymbol{\eta})) \underline{\mathbf{R}}_\alpha + \underline{\mathbf{R}}_\beta^* (\nu_\ell A_\ell(\boldsymbol{\eta})^T) \Sigma(\boldsymbol{\nu})^{-1} A_j(\boldsymbol{\nu})^T \underline{\mathbf{R}}_\alpha \\ &\quad + i \underline{\mathbf{R}}_\beta^* A_j(\boldsymbol{\nu}) \Sigma(\boldsymbol{\nu})^{-1} \underline{\mathbf{S}}_\alpha - i \underline{\mathbf{S}}_\beta^* \Sigma(\boldsymbol{\nu})^{-1} A_j(\boldsymbol{\nu})^T \underline{\mathbf{R}}_\alpha. \end{aligned}$$

Recall that we are interested here in indices α, β between 1 and N , so we can use the relations (which express that \mathbf{R}_γ is an eigenvector of $\mathbf{J}H$ for the eigenvalue $-\omega_\gamma$):

$$\underline{\mathbf{S}}_\alpha = (i \nu_\ell A_\ell(\boldsymbol{\eta}) - \underline{\omega}_\alpha \Sigma(\boldsymbol{\nu})) \underline{\mathbf{R}}_\alpha, \quad \underline{\mathbf{S}}_\beta = (i \nu_\ell A_\ell(\boldsymbol{\eta}) - \underline{\omega}_\beta \Sigma(\boldsymbol{\nu})) \underline{\mathbf{R}}_\beta.$$

This simplifies the above expression of $\underline{\mathbf{R}}_\beta^* \partial_{\eta_j} H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha$ accordingly:

$$\underline{\mathbf{R}}_\beta^* \partial_{\eta_j} H(\underline{\tau}, \underline{\eta}) \underline{\mathbf{R}}_\alpha = -\underline{\mathbf{R}}_\beta^* (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \underline{\mathbf{R}}_\alpha - i \underline{\omega}_\alpha \underline{\mathbf{R}}_\beta^* A_j(\boldsymbol{\nu}) \underline{\mathbf{R}}_\alpha + i \overline{\omega}_\beta \underline{\mathbf{R}}_\beta^* A_j(\boldsymbol{\nu})^T \underline{\mathbf{R}}_\alpha.$$

Plugging this expression in (2.38), we obtain

$$\begin{aligned}
\partial_{\eta_j} \Delta(\underline{\tau}, \underline{\boldsymbol{\eta}}) &= -\kappa \lambda_\alpha \overline{\lambda_\beta} \frac{\underline{R}_\beta^* (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \underline{R}_\alpha}{\underline{\omega}_\alpha + \overline{\underline{\omega}}_\beta} \\
&\quad - i \kappa \lambda_\alpha \overline{\lambda_\beta} \underline{\omega}_\alpha \frac{\underline{R}_\beta^* A_j(\boldsymbol{\nu}) \underline{R}_\alpha}{\underline{\omega}_\alpha + \overline{\underline{\omega}}_\beta} + i \kappa \lambda_\alpha \overline{\lambda_\beta} \overline{\underline{\omega}}_\beta \frac{\underline{R}_\beta^* A_j(\boldsymbol{\nu})^T \underline{R}_\alpha}{\underline{\omega}_\alpha + \overline{\underline{\omega}}_\beta} \\
&= -\kappa \int_0^{+\infty} \widehat{\mathbf{r}}(1, z)^* (A_j(\boldsymbol{\eta}) + A_j(\boldsymbol{\eta})^T) \widehat{\mathbf{r}}(1, z) dz - 2\kappa \operatorname{Im} \int_0^{+\infty} \widehat{\mathbf{r}}(1, z)^* A_j(\boldsymbol{\nu}) \partial_z \widehat{\mathbf{r}}(1, z) dz,
\end{aligned}$$

and we thus find the relation $\partial_{\eta_j} \Delta(\underline{\tau}, \underline{\boldsymbol{\eta}}) = -\kappa c_j$. In other words, we have found the relation

$$\forall j = 1, \dots, d-1, \quad \frac{c_j}{c_0} = \frac{\partial_{\eta_j} \Delta(\underline{\tau}, \underline{\boldsymbol{\eta}})}{\partial_\tau \Delta(\underline{\tau}, \underline{\boldsymbol{\eta}})},$$

and the claim of Proposition 2.1 follows from the factorization (which holds near the simple root $(\underline{\tau}, \underline{\boldsymbol{\eta}})$ of Δ , and here the sign convention for τ plays a role):

$$\Delta(\tau, \boldsymbol{\eta}) = \vartheta(\tau, \boldsymbol{\eta}) (\tau + \tau_r(\boldsymbol{\eta})), \quad \vartheta(\underline{\tau}, \underline{\boldsymbol{\eta}}) \neq 0.$$

We thus get $c_j/c_0 = \partial_{\eta_j} \tau_r(\underline{\boldsymbol{\eta}})$, which yields the final form of the amplitude equation (2.20). \square

Let us recall that our goal is to construct the asymptotic expansion (2.12) of the solution \mathbf{u}^ε to the high frequency problem (2.1), (2.11). At this stage, we have shown that the leading order amplitude $\mathbf{u}^{(1)}$ is given on the boundary $\partial\Omega$ by (2.15) and that for a ‘reasonable’ corrector $\mathbf{u}^{(2)}$ in (2.12) to exist, the amplitude function w should solve the equation (2.20). In order to proceed and construct the first term in the asymptotic expansion (2.12), the main question is to investigate the well-posedness properties of the equation (2.20). This will rely on earlier results by Hunter and Benzoni-Gavage [Hum06, BG09]. Once we have w , the leading order term $\mathbf{u}^{(1)}$ is defined on the boundary $\partial\Omega$ and, following [Les07, MA88, Mét04], one possibility for defining $\mathbf{u}^{(1)}$ in the whole space domain Ω is simply to let

$$\mathbf{u}^{(1)}(t, x, \theta, z) := \chi(x \cdot \boldsymbol{\nu}) \mathbf{u}^{(1)}(t, y, \theta, z),$$

where y denotes the orthogonal projection of x on $\partial\Omega$, and the function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ has compact support and satisfies $\chi(0) = 1$. There are two questions then: can we construct indeed a corrector $\mathbf{u}^{(2)}$ that satisfies (2.14)? and, provided that the exact solution \mathbf{u}^ε to (2.1), (2.11) exists on a fixed time interval independent of ε , does (2.12) provide with an accurate description of \mathbf{u}^ε on this time interval? These questions are answered positively in Chapters 3, 4, and 5. But before entering analytical issues, we are going to focus on how the previous analysis applies to the case of elastodynamics.

2.3 Isotropic elastodynamics

In this section we explain how the previous analysis applies to the system of elastodynamics for hyperelastic materials. The results generalize those of [Lar83] to any space dimension, and put the leading order amplitude equation in the form (2.20) which is more convenient in view of applying the well-posedness results of [Hum06, BG09]. We refer to [Cia83] for an introduction to elasticity and the physical background. More specifically, we consider an elastic material in the reference domain $\Omega = \{x \cdot \boldsymbol{\nu} > 0\} \subset \mathbb{R}^d$. The deformation gradient is $\nabla\varphi = I + \nabla\mathbf{u}$, where $\mathbf{u}(t, x) \in \mathbb{R}^d$ represents, at a given

time t , the displacement of the material at a point $x \in \Omega$. The space dimension d equals either 2 or 3. Due to frame indifference, the elastic energy W is a function of the so-called Cauchy-Green strain tensor $C := \nabla \varphi^T \nabla \varphi \in \mathcal{M}_d(\mathbb{R})$. We may equivalently rewrite W in terms of the Green - Saint Venant strain tensor $E := (C - I)/2$. In terms of the displacement gradient, this gives

$$\forall \alpha, j = 1, \dots, d, \quad E_{\alpha,j} = \frac{1}{2} (u_{\alpha,j} + u_{j,\alpha} + u_{\alpha,\ell} u_{j,\ell}).$$

When the material is *isotropic*, the energy W only depends on the principal invariants of the Cauchy-Green strain tensor C . Assuming that the energy W is a smooth function of E , its Taylor expansion at $E = 0$ then reads (see [Cia83, Chapter 1.4]):

$$(2.39) \quad W(E) = \frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } (E^2) + \alpha_1 (\text{tr } E)^3 + \alpha_2 (\text{tr } E) \text{tr } (E^2) + \alpha_3 \text{tr } (E^3) + O(\|E\|^4).$$

In the above formula, λ and μ stand for the so-called Lamé coefficients of the material, and $\alpha_1, \alpha_2, \alpha_3$ are constants. For future use, we introduce the Frobenius norm of a matrix $M \in \mathcal{M}_d(\mathbb{R})$:

$$\|M\|^2 := \sum_{\alpha,j=1}^d M_{\alpha j}^2 = \text{tr } (M M^T).$$

Using the expression of E in terms of the displacement gradient $\nabla \mathbf{u}$, we can rewrite the above Taylor expansion of W as

$$(2.40) \quad \begin{aligned} W(\nabla \mathbf{u}) &= \frac{\lambda}{2} (\text{tr } \nabla \mathbf{u})^2 + \frac{\mu}{4} \|\nabla \mathbf{u} + \nabla \mathbf{u}^T\|^2 \\ &+ \beta_1 (\text{tr } \nabla \mathbf{u})^3 + \beta_2 (\text{tr } \nabla \mathbf{u}) \|\nabla \mathbf{u} + \nabla \mathbf{u}^T\|^2 + \beta_3 \text{tr } (\nabla \mathbf{u} \nabla \mathbf{u}^T \nabla \mathbf{u}) + \beta_4 \text{tr } (\nabla \mathbf{u}^3) + O(\|\nabla \mathbf{u}\|^4). \end{aligned}$$

For the sake of completeness, the coefficients β_1, \dots, β_4 in (2.40) are given by:

$$\beta_1 := \alpha_1, \quad \beta_2 := \frac{\lambda}{2} + \frac{\alpha_2}{4}, \quad \beta_3 := \mu + \frac{3\alpha_3}{4}, \quad \beta_4 := \frac{\alpha_3}{4}.$$

Their precise expression is not relevant for our purpose though. Let us note however that, even in the simplest case of the so-called Saint Venant - Kirchhoff materials, for which one has

$$(2.41) \quad \forall E, \quad W(E) = \frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } (E^2),$$

the coefficients β_1, \dots, β_4 in the third order terms of the Taylor expansion of $W(\nabla \mathbf{u})$ are not all zero. As a matter of fact, there is not much simplification in the final expression of the amplitude equation if one assumes that one/some of the coefficients β_1, \dots, β_4 are zero, so we keep them all in what follows⁸.

From now on, we consider the previous derivation of the amplitude equation (2.20) when $N = d \geq 2$ and the energy W satisfies (2.40) near $\nabla \mathbf{u} = 0$. Let us first take a look at assumptions (H1), (H2), (H3). Assumption (H1) is trivially fulfilled since the Taylor expansion of W starts at the second order. Furthermore, we compute⁹:

$$c_{\alpha j \beta \ell} = \lambda \delta_{\alpha j} \delta_{\beta \ell} + \mu (\delta_{\alpha \beta} \delta_{j \ell} + \delta_{\alpha \ell} \delta_{\alpha j}),$$

⁸The only real simplification occurs when $\beta_2, \beta_3, \beta_4$ are all zero, but this is incompatible with the definition of these coefficients since μ is nonzero and therefore β_3, β_4 cannot vanish simultaneously.

⁹Some of the expressions below already appeared in [BGC12] so we do not reproduce all the computations but rather give the expressions that are useful for our purpose.

with δ the Kronecker symbol ($\delta_{n_1 n_2} = 1$ if $n_1 = n_2$, zero otherwise). The above expression of the coefficients $c_{\alpha j \beta \ell}$ gives:

$$\forall \boldsymbol{\xi}, v \in \mathbb{R}^d, \quad c_{\alpha j \beta \ell} v_\alpha \xi_j v_\beta \xi_\ell = \mu |\boldsymbol{\xi}|^2 |v|^2 + (\lambda + \mu) (v \cdot \boldsymbol{\xi})^2.$$

The fulfillment of Assumption (H2) is then equivalent to the well-known inequalities

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0.$$

One can then define the velocity of ‘shear’ and ‘pressure’ waves by

$$c_S := \sqrt{\mu}, \quad c_P := \sqrt{\lambda + 2\mu}.$$

As has now long been known, see e.g. [Lar83, Ser06], the fulfillment of Assumption (H3) is equivalent to the additional requirement $c_P > c_S$, or equivalently $\lambda + \mu > 0$. In that case, one may uniquely define a velocity $c_R \in (0, c_S)$ by solving the polynomial equation

$$(2.42) \quad \left(\frac{c_R^2}{2c_S^2} - 1 \right)^4 = \left(1 - \frac{c_R^2}{c_S^2} \right) \left(1 - \frac{c_R^2}{c_P^2} \right).$$

Assumption (H3) is then satisfied with $\tau_r(\boldsymbol{\eta}) := c_R |\boldsymbol{\eta}|$ for all tangential wave vector $\boldsymbol{\eta}$ (which obviously satisfies the homogeneity property $\tau_r(k\boldsymbol{\eta}) = k\tau_r(\boldsymbol{\eta})$ for any $k > 0$). Given a nonzero tangential wave vector $\boldsymbol{\eta}$, the one-dimensional family of surface waves solution to (2.4) is spanned by

$$(2.43) \quad e^{\pm i \tau_r(\boldsymbol{\eta}) t + i \boldsymbol{\eta} \cdot \boldsymbol{x}} V(\boldsymbol{\nu} \cdot \boldsymbol{x}), \quad V(z) := e^{-\omega_1 z} (-2i\omega_1 \omega_2 \boldsymbol{\eta} + 2\omega_2 |\boldsymbol{\eta}|^2 \boldsymbol{\nu}) + e^{-\omega_2 z} (\omega_1^2 + |\boldsymbol{\eta}|^2) (i\boldsymbol{\eta} - \omega_2 \boldsymbol{\nu}).$$

The eigenmodes $\omega_{1,2}$ in (2.43) are defined by:

$$(2.44) \quad \omega_1 := |\boldsymbol{\eta}| \sqrt{1 - \frac{c_R^2}{c_S^2}}, \quad \omega_2 := |\boldsymbol{\eta}| \sqrt{1 - \frac{c_R^2}{c_P^2}},$$

and thus satisfy

$$(2.45) \quad 0 < \omega_1 < \omega_2, \quad 4|\boldsymbol{\eta}|^2 \omega_1 \omega_2 = (\omega_1^2 + |\boldsymbol{\eta}|^2)^2.$$

The latter equality is an equivalent form of (2.42).

Our goal now is to identify the amplitude equation (2.20) with the function V given by (2.43) (and the corresponding $\hat{r}(k, z)$ defined by (2.15)). We first compute the group velocity

$$\partial_{\eta_j} \tau_r(\boldsymbol{\eta}) = c_R \frac{\eta_j}{|\boldsymbol{\eta}|}.$$

Specifying from now on to the case $\Omega = \{x_d > 0\}$, that is $\boldsymbol{\nu} = (0, \dots, 0, 1)^T$, the amplitude equation (2.20) reads:

$$\partial_t w + c_R \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|} \cdot \nabla_x w + \mathcal{H}(\mathcal{B}(w, w)) = g,$$

with \mathcal{B} given by (2.21). Let us recall that the tangential wave vector $\boldsymbol{\eta}$ has the form $(\eta_1, \dots, \eta_{d-1}, 0)^T$, and the amplitude w , which is defined on the boundary $\partial\Omega$ depends on $(t, x_1, \dots, x_{d-1}, \theta)$. Let us examine more closely at the expression of the bilinear Fourier multiplier \mathcal{B} .

- The constant c_0 . It is defined by (2.22). With the definition (2.43) of V , we compute¹⁰

$$\begin{aligned}
c_0 &= -2\tau \int_0^{+\infty} |V(z)|^2 dz \\
&= -2\tau \left\{ \frac{4\omega_2^2 |\boldsymbol{\eta}|^2 (\omega_1^2 + |\boldsymbol{\eta}|^2)}{2\omega_1} + \frac{(\omega_1^2 + |\boldsymbol{\eta}|^2)^2 (\omega_2^2 + |\boldsymbol{\eta}|^2)}{2\omega_2} - 4\omega_2 |\boldsymbol{\eta}|^2 (\omega_1^2 + |\boldsymbol{\eta}|^2) \right\} \\
(2.46) \quad &= -4\tau |\boldsymbol{\eta}|^2 (\omega_2 - \omega_1) \left\{ |\boldsymbol{\eta}|^2 \left(\frac{\omega_2}{\omega_1} - 1 \right) + 2\omega_1 \omega_2 \right\},
\end{aligned}$$

where we have used the dispersion relation (2.45) to obtain the final expression of c_0 . The expression (2.46) gives the constant in the definition (2.21) of \mathcal{B} . It remains to clarify the expression of the kernel b , which is given by (2.29). The first task is to compute the coefficients $d_{\alpha j \beta \ell \gamma m}$.

- The coefficients $d_{\alpha j \beta \ell \gamma m}$. They are obtained as the third order derivatives of W at 0, see (2.3). Using the Taylor expansion (2.40) of W , we get the decomposition

$$d_{\alpha j \beta \ell \gamma m} = 6\beta_1 d_{\alpha j \beta \ell \gamma m}^1 + 2\beta_2 d_{\alpha j \beta \ell \gamma m}^2 + \beta_3 d_{\alpha j \beta \ell \gamma m}^3 + 3\beta_4 d_{\alpha j \beta \ell \gamma m}^4,$$

with

$$(2.47) \quad d_{\alpha j \beta \ell \gamma m}^1 = \delta_{\alpha j} \delta_{\beta \ell} \delta_{\gamma m},$$

$$(2.48) \quad d_{\alpha j \beta \ell \gamma m}^2 = \delta_{\alpha j} \delta_{\beta \gamma} \delta_{\ell m} + \delta_{\beta \ell} \delta_{\alpha \gamma} \delta_{j m} + \delta_{\gamma m} \delta_{\alpha \beta} \delta_{j \ell},$$

$$(2.49) \quad d_{\alpha j \beta \ell \gamma m}^3 = \delta_{\alpha \beta} (\delta_{\gamma j} \delta_{\ell m} + \delta_{\gamma \ell} \delta_{j m}) + \delta_{\alpha \gamma} (\delta_{\beta j} \delta_{\ell m} + \delta_{\beta m} \delta_{j \ell}) + \delta_{\beta \gamma} (\delta_{\alpha \ell} \delta_{j m} + \delta_{\alpha m} \delta_{j \ell}),$$

$$(2.50) \quad d_{\alpha j \beta \ell \gamma m}^4 = \delta_{\alpha \ell} \delta_{\beta m} \delta_{\gamma j} + \delta_{\alpha m} \delta_{\beta j} \delta_{\gamma \ell}.$$

Each of these four coefficients is invariant through the permutations $(\alpha, j) \leftrightarrow (\beta, \ell)$, $(\alpha, j) \leftrightarrow (\gamma, m)$ and $(\beta, \ell) \leftrightarrow (\gamma, m)$, and therefore through any permutation of (α, j) , (β, ℓ) , (γ, m) . Associated with the above decomposition for $d_{\alpha j \beta \ell \gamma m}$, we have a decomposition of the kernel $b(\xi_1, \xi_2, \xi_3)$ in (2.29). We thus find that in the case of isotropic elasticity, the kernel $b(\xi_1, \xi_2, \xi_3)$ is a linear combination with real coefficients of the ‘elementary’ kernels

$$(2.51) \quad b^k(\xi_1, \xi_2, \xi_3) := \int_0^{+\infty} d_{\alpha j \beta \ell \gamma m}^k (\nu_j \rho_{\alpha, z} + i \xi_1 \eta_j \rho_{\alpha}) (\nu_{\ell} \rho_{\beta, z} + i \xi_2 \eta_{\ell} \rho_{\beta}) (\nu_m \rho_{\gamma, z} + i \xi_3 \eta_m \rho_{\gamma}) dz,$$

where k runs through $\{1, \dots, 4\}$ and the coefficients $d_{\alpha j \beta \ell \gamma m}^k$ are given in (2.47), (2.48), (2.49), (2.50). Let us recall eventually that the functions $\rho_{\alpha}, \rho_{\beta}, \rho_{\gamma}$ are given in (2.29).

In the remaining of this Section, we are going to make the kernels b^1, \dots, b^4 explicit, meaning that we are going to express them as linear combinations of explicitly computable kernels. This will give the final expression of b as a linear combination of explicit kernels. In order to avoid lengthy and somehow useless computations, we shall not try to make the coefficients in the linear combinations explicit. The interested reader will achieve this by using the expressions below and expanding all trilinear expressions of $\exp(-\omega_{1,2} |\xi| z)$ below explicitly.

From the definition (2.51), it appears that we need to compute the quantity

$$\nu_j \frac{\partial}{\partial z} (\widehat{r}_{\alpha}(\xi, z)) + i \xi \eta_j \widehat{r}_{\alpha}(\xi, z),$$

¹⁰Recall that we consider here the case $\tau = -\tau_r(\boldsymbol{\eta}) = -c_R |\boldsymbol{\eta}|$.

for all possible values of α, j and ξ . Using (2.43), we get (keeping the subscript ‘, z ’ for denoting partial differentiation with respect to z):

$$\nu_j \widehat{r}_{\alpha,z}(\xi, z) + i \xi \eta_j \widehat{r}_\alpha(\xi, z) = \begin{cases} |\xi| \eta_\alpha \eta_j \mathbb{T}_{11}(\xi, z), & \alpha, j \leq d-1, \\ i \xi \eta_j \omega_2 \mathbb{T}_{d1}(\xi, z), & j \leq d-1, \alpha = d, \\ i \xi \eta_\alpha \omega_2 \mathbb{T}_{1d}(\xi, z), & \alpha \leq d-1, j = d, \\ -|\xi| \omega_2 \mathbb{T}_{dd}(\xi, z), & \alpha = j = d. \end{cases}$$

with

$$(2.52) \quad \begin{cases} \mathbb{T}_{11}(\xi, z) := 2 \omega_1 \omega_2 e^{-\omega_1 |\xi| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi| z}, \\ \mathbb{T}_{d1}(\xi, z) := 2 |\boldsymbol{\eta}|^2 e^{-\omega_1 |\xi| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi| z}, \\ \mathbb{T}_{1d}(\xi, z) := 2 \omega_1^2 e^{-\omega_1 |\xi| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi| z}, \\ \mathbb{T}_{dd}(\xi, z) := 2 \omega_1 |\boldsymbol{\eta}|^2 e^{-\omega_1 |\xi| z} - \omega_2 (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi| z}. \end{cases}$$

In particular, there holds

$$(2.53) \quad \nu_j \widehat{r}_{j,z}(\xi, z) + i \xi \eta_j \widehat{r}_j(\xi, z) = |\xi| (\omega_1^2 + |\boldsymbol{\eta}|^2) (\omega_2^2 - |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi| z} = -|\xi| \frac{c_R^2}{c_P^2} |\boldsymbol{\eta}|^2 (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi| z}.$$

• The kernel b^1 . This is by far the simplest case. We start from the definition (2.51) and use the expression (2.47) for the coefficients $d_{\alpha j \beta \ell \gamma m}^1$. We get

$$b^1(\xi_1, \xi_2, \xi_3) = \int_0^{+\infty} (\nu_j \rho_{j,z} + i \xi_1 \eta_j \rho_j) (\nu_\ell \rho_{\ell,z} + i \xi_2 \eta_\ell \rho_\ell) (\nu_m \rho_{m,z} + i \xi_3 \eta_m \rho_m) dz,$$

where we warn the reader that, in the first term in the integral, ρ_j is a short notation for $\widehat{r}_j(\xi_1, z)$, in the second term, ρ_ℓ is a short notation for $\widehat{r}_\ell(\xi_2, z)$ and so on. We now use (2.53), and get (here the sum of products decouples as the product of sums over j, ℓ and m):

$$b^1(\xi_1, \xi_2, \xi_3) = \star \int_0^{+\infty} |\xi_1| |\xi_2| |\xi_3| \exp(-\omega_2 (|\xi_1| + |\xi_2| + |\xi_3|) z) dz,$$

where, here and from now on, \star denotes any *real* constant that depends only on $\boldsymbol{\eta}$ and that can be explicitly computed from (2.52) or (2.53) (though we shall not keep track of any such constant). We thus obtain the expression:

$$(2.54) \quad b^1(\xi_1, \xi_2, \xi_3) = \star \frac{|\xi_1| |\xi_2| |\xi_3|}{|\xi_1| + |\xi_2| + |\xi_3|},$$

which corresponds to the simplified kernel introduced in [HIZ95] and that also arises in incompressible magnetohydrodynamics [AH03]. As shown in [AHP02], this kernel corresponds in the physical space to the operator (here we forget about harmless multiplicative real constants):

$$\frac{1}{2} \partial_{\theta\theta} \left(\mathcal{H} \left((\mathcal{H} w)^2 \right) \right) + (\mathcal{H} w) \partial_{\theta\theta} w.$$

- The kernel b^2 . We now use the expression (2.48) and derive

$$\begin{aligned}
b^2(\xi_1, \xi_2, \xi_3) &= \int_0^{+\infty} (\nu_j \rho_{j,z} + i \xi_1 \eta_j \rho_j) (\nu_\ell \rho_{\beta,z} + i \xi_2 \eta_\ell \rho_\beta) (\nu_\ell \rho_{\beta,z} + i \xi_3 \eta_\ell \rho_\beta) dz \\
&\quad + \int_0^{+\infty} (\nu_j \rho_{\alpha,z} + i \xi_1 \eta_j \rho_\alpha) (\nu_\ell \rho_{\ell,z} + i \xi_2 \eta_\ell \rho_\ell) (\nu_j \rho_{\alpha,z} + i \xi_3 \eta_j \rho_\alpha) dz \\
&\quad + \int_0^{+\infty} (\nu_j \rho_{\alpha,z} + i \xi_1 \eta_j \rho_\alpha) (\nu_j \rho_{\alpha,z} + i \xi_2 \eta_j \rho_\alpha) (\nu_m \rho_{m,z} + i \xi_3 \eta_m \rho_m) dz.
\end{aligned}$$

Changing indices in the last two integrals, we have written $b^2(\xi_1, \xi_2, \xi_3)$ under the form¹¹

$$(2.55) \quad b^2(\xi_1, \xi_2, \xi_3) = \mathbb{B}^2(\xi_1, \xi_2, \xi_3) + \mathbb{B}^2(\xi_2, \xi_3, \xi_1) + \mathbb{B}^2(\xi_3, \xi_1, \xi_2) = \sum_{\substack{\sigma \in \mathfrak{S}_3, \\ \varepsilon(\sigma)=1}} \mathbb{B}^2(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}),$$

where \mathbb{B}^2 is symmetric with respect to its last two arguments (that is, $\mathbb{B}^2(\xi_1, \xi_3, \xi_2) = \mathbb{B}^2(\xi_1, \xi_2, \xi_3)$ for all ξ_1, ξ_2, ξ_3), and the formula (2.55) shows that one ‘symmetrizes’ the function \mathbb{B}^2 over the alternating group $\mathfrak{A}_3 \subset \mathfrak{S}_3$. Using (2.52) and (2.53), we have

$$\begin{aligned}
\mathbb{B}^2(\xi_1, \xi_2, \xi_3) &= \star |\xi_1| \int_0^{+\infty} e^{-\omega_2 |\xi_1| z} (\nu_\ell \rho_{\beta,z} + i \xi_2 \eta_\ell \rho_\beta) (\nu_\ell \rho_{\beta,z} + i \xi_3 \eta_\ell \rho_\beta) dz \\
&= \star |\xi_1| |\xi_2| |\xi_3| \int_0^{+\infty} e^{-\omega_2 |\xi_1| z} \mathbb{T}_{11}(\xi_2, z) \mathbb{T}_{11}(\xi_3, z) dz \\
&\quad + \star |\xi_1| \xi_2 \xi_3 \int_0^{+\infty} e^{-\omega_2 |\xi_1| z} \mathbb{T}_{d1}(\xi_2, z) \mathbb{T}_{d1}(\xi_3, z) dz \\
&\quad + \star |\xi_1| \xi_2 \xi_3 \int_0^{+\infty} e^{-\omega_2 |\xi_1| z} \mathbb{T}_{1d}(\xi_2, z) \mathbb{T}_{1d}(\xi_3, z) dz \\
&\quad + \star |\xi_1| |\xi_2| |\xi_3| \int_0^{+\infty} e^{-\omega_2 |\xi_1| z} \mathbb{T}_{dd}(\xi_2, z) \mathbb{T}_{dd}(\xi_3, z) dz.
\end{aligned}$$

Let us examine the first integral in the latter sum of four. We need to compute

$$\int_0^{+\infty} e^{-\omega_2 |\xi_1| z} \left(2\omega_1 \omega_2 e^{-\omega_1 |\xi_2| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi_2| z} \right) \left(2\omega_1 \omega_2 e^{-\omega_1 |\xi_3| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi_3| z} \right) dz,$$

then multiply by $|\xi_1| |\xi_2| |\xi_3|$ and eventually ‘symmetrize’ with respect to ξ_1, ξ_2, ξ_3 by summing as in (2.55), and this will give part of the kernel b^2 . As reported in [Hun89, Hun06], this first integral in \mathbb{B}^2 gives rise in the expression of b^2 to a linear combination with real coefficients of the following three kernels:

$$(2.56) \quad \frac{|\xi_1| |\xi_2| |\xi_3|}{|\xi_1| + |\xi_2| + |\xi_3|},$$

$$(2.57) \quad |\xi_1| |\xi_2| |\xi_3| \left\{ \frac{1}{r |\xi_1| + |\xi_2| + |\xi_3|} + \frac{1}{|\xi_1| + r |\xi_2| + |\xi_3|} + \frac{1}{|\xi_1| + |\xi_2| + r |\xi_3|} \right\},$$

$$(2.58) \quad |\xi_1| |\xi_2| |\xi_3| \left\{ \frac{1}{|\xi_1| + r |\xi_2| + r |\xi_3|} + \frac{1}{r |\xi_1| + |\xi_2| + r |\xi_3|} + \frac{1}{r |\xi_1| + r |\xi_2| + |\xi_3|} \right\},$$

¹¹Here, and here only, ε denotes the signature of a permutation.

where $r := \omega_1/\omega_2 \in (0, 1)$ can be computed from the Lamé coefficients of the material. The kernel in (2.56) already appears in (2.54), so the b^1 part of the overall kernel b contributes to a term that is similar to one of the many in the decomposition of b^2 .

Using the expression of \mathbb{T}_{dd} in (2.52), one can check that the last integral in the above decomposition of \mathbb{B}^2 also gives rise, after multiplication by $|\xi_1| |\xi_2| |\xi_3|$ and symmetrization with respect to ξ_1, ξ_2, ξ_3 , to a linear combination with real coefficients of the three kernels in (2.56), (2.57), (2.58).

We now examine the second and third integrals in the decomposition of \mathbb{B}^2 , which involve the expressions $\mathbb{T}_{d1}, \mathbb{T}_{1d}$ in (2.52), and that are multiplied by $|\xi_1| \xi_2 \xi_3$ (no absolute value for $\xi_2 \xi_3$ here !). For instance, we compute the integral

$$\int_0^{+\infty} e^{-\omega_2 |\xi_1| z} \left(2 |\boldsymbol{\eta}|^2 e^{-\omega_1 |\xi_2| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi_2| z} \right) \left(2 |\boldsymbol{\eta}|^2 e^{-\omega_1 |\xi_3| z} - (\omega_1^2 + |\boldsymbol{\eta}|^2) e^{-\omega_2 |\xi_3| z} \right) dz,$$

multiply by $|\xi_1| \xi_2 \xi_3$ and symmetrize with respect to ξ_1, ξ_2, ξ_3 as in (2.55). This computation gives rise in the expression of b^2 to a linear combination with real coefficients of the following kernels:

$$(2.59) \quad \frac{1}{|\xi_1| + |\xi_2| + |\xi_3|} \left\{ |\xi_1| \xi_2 \xi_3 + \xi_1 |\xi_2| \xi_3 + \xi_1 \xi_2 |\xi_3| \right\},$$

$$(2.60) \quad \frac{|\xi_1| \xi_2 \xi_3}{|\xi_1| + r |\xi_2| + r |\xi_3|} + \frac{\xi_1 |\xi_2| \xi_3}{r |\xi_1| + |\xi_2| + r |\xi_3|} + \frac{\xi_1 \xi_2 |\xi_3|}{r |\xi_1| + r |\xi_2| + |\xi_3|},$$

$$(2.61) \quad |\xi_1| \xi_2 \xi_3 \left\{ \frac{1}{|\xi_1| + r |\xi_2| + |\xi_3|} + \frac{1}{|\xi_1| + |\xi_2| + r |\xi_3|} \right\} \\ + \xi_1 |\xi_2| \xi_3 \left\{ \frac{1}{r |\xi_1| + |\xi_2| + |\xi_3|} + \frac{1}{|\xi_1| + |\xi_2| + r |\xi_3|} \right\} \\ + \xi_1 \xi_2 |\xi_3| \left\{ \frac{1}{r |\xi_1| + |\xi_2| + |\xi_3|} + \frac{1}{|\xi_1| + r |\xi_2| + |\xi_3|} \right\}.$$

The reader can check that, in the decomposition of \mathbb{B}^2 , the integral that involves \mathbb{T}_{1d} also gives rise, after multiplication by $|\xi_1| \xi_2 \xi_3$ and symmetrization, to a linear combination with real coefficients of the three kernels in (2.59), (2.60), (2.61).

At this stage, we have proved that, with the expression (2.48) for $d_{\alpha j \beta \ell \gamma m}^2$, the corresponding kernel b^2 in (2.51) is a linear combination with real coefficients of the six kernels given in (2.56), (2.57), (2.58), (2.59), (2.60) and (2.61).

• The kernels b^3 and b^4 . Using the expression of $d_{\alpha j \beta \ell \gamma m}^3$ in (2.49), we can decompose the kernel b^3 under the form

$$b^3(\xi_1, \xi_2, \xi_3) = \mathbb{B}^3(\xi_1, \xi_2, \xi_3) + \mathbb{B}^3(\xi_2, \xi_1, \xi_3) + \mathbb{B}^3(\xi_1, \xi_3, \xi_2) + \mathbb{B}^3(\xi_3, \xi_1, \xi_2) + \mathbb{B}^3(\xi_2, \xi_3, \xi_1) + \mathbb{B}^3(\xi_3, \xi_2, \xi_1) \\ = \sum_{\sigma \in \mathfrak{S}_3} \mathbb{B}^3(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}),$$

where the function \mathbb{B}^3 is defined by

$$\mathbb{B}^3(\xi_1, \xi_2, \xi_3) := \int_0^{+\infty} (\nu_j \rho_{\alpha, z} + i \xi_1 \eta_j \rho_\alpha) (\nu_\ell \rho_{\alpha, z} + i \xi_2 \eta_\ell \rho_\alpha) (\nu_\ell \rho_{j, z} + i \xi_3 \eta_\ell \rho_j) dz,$$

and is symmetrized over the symmetric group \mathfrak{S}_3 to obtain the kernel b^3 . Considering all possible indices α, j, ℓ , we end up with the following decomposition of \mathbb{B}^3 :

$$\begin{aligned}
\mathbb{B}^3(\xi_1, \xi_2, \xi_3) = & \star |\xi_1| |\xi_2| |\xi_3| \int_0^{+\infty} \mathbb{T}_{11}(\xi_1, z) \mathbb{T}_{11}(\xi_2, z) \mathbb{T}_{11}(\xi_3, z) dz \\
& + \star |\xi_1| |\xi_2| |\xi_3| \int_0^{+\infty} \mathbb{T}_{dd}(\xi_1, z) \mathbb{T}_{dd}(\xi_2, z) \mathbb{T}_{dd}(\xi_3, z) dz \\
& + \star \xi_1 \xi_2 |\xi_3| \int_0^{+\infty} \mathbb{T}_{d1}(\xi_1, z) \mathbb{T}_{d1}(\xi_2, z) \mathbb{T}_{11}(\xi_3, z) dz \\
& + \star \xi_1 \xi_2 |\xi_3| \int_0^{+\infty} \mathbb{T}_{1d}(\xi_1, z) \mathbb{T}_{1d}(\xi_2, z) \mathbb{T}_{dd}(\xi_3, z) dz \\
& + \star \xi_1 |\xi_2| \xi_3 \int_0^{+\infty} \mathbb{T}_{1d}(\xi_1, z) \mathbb{T}_{11}(\xi_2, z) \mathbb{T}_{d1}(\xi_3, z) dz \\
& + \star \xi_1 |\xi_2| \xi_3 \int_0^{+\infty} \mathbb{T}_{d1}(\xi_1, z) \mathbb{T}_{dd}(\xi_2, z) \mathbb{T}_{1d}(\xi_3, z) dz \\
& + \star |\xi_1| \xi_2 \xi_3 \int_0^{+\infty} \mathbb{T}_{11}(\xi_1, z) \mathbb{T}_{1d}(\xi_2, z) \mathbb{T}_{1d}(\xi_3, z) dz \\
& + \star |\xi_1| \xi_2 \xi_3 \int_0^{+\infty} \mathbb{T}_{dd}(\xi_1, z) \mathbb{T}_{d1}(\xi_2, z) \mathbb{T}_{d1}(\xi_3, z) dz.
\end{aligned}$$

Computing the first two integrals in the latter decomposition of \mathbb{B}^3 , we get linear combinations with real coefficients of the kernels in (2.56), (2.57), (2.58) (the first two lines are already symmetric with respect to ξ_1, ξ_2, ξ_3 so symmetrizing over \mathfrak{S}_3 only yields the harmless multiplicative factor 6).

We now compute the third and fourth integrals in the decomposition of \mathbb{B}^3 , which are entirely similar. Expanding the products in the integrals, then multiplying by $\xi_1 \xi_2 |\xi_3|$ and symmetrizing over \mathfrak{S}_3 yields¹² a linear combination of the kernels (2.59), (2.60), (2.61), and the (hopefully last!) two kernels:

$$(2.62) \quad \frac{|\xi_1| \xi_2 \xi_3}{r |\xi_1| + |\xi_2| + |\xi_3|} + \frac{\xi_1 |\xi_2| \xi_3}{|\xi_1| + r |\xi_2| + |\xi_3|} + \frac{\xi_1 \xi_2 |\xi_3|}{|\xi_1| + |\xi_2| + r |\xi_3|},$$

$$(2.63) \quad |\xi_1| \xi_2 \xi_3 \left\{ \frac{1}{r |\xi_1| + r |\xi_2| + |\xi_3|} + \frac{1}{r |\xi_1| + |\xi_2| + r |\xi_3|} \right\}$$

$$\begin{aligned}
& + \xi_1 |\xi_2| \xi_3 \left\{ \frac{1}{r |\xi_1| + r |\xi_2| + |\xi_3|} + \frac{1}{|\xi_1| + r |\xi_2| + r |\xi_3|} \right\} \\
& + \xi_1 \xi_2 |\xi_3| \left\{ \frac{1}{r |\xi_1| + |\xi_2| + r |\xi_3|} + \frac{1}{|\xi_1| + r |\xi_2| + r |\xi_3|} \right\}.
\end{aligned}$$

The reader can check that the remaining four integrals in the decomposition of \mathbb{B}^3 yield, after multiplication by either $\xi_1 |\xi_2| \xi_3$ or $|\xi_1| \xi_2 \xi_3$ and symmetrization over \mathfrak{S}_3 , a linear combination of the eight kernels in (2.56), (2.57), (2.58), (2.59), (2.60), (2.61), (2.62) and (2.63).

We leave it as an exercise to the interested reader to verify that with the definition (2.50), the corresponding kernel b^4 can be also written as a linear combination with real coefficients of the eight kernels given in (2.56), (2.57), (2.58), (2.59), (2.60), (2.61), (2.62) and (2.63).

¹²As a matter of fact, symmetrizing over \mathfrak{A}_3 is sufficient here because the third and fourth lines are already symmetric with respect to ξ_1, ξ_2 .

• Final simplifications. In [Hun89, Hun06], it is reported that only the kernels in (2.56), (2.57), (2.58) are necessary for computing the leading order amplitude equation. This is not necessarily in contradiction with the above computations since, according to (2.20), what we really need is the value of b over the triplets (ξ_1, ξ_2, ξ_3) verifying $\xi_1 + \xi_2 + \xi_3 = 0$. Namely, we now introduce the symmetric kernel

$$\Lambda(k, k') := b(- (k + k'), k, k'),$$

which allows us to rewrite \mathcal{B} as

$$\widehat{\mathcal{B}(w, w)}(k) := -\frac{1}{4\pi c_0} \int_{\mathbb{R}} \Lambda(k - \xi, \xi) \widehat{w}(k - \xi) \widehat{w}(\xi) d\xi.$$

It remains to examine whether for the kernels we have found in (2.56), (2.57), (2.58), (2.59), (2.60), (2.61), (2.62) and (2.63), the corresponding functions Λ are linearly independent.

Let us start with the easiest case of the kernel in (2.59). Let us first observe that for all (k, k') , we have the relation

$$|k + k'| (|k k'| + k k') = (k + k') (|k k'| + k k'),$$

and therefore

$$\frac{|\xi_1| |\xi_2| |\xi_3| + \xi_1 |\xi_2| |\xi_3| + \xi_1 \xi_2 |\xi_3|}{|\xi_1| + |\xi_2| + |\xi_3|} \Big|_{(\xi_1, \xi_2, \xi_3) = (- (k+k'), k, k')} = -\frac{|\xi_1| |\xi_2| |\xi_3|}{|\xi_1| + |\xi_2| + |\xi_3|} \Big|_{(\xi_1, \xi_2, \xi_3) = (- (k+k'), k, k')},$$

which means that the kernel in (2.59) is useless in our list for decomposing the bilinear operator (2.21). In the same spirit, if we let b_r , resp. \tilde{b}_r , denote the kernel in (2.58), resp. (2.60), we then compute

$$(b_r + \tilde{b}_r)(- (k + k'), k, k') = \frac{4}{1 + r} \frac{|\xi_1| |\xi_2| |\xi_3|}{|\xi_1| + |\xi_2| + |\xi_3|} \Big|_{(\xi_1, \xi_2, \xi_3) = (- (k+k'), k, k')},$$

which means that the kernel in (2.60) is also useless. We now leave it as exercise to the reader to verify that the kernels in (2.61), (2.62), (2.63), when evaluated at $(-k - k', k, k')$, can be written as linear combinations with real coefficients of the expressions in (2.56), (2.57), (2.58).

We summarize our findings in the following Proposition. Let us emphasize that our result is independent of the space dimension $d \geq 2$.

Proposition 2.2. *Consider a hyperelastic isotropic material with Lamé coefficients satisfying $\mu > 0$, $\lambda + \mu > 0$. Then weakly nonlinear Rayleigh waves traveling along the boundary of the half-space $\Omega := \{x \cdot \nu > 0\}$ are governed on the boundary of Ω by the amplitude equation*

$$(2.64) \quad \partial_t w + c_R \frac{\eta}{|\eta|} \cdot \nabla_x w + \mathcal{H}(\mathcal{B}(w, w)) = g,$$

where the bilinear Fourier multiplier \mathcal{B} in the variable θ has the expression

$$\forall k \in \mathbb{R}, \quad \widehat{\mathcal{B}(w, w)}(k) = \int_{\mathbb{R}} \Lambda(k - k', k') \widehat{w}(k - k') \widehat{w}(k') dk',$$

and the kernel Λ is a linear combination with real coefficients of the expressions in (2.56), (2.57), (2.58) evaluated at $(-k - k', k, k')$.

2.4 Well-posedness of the amplitude equation

In this Section, we show a well-posedness result for the amplitude equation (2.20), with the bilinear operator \mathcal{B} defined by (2.21). In the absence of slow spatial variables $y \in \mathbb{R}^{d-1}$, well-posedness results for (2.20) have been obtained in [Hun06] when the variable θ lies in the torus $\mathbb{R}/(2\pi\mathbb{Z})$ and in [Sec15] when the variable θ lies in \mathbb{R} . Namely, in both [Hun06] and [Sec15], well-posedness is proved for an equation of the form

$$\partial_t w + \partial_\theta \mathcal{A}(w, w) = 0,$$

that is obtained from (2.20) by applying a half-derivative in θ (no slow spatial variable $y \in \mathbb{R}^{d-1}$ is considered here). Applying such a half-derivative implies that the bilinear operator \mathcal{A} is defined by means of a kernel a that is homogeneous degree 1/2 while the kernel b in (2.21) is homogeneous degree 2, see [BGC12]. Let us also recall that for piecewise smooth kernels homogeneous degree 0, well-posedness has been proved in [BG09] under a suitable stability condition exhibited in [Hun89]. Our goal here is in some sense to encompass the results from these previous works by considering the original form (2.20) of the amplitude equation and by including the slow spatial variables $y \in \mathbb{R}^{d-1}$.

Proposition 2.3. *Let $\mathbf{v} \in \mathbb{R}^{d-1}$ be a fixed velocity vector, and assume that the kernel b in (2.21) is symmetric with respect to its arguments and that there exists a constant $C > 0$ such that*

$$(2.65) \quad \forall (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \quad |b(\xi_1, \xi_2, \xi_3)| \leq C |\xi_1|^{1/2} |\xi_2|^{1/2} |\xi_3|^{1/2} \min(|\xi_1|^{1/2}, |\xi_2|^{1/2}, |\xi_3|^{1/2}).$$

Then there exists an integer \bar{m} that only depends on the space dimension d such that for all $m \in \mathbb{N}$ with $m \geq \bar{m}$ and for all $R > 0$, there exists a time $T = T(m, R)$ such that if the initial condition $u_0 \in H^m(\mathbb{R}_y^{d-1} \times \mathbb{R}_\theta; \mathbb{R})$ satisfies $\|u_0\|_{H^m} \leq R$, then there exists a unique $u \in \mathcal{C}([0, T]; H^m(\mathbb{R}_y^{d-1} \times \mathbb{R}_\theta; \mathbb{R}))$ solution to the Cauchy problem

$$(2.66) \quad \partial_t u + \sum_{j=1}^{d-1} v_j \partial_j u + \mathcal{H}(\mathcal{B}(u, u)) = 0, \quad u|_{t=0} = u_0.$$

When the initial condition for (2.20) vanishes but the source term g in (2.20) is nonzero, one solves (2.20) by using the Duhamel formula. We omit the details here and focus on the solvability of the pure Cauchy problem for nonzero initial data and zero forcing term. In isotropic elastodynamics, the kernel b whose expression is given in Proposition 2.2 satisfies the bound (2.65), as shown in [Hun06].

Proof. Following the previous works [Hun06, BG09, Sec15], we only show here an a priori estimate for the solutions to the Cauchy problem. By standard arguments, see for instance [Tay11b], a priori estimates can be turned into a well-posedness result as stated in Proposition 2.3 by using convenient Fourier truncation approximations (recall here that the underlying space domain is $\mathbb{R}_y^{d-1} \times \mathbb{R}_\theta$ so Fourier analysis is readily available).

We therefore consider a solution $u \in \mathcal{C}([0, T]; H^m(\mathbb{R}_y^{d-1} \times \mathbb{R}_\theta; \mathbb{R}))$ to the Cauchy problem (2.66) and try to derive an estimate for the evolution of the H^m norm of u . We shall make as if u were sufficiently smooth for all manipulations below to be rigorous. As a matter of fact, using the Fourier expression of the H^m norm, we even only deal with the L^2 norm of the functions u , $\partial_\theta^m u$, $\partial_y^\alpha u$ with $|\alpha| = m$ ($\alpha \in \mathbb{N}^{d-1}$). All other partial derivatives of u can be dealt with by interpolating between such ‘extreme’ cases. Once again, we refer to [Tay11b] for all details on such arguments. Let us first prove the following bounds on the operator \mathcal{B} .

Lemma 2.4. *Under the assumptions of Proposition 2.3, the bilinear operator \mathcal{B} is symmetric. It satisfies the Leibniz rule*

$$\partial_\theta \mathcal{B}(u, v) = \mathcal{B}(\partial_\theta u, v) + \mathcal{B}(u, \partial_\theta v),$$

and more generally the Leibniz rule at any order of differentiation in θ , as well as the bounds¹³

$$\begin{aligned} \forall u, v, w \in \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; \mathbb{R}), \quad & \left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(v, w)) \, dy \, d\theta \right| \leq C \|u\|_{L^2} \|v\|_{H^1} \|w\|_{H^{m_0}}, \\ \forall u, v \in \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; \mathbb{R}), \quad & \left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(u, v)) \, dy \, d\theta \right| \leq C \|v\|_{H^{m_0+1}} \|u\|_{L^2}^2, \end{aligned}$$

for a suitable constant C and any integer m_0 satisfying $m_0 > (d-1)/2 + 2$. (The Sobolev norms refer to the space domain $\mathbb{R}_y^{d-1} \times \mathbb{R}_\theta$.)

Proof. The fact that \mathcal{B} is symmetric comes from the symmetry of the kernel b with respect to its three arguments. We now consider three functions u, v, w in the Schwartz space $\mathcal{S}(\mathbb{R}_y^{d-1} \times \mathbb{R}_\theta; \mathbb{R})$. Below the ‘hat’ notation stands for the partial Fourier transform with respect to $\theta \in \mathbb{R}$. Applying Plancherel’s Theorem, we get¹⁴

$$\int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(v, w)) \, dy \, d\theta = i \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} \overline{\widehat{u}(y, k)} \widehat{v}(y, k - k') \widehat{w}(y, k') \operatorname{sgn}(-k) b(-k, k - k', k') \, dy \, dk \, dk'.$$

We use the bound on b together with the inequality

$$|k|^{1/2} \leq C \left(|k - k'|^{1/2} + |k'|^{1/2} \right),$$

and obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(v, w)) \, dy \, d\theta \right| &\leq C \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} |\widehat{u}(y, k)| |\widehat{v}(y, k - k')| |\widehat{w}(y, k')| \times \\ &\quad \left(|k - k'| |k'|^{1/2} + |k - k'|^{1/2} |k'| \right) \min(|k|^{1/2}, |k - k'|^{1/2}, |k'|^{1/2}) \, dy \, dk \, dk' \\ &\leq C \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} |\widehat{u}(y, k)| |k - k'| |\widehat{v}(y, k - k')| |k'| |\widehat{w}(y, k')| \, dy \, dk \, dk'. \end{aligned}$$

We then apply the classical Young inequality $L^2 * L^1 \rightarrow L^2$, use Plancherel Theorem again and get

$$\begin{aligned} \left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(v, w)) \, dy \, d\theta \right| &\leq C \int_{\mathbb{R}^{d-1}} \|\widehat{u}(y, \cdot)\|_{L^2} \|k \widehat{v}(y, \cdot)\|_{L^2} \|k \widehat{w}(y, \cdot)\|_{L^1} \, dy \\ &\leq C \int_{\mathbb{R}^{d-1}} \|u(y, \cdot)\|_{L^2} \|\partial_\theta v(y, \cdot)\|_{L^2} \|(1 + k^2)^{1/2} k \widehat{w}(y, \cdot)\|_{L^2} \, dy \\ &\leq C \|u\|_{L^2} \|\partial_\theta v\|_{L^2} \sup_{y \in \mathbb{R}^{d-1}} \|w(y, \cdot)\|_{H^2(\mathbb{R})}. \end{aligned}$$

We then apply the Sobolev imbedding Theorem and obtain the first estimate of Lemma 2.4.

¹³The bounds obviously extend by continuity to functions in appropriate Sobolev spaces and are not restricted to functions in the Schwartz class.

¹⁴In the computations below, we do not use the fact that u, v, w are real valued.

Let us now turn to the case $u = v$ where we expect some cancelation arising from the skew-symmetric operator \mathcal{H} . We compute

$$\begin{aligned} \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(u, v)) \, dy \, d\theta &= i \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} \overline{\widehat{u}(y, k)} \widehat{u}(y, k - k') \widehat{v}(y, k') \operatorname{sgn}(-k) b(-k, k - k', k') \, dy \, dk \, dk' \\ &= i \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} \widehat{u}(y, -k) \widehat{u}(y, k - k') \widehat{v}(y, k') \operatorname{sgn}(-k) b(-k, k - k', k') \, dy \, dk \, dk' \\ &= \frac{i}{2} \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} \widehat{u}(y, -k) \widehat{u}(y, k - k') \widehat{v}(y, k') \left(\operatorname{sgn}(-k) + \operatorname{sgn}(k - k') \right) b(-k, k - k', k') \, dy \, dk \, dk', \end{aligned}$$

where we have now used the fact that u is real valued, and the symmetry of b . Let us observe that if $-k$ and $k - k'$ have opposite signs, then the quantity $\operatorname{sgn}(-k) + \operatorname{sgn}(k - k')$ vanishes. If $-k$ and $k - k'$ have the same sign, then the sum of signs is either 2 or -2 , and there holds

$$|k| \leq |k'|, \quad \text{and} \quad |k - k'| \leq |k'|.$$

This yields

$$\begin{aligned} &\left| \left(\operatorname{sgn}(-k) + \operatorname{sgn}(k - k') \right) b(-k, k - k', k') \right| \\ &\leq C \left| \operatorname{sgn}(-k) + \operatorname{sgn}(k - k') \right| |k|^{1/2} |k - k'|^{1/2} |k'|^{1/2} \min(|k|^{1/2}, |k - k'|^{1/2}, |k'|^{1/2}) \leq C |k'|^2. \end{aligned}$$

Using similar inequalities as above (convolution, Cauchy-Schwarz), we can then derive the bound

$$\begin{aligned} \left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(u, v)) \, dy \, d\theta \right| &\leq C \int_{\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}} |\widehat{u}(y, -k)| |\widehat{u}(y, k - k')| |k'|^2 |\widehat{v}(y, k')| \, dy \, dk \, dk' \\ &\leq C \int_{\mathbb{R}^{d-1}} \|\widehat{u}(y, \cdot)\|_{L^2} \|\widehat{u}(y, \cdot)\|_{L^2} \|k^2 \widehat{v}(y, \cdot)\|_{L^1} \, dy \\ &\leq C \|u\|_{L^2}^2 \sup_{y \in \mathbb{R}^{d-1}} \|v(y, \cdot)\|_{H^3(\mathbb{R})}. \end{aligned}$$

Applying Sobolev imbedding Theorem completes the proof of Lemma 2.4. \square

Let us consider an integer $m \geq m_0 + 1$ with m_0 as in Lemma 2.4. We consider a sufficiently smooth solution u to (2.66) and compute (the transport terms with respect to the variables y are harmless here):

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 = -2 \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(\mathcal{B}(u, u)) \, dy \, d\theta.$$

Applying the first bound in Lemma 2.4, we get

$$(2.67) \quad \left| \frac{d}{dt} \|u(t)\|_{L^2}^2 \right| \leq C \|u(t)\|_{L^2} \|u(t)\|_{H^1} \|u(t)\|_{H^{m_0}} \leq C \|u(t)\|_{H^m}^3.$$

Let us now differentiate m times (2.66) with respect to θ , and get

$$\partial_t \partial_\theta^m u + \sum_{j=1}^{d-1} v_j \partial_j \partial_\theta^m u + 2 \mathcal{H}(\mathcal{B}(\partial_\theta^m u, u)) = - \sum_{m'=1}^{m-1} \binom{m}{m'} \mathcal{H}(\mathcal{B}(\partial_\theta^{m'} u, \partial_\theta^{m-m'} u)).$$

Taking the L^2 scalar product with $\partial_\theta^m u$, we get

$$\begin{aligned} \frac{d}{dt} \|\partial_\theta^m u(t)\|_{L^2}^2 &= -4 \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \partial_\theta^m u \mathcal{H}(\mathcal{B}(\partial_\theta^m u, u)) \, dy \, d\theta \\ &\quad - 2 \sum_{m'=1}^{m-1} \binom{m}{m'} \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \partial_\theta^m u \mathcal{H}(\mathcal{B}(\partial_\theta^{m'} u, \partial_\theta^{m-m'} u)) \, dy \, d\theta. \end{aligned}$$

For the first integral, we apply the second estimate of Lemma 2.4 and get

$$\left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \partial_\theta^m u \mathcal{H}(\mathcal{B}(\partial_\theta^m u, u)) \, dy \, d\theta \right| \leq C \|u(t)\|_{H^{m_0+1}} \|u(t)\|_{H^m}^2 \leq C \|u(t)\|_{H^m}^3.$$

For the remaining terms, we apply the first estimate of Lemma 2.4. Assuming without loss of generality $m-1 \geq m' \geq m-m' \geq 1$, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \partial_\theta^m u \mathcal{H}(\mathcal{B}(\partial_\theta^{m'} u, \partial_\theta^{m-m'} u)) \, dy \, d\theta \right| &\leq C \|u(t)\|_{H^m} \|u(t)\|_{H^{m'+1}} \|u(t)\|_{H^{m-m'+m_0}} \\ &\leq C \|u(t)\|_{H^m}^2 \|u(t)\|_{H^{m-m'+m_0}}. \end{aligned}$$

We now choose $m \geq \bar{m}$ with $\bar{m} := 2m_0$, m_0 as in Lemma 2.4. Then $m-m' \leq m/2$ and $m_0 \leq m/2$ so that collecting all previous inequalities we get

$$(2.68) \quad \left| \frac{d}{dt} \|\partial_\theta^m u(t)\|_{L^2}^2 \right| \leq C \|u(t)\|_{H^m}^3.$$

The y -partial derivatives of u are estimated in an entirely similar way, using the Leibniz rule (with respect to y) for the bilinear term $\mathcal{B}(u, u)$. We omit the details, which are entirely similar to what has been done above for the θ m -th derivative. Collecting (2.67), (2.68) and the analogous estimates for the y -derivatives and all cross/intermediate θ, y derivatives, we end up with

$$\left| \frac{d}{dt} \|u(t)\|_{H^m}^2 \right| \leq C \|u(t)\|_{H^m}^3,$$

as long as $m \geq \bar{m} = 2m_0$, and u is a sufficiently smooth solution to (2.66). The latter differential inequality provides with a control of $\|u(t)\|_{H^m}$ in terms of $\|u_0\|_{H^m}$ on a time interval that depends on m (because the above constant C does depend on m). Applying standard regularization procedures as in [Tay11b], we can prove well-posedness for (2.66) in the Sobolev space H^m , $m \geq \bar{m}$, as stated in Proposition 2.3. \square

Chapter 3

Existence of exact solutions

3.1 Introduction

Our main focus in the remaining chapters is to provide rigorous answers to the basic questions of geometric optics for surface *pulses* in isotropic hyperelastic materials. We will describe how the results extend to wavetrains in section 6.2. Although many pieces of the argument work just as well in higher dimensions, there is one piece that requires us to assume $d = 2$ in our main results, Theorems 3.7 and 3.8. At several points our estimates rely on the use of Kreiss symmetrizers and, as we explain in section 6.3, there is a serious difficulty with the construction of Kreiss symmetrizers for linearized elasticity in $d \geq 3$.

We let the unknown $\phi = (\phi_1, \phi_2)(t, x)$ represent the deformation of an isotropic, hyperelastic, Saint Venant-Kirchhoff (SVK) material in the reference configuration $\omega = \{x = (x_1, x_2) : x_2 > 0\}$, which is subjected to a surface force $g = g(t, x)$. Here $\phi(t, \cdot) : \omega \rightarrow \mathbb{R}^2$ and $g(t, \cdot) : \partial\omega \rightarrow \mathbb{R}^2$. The equations are a second-order, quasilinear 2×2 system

$$(3.1) \quad \begin{aligned} \partial_t^2 \phi - \operatorname{Div}(\nabla \phi \sigma(\nabla \phi)) &= 0 \text{ in } x_2 > 0 \\ \nabla \phi \sigma(\nabla \phi) n &= g \text{ on } x_2 = 0, \\ \phi(t, x) &= x \text{ and } g = 0 \text{ in } t \leq 0, \end{aligned}$$

where $n = (-1, 0)$ is the outer unit normal to the boundary of ω , $\nabla \phi = (\partial_{x_j} \phi_i)_{i,j=1,2}$ is the spatial gradient matrix, σ is the stress $\sigma(\nabla \phi) = \lambda \operatorname{tr} E \cdot I + 2\mu E$ with Lamé constants λ and μ strictly positive, and E is the strain $E(\nabla \phi) = \frac{1}{2}({}^t \nabla \phi \cdot \nabla \phi - I)$. Here ${}^t \nabla \phi$ denotes the transpose of $\nabla \phi$, $\operatorname{tr} E$ is the trace of the matrix E , and

$$(3.2) \quad \operatorname{Div} M = \left(\sum_{j=1}^2 \partial_j m_{i,j} \right)_{i=1,2} \text{ for a matrix } M = (m_{i,j})_{i,j=1,2}.$$

We recall that the stored energy function $W(E)$ for an SVK material (2.41) is the leading part, quadratic in E , of the general isotropic hyperelastic energy given by (2.39).¹ In section 6.1 we explain how our results extend to this more general case. It will simplify the exposition to work initially with the SVK problem, which contains all the main difficulties.

¹When rewritten in terms of $\nabla \phi$ or $\nabla U = \nabla \phi - I$, the stored energy function is fourth order in those arguments.

The system (3.1) has the form

$$(3.3) \quad \begin{aligned} \partial_t^2 \phi + \sum_{|\alpha|=2} \mathcal{A}_\alpha(\nabla \phi) \partial_x^\alpha \phi &= 0 \text{ in } x_2 > 0 \\ B(\nabla \phi) = g \text{ or, equivalently, } \partial_{x_2} \phi &= \mathcal{H}(\partial_{x_1} \phi, g) \text{ on } x_2 = 0 \\ \phi(t, x) = x \text{ and } g = 0 \text{ in } t &\leq 0, \end{aligned}$$

where the matrices \mathcal{A}_α are symmetric, and the real functions $\mathcal{A}_\alpha(\cdot)$, $B(\cdot)$, and $\mathcal{H}(\cdot)$ are C^∞ (in fact, analytic) in their arguments. The second form of the boundary condition is obtained by an application of the implicit function theorem to the equation $B(\nabla \phi) = g$, and is valid for $|\nabla \phi - I_{2 \times 2}|_{L^\infty}$ small. Set $x' = (t, x_1)$.

Defining the displacement $U(t, x) = \phi(t, x) - x$, we rewrite (3.3) as

$$(3.4) \quad \begin{aligned} \partial_t^2 U + \sum_{|\alpha|=2} A_\alpha(\nabla U) \partial_x^\alpha U &= 0 \text{ in } x_2 > 0 \\ h(\nabla U) = g \text{ or } \partial_{x_2} U = H(\partial_{x_1} U, g) &\text{ on } x_2 = 0 \\ U(t, x) = 0 \text{ and } g(t, x_1) = 0 \text{ in } t &\leq 0, \end{aligned}$$

where the functions A_α , h , and H are related to \mathcal{A}_α , B , and \mathcal{H} in the obvious way. Here H is defined near $(0, 0)$ and satisfies

$$(3.5) \quad H(0, 0) = 0.$$

We take $g(t, x_1)$ to be an H^s pulse with the weakly nonlinear scaling:

$$(3.6) \quad g = g^\varepsilon(t, x_1) = \varepsilon^2 G \left(t, x_1, \frac{\beta \cdot (t, x_1)}{\varepsilon} \right),$$

where $G(x', \theta) \in H^s(\mathbb{R}^2 \times \mathbb{R})$ for some large enough s to be specified, and $\beta \in \mathbb{R}^2 \setminus 0$ is a frequency in the elliptic region of (the linearization at 0 of) (3.4), chosen so that the uniform Lopatinskii condition fails at β .² We will refer to β as a *Rayleigh frequency*. We expect the response $U^\varepsilon(t, x)$ to be a Rayleigh wave, or rather, a Rayleigh *pulse*, propagating in the boundary.

The main step in constructing the leading term of a geometric optics approximation to the pulse U^ε was given in Proposition 2.3 of Chapter 2. In chapter 4 we complete the construction of the leading term and first corrector, thereby producing an approximate solution to the SVK system. The basic questions arise of whether an exact pulse solution exists on a fixed time interval independent of ε , and, if so, whether the approximate solution is “close” in some sense to the exact solution on such a time interval. These questions are answered in Theorems 3.7 and 3.8. In this chapter we resolve the first question in the affirmative and prove Theorem 3.7. In chapter 5 we complete the proof of Theorem 3.8, which shows that the approximate solution is close in a precise sense to the exact solution for small ε .³

We look for $U^\varepsilon(t, x)$ in the form

$$(3.7) \quad U^\varepsilon(t, x) = u^\varepsilon(t, x, \theta) \Big|_{\theta = \frac{\beta \cdot (t, x_1)}{\varepsilon}},$$

²See definition 3.4. We may take $\beta = (\pm \tau_r(\eta), \eta)$, for $\tau_r(\eta)$ as in (H3) of chapter 2 and $\eta = \mathbf{1}$.

³We note that for a fixed ε the existence of U^ε on a time interval $(-\infty, T_\varepsilon]$ that may depend on ε follows from the main result of [ST88].

where $u^\varepsilon(t, x, \theta)$ satisfies the *singular system* obtained from (3.4) by plugging in the ansatz (3.7):

$$(3.8) \quad \begin{aligned} \partial_{t,\varepsilon}^2 u^\varepsilon + \sum_{|\alpha|=2} A_\alpha(\nabla_\varepsilon u^\varepsilon) \partial_{x,\varepsilon}^\alpha u^\varepsilon &= 0 \text{ in } \{(t, x, \theta) : x_2 > 0\} \\ \partial_{x_2} u^\varepsilon &= H(\partial_{x_1,\varepsilon} u^\varepsilon, \varepsilon^2 G(x', \theta)) \text{ on } x_2 = 0 \\ u^\varepsilon(t, x) &= 0 \text{ and } G(x', \theta) = 0 \text{ in } t \leq 0. \end{aligned}$$

With $\beta = (\beta_0, \beta_1)$ and $\alpha = (\alpha_1, \alpha_2)$ the notation in (3.8) is:

$$(3.9) \quad \begin{aligned} \partial_{t,\varepsilon} &= \partial_t + \frac{\beta_0}{\varepsilon} \partial_\theta, \quad \partial_{x_1,\varepsilon} = \partial_{x_1} + \frac{\beta_1}{\varepsilon} \partial_\theta \\ \nabla_\varepsilon &= (\partial_{x_1,\varepsilon}, \partial_{x_2}), \quad \partial_{x,\varepsilon}^\alpha = \partial_{x_1,\varepsilon}^{\alpha_1} \partial_{x_2}^{\alpha_2}. \end{aligned}$$

The main effort of this paper is devoted to proving existence of a solution to the singular system (3.8) on a fixed time interval independent of ε . We refer to the system as “singular” not only because of the factors of $\frac{1}{\varepsilon}$ that appear, but also because $\partial_{x'}$ and ∂_θ derivatives occur in the linear combinations (3.9). To study such systems we require pseudodifferential operators that are singular in the same sense; a calculus of such operators is described in the appendix.

Using an idea employed by [ST88] in the nonsingular case, we set $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$, formally make the substitutions

$$(3.10) \quad v_1^\varepsilon = \partial_{x_1,\varepsilon} u^\varepsilon \text{ and } v_2^\varepsilon = \partial_{x_2} u^\varepsilon,$$

in the system obtained by differentiating the interior and boundary equations of (3.8) with respect to $\partial_{x_1,\varepsilon}$:

$$(3.11) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_1^\varepsilon + \sum_{|\alpha|=2} A_\alpha(v^\varepsilon) \partial_{x,\varepsilon}^\alpha v_1^\varepsilon &= - \sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_1,\varepsilon} (A_\alpha(v^\varepsilon)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_1^\varepsilon - \partial_{x_1,\varepsilon} (A_{(0,2)}(v^\varepsilon)) \partial_{x_2} v_2^\varepsilon \\ \partial_{x_2} v_1^\varepsilon - d_{v_1} H(v_1^\varepsilon, h(v^\varepsilon)) \partial_{x_1,\varepsilon} v_1^\varepsilon &= d_g H(v_1^\varepsilon, \varepsilon^2 G) \partial_{x_1,\varepsilon} (\varepsilon^2 G) \text{ on } x_2 = 0. \end{aligned}$$

Similarly, differentiating the interior equation of (3.8) with respect to x_2 and making the same substitutions we obtain

$$(3.12) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_2^\varepsilon + \sum_{|\alpha|=2} A_\alpha(v^\varepsilon) \partial_{x,\varepsilon}^\alpha v_2^\varepsilon &= - \sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_2} (A_\alpha(v^\varepsilon)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_1^\varepsilon - \partial_{x_2} (A_{(0,2)}(v^\varepsilon)) \partial_{x_2} v_2^\varepsilon \\ v_2^\varepsilon &= H(v_1^\varepsilon, \varepsilon^2 G) \text{ on } x_2 = 0. \end{aligned}$$

Equations (3.11) and (3.12) are a coupled nonlinear system for the unknown $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$. Once v^ε is determined, one can solve the following linear system for u^ε . By Remark 3.1 this system is equivalent to (3.8):

$$(3.13) \quad \begin{aligned} \partial_{t,\varepsilon}^2 u^\varepsilon + \sum_{|\alpha|=2} A_\alpha(v^\varepsilon) \partial_{x,\varepsilon}^\alpha u^\varepsilon &= 0 \\ \partial_{x_2} u^\varepsilon - d_{v_1} H(v_1^\varepsilon, h(v^\varepsilon)) \partial_{x_1,\varepsilon} u^\varepsilon &= H(v_1^\varepsilon, \varepsilon^2 G(x', \theta)) - d_{v_1} H(v_1^\varepsilon, \varepsilon^2 G) v_1^\varepsilon \text{ on } x_2 = 0. \end{aligned}$$

Recall that $u^\varepsilon, v^\varepsilon$ and G all vanish in $t < 0$.

Remark 3.1. *We show in section 3.6 that for each fixed $\varepsilon \in (0, 1]$, the problems (3.11), (3.12), and (3.13) have solutions on a time interval $(-\infty, T_\varepsilon]$ that may depend on ε . Moreover, as a consequence of Proposition 3.83, we have $v_1^\varepsilon = \partial_{x_1,\varepsilon} u^\varepsilon$ and $v_2^\varepsilon = \partial_{x_2} u^\varepsilon$ on $(-\infty, T_\varepsilon]$. The work in sections 3.2 to 3.5 will show that T_ε can be chosen independent of ε .*

3.1.1 Assumptions

We make only one assumption, namely, that we are considering the equations of nonlinear elasticity with “traction” boundary conditions for a Saint Venant-Kirchhoff material on a 2D half-space. More precisely, we assume:

(A1) We study the equations (3.4) with Lamé constants satisfying $\mu > 0$, $\lambda + \mu > 0$. The interior and boundary operators in these equations are the same as the operators appearing in (2.1), (2.2), where $N = d = 2$ and $W(\nabla u)$ is given by (2.40) for the Saint Venant-Kirchhoff stored energy $W(E)$ as in (2.41).

In section (6.1) we will replace assumption (A1) by the more general assumption:

(A1g) We study the equations (3.4) with Lamé constants satisfying $\mu > 0$, $\lambda + \mu > 0$. The interior and boundary operators in these equations are the operators appearing in (2.1), (2.2), where $N = d = 2$ and $W(\nabla u)$ is given by (2.40) for $W(E)$ as in (2.39), the general isotropic hyperelastic stored energy. Moreover, we assume that $W(E)$ is an analytic function.⁴

Remark 3.2. *The discussion in section 2.3 shows that under either of these assumptions, the hypotheses (H1), (H2), (H3) of chapter 2 are satisfied.*

We recall from (3.4) that the system satisfied by the displacement $U(t, x) = \phi(t, x) - x$ is:

$$(3.14) \quad \begin{aligned} \partial_t^2 U + \sum_{|\alpha|=2} A_\alpha(\nabla U) \partial_x^\alpha U &= 0 \text{ in } x_2 > 0 \\ \partial_{x_2} U &= H(\partial_{x_1} U, g) \text{ on } x_2 = 0 \\ U(t, x) &= 0 \text{ and } g(t, x_1) = 0 \text{ in } t \leq 0, \end{aligned}$$

The “background state” is $\nabla U = 0$ and corresponds to the identity deformation $\phi(t, x) = x$. Let us write the arguments of H as (v_1, g) . The linearized system at the background state is (P^0, B^0) , where

$$(3.15) \quad \begin{aligned} P^0 &= \partial_t^2 + \sum_{|\alpha|=2} A_\alpha(0) \partial_x^\alpha \\ B^0 &= \partial_{x_2} - d_{v_1} H(0, 0) \partial_{x_1}. \end{aligned}$$

The perturbed linearized system is, with $h(v)$ as in (3.4),

$$(3.16) \quad \begin{aligned} P^v u &= \partial_t^2 u + \sum_{|\alpha|=2} A_\alpha(v) \partial_x^\alpha u = f \\ B^v u &= \partial_{x_2} u - d_{v_1} H(v_1, h(v)) \partial_{x_1} u = g. \end{aligned}$$

Remark 3.3. *For (v, G) near $(0, 0)$ we have $h(v_1, v_2) = G \Leftrightarrow v_2 = H(v_1, G)$. The linearized boundary operator at v can be written*

$$(3.17) \quad h_{v_1}(v) \partial_{x_1} + h_{v_2}(v) \partial_{x_2}$$

By the chain rule applied to $h(v_1, H(v_1, G)) = G$ we have

$$(3.18) \quad (h_{v_2}(v))^{-1} h_{v_1}(v) = -H_{v_1}(v_1, h(v)),$$

which explains the form of B^μ .

⁴We consider only displacements with ∇u small, so it is enough to assume that $W(E)$ is analytic near $E = 0$.

Lopatinskii determinant.⁵ Below we let (σ, ξ_1, ξ_2) denote real variables dual to (t, x_1, x_2) , set $\tau = \sigma - i\gamma$ for $\gamma \geq 0$, and let $\Lambda_{\mathcal{D}}$ denote the (nonsingular) operator defined by the Fourier multiplier $\Lambda(\sigma, \xi_1, \gamma) := (\sigma^2 + \xi_1^2 + \gamma^2)^{1/2}$.

In order to define the Lopatinskii determinant we set $U = (e^{\gamma t} \Lambda_{\mathcal{D}} e^{-\gamma t} u, D_{x_2} u) = (\Lambda_{\mathcal{D}, \gamma} u, D_{x_2} u)$, $U^\gamma = e^{-\gamma t} U$, and rewrite (3.16) as a 4×4 first-order system:

$$(3.19) \quad D_{x_2} U^\gamma - \mathcal{A}(v, \mathcal{D}) U^\gamma = \tilde{f}^\gamma, \quad \mathcal{B}(v, \mathcal{D}) U^\gamma = \tilde{g}^\gamma,$$

where, with the matrices A_α evaluated at v , we have

$$(3.20) \quad \begin{aligned} \tilde{f} &:= \begin{pmatrix} 0 \\ -A_{(0,2)}^{-1} f \end{pmatrix}, \quad \tilde{g} = -ig \\ \mathcal{A}(v, \mathcal{D}) &= - \begin{pmatrix} 0 & \Lambda_{\mathcal{D}} I_2 \\ \left(A_{(0,2)}^{-1} (D_t - i\gamma)^2 + A_{(0,2)}^{-1} A_{(2,0)} D_{x_1}^2 \right) \Lambda_{\mathcal{D}}^{-1} & A_{(0,2)}^{-1} A_{(1,1)} D_{x_1} \end{pmatrix} \\ \mathcal{B}(v, \mathcal{D}) &= (-d_{v_1} H(v_1, h(v)) D_{x_1} \Lambda_{\mathcal{D}}^{-1} \quad I_2). \end{aligned}$$

Denoting the matrix symbols of $\mathcal{A}(v, \mathcal{D})$ and $\mathcal{B}(v, \mathcal{D})$ by $\mathcal{A}(v, \sigma, \xi_1, \gamma)$, $\mathcal{B}(v, \sigma, \xi_1, \gamma)$ and setting $z = (\sigma, \xi_1, \gamma)$, when $\gamma > 0$ we let $E^+(v, z)$ denote the direct sum of the generalized eigenspaces of $\mathcal{A}(v, z)$ associated to the eigenvalues with positive imaginary part. This two-dimensional space has a continuous extension $E^+(\sigma, \xi_1, 0)$ to points $(\sigma, \xi_1, 0) \neq 0$.

Definition 3.4. Let $\mathcal{X} := \{z = (\sigma, \xi_1, \gamma) \neq 0 : \gamma \geq 0\}$.

1) The operators $(\mathcal{A}(v, \mathcal{D}), \mathcal{B}(v, \mathcal{D}))$ (or (P^v, B^v)) satisfy the Lopatinskii condition at (v, z) provided

$$(3.21) \quad \mathcal{B}(v, z) : E^+(v, z) \rightarrow \mathbb{C}^2$$

is an isomorphism.

2) One can locally choose a basis $\{r_1(v, z), r_2(v, z)\}$ for $E^+(v, z)$ that is C^∞ in $\gamma > 0$ and which extends continuously to $\gamma = 0$. We define the (2×2) Lopatinskii determinant

$$(3.22) \quad D(v, z) = \det(\mathcal{B}(v, z) r_1(v, z), \mathcal{B}(v, z) r_2(v, z)) := \det b^+(v, z).$$

Clearly, $D(v, z) \neq 0$ if and only if the Lopatinskii condition holds at (v, z) .

3) Let $\Delta(v, z, \xi_2)$ be the determinant of the principal symbol of $D_{x_2} - \mathcal{A}(v, \mathcal{D})$. The elliptic region at v is the set of $(\sigma, \xi_1, 0) \in \mathcal{X}$ such that $\Delta(v, \sigma, \xi_1, 0, \xi_2)$ has no real roots in ξ_2 .

Here we list the properties of (P^v, B^v) that are important for proving energy estimates and existence theorems.

(P1) The operator P^0 is strictly hyperbolic.

(P2) The boundary $x_2 = 0$ is noncharacteristic for P^0 .

(P3) (LU) The Lopatinskii determinant $D(0, \sigma, \xi_1, \gamma)$ for (P^0, B^0) is nonvanishing at all points (σ, ξ_1, γ) with $\gamma > 0$ and at all points $(\sigma, \xi_1, 0) \in \mathcal{X}$ that lie outside the elliptic region at $v = 0$.

(P4) (LC) If $\underline{z} = (\underline{\sigma}, \underline{\xi}_1, 0) \neq 0$ is such that $D(0, \underline{z}) = 0$, then $\partial_\sigma D(0, \underline{z}) \neq 0$.

⁵The remainder of this section discusses some consequences of Assumption (A1), and is somewhat more technical. It may be read after the statement of the main results, section 3.1.2, and after the survey of the proofs, section 3.1.3.

(P5) (R_v) Suppose $D(0, \underline{z}) = 0$. There exists a $\mu_0 > 0$ such that if $|v| < \mu_0$ the following holds: there exist 2×2 matrices $k_i(v, z)$, $i = 1, 2$, defined in a conic neighborhood Γ of \underline{z} , which are C^∞ , homogeneous of degree zero in z , and elliptic, such that for $z \in \Gamma$

$$(3.23) \quad b^+(v, z) = k_1(v, z) \begin{pmatrix} q(v, z)\Lambda^{-1}(z) & 0 \\ 0 & 1 \end{pmatrix} k_2(v, z), \text{ where } q(v, \sigma, \xi_1, \gamma) = \sigma - i\gamma - \nu(v, \xi_1).$$

Here, $\nu(v, \xi_1)$ is a *real*-valued C^∞ function homogeneous of degree one in ξ_1 ; we note that $\xi_1 \neq 0$.

The above five properties, which are verified in [S-T], (pp. 268-270 and p. 283), are sufficient for proving the a priori estimates of section 3.2. These properties allow us to cover the half sphere $\Sigma = \{z = (\sigma, \xi_1, \gamma) \in \mathcal{X} : |z|^2 = 1\}$ by a finite number of open sets \mathcal{O}_j , $j \in J_a \cup J_c$ such that for each $j \in J_a$, $D(0, z)$ is bounded away from 0 in \mathcal{O}_j , and for each $j \in J_c$, $D(0, z) = 0$ at some point of \mathcal{O}_j . Moreover, we can choose the \mathcal{O}_j such that if $\mu_1 > 0$ is small enough and $|v| \leq \mu_1$, we have:

(i) $D(v, z)$ is bounded away from zero for $z \in \mathcal{O}_j$, $j \in J_a$, while for $j \in J_c$, we have $D(0, \underline{z}) = 0$ for some $\underline{z} \in \mathcal{O}_j$ and $b^+(v, z)$ satisfies (3.23) for $z \in \mathcal{O}_j$.

(ii) We can write $J_a = J_h \cup J_e$, where J_h denotes the set of indices such that $\Delta(0, z, \xi_2)$ has at least one real root ξ_2 for some $z \in \mathcal{O}_j$, while $\Delta(v, z, \xi_2)$ has no real roots for $j \in J_e$, $z \in \mathcal{O}_j$.

(iii) For $j \in J_c$, $z \in \mathcal{O}_j$ the symbol $\Delta(v, z, \xi_2)$ has no real roots.

Remark 3.5. 1. Let $\mathcal{A}(v, z)$ be the symbol of the operator $\mathcal{A}(v, \mathcal{D})$ in (3.20). The operator P^0 is strictly hyperbolic; consequently, for $|v|$ small the system (3.19) can be conjugated microlocally to the form called “block structure” in the sense of Kreiss-Majda.⁶

In particular, for $|v|$ small there exists for each $\underline{z} \in \mathcal{X}$ a C^∞ invertible matrix $S(v, z)$, homogeneous of degree zero and defined in a conic neighborhood of \underline{z} , such that

$$(3.24) \quad S^{-1}\mathcal{A}S = \text{diag}(a_1, \dots, a_k, a^+, a^-),$$

where the blocks a^\pm satisfy $\pm \text{Im } a^\pm > 0$, and the blocks a_j have Jordan form at $(0, \underline{z})$ with the single real eigenvalue λ_j . We refer to Chapter 7 of [CP82] for the full definition.

2. If \underline{z} lies in the elliptic region at $v = 0$, then for $|v|$ small and z in a conic neighborhood of \underline{z} the 4×4 matrix $S(v, z) = [S^+ \ S^-]$ is chosen so that the 2 columns of $S^+(v, z)$ give a basis of $E^+(v, z)$. We define $b^\pm(v, z) = \mathcal{B}(v, z)S^\pm(v, z)$; thus, in this case the vectors r_1, r_2 in (3.22) satisfy $[r_1, r_2] = S^+$.

We let $\{\phi_j(\sigma, \xi_1, \gamma), j \in J_e \cup J_h \cup J_c\}$ be a partition of unity subordinate to the covering $\{\mathcal{O}_j\}$, consisting of symbols homogeneous of degree zero. The neighborhoods \mathcal{O}_j may be (and are) chosen so that each is a neighborhood in which some matrix $S(v, z)$ as above defined. In particular, we have

$$(3.25) \quad S^{-1}\mathcal{A}S = \text{diag}(a^+, a^-) \text{ for } |v| < \mu_1, z \in \mathcal{O}_j, \text{ when } j \in J_e \cup J_c.$$

For each S we will also denote by S an elliptic extension of S to all of \mathcal{X} .

The following additional properties, which are verified in [S-T], p. 286, are used in proving the existence theorems:⁷

⁶When $d \geq 3$, the operator P^0 fails to be strictly hyperbolic, and for v small P^v exhibits characteristics of variable multiplicity. As shown in [MZ05], smooth Kreiss symmetrizers can sometimes be constructed in such situations. Unfortunately, Proposition 6.2 shows that the results of [MZ05] do not apply to (P^v, B^v) when $d \geq 3$.

⁷The labels F.G. (“formule de Green”) and A.A. (“auto-adjoint”), like LU (“Lopatinskii uniform”) and LC (“controle de Lopatinskii”) above, are borrowed from [ST88] and used here to facilitate comparison with that paper.

(P6) (F.G.) : For $|v|$ small enough, (P^v, B^v) satisfies the following Green's formula: there exist differential operators \tilde{B} , A , \tilde{A} such that for all u, v in $C_0^\infty(\overline{\mathbb{R}_+^3})$ (we drop the superscript v here):

$$(3.26) \quad (Pu, w)_{L^2(t,x)} = (u, P^*w)_{L^2(t,x)} + \langle Bu(0), \tilde{A}w(0) \rangle_{L^2(t,x_1)} + \langle Au(0), \tilde{B}w(0) \rangle_{L(t,x_1)}.$$

The operator \tilde{B} has order 1, while A and \tilde{A} have order 0.

(P7) (A.A.) The problem (P^v, B^v) satisfies (F.G.) and the operators P^* and \tilde{B} have the same principal parts as P^v and B^v , respectively.

Remark 3.6. 1) An essential consequence of (P5) is that the Lopatinski determinant, when computed at states $v \neq 0$ but near 0, can only vanish when $\gamma = 0$. In other words for small enough perturbations v the Lopatinski determinant of the perturbed operators (P^v, B^v) , like that of (P^0, B^0) , vanishes only at points where $\gamma = 0$. In a weakly stable problem, where one has $D(0, \underline{z}) = 0$ for some \underline{z} with $\gamma = 0$, even if one knows that $D(0, z)$ can only vanish when $\gamma = 0$, for the nonlinear theory one still has to rule out the possibility that $D(v, z) = 0$ for some z with $\gamma > 0$ (an exponentially growing mode) for states v near 0. That is accomplished by verifying (P5) in this case.

Another consequence explained below is that (R_v) allows us to avoid the need for a sharp Garding inequality in the singular calculus.⁸

2) Caution: Glancing points as well as points in the hyperbolic and mixed hyperbolic-elliptic regions lie in the support of ϕ_j for some $j \in J_h$.

The paper [ST88] verified properties (P1)-(P7) for the SVK system with $\lambda > 0$, $\mu > 0$; in fact that verification works just as well when $\mu > 0$, $\lambda + \mu > 0$. In section 6.1 we explain why (P1)-(P7) also hold under the more general assumption (A1g).

3.1.2 Main results

Our first main result asserts the existence of exact solutions to the coupled singular problems (3.11)-(3.13) on a fixed time interval independent of ε . An immediate consequence is the existence of exact solutions to the original (nonsingular) problem (3.4) on such a time interval.

Theorem 3.7. (a) Assume (A1) and let β as in (3.6) be a Rayleigh frequency. Suppose $m > 3d + 4 + \frac{d+1}{2}$ and consider the coupled singular problems (3.11), (3.12), (3.13), where $G(x', \theta) \in H^{m+3}(b\Omega)$ and vanishes in $t < 0$.⁹ There exist positive constants ε_0 and T_1 and unique C^2 solutions $v^\varepsilon(t, x, \theta) = (v_1^\varepsilon, v_2^\varepsilon)$ and $u^\varepsilon(t, x, \theta)$ to the coupled problems on the time interval $[0, T_1]$ for $\varepsilon \in (0, \varepsilon_0]$; the constant T_1 is independent of $\varepsilon \in (0, \varepsilon_0]$. Moreover, we have $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_{T_1} .

(b) Consequently, for $\varepsilon \in (0, \varepsilon_0]$ the function u^ε gives the unique C^2 solution to the system (3.8) on the time interval $[-\infty, T_1]$, and

$$(3.27) \quad U^\varepsilon(t, x) = u^\varepsilon(t, x, \theta) \Big|_{\theta = \frac{\beta \cdot (t, x_1)}{\varepsilon}}$$

gives the unique C^2 Rayleigh pulse solution to the Saint Venant-Kirchhoff system (3.4) with boundary data

$$(3.28) \quad g = g^\varepsilon(t, x_1) = \varepsilon^2 G \left(t, x_1, \frac{\beta \cdot (t, x_1)}{\varepsilon} \right).$$

(c) The $E_{m, T_1}(v^\varepsilon)$ norm is uniformly bounded for $\varepsilon \in (0, \varepsilon_0]$; this norm is defined in (3.77).

⁸This is fortunate, since we have no such inequality.

⁹The choice $m > 3d + 4 + \frac{d+1}{2}$ is determined by the requirements of the singular calculus, Appendix A.

The main steps in the proof of Theorem 3.7 are Propositions 3.16 and 3.20 (which give a priori estimates uniform with respect to ε) for the singular systems (3.11)-(3.13), and Propositions 3.83 and 3.87, which are local existence and continuation theorems for a fixed ε .

The main technical tools used in the proof are the calculus of singular pseudodifferential operators for pulses constructed in [CGW14b] and summarized in Appendix A, and the new estimates of singular norms of nonlinear functions (including microlocal ‘‘Rauch-lemma’’-type estimates for singular norms) given in section 3.4. Section A.5 gives several commutator estimates that extend the results of [CGW14b].

Our second main result gives a precise sense in which the approximate solution constructed in Chapter 4 is close to the exact solution of Theorem 3.7, together with a rate of convergence. The proof of Theorem 3.8 depends on the results of chapters 3 and 4 and is completed in chapter 5.

Theorem 3.8. *We make the same assumptions as in Theorem 3.7, let $\delta > 0$, and let u^ε , $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ be as in Theorem 3.7. Let*

$$u_a^\varepsilon(t, x, \theta) = \varepsilon^2 u_\sigma^\varepsilon(t, x, \theta) + \varepsilon^3 u_\tau^\varepsilon(t, x, \theta)$$

be the approximate solution to the singular system (3.8) given in (5.1) and let $v_a^\varepsilon = \nabla_\varepsilon u_a^\varepsilon$. There exist positive constants $\varepsilon_1 \leq \varepsilon_0$ and $T_2 \leq T_1$ such that for $\varepsilon \in (0, \varepsilon_1]$ we have on Ω_{T_2} ,

$$(3.29) \quad E_{m, T_2}(v^\varepsilon - v_a^\varepsilon) \leq C_\delta \varepsilon^{\frac{1}{4} - \delta}.$$

The norm E_{m, T_2} (3.77) is a sum of 18 terms. As a corollary of Theorem 3.8 and the estimates of u_τ^ε given in section 5.2 we obtain, for example, the following result for the exact solution $U^\varepsilon(t, x)$ to the original problem (3.4).

Corollary 3.9. *Let $\delta > 0$ and set*

$$U^\varepsilon(t, x) = u^\varepsilon(t, x, \theta)|_{\theta = \frac{\beta \cdot (t, x_1)}{\varepsilon}} \quad \text{and} \quad U_\sigma^\varepsilon(t, x) = \varepsilon^2 u_\sigma^\varepsilon(t, x, \theta)|_{\theta = \frac{\beta \cdot (t, x_1)}{\varepsilon}}.$$

There exist positive constants $\varepsilon_1 \leq \varepsilon_0$ and $T_2 \leq T_1$ such that for $\varepsilon \in (0, \varepsilon_1]$, we have on $[-\infty, T_2] \times \mathbb{R}_+^2$:

$$(3.30) \quad \begin{aligned} (a) & \frac{1}{\varepsilon} |\nabla_{t, x}(U^\varepsilon(t, x) - \varepsilon^2 U_\sigma^\varepsilon(t, x))|_{L^\infty} \leq C_\delta \varepsilon^{\frac{1}{4} - \delta} \quad \text{and} \\ (b) & |\partial_{t, x}^\alpha (U^\varepsilon(t, x) - \varepsilon^2 U_\sigma^\varepsilon(t, x))|_{L^\infty} \leq C_\delta \varepsilon^{\frac{1}{4} - \delta}, \quad \text{where } \alpha = (\alpha_t, \alpha_{x_1}, \alpha_{x_2}) \text{ with } |\alpha| = 2, \alpha_t \leq 1. \end{aligned}$$

Proof of Corollary. The second estimate follows from (3.29) and the presence of the terms $\left| \begin{pmatrix} \Lambda v_j \\ D_{x_2} v_j \end{pmatrix} \right|_{\infty, m, T_2}^{(2)}$, $j = 1, 2$ in the definition of the E_{m, T_2} norm (3.78). Similarly, the first estimate follows from the presence of the term $\left| \frac{\nabla_\varepsilon u}{\varepsilon} \right|_{\infty, m, T_2}$ in the definition of the E_{m, T_2} norm, together with Remark 3.41. \square

3.1.3 Survey of the proofs.

In the setting of the original nonsingular problem (3.4) with non-oscillating data, Sable-Tougeron [ST88] has shown that if sufficiently precise estimates can be obtained for the linearized problems corresponding to (3.11)-(3.13), then the nonlinear problems can be solved by a standard fixed point iteration applied to (the nonsingular analogues of) the coupled problems (3.11)-(3.13), even though the linearized estimates for the v_1 problem (3.11) exhibit a loss of one derivative on the boundary.¹⁰ Roughly, the necessary

¹⁰With some abuse we shall refer to estimates for the linearized problems as ‘‘the linearized estimates’’.

precision is gained by taking advantage of the fact that the uniform Lopatinskii condition fails for (3.11) in the *elliptic* region to derive estimates that minimize the loss for pieces of the solution microlocalized to that region, and which exhibit no loss for pieces microlocalized away from the bad set. At the same time this argument takes advantage of two gains that derive from considering coupled differentiated problems like (3.11)-(3.13): the fact that the coefficients now depend on the unknown v itself rather than the gradient of the unknown as in (3.8), and the fact that the problem for v_2 (3.12) is a Dirichlet problem, so the associated linearized problem satisfies the uniform Lopatinskii condition and solutions exhibit no loss of derivatives. In our study of the singular problems in this chapter we shall make use of all these ideas. Naturally, new difficulties arise due to the singular nature of the problems, and our purpose here is to summarize what is new in our approach to dealing with these.

The first step, carried out in section 3.2, is to obtain the basic L^2 estimates that are satisfied by solutions of the linearized singular systems (3.34)-(3.35). This is done in Proposition 3.13, whose proof is a fairly straightforward adaptation to the singular setting of an analogous result of [ST88]. To see the loss on the boundary in the estimates (3.47) for the Neumann-type problem (3.34) relative to the estimates (3.48) for the Dirichlet problem (3.35), one must recall that \mathcal{G}_1 is the forcing term in a first-order boundary condition, while \mathcal{G}_2 is the forcing term in a zero-order boundary condition. Thus, there is a loss of one *singular* derivative (Λ_D) on the boundary in the estimate for v_1 relative to the corresponding estimate for v_2 .¹¹

The next step is to obtain the “slightly higher derivative” estimates of Proposition 3.11. A serious obstacle arises in the proof of (3.38) and (3.41), where one seeks to control the $L^\infty(x_2, L^2(t, x_1, \theta))$ norm uniformly with respect to ε . Such control is essential for later getting $L^\infty(t, x)$ bounds. In a nonsingular problem one would use the equation and the fact that the boundary is noncharacteristic to control the L^2 norm of one normal derivative $\partial_{x_2}v$ using prior L^2 control of first-order tangential derivatives $\partial_{(t, x_1, \theta)}v$. In a singular problem the factors of $\frac{1}{\varepsilon}$ in the singular tangential derivatives $\partial_{t, \varepsilon}$ and $\partial_{x_1, \varepsilon}$ wreck this argument. The resolution is to use a singular operator in the extended calculus of section A.4 to partition the solution into one piece with essential support in the region $|\sigma, \xi_1, \gamma| \geq \delta|k/\varepsilon|$ ¹² where the $\frac{1}{\varepsilon}$ factors are harmless and the above argument works, and another piece concentrated in the complementary region where $(\sigma, \xi_1) + \frac{\beta k}{\varepsilon}$ is nearly parallel to β . For the second piece one can use that fact that β lies in the elliptic region to block-diagonalize the singular system and control the $L^\infty(x_2, L^2(t, x_1, \theta))$ norm by an energy estimate. This kind of difficulty always arises in singular boundary problems (see [Wil02, CGW14a]), but this is the first time we have had to deal with it in a quasilinear weakly stable problem.

Another difficulty peculiar to singular problems occurs in the proof of the estimate (3.37), which one might hope to prove simply by commuting the singular operator $\Lambda_D^{\frac{1}{2}}$ through the linearized v_1 problem (3.35) and applying the estimate (3.47). It turns out that such an argument can be used only on certain microlocalized pieces of the solution, while other pieces require a separate new energy estimate. The two cases are distinguished by the size of the commutator errors that arise; those in the first case are controllable uniformly with respect to ε (and they appear as the final term on the right in (3.37)), while those in the second case are not. This analysis required a further development of the singular calculus of [CGW14b] that is given in section A.5. The points discussed in this and the above paragraph are treated mainly in steps 2 and 4 of the proof of Proposition 3.11. The estimates of Proposition 3.11 provide the foundation for Chapters 3 and 5.

These estimates introduce a technical problem into subsequent arguments that was already encoun-

¹¹The singular operator Λ_D and the singular norms that appear in the estimates of this chapter are defined in part (g) of Notations 3.10.

¹²Here k is the Fourier variable dual to θ and $\delta > 0$ is sufficiently small.

tered by [ST88] in the nonsingular setting. The presence of the microlocalized norms, that is the terms involving the singular operators $\phi_{j,D}$, forces us always to apply the linearized estimates to problems posed on the *full* half-space Ω , because there is no way to time-localize these norms. However, the other norms that appear in the linearized estimates, even the singular norms involving fractional powers of Λ_D , can be time-localized by interpolation, and because of that we are ultimately able to use the global estimates on Ω to prove a short-time nonlinear existence theorem.

In the proof of short-time existence for the nonlinear, nonsingular problem (3.4) with non-oscillating data, [ST88] used a clever iteration scheme to solve (the nonsingular analogues of) the coupled problems (3.11)-(3.13). Calling the iterates $v^k = (v_1^k, v_2^k)$ and u^k , Sablé-Tougeron found that the most difficult step was to get uniform (with respect to k) bounds for the v^k ; for this she used her precise estimates and the other “gains” described in the first paragraph of this section.¹³ It was possible to control the v^k on a fixed time interval by working just with the two equations (3.11), (3.12) and ignoring for the moment (3.13). Once the v^k were uniformly controlled, uniform bounds for the u^k were readily obtained, and the existence of limits v and u satisfying $v = \nabla u$ followed by relatively straightforward arguments.

We tried at first to prove the uniform (with respect to ε) existence theorem for the nonlinear singular systems, Theorem 3.7, by applying the iteration scheme of [ST88] to the coupled problems (3.11)-(3.13), and attempting to estimate the v^k , which now depend on ε , uniformly with respect to *both* k and ε , on a time interval independent of ε . The iteration scheme is the one given in (3.256)-(3.258); the problem for v^{k+1} is linear so one can apply the estimates of Proposition 3.11 to estimate v^{k+1} provided one has the control of v^k specified by (3.36), where v^k plays the role of w in (3.36). This approach failed. In particular, we were not able to bound the norm $|v^k/\varepsilon|_{\infty,T}$ of the k th iterate uniformly with respect to ε and k up to a fixed time T independent of ε . If one knew $v^k = \nabla_\varepsilon u^k$ (or could control the difference well enough), then this approach could have been made to work, but we had to abandon this idea. It seems that the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$, which allows one to use the u^ε equation (3.13) to control $|v^\varepsilon/\varepsilon|_{\infty,T}$, holds only in the limit $k \rightarrow \infty$.

In order to take advantage of the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ we use instead a continuous induction argument. This argument, given in the proof of Proposition 3.16, has three parts. First, one needs a *short-time existence theorem* for ε fixed for the nonlinear problems (3.11)-(3.13) on a domain Ω_{T_ε} that depends on ε ; this is given by Proposition 3.83. This result also establishes the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_{T_ε} .¹⁴ Second, one needs a *continuation theorem* for ε fixed, Proposition 3.87, which states that if one has a solution to (3.11)-(3.13) on some domain $\Omega_{T_{1,\varepsilon}}$ that is sufficiently small in an appropriate norm on that domain, then that solution can be extended with similar bounds to a later time $T_{2,\varepsilon} > T_{1,\varepsilon}$. The appropriate norm turns out to be the singular energy norm $E_{m,T_{1,\varepsilon}}(v)$ (3.77) for m sufficiently large. Finally, the argument requires an *a priori estimate* for the $E_{m,\gamma}$ norm of solutions to the modified systems (3.86)-(3.88) on Ω . This estimate is given in Proposition 3.20; the constants in the estimate depend only on M_0 as in (3.84), a number that is independent of the parameters ε and T on which (3.86)-(3.88) depend. A detailed explanation of how this estimate can be localized in time and combined with the other two results to obtain existence for (3.11)-(3.13) on a domain Ω_T with $T > 0$ independent of ε is given in section 3.3.2.¹⁵

To understand the form of the modified systems (3.86)-(3.88), recall that although we are not using an iteration scheme, we still need to work with systems to which we can apply the linearized estimates. Observe that with one important exception (the right side of the boundary condition in (3.87); see Remark 3.18), the coefficients in the modified systems all depend on v^s rather than the unknown v^ε . Here v^s is our

¹³Here we refer to bounds in a Sobolev norm stronger than the L^∞ norm.

¹⁴The notation Ω_T is defined in Notations 3.10(a).

¹⁵We switched to a “continuous induction” argument after studying the thesis of [Pad14], where such a strategy is used on a quite different problem.

notation with two parameters suppressed for $v_T^{\varepsilon,s}$, a carefully chosen Seeley extension (Proposition 3.15) to $t > T$ of $v^\varepsilon|_{\Omega_T}$, where v^ε is the solution on Ω_{T_ε} to the nonlinear problems (3.11)-(3.12) provided by the local existence result, Proposition 3.83, and where $0 < T < T_\varepsilon$. A consequence of the linear existence theorem, Theorem 3.84, is that the linearized singular problems exhibit *causality*; roughly, solutions in $t < T$ are unaffected by changing the forcing terms and coefficients in $t > T$. Thus, for any choice of $T \in (0, T_\varepsilon)$, we know that the solution v^ε to the modified problems (3.86)-(3.87) on the full half space Ω agrees with the already given $v_T^{s,\varepsilon} = v^\varepsilon$ on Ω_T . When we take into account the relations between the global norm $E_{m,\gamma}(v^\varepsilon)$ and the time-localized norm $E_{m,T}(v^\varepsilon)$ given by Proposition 3.15, we are then able to estimate solutions of the nonlinear problems (3.11)-(3.12) on Ω_T by estimating solutions of the linear problems (3.86)-(3.87) on the full space Ω .

The most difficult of the three steps is the proof of the a priori estimate. This is carried out in section 3.5, which is the heart of the rigorous analysis. To take advantage of the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_{T_ε} , we must estimate solutions to the three systems (3.86)-(3.88) “simultaneously” in the following sense. When we apply the linearized estimates of section 3.2.1 to the problem (3.86) or (3.87), we find that terms arise that can only be controlled by estimating solutions to (3.88). Similarly, estimates of solutions to (3.88) lead to terms that can only be controlled by estimating solutions to (3.86) and (3.87). An example of this occurs in the first interior commutator estimate (3.184), where the terms involving factors of $\frac{1}{\varepsilon}$ on the right can be controlled only by estimating solutions of (3.88) and using the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$. In a sense we have to estimate more than three systems simultaneously, since we estimate not only (3.86)-(3.88), but also $\frac{1}{\sqrt{\varepsilon}}$ (3.86), $\sqrt{\varepsilon}$ (3.87), $\frac{1}{\varepsilon}$ (3.88), and so on; the complete list is given in the outline of section 3.5.1. The norm $E_{m,\gamma}(v)$ that appears in the a priori estimate of Proposition 3.20 must incorporate all the time-localizable norms that arise in estimating all these systems; thus, it is not surprising that it contains a rather large number of terms. In fact each term $E_{m,\gamma}(v_j)$ (3.78) in the definition of $E_{m,\gamma}$ is a sum of 18 terms. It is perhaps surprising that any *finite* number of terms works! We note that the interior and boundary commutator estimates of section 3.5 can be (and are) applied without change in the error analysis of section 5. However, the boundary and interior forcing estimates of chapter 5 are substantially different.

Although we have used the singular calculus of [CGW14b] in several earlier papers, for example [CGW14a, CW13, CW14], this is the first time we have had to estimate singular norms $\langle \Lambda^r f(u) \rangle_{m,\gamma}$ (defined below in Notations 3.10(g)) of nonlinear functions of u . In fact, to take advantage of the extra microlocal precision in the linearized estimates of Proposition 3.11, we also need to show that in some cases even *microlocal* regularity of $u(t, x, \theta)$ in singular norms is preserved under nonlinear functions. The microlocal results are a singular analogue of the classical Rauch’s lemma [Rau79]. Both types of results are new for singular norms and are proved in section 3.3.1, for example in Propositions 3.25 and 3.28. We use these tools throughout chapters 3 and 5. In estimates of $\langle \Lambda^r f(u) \rangle_{m,\gamma}$ where $r > 0$, we have to assume that f is real-analytic; we do not know if this can be weakened to, say, C^∞ . This restriction still allows us to handle the SVK system, and is the reason for the condition of analyticity of $W(E)$ in Assumption (A1g). Both tame and (simpler) non-tame versions of these estimates are given. The tame estimates are used mainly in the proof of the continuation result, Proposition 3.87.

We conclude this survey with a few remarks on the local existence and continuation results for fixed ε of section 3.6. In this section we take advantage of the fact that when ε is fixed, one *can* use the equation to control normal derivatives $\partial_{x_2} v$ in terms of singular tangential derivatives $\partial_{t,\varepsilon} v, \partial_{x_1,\varepsilon} v$. However, even when ε is fixed the nonlinear systems being studied are still singular, because $\partial_{x'}$ and ∂_θ derivatives occur in the linear combination $\partial_{x'} + \beta \frac{\partial a}{\varepsilon}$. So the singular calculus is still needed to prove the linearized estimates on which the argument is based. Given those estimates and the results of section 3.3.1, the

proof of the local existence theorem is a matter of adapting the argument of [ST88] to singular problems. For example, the iteration scheme (3.256)-(3.258) is the singular analogue of the one used in [ST88]. This argument is based on the non-tame linearized estimates (3.254)-(3.255). The proof of the continuation result is based on a tame version of the linearized estimates given in Proposition 3.86. We first prove existence of a low regularity continuation essentially by repeating the local existence proof; we then use the tame linearized estimates to show that this continuation has the desired higher regularity.

3.2 The basic estimates for the linearized singular systems

With apologies to the reader for the following long list, we begin by gathering in one place most of the notation for spaces and norms that is needed below.

Notations 3.10. We take m to be a nonnegative integer, let $(\xi', k) = (\sigma, \xi_1, k)$ be the Fourier transform variables dual to $(x', \theta) = (t, x_1, \theta)$, and set $\langle \xi', k \rangle = (|\xi'|^2 + k^2 + 1)^{1/2}$, $\langle \xi', k, \gamma \rangle = (|\xi'|^2 + k^2 + \gamma^2)^{1/2}$.

(a) Let $\Omega := \{(t, x_1, x_2, \theta) \in \mathbb{R}^4 : x_2 > 0\}$, and for $T > 0$ set $\Omega_T := \Omega \cap \{-\infty < t < T\}$, $b\Omega := \{(t, x_1, \theta) \in \mathbb{R}^3, \text{ and } b\Omega_T := b\Omega \cap \{-\infty < t < T\}$.

(b) Let $H^m \equiv H^m(b\Omega)$ be the standard Sobolev space with norm $\langle V(x', \theta) \rangle_m = |\langle \xi', k \rangle^m \hat{V}(\xi', k)|_{L^2(\xi', k)}$. For $\gamma \geq 1$ and $V \in H^m$ we define $|V|_{H^m_\gamma} = |\langle \xi', k, \gamma \rangle^m \hat{V}(\xi', k)|_{L^2(\xi', k)}$. For $V \in e^{\gamma t} H^m$ we set $\langle V \rangle_{m, \gamma} := |e^{-\gamma t} V|_{H^m_\gamma}$. We sometimes write $V^\gamma = e^{-\gamma t} V$.

(c) $L^2 H^m \equiv L^2(\overline{\mathbb{R}}_+, H^m(b\Omega))$ with norm $|U(t, x, \theta)|_{L^2 H^m} \equiv |U|_{0, m}$ given by

$$|U|_{0, m}^2 = \int_0^\infty \langle U(x', x_2, \theta) \rangle_m^2 dx_2.$$

Similarly, for $U \in L^2 e^{\gamma t} H^m := L^2(\overline{\mathbb{R}}_+, e^{\gamma t} H^m(b\Omega))$ we set

$$(3.31) \quad |U|_{0, m, \gamma}^2 = \int_0^\infty \langle U(x', x_2, \theta) \rangle_{m, \gamma}^2 dx_2.$$

(d) $CH^m \equiv C(\overline{\mathbb{R}}_+, H^m(b\Omega))$ denotes the space of continuous bounded functions of x_2 with values in $H^m(b\Omega)$, with norm $|U(t, x, \theta)|_{CH^m} = |U|_{\infty, m} := \sup_{x_2 \geq 0} |U(\cdot, x_2, \cdot)|_{H^m(b\Omega)}$. The corresponding norm on $CH^m \equiv C(\overline{\mathbb{R}}_+, e^{\gamma t} H^m(b\Omega))$ is denoted $|V|_{\infty, m, \gamma}$.

(e) For a nonnegative integer M and define $C^{0, M} := C(\overline{\mathbb{R}}_+, C^M(b\Omega))$ as the space of continuous bounded functions of x_2 with values in $C^M(b\Omega)$, with norm $|U(x', x_2, \theta)|_{C^{0, M}} := |U|_{L^\infty W^{M, \infty}}$. Here $L^\infty W^{M, \infty}$ denotes the space $L^\infty(\overline{\mathbb{R}}_+; W^{M, \infty}(b\Omega))$.

(f) The corresponding spaces on Ω_T are denoted $L^2 H_T^m$, $L^2 e^{\gamma t} H_T^m$, CH_T^m , $Ce^{\gamma t} H_T^m$ and C_T^{0, M_0} with norms $|U|_{0, m, T}$, $|U|_{0, m, \gamma, T}$, $|U|_{\infty, m, T}$, $|U|_{\infty, m, \gamma, T}$, and $|U|_{C_T^{0, M_0}}$ respectively. These norms have clear meanings when m is a nonnegative integer; otherwise they are defined as usual by interpolation. On $b\Omega_T$ we use the spaces H_T^m and $e^{\gamma t} H_T^m$ with norms $\langle U \rangle_{m, T}$ and $\langle U \rangle_{m, \gamma, T}$.

(g) For $r \in \overline{\mathbb{R}}_+$ let Λ_D^r be the singular operator associated to the singular symbol $\Lambda^r(\xi' + \frac{\beta k}{\varepsilon}, \gamma) := (|\xi' + \frac{\beta k}{\varepsilon}|^2 + \gamma^2)^{r/2}$ (see (A.2)). With $\xi' = (\sigma, \xi_1)$ for $V = V(x', \theta)$ we define (with slight abuse of the

notation $\langle \cdot \rangle_{m,\gamma}$ defined in (b)¹⁶) the singular norms

$$(3.32) \quad \begin{aligned} \langle \Lambda^r V \rangle_{m,\gamma} &:= |\Lambda_D^r(e^{-\gamma t} V)|_{H_\gamma^m} = \left| (|\xi' + \frac{\beta k}{\varepsilon}|^2 + \gamma^2)^{r/2} \langle \xi', k, \gamma \rangle^m \hat{V}(\sigma - i\gamma, \xi_1, k) \right|_{L^2(\xi', k)} \\ \langle \Lambda_1^r V \rangle_m &:= \left| (|\xi' + \frac{\beta k}{\varepsilon}|^2 + 1)^{r/2} \langle \xi', k \rangle^m \hat{V}(\xi', k) \right|_{L^2(\xi', k)}. \end{aligned}$$

(h) When r is a nonnegative integer the norms defined in (g) can be localized to $b\Omega_T$ by replacing $\Lambda_D V$ by $((\partial_{x'} + \beta \frac{\partial}{\varepsilon})V, \gamma V)$. Denote the localized norms by $\langle \Lambda^r V \rangle_{m,\gamma,T}$ and $\langle \Lambda_1^r V \rangle_{m,T}$.

For non-integer $r > 0$, the localized norms are defined by interpolation from the integral case.¹⁷

(i) Having defined these singular norms of functions on $b\Omega$ and $b\Omega_T$, we define the corresponding norms for functions $U(x', x_2, \theta)$ on Ω and Ω_T by obvious analogy with (c) and (d). We denote these norms $|\Lambda^r U|_{0,m,\gamma}$, $|\Lambda^r U|_{\infty,m,\gamma}$, $|\Lambda_1^r U|_{0,m}$, $|\Lambda_1^r U|_{\infty,m}$; the corresponding time-localized norms are $|\Lambda^r U|_{0,m,\gamma,T}$, $|\Lambda^r U|_{\infty,m,\gamma,T}$, $|\Lambda_1^r U|_{0,m,T}$, $|\Lambda_1^r U|_{\infty,m,T}$.

(j) If ϕ_D is a singular pseudodifferential operator associated to the singular symbol $\phi(\varepsilon W(x', \theta), \xi' + \frac{\beta k}{\varepsilon}, \gamma)$, we set

$$(3.33) \quad \langle \phi V \rangle_{m,\gamma} := |\phi_D(e^{-\gamma t} V)|_{H_\gamma^m}.$$

We do not attempt to define a time-localized version of (3.33).

(k) We will sometimes (for example, in convolutions) write $X := \xi' + \beta \frac{k}{\varepsilon}$, $Y = \eta' + \beta \frac{l}{\varepsilon}$, where (η', l) also denotes variables dual to (x', θ) .

(l) All constants appearing in the estimates below are independent of ε , γ , and T unless such dependence is explicitly noted.

(m) We use ∂ to denote a tangential derivative with respect to one of the variables (x', θ) .

(n) We set $D_{x_2,\varepsilon} = \frac{1}{i} \partial_{x_2}$, $D_{t,\varepsilon} = \frac{1}{i} \partial_{t,\varepsilon}$, $D_{x_1,\varepsilon} = \frac{1}{i} \partial_{x_1,\varepsilon}$, $D_{x',\varepsilon} = (D_{t,\varepsilon}, D_{x_1,\varepsilon})$.

(o) Set $\Lambda_{D,\gamma}^r = e^{\gamma t} \Lambda_D^r e^{-\gamma t}$.

3.2.1 Statement of the estimates.

Here we give the main estimates for the ‘‘linearized versions’’ of (3.11) and (3.12), namely,

$$(3.34) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_1^\varepsilon + \sum_{|\alpha|=2} A_\alpha(w^\varepsilon) \partial_{x,\varepsilon}^\alpha v_1^\varepsilon &= \mathcal{F}_1 \text{ on } \Omega \\ \partial_{x_2} v_1^\varepsilon - d_{v_1} H(w_1^\varepsilon, h(w^\varepsilon)) \partial_{x_1,\varepsilon} v_1^\varepsilon &= \mathcal{G}_1 \text{ on } x_2 = 0 \end{aligned}$$

and

$$(3.35) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_2^\varepsilon + \sum_{|\alpha|=2} A_\alpha(w^\varepsilon) \partial_{x,\varepsilon}^\alpha v_2^\varepsilon &= \mathcal{F}_2 \text{ on } \Omega \\ v_2^\varepsilon &= \mathcal{G}_2 \text{ on } x_2 = 0. \end{aligned}$$

¹⁶Observe that for $\langle \Lambda^r V \rangle_{m,\gamma}$ as defined in (3.32) and $\langle w \rangle_{m,\gamma}$ as in (b), we have

$$\langle \Lambda^r V \rangle_{m,\gamma} = \langle w \rangle_{m,\gamma}, \text{ where } w = e^{\gamma t} \Lambda_D^r(e^{-\gamma t} V).$$

¹⁷A concise reference for the method of complex interpolation used here is [Tay11a], Chapter 4.

Here the functions v^ε , w^ε , \mathcal{F}_i , \mathcal{G}_i all vanish in $t < 0$.

The operators $\{\phi_{j,D}, j \in J_e \cup J_h \cup J_c\}$ that appear below form a (singular) pseudodifferential partition of the identity operator; they have symbols $\phi_j(X, \gamma)$, where the $\phi_j(\xi', \gamma)$ define the partition of unity chosen in section 3.1.1.

Proposition 3.11. *Suppose $n \geq 3d + 4$, $s_0 > \frac{d+1}{2} + 2$, and let δ and K be positive constants.¹⁸ Assume that for $\varepsilon \in (0, 1]$ the function w^ε appearing in the coefficients of (3.34), (3.35) satisfies*

$$(3.36) \quad \left| \frac{w^\varepsilon}{\varepsilon} \right|_{C^{0,n}} + \left| \frac{w^\varepsilon}{\varepsilon} \right|_{CH^{s_0}} + |\partial_{x_2} w^\varepsilon|_{L^\infty(\Omega)} < K, \quad |w^\varepsilon|_{L^\infty(\Omega)} < \delta.$$

The following a priori estimates are valid provided δ is small enough; the constants γ_0 and C that appear depend on K . The first estimates are for the weakly stable Neumann-type problem (3.34) satisfied by v_1 : for $\gamma \geq \gamma_0$ we have

$$(3.37) \quad \gamma \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right|_{0,0,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right\rangle_{0,\gamma}^2 + \sum_{J_h \cup J_e} \left\langle \phi_j \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right\rangle_{0,\gamma}^2 \leq \\ C\gamma^{-1} \left(|\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda \mathcal{G}_1 \rangle_{0,\gamma}^2 \right) + C\gamma^{-1} \left| \left(\begin{array}{c} \Lambda v_1 / \sqrt{\varepsilon} \\ D_{x_2} v_1 / \sqrt{\varepsilon} \end{array} \right) \right|_{0,0,\gamma}^2,$$

$$(3.38) \quad \gamma \left| \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right|_{\infty,0,\gamma}^2 \leq C\gamma^{-1} \left(|\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{0,0,\gamma}^2 + |\mathcal{F}_1|_{0,1,\gamma}^2 + \langle \Lambda \mathcal{G}_1 \rangle_{0,\gamma}^2 + \langle \Lambda^{\frac{1}{2}} \mathcal{G}_1 \rangle_{1,\gamma}^2 \right) + \\ C\gamma^{-1} \left| \left(\begin{array}{c} \Lambda v_1 / \sqrt{\varepsilon} \\ D_{x_2} v_1 / \sqrt{\varepsilon} \end{array} \right) \right|_{0,0,\gamma}^2.$$

$$(3.39) \quad \gamma \left| \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right|_{0,1,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda^{\frac{1}{2}} v_1 \\ D_{x_2} \Lambda^{-\frac{1}{2}} v_1 \end{array} \right) \right\rangle_{1,\gamma}^2 + \sum_{J_h \cup J_e} \left\langle \phi_j \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right\rangle_{1,\gamma}^2 \leq \\ C\gamma^{-1} \left(|\mathcal{F}_1|_{0,1,\gamma}^2 + \langle \Lambda^{\frac{1}{2}} \mathcal{G}_1 \rangle_{1,\gamma}^2 \right).$$

The next estimates are for the uniformly stable Dirichlet problem (3.35) satisfied by v_2 : for $\gamma \geq \gamma_0$

$$(3.40) \quad \gamma \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_2 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_2 \end{array} \right) \right|_{0,0,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda v_2 \\ D_{x_2} v_2 \end{array} \right) \right\rangle_{0,\gamma}^2 \leq \\ C \left(\gamma^{-1} |\Lambda^{\frac{1}{2}} \mathcal{F}_2|_{0,0,\gamma}^2 + \gamma \langle \Lambda \mathcal{G}_2 \rangle_{0,\gamma}^2 + \sum_{J_h} \left\langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \right\rangle_{0,\gamma}^2 \right) + C\gamma^{-1} \left| \left(\begin{array}{c} \Lambda v_2 / \sqrt{\varepsilon} \\ D_{x_2} v_2 / \sqrt{\varepsilon} \end{array} \right) \right|_{0,0,\gamma}^2.$$

$$(3.41) \quad \gamma \left| \left(\begin{array}{c} \Lambda v_2 \\ D_{x_2} v_2 \end{array} \right) \right|_{\infty,0,\gamma}^2 \leq C \left(\gamma^{-1} |\mathcal{F}_2|_{0,1,\gamma}^2 + \gamma \langle \Lambda \mathcal{G}_2 \rangle_{0,\gamma}^2 + \gamma \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{1,\gamma}^2 + \sum_{J_h} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{1,\gamma}^2 \right).$$

¹⁸See, for example, Propositions A.10 and A.16 to understand the restrictions on n and s_0 .

$$(3.42) \quad \gamma \left| \left(\begin{array}{c} \Lambda v_2 \\ D_{x_2} v_2 \end{array} \right) \right|_{0,1,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda^{\frac{1}{2}} v_2 \\ D_{x_2} \Lambda^{-\frac{1}{2}} v_2 \end{array} \right) \right\rangle_{1,\gamma}^2 \leq C \left(\gamma^{-1} |\mathcal{F}_2|_{0,1,\gamma}^2 + \gamma \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{1,\gamma}^2 + \sum_{J_h} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{1,\gamma}^2 \right).$$

Remark 3.12. *The terms on the right in (3.37) and (3.38) involving division by $\sqrt{\varepsilon}$ are needed to control the commutators with $\Lambda_D^{1/2}$ that appear when that operator is applied to the equation written as a first-order system.*

3.2.2 Proofs of the estimates.

Estimates (3.37), (3.39) and (3.40), (3.42) are proved by analyzing the singular systems (3.34), (3.35) using the singular pulse calculus of [CGW14b]. A first step is to rewrite the second-order differential systems as first order singular pseudo-differential systems to which we can readily apply singular Kreiss symmetrizers.

Estimates (3.38) and (3.41) require, in addition, a diagonalization argument using a cutoff χ (A.4) in the extended calculus similar to that given in [Wil02], but modified in the case of (3.38) to take into account the weakly stable nature of the problem (3.34). The Dirichlet problem (3.35) is uniformly stable, that is, the Lopatinskii determinant $D(0, z)$ (defined with $\mathcal{B}_2 = (I_2 \ 0)$ now) is nonvanishing for all $z \in \mathcal{X}$, so the set J_c is empty in this case.

To prove the estimates for v_1 we set $U = (e^{\gamma t} \Lambda_D e^{-\gamma t} v_1, D_{x_2} v_1) = (\Lambda_{D,\gamma} v_1, D_{x_2} v_1)$, $U^\gamma = e^{-\gamma t} U$, and rewrite (3.34) as a 4×4 singular first-order system:

$$(3.43) \quad D_{x_2} U^\gamma - \mathcal{A}(w, D_{x',\varepsilon}) U^\gamma = \tilde{\mathcal{F}}_1^\gamma, \quad \mathcal{B}_1(w, D_{x',\varepsilon}) U^\gamma = \tilde{\mathcal{G}}_1^\gamma,$$

where, with the matrices A_α evaluated at w , we have

$$(3.44) \quad \begin{aligned} \tilde{\mathcal{F}}_1 &:= \begin{pmatrix} 0 \\ -A_{(0,2)}^{-1} \mathcal{F}_1 \end{pmatrix}, \quad \tilde{\mathcal{G}}_1 = -i \mathcal{G}_1 \\ \mathcal{A}(w, D_{x',\varepsilon}) &:= - \begin{pmatrix} 0 & \Lambda_D I_2 \\ \left(A_{(0,2)}^{-1} (D_{t,\varepsilon} - i\gamma)^2 + A_{(0,2)}^{-1} A_{(2,0)} D_{x_1,\varepsilon}^2 \right) \Lambda_D^{-1} & A_{(0,2)}^{-1} A_{(1,1)} D_{x_1,\varepsilon} \end{pmatrix} \\ \mathcal{B}_1(w, D_{x',\varepsilon}) &:= \left(-d_{v_1} H(w_1, h(w)) D_{x_1,\varepsilon} \Lambda_D^{-1} \quad I_2 \right). \end{aligned}$$

This is the singular analogue of (3.20).

Writing the system (3.35) as a first order system for $U = (e^{\gamma t} \Lambda_D e^{-\gamma t} v_2, D_{x_2} v_2)$, we obtain

$$(3.45) \quad D_{x_2} U^\gamma - \mathcal{A}(w, D_{x',\varepsilon}) U^\gamma = \tilde{\mathcal{F}}_2^\gamma, \quad \mathcal{B}_2 U^\gamma = \tilde{\mathcal{G}}_2^\gamma,$$

where $\tilde{\mathcal{F}}_2$ is defined like $\tilde{\mathcal{F}}_1$, but now

$$(3.46) \quad \mathcal{B}_2 = (I_2 \ 0) \text{ and } \tilde{\mathcal{G}}_2 = \Lambda_{D,\gamma} \mathcal{G}_2.$$

The following proposition gives the L^2 estimates for the singular systems (3.43), (3.45).

Proposition 3.13. *With notation and assumptions as in Proposition 3.11, we have the following a priori estimates for the linearized singular Neumann and Dirichlet systems, (3.43) and (3.45). For $\gamma \geq \gamma_0$,*

$$(3.47) \quad \gamma \left| \left(\frac{\Lambda v_1}{D_{x_2} v_1} \right) \right|_{0,0,\gamma}^2 + \gamma \left\langle \left(\frac{\Lambda^{\frac{1}{2}} v_1}{D_{x_2} \Lambda^{-\frac{1}{2}} v_1} \right) \right\rangle_{0,\gamma}^2 + \sum_{J_h \cup J_e} \left\langle \phi_j \left(\frac{\Lambda v_1}{D_{x_2} v_1} \right) \right\rangle_{0,\gamma}^2 \leq C \gamma^{-1} \left(|\mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda^{\frac{1}{2}} \mathcal{G}_1 \rangle_{0,\gamma}^2 \right)$$

and

$$(3.48) \quad \gamma \left| \left(\frac{\Lambda v_2}{D_{x_2} v_2} \right) \right|_{0,0,\gamma}^2 + \gamma \left\langle \left(\frac{\Lambda^{\frac{1}{2}} v_2}{D_{x_2} \Lambda^{-\frac{1}{2}} v_2} \right) \right\rangle_{0,\gamma}^2 \leq C \left(\gamma^{-1} |\mathcal{F}_2|_{0,0,\gamma}^2 + \gamma \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{0,\gamma}^2 + \sum_{J_h} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{0,\gamma}^2 \right).$$

Proof. 1. The following estimate holds for $\phi_{j,D}(e^{-\gamma t} U)$ with $j \in J_h \cup J_e$.

$$(3.49) \quad \gamma |\phi_j U|_{0,0,\gamma}^2 + \langle \phi_j U \rangle_{0,\gamma}^2 \leq C \left(\gamma^{-1} |\phi_j \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \phi_j \mathcal{G}_1 \rangle_{0,\gamma}^2 + \gamma^{-1} |U|_{0,0,\gamma}^2 + \langle \Lambda^{-1} U(0) \rangle_{0,\gamma}^2 \right).$$

This Kreiss-type estimate is the singular analogue of estimate (1.6) in [ST88], p. 268, and it is proved by essentially the same argument; the only difference is that all symbols, including the symbol of the Kreiss symmetrizer, are now quantized in the singular calculus by the procedure described in section A.2. For example, if we let $r(v, \xi', \gamma)$ denote the symbol of the (nonsingular) Kreiss symmetrizer used in the proof of (1.6) in [ST88], the corresponding singular operator that must be used in the proof of (3.49) is the operator associated to $r(w^\varepsilon(x, \theta), X, \gamma)$.¹⁹ In view of the assumption (3.36) the function $w^\varepsilon(x, \theta)$ can play the role of $\varepsilon V(x, \theta)$ in the integral (A.2). The last two terms on the right in (3.49) arise from commutators by application of Proposition A.15.

2. The following Kreiss-type estimate is a variant of (3.49) that holds for $\phi_{j,D} U^\gamma$ with $j \in J_e$.

$$(3.50) \quad \gamma |\phi_j U|_{0,0,\gamma}^2 + \gamma \langle \phi_j \Lambda^{-\frac{1}{2}} U \rangle_{0,\gamma}^2 \leq C \left(\gamma^{-1} |\phi_j \mathcal{F}_1|_{0,0,\gamma}^2 + \gamma \langle \phi_j \Lambda^{-\frac{1}{2}} \mathcal{G}_1 \rangle_{0,\gamma}^2 + \gamma^{-1} |U|_{0,0,\gamma}^2 + \langle \Lambda^{-1} U(0) \rangle_{0,\gamma}^2 \right).$$

This is the analogue of estimate (1.7) in [ST88], p. 268, and is again proved by using singular operators in the argument given there.

3. For $\phi_{j,D} U^\gamma$ with $j \in J_c$ (the “bad” set where LU fails) we have

$$(3.51) \quad \gamma \left(|\phi_j U|_{0,0,\gamma}^2 + \langle \Lambda^{-\frac{1}{2}} \phi_j U \rangle_{0,\gamma}^2 \right) \leq C \gamma^{-1} \left(|\phi_j \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda^{\frac{1}{2}} \phi_j \mathcal{G}_1 \rangle_{0,\gamma}^2 + |U|_{0,0,\gamma}^2 + \langle \Lambda^{-\frac{1}{2}} U(0) \rangle_{0,\gamma}^2 \right).$$

This is the analogue of estimate (1.10) in [ST88], p. 272, and, except for an important simplification that we now discuss, is again proved by implementing the argument given there with singular operators.

Consider a zero order symbol $b^+(v, z)$ as in Remark 3.5, defined for $|v|$ small and $z \in \mathcal{O}_j$, $j \in J_c$. Let $\tilde{b}^+(v, z)$ be an extension to \mathcal{X} defined by taking extensions \tilde{k}_1 , \tilde{k}_2 , and \tilde{q} of the factors in (3.23), where the \tilde{k}_i are elliptic of order zero and we still have \tilde{q} of order one satisfying

$$(3.52) \quad \Im \tilde{q}(v, z) = -\gamma.$$

¹⁹Full details of how the singular calculus is implemented to prove estimates like (3.49) are given in section 5 of [Wil02].

Dropping tildes on extensions and letting q_D denote the singular operator with symbol $q(w^\varepsilon, X, \gamma)$, we see that the estimate

$$(3.53) \quad \Re(iq_D \Lambda_D^{-1} v, v) \geq C\gamma |\Lambda_D^{-\frac{1}{2}} v|_{L^2}^2$$

follows immediately from (3.52); there is no need for a singular sharp Garding inequality to replace the use of the standard sharp Garding inequality in [ST88]. With (3.53) one obtains the key estimate,

$$(3.54) \quad |\Lambda_D^{\frac{1}{2}} b_D^+ w|_{L^2} \geq C\gamma |\Lambda_D^{-\frac{1}{2}} w|_{L^2},$$

which is analogous to (1.8) of [ST88], by arguing as on p. 270 of [ST88].

4. For later use we observe that since $q_D \Lambda_D^{-1} = (q \Lambda^{-1})_D$, we can apply (3.52) and Proposition A.10(b) to obtain

$$(3.55) \quad \Re(iq_D \Lambda_D^{-1} v, \Lambda_D v) \geq C\gamma |v|_{L^2}^2.$$

A minor variation of the argument that gave (3.54) now yields²⁰

$$(3.56) \quad |\Lambda_D b_D^+ w|_{L^2} \geq C\gamma |w|_{L^2}.$$

5. Adding the estimates (3.49)-(3.51) and absorbing errors on the left yields for large enough γ :

$$(3.57) \quad \gamma \left| \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right|_{0,0,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda^{\frac{1}{2}} v_1 \\ D_{x_2} \Lambda^{-\frac{1}{2}} v_1 \end{array} \right) \right\rangle_{0,\gamma}^2 + \sum_{J_h \cup J_e} \left\langle \phi_j \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right\rangle_{0,\gamma}^2 \leq \\ C\gamma^{-1} \left(|\mathcal{F}_1|_{0,0,\gamma}^2 + \left\langle \Lambda^{\frac{1}{2}} \mathcal{G}_1 \right\rangle_{0,\gamma}^2 \right).$$

6. To prove the estimates for v_2 we use the first-order system (3.45). For the Dirichlet problem (3.35) the bad set J_c is empty, so we can apply the estimates (3.49), (3.50) with $\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2$ in place of $\mathcal{F}_1, \mathcal{G}_1$. Adding these estimates and absorbing terms yields

$$(3.58) \quad \gamma \left| \left(\begin{array}{c} \Lambda v_2 \\ D_{x_2} v_2 \end{array} \right) \right|_{0,0,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda^{\frac{1}{2}} v_2 \\ D_{x_2} \Lambda^{-\frac{1}{2}} v_2 \end{array} \right) \right\rangle_{0,\gamma}^2 \leq C \left(\gamma^{-1} |\mathcal{F}_2|_{0,0,\gamma}^2 + \gamma \left\langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \right\rangle_{0,\gamma}^2 + \sum_{J_h} \left\langle \phi_j \Lambda \mathcal{G}_2 \right\rangle_{0,\gamma}^2 \right).$$

□

Next we turn to the proof of Proposition 3.11. The L^2 estimate of Proposition (3.13) for v_1 can easily be applied (as in step 1 of the proof below) to yield the estimates (3.37) and (3.39). But these estimates by themselves provide no way to estimate $\left| \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right|_{\infty,0,\gamma}$; the presence of the factors of $1/\varepsilon$ in the components of $\mathcal{A}(w, D_{x',\varepsilon})$ prevent us from employing the usual argument that uses the equation to control that norm by controlling $|D_{x_2} U|_{0,0,\gamma}$. Instead, we must adapt to this weakly stable problem the type of argument that was used in [Wil02] to control $|U|_{\infty,0,\gamma}$ in the uniformly stable case. That is done for v_1 in steps 2 and 4 of the following proof.

²⁰We are not able to prove (3.56) simply by applying (3.54) with w replace by $\Lambda_D^{\frac{1}{2}} w$.

Proof of Proposition 3.11. 1. Proof of (3.39). The estimate (3.39) follows from (3.57) applied to the problem satisfied by $\partial_{(t,x_1,\theta)}U$. However, we are *not* able to prove (3.37) simply by applying (3.57) to the problem satisfied by $\mathcal{U} := \Lambda_{D,\gamma}^{1/2}U$. The norm of the commutator $\langle \Lambda_D^{1/2}[\mathcal{B}_1, \Lambda_D^{1/2}]U^\gamma \rangle_{L^2}$, introduces error terms that are unacceptably large.²¹ We get around this in the next step by microlocalizing the estimates.

2. Proof of (3.37). The next two estimates are proved by applying (3.49) and (3.50) to the problem satisfied by \mathcal{U} ; the less singular boundary norms in these microlocal estimates, as compared with (3.57), give rise to acceptable commutator errors. For $j \in J_h \cup J_e$ we have

$$(3.59) \quad \begin{aligned} & \gamma |\Lambda_D^{1/2} \phi_j U|_{0,0,\gamma}^2 + \langle \Lambda_D^{1/2} \phi_j U \rangle_{0,\gamma}^2 \lesssim \\ & \gamma^{-1} |\Lambda_D^{1/2} \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda_D^{1/2} \mathcal{G}_1 \rangle_{0,\gamma}^2 + \gamma^{-1} |\Lambda_D^{1/2} U|_{0,0,\gamma}^2 + \gamma^{-1} |U/\sqrt{\varepsilon}|_{0,0,\gamma}^2 + \gamma^{-1} \langle U(0) \rangle_{0,\gamma}^2, \end{aligned}$$

while for $j \in J_e$ we obtain

$$(3.60) \quad \begin{aligned} & \gamma |\Lambda_D^{1/2} \phi_j U|_{0,0,\gamma}^2 + \gamma \langle \phi_j U \rangle_{0,\gamma}^2 \lesssim \\ & \gamma^{-1} |\Lambda_D^{1/2} \mathcal{F}_1|_{0,0,\gamma}^2 + \gamma \langle \mathcal{G}_1 \rangle_{0,\gamma}^2 + \gamma^{-1} |\Lambda_D^{1/2} U|_{0,0,\gamma}^2 + \gamma^{-1} |U/\sqrt{\varepsilon}|_{0,0,\gamma}^2 + \gamma^{-1} \langle U(0) \rangle_{0,\gamma}^2. \end{aligned}$$

The terms $|U/\sqrt{\varepsilon}|_{0,0,\gamma}^2$ arise from Proposition A.16(a) applied to the interior commutator, while the terms $\gamma^{-1} \langle U(0) \rangle_{0,\gamma}^2$ come from Proposition A.16(d) applied to the boundary commutator.

Next we show that for $j \in J_c$

$$(3.61) \quad \begin{aligned} & \gamma \left(|\Lambda_D^{1/2} \phi_j U|_{0,0,\gamma}^2 + \langle \phi_j U(x_2) \rangle_{0,\gamma}^2 \right) \lesssim \\ & \gamma^{-1} \left(|\Lambda_D^{1/2} \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda_D \mathcal{G}_1 \rangle_{0,\gamma}^2 + |\Lambda_D^{1/2} U|_{0,0,\gamma}^2 + |U/\sqrt{\varepsilon}|_{0,0,\gamma}^2 + \langle U(0) \rangle_{0,\gamma}^2 \right). \end{aligned}$$

In this case (3.24) becomes $S^{-1} \mathcal{A} S = \text{diag}(a^+, a^-)$ in \mathcal{O}_j . Letting a^\pm denote C^∞ , homogeneous of degree one extensions to \mathcal{X} satisfying $\pm \text{Im } a^\pm > 0$, we consider the problems

$$(3.62) \quad \begin{aligned} & D_{x_2} V_- - a_D^- V_- = F_- \\ & D_{x_2} V_+ - a_D^+ V_+ = F_+, \quad b_D^+ V_+(0) = G_+. \end{aligned}$$

Writing

$$(3.63) \quad \begin{aligned} & \text{Im} \int_{x_2}^\infty \langle (D_{x_2} - a_D^-) V_-, \Lambda_D^2 V_- \rangle dx_2 = \text{Im} \int_{x_2}^\infty \langle F_-, \Lambda_D^2 V_- \rangle dx_2 \\ & \text{Im} \int_0^{x_2} \langle (D_{x_2} - a_D^+) V_+, V_+ \rangle dx_2 = \text{Im} \int_0^{x_2} \langle F_+, V_+ \rangle dx_2, \end{aligned}$$

integrating terms involving D_{x_2} by parts in x_2 , and applying the singular Garding's inequality of Theorem A.12 to terms involving a_D^\pm , we obtain

$$(3.64) \quad \begin{aligned} & \langle \Lambda_D V_-(x_2) \rangle_{L^2(x',\theta)} + |\Lambda_D^{3/2} V_-|_{L^2(x,\theta)} \lesssim |\Lambda_D^{1/2} F_-|_{L^2(x,\theta)} + |V_-/\sqrt{\varepsilon}|_{L^2(x,\theta)} \\ & \sqrt{\gamma} \langle V_+(x_2) \rangle_{L^2(x',\theta)} + \sqrt{\gamma} |\Lambda_D^{1/2} V_+|_{L^2(x,\theta)} \lesssim \sqrt{\gamma} \langle V_+(0) \rangle_{L^2(x',\theta)} + |F_+|_{L^2(x,\theta)} \end{aligned}$$

²¹These error terms are exhibited in Proposition A.16(b).

Adding the estimates (3.64) and using (3.56), we find

$$\begin{aligned}
(3.65) \quad & \sqrt{\gamma} \left(|\Lambda_D^{\frac{1}{2}} V|_{L^2} + \langle V(x_2) \rangle_{L^2} \right) \lesssim \sqrt{\gamma} \langle V_+(0) \rangle_{L^2} + \frac{1}{\sqrt{\gamma}} \left(|\Lambda_D^{\frac{1}{2}} F|_{L^2} + |V_-/\sqrt{\varepsilon}|_{L^2} \right) \\
& \lesssim \frac{1}{\sqrt{\gamma}} \left(\langle \Lambda_D b_D^+ V_+(0) \rangle_{L^2} + |\Lambda_D^{\frac{1}{2}} F|_{L^2} + |V_-/\sqrt{\varepsilon}|_{L^2} \right) \lesssim \\
& \frac{1}{\sqrt{\gamma}} \left(\langle \Lambda_D b_D V(0) \rangle_{L^2} + \langle \Lambda_D b_D^- V_-(0) \rangle_{L^2} + |\Lambda_D^{\frac{1}{2}} F|_{L^2} + |V_-/\sqrt{\varepsilon}|_{L^2} \right) \lesssim \\
& \frac{1}{\sqrt{\gamma}} \left(\langle \Lambda_D b_D V(0) \rangle_{L^2} + |\Lambda_D^{\frac{1}{2}} F|_{L^2} + |V_-/\sqrt{\varepsilon}|_{L^2} \right) \lesssim
\end{aligned}$$

By applying (3.65) to $V = (V_+, V_-) := S_D^{-1} \phi_{j,D} U^\gamma$ and using the commutator estimates of section 7, in particular Proposition A.17 and its corollary, we obtain (3.61).

Combining the estimates (3.59), (3.60), and (3.61) yields (3.37).

3. For later use we observe that a similar argument yields for $j \in J_c$ the following extension of (3.51):

$$\begin{aligned}
(3.66) \quad & \gamma \left(|\phi_j U|_{0,0,\gamma}^2 + \langle \Lambda^{-\frac{1}{2}} \phi_j U(x_2) \rangle_{0,\gamma}^2 \right) \leq \\
& C \gamma^{-1} \left(|\phi_j \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda^{\frac{1}{2}} \phi_j \mathcal{G}_1 \rangle_{0,\gamma}^2 + |U|_{0,0,\gamma}^2 + \langle \Lambda^{-\frac{1}{2}} U(0) \rangle_{0,\gamma}^2 \right).
\end{aligned}$$

For this one takes L^2 pairings with $\Lambda_D V_-$ and $\Lambda_D^{-1} V_+$ in (3.63), instead of with $\Lambda_D^2 V_-$ and V_+ . Similarly, one derives the following extension of (3.50) for $j \in J_e$:

$$\begin{aligned}
(3.67) \quad & \gamma |\phi_j U|_{0,0,\gamma}^2 + \gamma \langle \phi_j \Lambda^{-\frac{1}{2}} U(x_2) \rangle_{0,\gamma}^2 \leq \\
& C \left(\gamma^{-1} |\phi_j \mathcal{F}_1|_{0,0,\gamma}^2 + \gamma \langle \phi_j \Lambda^{-\frac{1}{2}} \mathcal{G}_1 \rangle_{0,\gamma}^2 + \gamma^{-1} |U|_{0,0,\gamma}^2 + \langle \Lambda^{-1} U(0) \rangle_{0,\gamma}^2 \right).
\end{aligned}$$

4. Proof of (3.38). For $j \in J_c$ the estimate (3.61) implies

$$(3.68) \quad \gamma |\phi_j U|_{\infty,0,\gamma}^2 \lesssim \gamma^{-1} \left(|\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda \mathcal{G}_1 \rangle_{0,\gamma}^2 + |\Lambda^{\frac{1}{2}} U|_{0,0,\gamma}^2 + |U/\sqrt{\varepsilon}|_{0,0,\gamma}^2 + \langle U(0) \rangle_{0,\gamma}^2 \right).$$

Let χ_D denote a cutoff in the extended calculus, with the support property (A.4), chosen so that $\chi = \sum_{j \in J_c} \phi_j \chi$. The estimate (3.68) yields

$$(3.69) \quad \gamma |\chi U|_{\infty,0,\gamma}^2 \lesssim \gamma^{-1} \left(|\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{0,0,\gamma}^2 + \langle \Lambda \mathcal{G}_1 \rangle_{0,\gamma}^2 + |\Lambda^{\frac{1}{2}} U|_{0,0,\gamma}^2 + |U/\sqrt{\varepsilon}|_{0,0,\gamma}^2 + \langle U(0) \rangle_{0,\gamma}^2 \right).$$

On the other hand from the support property of χ and the equation we obtain

$$(3.70) \quad |(1 - \chi)U|_{\infty,0,\gamma} \leq |D_{x_2}(1 - \chi)U|_{0,0,\gamma} + |U|_{0,0,\gamma} \leq C(|U|_{0,1,\gamma} + |\mathcal{F}_1|_{0,0,\gamma}).$$

The estimate (3.38) for v_1 is then a consequence of (3.69), (3.37), and (3.39).

5. Proof of (3.40) and (3.42). The proof of (3.42) is like that of (3.39) but simpler, since boundary commutators are zero. The estimate (3.40) is proved by applying (3.48) to the problem satisfied by $\Lambda_D^{\frac{1}{2}} U^\gamma$, where U^γ is defined using v_2 now.

6. Proof of (3.41). The argument is similar to step **2**, but the set J_c is empty now. We show first that for $j \in J_e$ we have

$$(3.71) \quad |\phi_j U|_{\infty,0,\gamma}^2 \leq C (\gamma^{-1} |\mathcal{F}_2|_{0,0,\gamma}^2 + \langle \Lambda \mathcal{G}_2 \rangle_{0,\gamma}^2 + \gamma^{-1} |U|_{0,0,\gamma}^2 + \langle \Lambda^{-1} U(0) \rangle_{0,\gamma}^2).$$

In place of (3.62), (3.63) we have

$$(3.72) \quad \begin{aligned} \operatorname{Im} \int_{x_2}^{\infty} \langle (D_{x_2} - a_D^-) V_-, V_- \rangle dx_2 &= \operatorname{Im} \int_{x_2}^{\infty} \langle F_-, V_- \rangle dx_2 \\ \operatorname{Im} \int_0^{x_2} \langle (D_{x_2} - a_D^+) V_+, V_+ \rangle dx_2 &= \operatorname{Im} \int_0^{x_2} \langle F_+, V_+ \rangle dx_2 \end{aligned}$$

and

$$(3.73) \quad \begin{aligned} \sqrt{\gamma} \langle V_-(x_2) \rangle_{L^2(x',\theta)} + \gamma |V_-|_{L^2(x,\theta)} &\lesssim |F_-|_{L^2(x,\theta)} \\ \sqrt{\gamma} \langle V_+(x_2) \rangle_{L^2(x',\theta)} + \gamma |V_+|_{L^2(x,\theta)} &\lesssim \sqrt{\gamma} \langle V_+(0) \rangle_{L^2(x',\theta)} + |F_+|_{L^2(x,\theta)}, \end{aligned}$$

which implies

$$(3.74) \quad \sqrt{\gamma} \langle V(x_2) \rangle_{L^2} + \gamma |V|_{L^2} \lesssim |F|_{L^2} + \sqrt{\gamma} \langle b_D V(0) \rangle_{L^2}.$$

Here in place of (3.54) we have used

$$(3.75) \quad \langle b_D^+ w \rangle_{L^2} \geq C \langle w \rangle_{L^2}.$$

Applying (3.74) to the problem satisfied by $V = S_D^{-1} \phi_{j,D} U^\gamma$, we obtain (3.71).

Let χ denote a cutoff in the extended calculus with the support property (A.4). Since χ can be chosen now so that $\chi = \sum_{j \in J_e} \phi_j \chi$, we have

$$(3.76) \quad |\chi U|_{\infty,0,\gamma}^2 \leq C (\gamma^{-1} |\mathcal{F}_2|_{0,0,\gamma}^2 + \langle \Lambda \mathcal{G}_2 \rangle_{0,\gamma}^2 + \gamma^{-1} |U|_{0,0,\gamma}^2 + \langle \Lambda^{-1} U(0) \rangle_{0,\gamma}^2).$$

Together with (3.70) (for U corresponding to v_2 now) and (3.42) this yields (3.41). □

3.3 Uniform time of existence for the nonlinear singular systems

In this section we describe our strategy for proving the existence of solutions to the coupled, nonlinear, singular systems (3.11)-(3.13) on a fixed time interval independent of small $\varepsilon > 0$. First, we define a ‘singular energy norm’ $E_{m,\gamma}(v)$, a time-localized variant of which, namely $E_{m,T}(v)$, will be shown to be uniformly bounded on a time interval independent of $\varepsilon \in (0, 1]$.

3.3.1 Singular energy norms

Let m be a nonnegative integer. Suppressing the epsilon superscript on v and the subscript D on the singular operator Λ_D , we define for $v = (v_1, v_2)$ the singular energy norm =

$$(3.77) \quad E_{m,\gamma}(v) := E_{m,\gamma}(v_1) + E_{m,\gamma}(v_2),$$

where $E_{m,\gamma}(v_j) =$

$$(3.78) \quad \begin{aligned} & \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_j \\ D_{x_2} \Lambda^{\frac{1}{2}} v_j \end{array} \right) \right|_{0,m,\gamma}^{(2)} + \left| \left(\begin{array}{c} \Lambda v_j \\ D_{x_2} v_j \end{array} \right) \right|_{\infty,m,\gamma}^{(2)} + \left| \left(\begin{array}{c} \Lambda v_j \\ D_{x_2} v_j \end{array} \right) \right|_{0,m+1,\gamma}^{(2)} + \left\langle \left(\begin{array}{c} \Lambda^{\frac{1}{2}} v_j \\ D_{x_2} \Lambda^{-\frac{1}{2}} v_j \end{array} \right) \right\rangle_{m+1,\gamma} \\ & + \left| \left(\begin{array}{c} \sqrt{\varepsilon} \Lambda^{\frac{3}{2}} v_j \\ \sqrt{\varepsilon} \Lambda^{\frac{1}{2}} D_{x_2} v_j \end{array} \right) \right|_{0,m+1,\gamma}^{(2)} + \left\langle \left(\begin{array}{c} \sqrt{\varepsilon} \Lambda v_j \\ \sqrt{\varepsilon} D_{x_2} v_j \end{array} \right) \right\rangle_{m+1,\gamma} + \left| \left(\begin{array}{c} \Lambda v_j / \sqrt{\varepsilon} \\ D_{x_2} v_j / \sqrt{\varepsilon} \end{array} \right) \right|_{0,m,\gamma}^{(2)} + \left\langle \left(\begin{array}{c} \Lambda^{\frac{1}{2}} v_j / \sqrt{\varepsilon} \\ D_{x_2} \Lambda^{-\frac{1}{2}} v_j / \sqrt{\varepsilon} \end{array} \right) \right\rangle_{m,\gamma} \\ & + \left| \frac{\nabla_\varepsilon u}{\varepsilon} \right|_{\infty,m,\gamma} + \left| \frac{\nabla_\varepsilon u}{\varepsilon} \right|_{0,m+1,\gamma} + \left| \frac{\Lambda^{\frac{1}{2}} \nabla_\varepsilon u}{\sqrt{\varepsilon}} \right|_{0,m+1,\gamma} + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} + \left| \frac{\Lambda^{\frac{1}{2}} \nabla_\varepsilon u}{\varepsilon} \right|_{0,m,\gamma}. \end{aligned}$$

Here the superscripts (2) on some terms have the following meanings:²²

$$(3.79) \quad \begin{aligned} & \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_j \\ D_{x_2} \Lambda^{\frac{1}{2}} v_j \end{array} \right) \right|_{0,m,\gamma}^{(2)} = \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_j \\ D_{x_2} \Lambda^{\frac{1}{2}} v_j \end{array} \right) \right|_{0,m,\gamma} + \left| \left(\begin{array}{c} \varepsilon D_{x_2} \Lambda^{\frac{3}{2}} v_j \\ \varepsilon \Lambda^{\frac{1}{2}} D_{x_2}^2 v_j \end{array} \right) \right|_{0,m-1,\gamma}, \\ & \left| \left(\begin{array}{c} \sqrt{\varepsilon} \Lambda^{\frac{3}{2}} v_j \\ \sqrt{\varepsilon} \Lambda^{\frac{1}{2}} D_{x_2} v_j \end{array} \right) \right|_{0,m+1,\gamma}^{(2)} = \left| \left(\begin{array}{c} \sqrt{\varepsilon} \Lambda^{\frac{3}{2}} v_j \\ \sqrt{\varepsilon} \Lambda^{\frac{1}{2}} D_{x_2} v_j \end{array} \right) \right|_{0,m+1,\gamma} + \left| \left(\begin{array}{c} \varepsilon^{\frac{3}{2}} D_{x_2} \Lambda^{\frac{3}{2}} v_j \\ \varepsilon^{\frac{3}{2}} \Lambda^{\frac{1}{2}} D_{x_2}^2 v_j \end{array} \right) \right|_{0,m,\gamma}. \end{aligned}$$

Remark 3.14. 1. It would be more proper to denote the norms in (3.77) and (3.78) by $E_{m,\gamma}(v, \nabla_\varepsilon u)$ and $E_{m,\gamma}(v_j, \nabla_\varepsilon u)$, so there is an abuse of notation here that we have introduced in order to lighten many of the expressions that occur throughout the paper. The fact that $\nabla_\varepsilon u$ rather than v appears in the third line of (3.78) is related to the fact that we will later have $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ only on Ω_{T_ε} .

2. Note that all interior norms in the first two lines of the definition of $E_{m,\gamma}(v)$ have “total weight” $m + 3$ if one assigns a weight of 1 to εD_{x_2} , ∞ , $\Lambda^{\frac{1}{2}}$, $\frac{1}{\sqrt{\varepsilon}}$, and to each tangential derivative ∂ , and assigns a weight of 2 to D_{x_2} . All boundary norms in those lines have weight $m + 2$.²³ The same applies to the third line if $\nabla_\varepsilon u$ is treated like v .

3. The second terms on the right in (3.79) arise in the interior commutator estimates. See, for example, part (b) of Proposition 3.61.

4. For each j , $E_{m,\gamma}(v_j)$ is a sum of 18 terms. This is the smallest number of terms for which we have been able to obtain an estimate that “closes” like the one in Proposition 3.20.

We also define singular energy norms localized in time for functions defined on Ω_T . Set

$$(3.80) \quad E_{m,T}(v) = E_{m,T}(v_1) + E_{m,T}(v_2),$$

where the norms $E_{m,T}(v_i)$ are defined by the right side of (3.78), except that norms $|\Lambda_1^r v_i|_{0,m,T}$, $|\Lambda_1^r v_i|_{\infty,m,T}$, etc., are now used in place of $|\Lambda^r v_i|_{0,m,\gamma}$, $|\Lambda^r v_i|_{\infty,m,\gamma}$, etc..²⁴ Similarly, norms $E_m(v_i)$ are defined using norms $|\Lambda_1^r v_i|_{0,m}$, $|\Lambda_1^r v_i|_{\infty,m}$, etc..

Finally, we define in the obvious way $E_{m,\gamma,T}(v)$, the time-localized version of $E_{m,\gamma}(v)$, using $|\Lambda^r v_i|_{0,m,\gamma,T}$, $|\Lambda^r v_i|_{\infty,m,\gamma,T}$, etc..

²²In each case the second term on the right is obtained from the first term by trading εD_{x_2} for a tangential derivative; this applies as well to the other three terms of (3.78) with (2) as a superscript.

²³In this weighting, εD_{x_2} must always be viewed like a tangential derivative of weight 1, not as the “composite” of ε (weight -2) and D_{x_2} (weight 2)

²⁴Recall parts (g)-(j) of Notations 3.10.

Proposition 3.15. *a) Let $T > 0$. For v and u defined on Ω we have*

$$(3.81) \quad E_{m,\gamma}(v) \geq E_{m,\gamma,T}(v) \geq C e^{-\gamma T} E_{m,T}(v).$$

b) For v and u initially defined in Ω_T with $v = 0$ in $t < 0$, one can choose a Seeley extension v_T^s to Ω [CP82] such that

$$(3.82) \quad \begin{aligned} E_{m,\gamma}(v_T^s) &\leq C E_{m,\gamma,T}(v_T^s) \leq C E_{m,T}(v_T^s), \\ E_{m,T}(v_T^s) &\leq C E_{m,T}(v_T^s), \text{ and} \\ \text{the } t\text{-support of } v^s &\text{ is contained in } [0, \frac{3}{2}T]. \end{aligned}$$

Here all constants C are independent of γ , ε , and T , and our Seeley extensions v_T^s have t -support in a fixed compact subset of $[0, 2)$ when $0 < T \leq 1$.

(c) The exact analogues of (a) and (b) hold for the norms $|w|_{m,\gamma}^$, $|w|_{m,\gamma,T}^*$, $|w|_{m,T}^*$, and $|w|_m^*$ defined in Definition 3.249.*

Proof. 1. For $T > 0$ the Seeley extension used here is defined first on sufficiently smooth functions $v(t, x, \theta)$, defined in $t \leq T$ and supported in $t \geq 0$, by

$$(3.83) \quad v^s(t, x, \theta) = \begin{cases} v(t, x, \theta), & t \leq T \\ \sum_{k=1}^M \lambda_k v(T + 2^k(T - t), x, \theta), & t > T \end{cases},$$

where M is sufficiently large (depending on m) and the λ_k are chosen to satisfy $\sum_{k=1}^M \lambda_k 2^{kj} = (-1)^j$ for $j = 0, \dots, M-1$. Observe that all terms in the sum have t -support in $[0, \frac{3}{2}T]$. The definition is extended by continuity to more general functions.

2. Inequalities corresponding to (3.81) and (3.82) are proved for the individual norms appearing in the definition of $E_{m,\gamma}(v)$. Since we can replace $\Lambda_D w$ by $((\partial_{x'} + \beta \frac{\partial \theta}{\varepsilon})w, \gamma w)$, such inequalities are obvious for the norms that involve only integral powers of Λ_D . The inequalities in the non-integral cases then follow by interpolation. \square

3.3.2 Strategy

By Proposition 3.83 for each fixed $\varepsilon \in (0, 1]$ we have classical solutions $v^\varepsilon, u^\varepsilon$ to the quasilinear singular systems (3.11), (3.12), and (3.13) on Ω_{T_ε} for some $T_\varepsilon > 0$. Moreover, we have $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_{T_ε} . Below we often suppress the superscript ε on u and v .

Let $T_0 < 1$ and M_0 be positive constants that will be chosen later to be sufficiently small (independently of ε, γ , and T). For a fixed $m > 3d + 4 + \frac{d+1}{2}$ define for each $\varepsilon \in (0, 1]$

$$(3.84) \quad T_\varepsilon^* = \sup\{T \in (0, \min(T_0, \frac{T_\varepsilon}{2})] : E_{m,T}(v^\varepsilon) \leq M_0\},$$

a number that might converge to 0 as $\varepsilon \rightarrow 0$, as far as we know now. We will use the a priori estimate of Proposition 3.20 and a continuation argument to prove the following proposition.

Proposition 3.16. *We make the same assumptions as in Theorem 3.7, and let $v^\varepsilon, u^\varepsilon$ on Ω_{T_ε} satisfying $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ be solutions to the quasilinear singular systems (3.11), (3.12), and (3.13) provided by Proposition 3.83. Define T_ε^* as in (3.84). There exist $\varepsilon_0 > 0$ and T_1 independent of $\varepsilon \in (0, \varepsilon_0]$ such that*

$$(3.85) \quad T_\varepsilon^* \geq T_1 > 0 \text{ for all } \varepsilon \in (0, \varepsilon_0].$$

For any T satisfying $0 < T \leq T_\varepsilon^*$ we let v_T^s denote a Seeley extension of $v|_{\Omega_T}$ to Ω chosen as in Proposition 3.15. We suppress the T -dependence of v_T^s below and write v^s for v_T^s .

Consider now the following three *linear* systems for the respective unknowns v_1 , v_2 , and u . We choose $\chi_0(t) \geq 0$ to be a C^∞ function that is equal to 1 on a neighborhood of $[0, 1]$ and supported in $(-1, 2)$.

$$(3.86) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_1 + \sum_{|\alpha|=2} A_\alpha(v^s) \partial_{x,\varepsilon}^\alpha v_1 &= - \left[\sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_1,\varepsilon} (A_\alpha(v^s)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_1^s - \partial_{x_1,\varepsilon} (A_{(0,2)}(v^s)) \partial_{x_2} v_2^s \right] \\ \partial_{x_2} v_1 - d_{v_1} H(v_1^s, h(v^s)) \partial_{x_1,\varepsilon} v_1 &= d_g H(v_1^s, \varepsilon^2 G) \partial_{x_1,\varepsilon} (\varepsilon^2 G) \text{ on } x_2 = 0. \end{aligned}$$

$$(3.87) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_2 + \sum_{|\alpha|=2} A_\alpha(v^s) \partial_{x,\varepsilon}^\alpha v_2 &= - \left[\sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_2} (A_\alpha(v^s)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_1^s - \partial_{x_2} (A_{(0,2)}(v^s)) \partial_{x_2} v_2^s \right] \\ v_2 &= \chi_0(t) H(v_1, \varepsilon^2 G) \text{ on } x_2 = 0. \end{aligned}$$

$$(3.88) \quad \begin{aligned} \partial_{t,\varepsilon}^2 u + \sum_{|\alpha|=2} A_\alpha(v^s) \partial_{x,\varepsilon}^\alpha u &= 0 \\ \partial_{x_2} u - d_{v_1} H(v_1^s, h(v^s)) \partial_{x_1,\varepsilon} u &= [H(v_1^s, \varepsilon^2 G(x', \theta)) - d_{v_1} H(v_1^s, \varepsilon^2 G) v_1^s] \text{ on } x_2 = 0. \end{aligned}$$

Remark 3.17. 1. The above three systems are solved on the full domain Ω . Each system depends on the parameters ε and T .

2. The definition of $v^s = v_T^s$ and causality (see the remarks right after Theorem 3.84) imply that the solutions v_1 , v_2 , and u of (3.86), (3.87), and (3.88) are equal to the corresponding solutions of the systems (3.11), (3.12), and (3.13) on Ω_T . The right sides of these equations depend on v_T^s , so the solutions v and u change as T changes, but we suppress this T -dependence in the notation.

3. We recall that the t -support of v^s is contained in $[0, \frac{3}{2}T]$. Since we always have $0 \leq 2T \leq T_\varepsilon$ and $v = \nabla_\varepsilon u$ on Ω_{T_ε} , it follows that $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on $\text{supp } v^s$ and $v^s = \nabla_\varepsilon u^s$.

We will estimate solutions to (3.86) and (3.87) by applying the estimates of section 3.2, where \mathcal{F}_1 and \mathcal{G}_1 are given by the right sides of the interior and boundary equations of (3.86), and where \mathcal{F}_2 and \mathcal{G}_2 are given by the right sides of (3.87). Note that the factor $\partial_{x_1,\varepsilon}(\varepsilon^2 G)$ occurs in \mathcal{G}_1 , and that the singular derivative here “uses up” one of the two factors of ε on G . The remaining factor is used up by Λ_D in a term like $\langle \Lambda \mathcal{G}_1 \rangle_{0,\gamma}^2$, which occurs on the right side of (3.37). On the other hand there is still a factor of $\varepsilon^{1/2}$ “to spare” in a term like $\langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{0,\gamma}^2$ which occurs on the right side of (3.40).

Remark 3.18. 1. Observe that v_1 , not v_1^s , occurs on the right in the boundary condition for (3.87); the reason is the following. When we apply the estimate (3.40) to estimate (say) $|\Lambda^{\frac{3}{2}} v_2|_{0,m,\gamma}$, we need to bound $\langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m,\gamma}$, for $j \in J_h$. Since \mathcal{G}_2 depends on v_1 , Proposition 3.29 implies that to estimate $\langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m,\gamma}$ we need control of $\langle \phi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m,\gamma}$, $j \in J_h$. That control comes from the estimate (3.37). If v_1^s appeared as an argument of \mathcal{G}_2 in the boundary condition for (3.87), we would have to estimate $\langle \phi_j \Lambda^{\frac{3}{2}} v_1^s \rangle_{m,\gamma}$ instead, but we know of no way to do this. The estimate (3.37) does not apply to v_1^s , since v_1^s is not a solution of (3.86) on Ω (just on Ω_T). In addition, we cannot use a result like Proposition 3.15 to deduce control of $\langle \phi_j \Lambda^{\frac{3}{2}} v_1^s \rangle_{m,\gamma}$ from control of $\langle \phi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m,\gamma}$, since we know of no such result that applies to norms involving $\phi_{j,D}$; we have no way to localize such norms in time.

On the other hand, since $v = v^s = v_T^s$ on Ω_T , we have $E_{m,T}(v) = E_{m,T}(v^s)$, so we can use Proposition 3.15 to deduce control of $E_{m,\gamma}(v^s)$ from control of $E_{m,\gamma}(v)$. There is no version of Proposition 3.15 for norms involving $\phi_{j,D}$, and that is one reason we do not include such norms in the definition of $E_{m,\gamma}(v)$.

2. The estimate $E_{m,T}(v^\varepsilon) \leq M_0$, valid for $0 < T \leq T_\varepsilon^*$, and the fact that the coefficients of our systems are functions of $v^s = v_T^s$, imply that the constants appearing in the estimates will be uniform with respect to ε .

We suppose that $G \in H^{m+3}(b\Omega)$ and for $T_0 > 0$ define the norm

$$(3.89) \quad N_{m,T_0}(\varepsilon^2 G) := \langle \Lambda_1^2 \varepsilon^2 G \rangle_{m,T_0} + \langle \Lambda_1 \varepsilon G \rangle_{m,T_0} + \langle \Lambda_1^{\frac{3}{2}} \varepsilon^{\frac{3}{2}} G \rangle_{m+1,T_0}.$$

We choose a Seeley extension of $G|_{\Omega_{T_0}}$ (also denoted G) to $H^{m+3}(b\Omega)$, compactly supported in t , such that the obvious analogue of (3.82) is satisfied. Observe that

$$(3.90) \quad N_{m,T_0}(\varepsilon^2 G) \lesssim \langle G \rangle_{m+3,T_0} := M_G \text{ for } \varepsilon \in (0, 1].$$

Remark 3.19. We will eventually need to take M_G small. This can be arranged for a given choice of G by taking T_0 small, since G vanishes in $t < 0$. Alternatively, for a given T_0 , one can adjust the choice of G to make M_G small. In this paper we use the first option. We can and do always suppose that $M_G \leq 1$.

The main step in showing Proposition 3.16 is the following a priori estimate, whose proof is concluded at the end of section 3.5.

Proposition 3.20. Let $m > 3d + 4 + \frac{d+1}{2}$ and $G \in H^{m+3}(b\Omega)$. There exist positive constants M_0 as in (3.84) and M_G as in (3.90) such that the following is true. There exist positive constants ε_0, γ_0 , and there exist increasing functions $Q_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2$, with $Q_i(z) \geq z$ such that for $\varepsilon \in (0, \varepsilon_0]$ and each T with $0 < T \leq T_\varepsilon^*$, the solution to (3.86)-(3.88) satisfies

$$(3.91) \quad E_{m,\gamma}(v) \leq \gamma^{-1} E_{m,\gamma}(v) Q_1(E_{m,T}(v^s)) + (\gamma^{-\frac{1}{2}} + \sqrt{\varepsilon}) Q_2(E_{m,T}(v^s)) \text{ for } \gamma \geq \gamma_0.$$

Assuming this proposition, we now prove Proposition 3.16.

Proof of Proposition 3.16. Since $E_{m,T}(v^s) \leq M_0$ we can choose γ_0 so that

$$(3.92) \quad \gamma^{-1} E_{m,\gamma}(v) Q_1(E_{m,T}(v^s)) \leq E_{m,\gamma}(v)/2 \text{ for } \gamma \geq \gamma_0.$$

Thus, for some $C > 0$

$$(3.93) \quad C e^{-\gamma T} E_{m,T}(v) \leq E_{m,\gamma}(v) \leq 2(\gamma^{-\frac{1}{2}} + \sqrt{\varepsilon}) Q_2(E_{m,T}(v)) \text{ for } \gamma \geq \gamma_0, \varepsilon \in (0, \varepsilon_0].$$

For any $T \in (0, T_\varepsilon^*]$ such that $\frac{1}{T} \geq \gamma_0$, we can take $\gamma = 1/T$ in (3.93) to obtain

$$(3.94) \quad E_{m,T}(v) \leq 2 \frac{e}{C} (\sqrt{T} + \sqrt{\varepsilon}) Q_2(M_0) \text{ for } \varepsilon \in (0, \varepsilon_0].$$

Clearly, for $N \in \mathbb{N}$ one can choose ε_0 and $T_1 = T_1(M_0, M_G)$ independent of ε such that the right side of (3.94) is $< M_0/N$. However, to finish we need to know that $T_1 \leq T_\varepsilon^*$ for all $\varepsilon \in (0, \varepsilon_0]$. If not, there is some $\varepsilon \in (0, \varepsilon_0]$ such that $T_1 > T_\varepsilon^*$. But then $E_{m,T_\varepsilon^*,-}(v^\varepsilon) < M_0/N$, where $T_\varepsilon^{*,-} < T_\varepsilon^*$ is as close as we like to T_ε^* . For N large enough we can then use our local continuation result for fixed ε , Proposition 3.87, to continue v^ε to a time $T_c > T_\varepsilon^*$ such that $E_{m,T_c}(v^\varepsilon) < M_0$, a contradiction.²⁵

□

²⁵The proof of Prop. 3.87 shows that the size of N depends on the norm of the Seeley extension operator used there.

Clearly, we need to study how the singular norms appearing on the right in (3.37)-(3.42) act on nonlinear functions of v , v_2 , and $\varepsilon^2 G$. We also need to show that microlocal regularity of functions like $\phi_{j,D} \Lambda_D v_1^\gamma$ is preserved under nonlinear functions. This study is carried out in section 3.4, and Proposition 3.20 is proved in section 3.5.

3.4 Singular norms of nonlinear functions

Here we take $x' = (x_0, x'') = (t, x'') \in \mathbb{R}^d$, $\theta \in \mathbb{R}$ and dual variables $\xi \in \mathbb{R}^d$, $k \in \mathbb{R}$, and (in this section) we drop the prime on x . We also fix $\beta \in \mathbb{R}^d \setminus 0$, set $X = \xi + \beta \frac{k}{\varepsilon}$, and write

$$(3.95) \quad \partial_\varepsilon := \partial_x + \beta \frac{\partial_\theta}{\varepsilon}, \quad \partial_{\varepsilon, \gamma} := (\partial_t + \gamma, \partial_{x''}) + \beta \frac{\partial_\theta}{\varepsilon}.$$

Observe that

$$(3.96) \quad \partial_{\varepsilon, \gamma}(e^{-\gamma t} u) = e^{-\gamma t} \partial_\varepsilon u.$$

We let Λ denote the singular symbol $\Lambda = \sqrt{|X|^2 + \gamma^2}$ and $\Lambda_1 = \sqrt{|X|^2 + 1}$.

Sometimes, for example in convolutions, we will use the variables $y \in \mathbb{R}^d$, $\omega \in \mathbb{R}$, dual variables $\eta \in \mathbb{R}^d$, $l \in \mathbb{R}$, and set $Y = \eta + \beta \frac{l}{\varepsilon}$.

In this section we consider functions $u = u(x, \theta)$. Unless otherwise noted, all constants below are independent of $\gamma \geq 1$ and $\varepsilon \in (0, 1]$. In every estimate we assume that the functions in question are such that the norms appearing on the right are finite.

The following proposition is used repeatedly in this section.

Proposition 3.21 ([RR82], Lemma 1.2.2). *Let $\varepsilon \in (0, 1]$ and take $\gamma \geq 1$. Suppose $G_{\varepsilon, \gamma} : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \rightarrow \mathbb{C}$ is a locally integrable measurable function that can be decomposed into a finite sum (suppress ε, γ)*

$$(3.97) \quad G(\xi, k, \eta, l) = \sum_{j=1}^K G_j(\xi, k, \eta, l)$$

such that for each j we have either

$$(3.98) \quad \sup_{\xi, k} \int |G_j(\xi, k, \eta, l)|^2 d\eta dl < C \text{ or } \sup_{\eta, l} \int |G_j(\xi, k, \eta, l)|^2 d\xi dk < C,$$

where C is independent of (ε, γ) . Then

$$(3.99) \quad (f, g) \rightarrow \int G(\xi, k, \eta, l) f(\xi - \eta, k - l) g(\eta, l) d\eta dl$$

defines a continuous bilinear map of $L^2 \times L^2 \rightarrow L^2$, and

$$(3.100) \quad \left| \int G(\xi, k, \eta, l) f(\xi - \eta, k - l) g(\eta, l) d\eta dl \right|_{L^2} \leq C |f|_{L^2} |g|_{L^2},$$

with C independent of (ε, γ) .

3.4.1 Non-tame estimates

We begin with the non-tame estimates that are used in the Proof of Proposition 3.20.

Proposition 3.22. *Let $m > (d+1)/2$, $r \geq 0$, and set $\langle u \rangle_m := |u|_{H^m}$. Assume the functions u and v are real-valued. Then*

- a) $\langle uv \rangle_m \leq C_1 \langle u \rangle_m \langle v \rangle_m$.
- b) $\langle \Lambda_1^r(uv) \rangle_m \leq C_2 [\langle \Lambda_1^r u \rangle_m \langle v \rangle_m + \langle u \rangle_m \langle \Lambda_1^r v \rangle_m]$.
- c) Let $C_3 = \max(C_1, C_2)$. For $n \in \mathbb{N}$, $\langle \Lambda_1^r(u^n) \rangle_m \leq n \langle \Lambda_1^r u \rangle_m \cdot (C_3 \langle u \rangle_m)^{n-1}$.
- d) Let $f(u)$ be a real-valued, (real-)analytic function satisfying $f(0) = 0$ with radius of convergence R at $u = 0$. If $C_3 \langle u \rangle_m < R$, then

$$(3.101) \quad \langle \Lambda_1^r f(u) \rangle_m \leq \langle \Lambda_1^r u \rangle_m \cdot h(C_3 \langle u \rangle_m),$$

where h is an analytic function with radius of convergence R at 0 with nonnegative coefficients.

Proof. **(a,b)** Part (a) follows directly from Proposition 3.21 by taking

$$(3.102) \quad G(\xi, k, \eta, l) = \frac{\langle \xi, k \rangle^m}{\langle \xi - \eta, k - l \rangle^m \langle \eta, l \rangle^m}$$

and consider the cases $|\eta, l| \leq \frac{1}{2}|\xi, k|$ and $|\eta, l| > \frac{1}{2}|\xi, k|$. To prove (b), use Proposition 3.21 in a similar way, together with the inequality

$$(3.103) \quad |X, 1|^r \leq C(|X - Y, 1|^r + |Y, 1|^r).$$

(c,d) Part (c) is a direct consequence of part (b), and (d) is proved by writing $f(z) = \sum_{n=1}^{\infty} a_n z^n$, applying (c), and taking $h(z) = \sum_{n=1}^{\infty} n|a_n|z^{n-1}$. □

Proposition 3.23. *Let $m > (d+1)/2$, $r \geq 0$. Assume that the functions u, v are real-valued. Then*

- a) $|uv|_{H_\gamma^m} \leq D_1 |u|_{H_\gamma^m} |v|_{H^m}$.
- b) $|\Lambda_D^r(uv)|_{H_\gamma^m} \leq D_2 \left[|\Lambda_D^r u|_{H_\gamma^m} |v|_{H^m} + |u|_{H_\gamma^m} |\Lambda_{1,D}^r v|_{H^m} \right]$.
- c) Let $D_3 = \max(C_1, D_2)$. Let $f(u)$ be a real-valued, (real-)analytic function satisfying $f(0) = 0$ with radius of convergence R at $u = 0$. If $D_3 \langle u \rangle_m < R$, then

$$(3.104) \quad |\Lambda_D^r f(u)|_{H_\gamma^m} \leq |\Lambda_D^r u|_{H_\gamma^m} h_1(D_3 \langle u \rangle_m) + |u|_{H_\gamma^m} \langle \Lambda_1^r u \rangle_m h_2(D_3 \langle u \rangle_m),$$

where the h_i are analytic functions with radius of convergence R at 0 with nonnegative coefficients.

Proof. **(a,b)** Part (a) is proved just like Proposition 3.22 (a). To prove (b) use Proposition 3.21 with the inequality

$$(3.105) \quad |X, \gamma|^r \leq C(|X - Y, \gamma|^r + |Y, 1|^r),$$

and proceed as in the proof of Proposition 3.22, part (b).

(c) Using part (b) we obtain

$$(3.106) \quad \begin{aligned} |\Lambda_D^r f(u)|_{H_\gamma^m} &= |\Lambda_D^r (f'(0)u + g(u)u^2)|_{H_\gamma^m} \leq \\ &C |\Lambda_D^r u|_{H_\gamma^m} + D_2 \left[|\Lambda_D^r u|_{H_\gamma^m} \langle ug(u) \rangle_m + |u|_{H_\gamma^m} |\Lambda_{1,D}(ug(u))|_{H^m} \right]. \end{aligned}$$

To finish, apply Proposition 3.22(a),(d) to the D_2 term. □

²⁶The proof shows that we could replace D_3 by C_1 in the h_1 term.

Recalling that $\langle \Lambda^r u \rangle_{m,\gamma} := |\Lambda_D^r(e^{-\gamma t} u)|_{H_\gamma^m}$ and $\langle \Lambda_1^r u \rangle_m := |\Lambda_{1,D}^r u|_{H^m}$ (as in Notations 3.10), we have the following immediate corollary of Proposition 3.23.

Corollary 3.24. *Let $m > (d+1)/2$, $r \geq 0$. Then*

- a) $\langle \Lambda^r(uv) \rangle_{m,\gamma} \leq C_2 [\langle \Lambda^r u \rangle_{m,\gamma} \langle v \rangle_m + \langle u \rangle_{m,\gamma} \langle \Lambda_1^r v \rangle_m]$.
- b) For f , h_1 , h_2 as in Proposition 3.23:

$$(3.107) \quad \langle \Lambda^r f(u) \rangle_{m,\gamma} \leq \langle \Lambda^r u \rangle_{m,\gamma} h_1(D_3 \langle u \rangle_m) + \langle u \rangle_{m,\gamma} \langle \Lambda_1^r u \rangle_m h_2(D_3 \langle u \rangle_m).$$

It is now straightforward to generalize these results to real-valued, real-analytic functions $f(u) = f(u_1, \dots, u_K) = u_1 g(u)$ of K real variables converging in a polydisk $\mathbb{D} = \{u : |u_i| < r_i\}$ for some $r_i > 0$.

Proposition 3.25. *Let $f(u) = f(u_1, \dots, u_K)$ be as just described. Let C_3 , D_3 be as in Propositions 3.22, 3.23 respectively. Then*

(3.108)

$$(a) \langle \Lambda_1^r f(u_1, \dots, u_K) \rangle_m \leq \sum_{j=1}^K \langle \Lambda_1^r u_j \rangle_m h_j(\langle C_3 u_1 \rangle_m, \dots, \langle C_3 u_K \rangle_m),$$

$$(b) |\Lambda_D^r f(u)|_{H_\gamma^m} \leq |\Lambda_D^r u_1|_{H_\gamma^m} k(\langle D_3 u_1 \rangle_m, \dots, \langle D_3 u_K \rangle_m) + |u_1|_{H_\gamma^m} \sum_{j=1}^K \langle \Lambda_1^r u_j \rangle_m k_j(\langle D_3 u_1 \rangle_m, \dots, \langle D_3 u_K \rangle_m),$$

$$(c) \langle \Lambda^r f(u) \rangle_{m,\gamma} \leq \langle \Lambda^r u_1 \rangle_{m,\gamma} k(\langle D_3 u_1 \rangle_m, \dots, \langle D_3 u_K \rangle_m) + \langle u_1 \rangle_{m,\gamma} \sum_{j=1}^K \langle \Lambda_1^r u_j \rangle_m k_j(\langle D_3 u_1 \rangle_m, \dots, \langle D_3 u_K \rangle_m),$$

where the functions h_j , k , k_j are real-analytic functions of u with nonnegative coefficients converging in the polydisk \mathbb{D} .

Next we give microlocal estimates reminiscent of ‘‘Rauch’s Lemma’’ [Rau79], except that they involve singular norms and microlocalization by singular pseudodifferential operators associated to symbols $\phi = \phi(X, \gamma)$. In these estimates we continue to assume that the functions in question are such that the norms appearing on the right are finite. We also make the following assumption on the \mathbb{R} -valued functions u , v , or u_j that appear.

Assumption 3.26. *Let $u(x, \theta)$ be \mathbb{R} -valued. Suppose $r \geq 1$, $m > (d+1)/2$ and let $(\xi_0, \gamma_0) \in \mathbb{R}^{d+1} \setminus 0$. We suppose that there is a conic neighborhood $\Gamma \subset \mathbb{R}^{d+1} \setminus 0$ of (ξ_0, γ_0) such that if $\psi(X, \gamma)$ is any singular symbol of order 0 (Definition A.2) with $\text{supp } \psi \subset\subset \Gamma$, we have*

$$(3.109) \quad \psi_D \Lambda_D^r e^{-\gamma t} u \in H^m \text{ and } \Lambda_D^{r-\frac{1}{2}} e^{-\gamma t} u \in H^m.$$

Notations 3.27. *In the microlocal estimates below we use operators defined by singular symbols $\psi(X, \gamma)$ and $\phi(X, \gamma)$ of order 0 with the following properties. For Γ as in Assumption 3.26 there is a conic neighborhood $\Gamma_1 \subset\subset \Gamma$ such that:*

- a) $\text{supp } \psi \subset\subset \Gamma$ and $\psi = 1$ on $\Gamma_1 \cap \{|X, \gamma| \geq 1\}$.
- b) $\text{supp } \phi \subset\subset \Gamma_1$.

Proposition 3.28. *Let $g(u)$ be a real-valued, real-analytic function with radius of convergence R and set $f(u) = u g(u)$. Under Assumption 3.26 on u and with ϕ , ψ as in Notations 3.27, we have $\phi_D \Lambda_D^r e^{-\gamma t} f(u) \in H^m$ and there exists $C_1 > 0$ such that (suppress subscripts D on ϕ , ψ , Λ , Λ_1):*

$$(3.110) \quad \langle \phi \Lambda^r f(u) \rangle_{m,\gamma} \leq A + B, \text{ where}$$

$$(3.111) \quad \begin{aligned} A &= \langle \Lambda^{r-\frac{1}{2}}u \rangle_{m,\gamma} \langle \Lambda_1^{r-\frac{1}{2}}u \rangle_m k_1 \\ B &= \langle \psi \Lambda^r u \rangle_{m,\gamma} k_2 + \langle \psi \Lambda u \rangle_{m,\gamma} \langle \Lambda_1^{r-1}u \rangle_m k_3, \end{aligned}$$

where the k_i are analytic functions with power series having radius of convergence R at 0 and nonnegative coefficients, and evaluated at $(C_1\langle u \rangle_m)$.

Proof. 1. Using (3.96) we have $\langle \phi \Lambda^r (ug(u)) \rangle_{m,\gamma} =$

$$(3.112) \quad |\phi \Lambda^{r-1} \Lambda(e^{-\gamma t} ug(u))|_{H_\gamma^m} \sim |\phi \Lambda^{r-1} \partial_{\varepsilon,\gamma}(e^{-\gamma t} ug(u))|_{H_\gamma^m} = |\phi \Lambda^{r-1}(e^{-\gamma t} \partial_\varepsilon(ug(u)))|_{H_\gamma^m},$$

which is $\leq A_1 + A_2$, where

$$(3.113) \quad A_1 = |\phi \Lambda^{r-1}(e^{-\gamma t} \partial_\varepsilon u g(u))|_{H_\gamma^m}, \quad A_2 = |\phi \Lambda^{r-1}(ug'(u)e^{-\gamma t} \partial_\varepsilon u)|_{H_\gamma^m}.$$

2. Write $e^{-\gamma t} \partial_\varepsilon u = \psi_D(e^{-\gamma t} \partial_\varepsilon u) + (1 - \psi_D)(e^{-\gamma t} \partial_\varepsilon u) := w_1 + w_2$. Then $A_1 \leq A_{1,1} + A_{1,2}$, where

$$(3.114) \quad A_{1,1} = |\phi \Lambda^{r-1}(w_1 g(u))|_{H_\gamma^m} \leq |\Lambda^{r-1}(w_1 g(u))|_{H_\gamma^m} \leq \langle \psi \Lambda^r u \rangle_{m,\gamma} k_1 + \langle \psi \Lambda u \rangle_{m,\gamma} \langle \Lambda_1^{r-1}u \rangle_m k_2$$

by Proposition 3.23, and $A_{1,2} = |\phi \Lambda^{r-1}(w_2 g(u))|_{H_\gamma^m} \leq^{27}$

$$(3.115) \quad |\phi \Lambda^{r-1}(w_2 g(0))|_{H_\gamma^m} + |\phi \Lambda^{r-1}(w_2 uh(u))|_{H_\gamma^m} = |\phi \Lambda^{r-1}(w_2 uh(u))|_{H_\gamma^m}.$$

We claim

$$(3.116) \quad |\phi \Lambda^{r-1}(w_2 uh(u))|_{H_\gamma^m} \leq C \langle \Lambda_1^{r-\frac{1}{2}}(uh(u)) \rangle_m \langle \Lambda^{r-\frac{1}{2}}u \rangle_{m,\gamma}$$

To see this we apply Proposition 3.21 with $G_{\varepsilon,\gamma}(\xi, k, \eta, l) =$

$$(3.117) \quad \frac{\phi(X, \gamma)(1 - \psi(Y, \gamma)) \langle X, \gamma \rangle^{r-1} \langle \xi, k, \gamma \rangle^m}{\langle X - Y, 1 \rangle^{r-\frac{1}{2}} \langle \xi - \eta, k - l, 1 \rangle^m \langle \eta, l, \gamma \rangle^m |Y, \gamma|^{r-\frac{3}{2}}},$$

noting that $r - \frac{3}{2} \geq -\frac{1}{2}$ and that on the support of $G_{\varepsilon,\gamma}$ we have

$$(3.118) \quad |(X, \gamma) - (Y, \gamma)| = |(X - Y, 0)| \geq C|X, \gamma| \text{ and } |(X - Y, 0)| \geq C|Y, \gamma|.$$

By Proposition 3.22(d) the right side of (3.116) is bounded by $\langle \Lambda^{r-\frac{1}{2}}u \rangle_{m,\gamma} \langle \Lambda_1^{r-\frac{1}{2}}u \rangle_m k_3$.

3. The estimate of A_2 clearly yields terms of the same form. □

Easy modifications of the previous proof yield the following analogue of Proposition 3.28 for $f(u_1, \dots, u_K)$.

Proposition 3.29. *Let $g(u) = g(u_1, \dots, u_K)$ be a real-valued, real-analytic function of K real variables that converges in a polydisk $\mathbb{D} = \{u : |u_i| < R_i\}$ for some $R_i > 0$. Let $r \geq 1$ and suppose $m > (d+1)/2$. Set $f(u) = u_1 g(u)$. Under Assumption 3.26 on the u_j and with ϕ, ψ as in Notations 3.27, we have $\phi_D \Lambda_D^r e^{-\gamma t} f(u) \in H^m$ and there exists $C > 0$ such that (suppress subscripts D on ϕ, ψ, Λ_1 , and Λ)*

$$(3.119) \quad \langle \phi \Lambda^r f(u) \rangle_{m,\gamma} \leq A + B, \text{ where}$$

²⁷We use $\phi(1 - \psi) = 0$ in (3.115).

$$(3.120) \quad \begin{aligned} A &= \langle \Lambda^{r-\frac{1}{2}} u_1 \rangle_{m,\gamma} \left(\sum_{j=1}^K \langle \Lambda_1^{r-\frac{1}{2}} u_j \rangle_m h_j \right) \\ B &= \sum_{i=1}^K \langle \psi \Lambda^r u_i \rangle_{m,\gamma} k_i + \sum_{i=1}^K \sum_{j=1}^K \langle \psi \Lambda u_i \rangle_{m,\gamma} \langle \Lambda_1^{r-1} u_j \rangle_m k_{i,j}. \end{aligned}$$

Here the functions h_j , k_i , and $k_{i,j}$ are real-analytic functions of u with nonnegative coefficients, converging in the polydisk \mathbb{D} , and evaluated at $(\langle Cu_1 \rangle_m, \dots, \langle Cu_K \rangle_m)$.

Remark 3.30. 1.) We shall apply Proposition 3.29 with $r = 3/2$ or $r = 1$.

2.) To estimate $\langle \phi \Lambda^r f(u) \rangle_{m,\gamma}$ for any real-analytic function $f(u)$ of $u = (u_1, \dots, u_K)$ such that $f(0) = 0$, we write

$$(3.121) \quad f(u) = \sum_{i=1}^K u_i g_i(u)$$

and apply Proposition 3.29 to the individual terms in this sum, letting the various u_i consecutively play the role of u_1 .

3) Observe that the real-analyticity of the function $f(u)$ in Proposition 3.29 is needed only for the parts of the proof that use Proposition 3.25. If one knew that Proposition 3.25 held for (say) $f \in C^\infty$, then Proposition 3.29 would be deducible by the same ‘‘chain rule’’ proof for such f .

4) When $r \geq 1$, one can prove Proposition 3.25 for $f \in C^\infty$ inductively by writing $|\Lambda_D^r e^{-\gamma t} f(u)|_{H_\gamma^m} \sim |\Lambda^{r-1}(\partial_{\varepsilon,\gamma} e^{-\gamma t} f(u))|_{H_\gamma^m}$. However, we need Proposition 3.25 for $r = 1/2$ as well as $r = 1$ and $3/2$ in the applications here. See, for example, the estimate (3.37).²⁸

In estimates of commutators we need the following refinement of Proposition 3.23(b).

Proposition 3.31. Let $r \geq 0$, and suppose $0 \leq \sigma \leq \min(s, t)$, where $s + t - \sigma > \frac{d+1}{2}$. Then

$$(3.122) \quad \begin{aligned} (a) \quad & |\Lambda_D^r(uv)|_{H_\gamma^\sigma} \lesssim |\Lambda_D^r u|_{H_\gamma^s} |v|_{H^t} + |u|_{H_\gamma^s} |\Lambda_{1,D}^r v|_{H^t} \\ (b) \quad & \langle \Lambda^r(uv) \rangle_{\sigma,\gamma} \lesssim \langle \Lambda^r u \rangle_{s,\gamma} \langle v \rangle_t + \langle u \rangle_{s,\gamma} \langle \Lambda_1^r v \rangle_t. \end{aligned}$$

Proof. Using $|X, \gamma|^r \lesssim |X - Y, \gamma|^r + |Y, 1|^r$, we have $|\Lambda_D^r(uv)|_{H_\gamma^\sigma} =$

$$(3.123) \quad \begin{aligned} & \left\| |X, \gamma|^r \langle \xi, k, \gamma \rangle^\sigma (\hat{u} * \hat{v}) \right\|_{L^2(\xi,k)} \lesssim \\ & \left\| |X - Y, \gamma|^r \langle \xi, k, \gamma \rangle^\sigma (|\hat{u}| * |\hat{v}|) \right\|_{L^2(\xi,k)} + \left\| |Y, 1|^r \langle \xi, k, \gamma \rangle^\sigma (|\hat{u}| * |\hat{v}|) \right\|_{L^2(\xi,k)} := A + B. \end{aligned}$$

We estimate A and B by applying Proposition 3.21 to $G(\xi, k, \eta, l) =$

$$(3.124) \quad \frac{|X - Y, \gamma|^r \langle \xi, k, \gamma \rangle^\sigma}{|X - Y, \gamma|^r \langle \xi - \eta, k - l, \gamma \rangle^s \langle \eta, l \rangle^t} \quad \text{and} \quad \frac{|Y, 1|^r \langle \xi, k, \gamma \rangle^\sigma}{\langle \xi - \eta, k - l, \gamma \rangle^s |Y, 1|^r \langle \eta, l \rangle^t}$$

respectively. □

²⁸We have been unable to adapt either Gagliardo-Nirenberg type arguments or arguments involving (double) dyadic decompositions to estimate singular norms of nonlinear C^∞ functions of u . The various estimates of this section work well, though, for singular norms of analytic functions of u .

In section 3.6 we work with the following norms on the half space that involve higher normal derivatives.

Definition 3.32. Let $\mathbb{R}_+^{d+1} = \{x = (x_0, x_1, \dots, x_d) : x_d \geq 0\}$ For functions $u(x, \theta)$ with $(x, \theta) \in \mathbb{R}^{d+1} \times \mathbb{R}$, $r \geq 0$, $m \in \mathbb{N}$, we define

$$(3.125) \quad \begin{aligned} |\Lambda^r u|_{m, \gamma} &:= \sum_{j=0}^m |\Lambda^r \partial_{x_d}^j u|_{0, m-j, \gamma}, \\ |\Lambda_1^r u|_m &:= \sum_{j=0}^m |\Lambda_1^r \partial_{x_d}^j u|_{0, m-j}. \end{aligned}$$

Time localized versions of these norms, $|\Lambda^r u|_{m, \gamma, T}$, $|\Lambda_1^r u|_{m, T}$, are defined in the usual way.

We will need analogues of some of the above results for these norms.

Proposition 3.33. Let $m > (d+2)/2$, $r \geq 0$. Then

- a) $|\Lambda^r(uv)|_{m, \gamma} \lesssim |\Lambda^r u|_{m, \gamma} |v|_m + |u|_{m, \gamma} |\Lambda_1^r v|_m$.
- b) For f , h_1 , h_2 as in Proposition 3.23 and real-valued $u(x, \theta)$:

$$(3.126) \quad |\Lambda^r f(u)|_{m, \gamma} \leq |\Lambda^r u|_{m, \gamma} h_1(C|u|_m) + |u|_{m, \gamma} |\Lambda_1^r u|_m h_2(C|u|_m).$$

- (c) Similarly, the analogue of Proposition 3.25 holds for $|\Lambda^r f(u_1, \dots, u_k)|_{m, \gamma}$.

After taking extensions of u and v into $x_d < 0$, one can use the Fourier transform in all variables to prove this Proposition just as before.

3.4.2 Tame estimates

Next we provide the tame estimates that we need for the proof of the continuation argument of section 3.6. In these estimates a lower Sobolev-type norm takes the role sometimes played by an L^∞ or Lipschitz norm in tame estimates. We prove the first four results below after the statement of Proposition 3.37.

Proposition 3.34. Let $m_0 > (d+1)/2$, $m \geq 0$, $r \geq 0$, and set $\langle u \rangle_m := |u|_{H^m}$. Assume the functions u and v are real-valued. Then

- a) $\langle uv \rangle_m \leq C_1(\langle u \rangle_m \langle v \rangle_{m_0} + \langle u \rangle_{m_0} \langle v \rangle_m)$.
- b) $\langle \Lambda_1^r(uv) \rangle_m \leq C_2[\langle \Lambda_1^r u \rangle_m \langle v \rangle_{m_0} + \langle u \rangle_m \langle \Lambda_1^r v \rangle_{m_0} + \langle \Lambda_1^r u \rangle_{m_0} \langle v \rangle_m + \langle u \rangle_{m_0} \langle \Lambda_1^r v \rangle_m]$.
- c) For $n \in \mathbb{N}$, $\langle \Lambda_1^r(u^n) \rangle_m \leq [\langle \Lambda_1^r u \rangle_m \cdot n(C_3 \langle u \rangle_{m_0})^{n-1} + \langle \Lambda_1^r u \rangle_{m_0} \langle u \rangle_m \cdot n(n-1)(C_3 \langle u \rangle_{m_0})^{n-2}]$.
- d) Let $f(u)$ be a real-valued, (real-)analytic function satisfying $f(0) = 0$ with radius of convergence R at $u = 0$. If $C_3 \langle u \rangle_m < R$, then

$$(3.127) \quad \langle \Lambda_1^r f(u) \rangle_m \leq [\langle \Lambda_1^r u \rangle_m + \langle u \rangle_m \langle \Lambda_1^r u \rangle_{m_0}] \cdot h(C_3 \langle u \rangle_{m_0}),$$

where h is an analytic function with nonnegative coefficients and radius of convergence R at 0.

Proposition 3.35. Let $m_0 > (d+1)/2$, $m \geq 0$, $r \geq 0$. Assume that the functions u , v are real-valued. Then

- a) $|uv|_{H_\gamma^m} \leq C[|u|_{H_\gamma^m} \langle v \rangle_{m_0} + \langle u \rangle_{m_0} |v|_{H_\gamma^m}]$.
- b) $|\Lambda_D^r(uv)|_{H_\gamma^m} \leq C[|\Lambda_D^r u|_{H_\gamma^m} \langle v \rangle_{m_0} + |u|_{H_\gamma^m} \langle \Lambda_1^r v \rangle_{m_0} + |\Lambda_D^r v|_{H_\gamma^m} \langle u \rangle_{m_0} + |v|_{H_\gamma^m} \langle \Lambda_1^r u \rangle_{m_0}]$.
- c) Let $f(u)$ be a real-valued, (real-)analytic function satisfying $f(0) = 0$ with radius of convergence R at $u = 0$. If $C \langle u \rangle_m < R$, then

$$(3.128) \quad |\Lambda_D^r f(u)|_{H_\gamma^m} \leq [|\Lambda_D^r u|_{H_\gamma^m} + |u|_{H_\gamma^m} \langle \Lambda_1^r u \rangle_{m_0}] \cdot h(C \langle u \rangle_{m_0}),$$

where h is an analytic function with nonnegative coefficients and radius of convergence R at 0.

Corollary 3.36. *Let $m_0 > (d+1)/2$, $m \geq 0$, $r \geq 0$. Then*

- (a) $\langle \Lambda^r(uv) \rangle_{m,\gamma} \leq C [\langle \Lambda^r u \rangle_{m,\gamma} \langle v \rangle_{m_0} + \langle u \rangle_{m,\gamma} \langle \Lambda_1^r v \rangle_{m_0} + \langle \Lambda^r v \rangle_{m,\gamma} \langle u \rangle_{m_0} + \langle v \rangle_{m,\gamma} \langle \Lambda_1^r u \rangle_{m_0}]$.
(b) *For f and h as in Proposition 3.35:*

$$(3.129) \quad \langle \Lambda^r f(u) \rangle_{m,\gamma} \leq [\langle \Lambda^r u \rangle_{m,\gamma} + \langle u \rangle_{m,\gamma} \langle \Lambda_1^r u \rangle_{m_0}] h(C \langle u \rangle_{m_0}).$$

Proposition 3.37. *Let $m_0 > (d+1)/2$, $m \geq 0$, $r \geq 0$. Suppose $f(u) = f(u_1, \dots, u_K)$ is a real-valued, real-analytic function of K real variables converging in a polydisk $\mathbb{D} = \{u : |u_i| < r_i\}$ for some $r_i > 0$ and such that $f(0) = 0$. Then*

$$(3.130) \quad \begin{aligned} (a) \langle \Lambda_1^r f(u_1, \dots, u_K) \rangle_m &\leq \sum_{j=1}^K \langle \Lambda_1^r u_j \rangle_m h_j + \sum_{j,k=1}^K \langle u_j \rangle_m \langle \Lambda_1^r u_k \rangle_{m_0} h_{j,k}, \\ (b) |\Lambda_D^r f(u_1, \dots, u_K)|_{H_\gamma^m} &\leq \sum_{j=1}^K |\Lambda_D^r u_j|_{H_\gamma^m} h_j + \sum_{j,k=1}^K |u_j|_{H_\gamma^m} \langle \Lambda_1^r u_k \rangle_{m_0} h_{j,k}, \\ (c) \langle \Lambda^r f(u_1, \dots, u_K) \rangle_{m,\gamma} &\leq \sum_{j=1}^K \langle \Lambda^r u_j \rangle_{m,\gamma} h_j + \sum_{j,k=1}^K \langle u_j \rangle_{m,\gamma} \langle \Lambda_1^r u_k \rangle_{m_0} h_{j,k}, \end{aligned}$$

where the functions $h_j, h_{j,k}$ are real-analytic functions of u with nonnegative coefficients converging in the polydisk \mathbb{D} , and evaluated at $(\langle Cu_1 \rangle_{m_0}, \dots, \langle Cu_K \rangle_{m_0})$ for some $C > 0$. They may change from line to line.

Proof. **1.** The previous four results all follow from Proposition 3.35(b). Proposition 3.35(a) follows by taking $r = 0$, while Proposition 3.34 (b) follows by taking $\gamma = 1$; moreover, Proposition 3.34(b) implies Proposition 3.34(a). Corollary 3.36(a) follows from Proposition 3.35(b) because of our freedom to switch $e^{-\gamma t}$ from u to v . The remaining parts of the above propositions, which concern $f(u)$, follow by repeated application of the parts already proved by using analyticity of f .

2. Proof of Proposition 3.35(b). We have

$$(3.131) \quad |\Lambda^r(uv)|_{H_\gamma^m} \lesssim \sum_{i=1}^2 \| |X, \gamma|^r \langle \xi, k, \gamma \rangle^m \chi_i |\hat{u}(\xi - \eta, k - l)| * |\hat{v}(\eta, l)| \|_{L^2(\xi, k)} = A_1 + A_2,$$

where $\chi_1(\xi, k, \eta, l)$ (resp. χ_2) is the characteristic function of $\{|\eta, l| \leq \frac{1}{2}|\xi, k|\}$ (resp. $\{|\eta, l| \geq \frac{1}{2}|\xi, k|\}$). To see that $A_1 \lesssim |\Lambda_D^r u|_{H_\gamma^m} \langle v \rangle_{m_0} + |u|_{H_\gamma^m} \langle \Lambda_1^r v \rangle_{m_0}$, we write $|X, \gamma|^r \lesssim |X - Y, \gamma|^r + |Y, 1|^r$ and apply Proposition 3.21 with $G(\xi, k, \eta, l)$ first equal to

$$(3.132) \quad \frac{|X - Y, \gamma|^r \langle \xi, k, \gamma \rangle^m \chi_1}{|X - Y, \gamma|^r \langle \xi - \eta, k - l, \gamma \rangle^m \langle \eta, l \rangle_{m_0}}, \text{ and then } \frac{|Y, 1|^r \langle \xi, k, \gamma \rangle^m \chi_1}{|Y, 1|^r \langle \xi - \eta, k - l, \gamma \rangle^m \langle \eta, l \rangle_{m_0}}.$$

The proof is completed by showing similarly that $A_2 \lesssim |\Lambda_D^r v|_{H_\gamma^m} \langle u \rangle_{m_0} + |v|_{H_\gamma^m} \langle \Lambda_1^r u \rangle_{m_0}$. \square

The next estimates will be used in section 3.6 to get tame estimates of commutators for ε fixed.

Proposition 3.38. *Suppose $m_0 > \frac{d+1}{2}$, $0 \leq m \leq 2m_0$, and $m_1 + m_2 \leq m$ with $m_i \geq 0$ for $i = 1, 2$. Let ∂^k denote any operator of the form $\partial_{x', \theta}^\alpha$ with $|\alpha| = k$. Then for $r \geq 0$*

$$(3.133) \quad \begin{aligned} (a) |\Lambda^r(\partial^{m_1} u \cdot \partial^{m_2} v)|_{L^2(x', \theta)} &\lesssim |\Lambda^r u|_{H_\gamma^m} \langle v \rangle_{m_0} + |u|_{H_\gamma^m} \langle \Lambda_1^r v \rangle_{m_0} + |\Lambda^r v|_{H_\gamma^m} \langle u \rangle_{m_0} + |v|_{H_\gamma^m} \langle \Lambda_1^r u \rangle_{m_0} \\ (b) \langle \Lambda^r(\partial^{m_1} u \cdot \partial^{m_2} v) \rangle_{0,\gamma} &\lesssim \langle \Lambda^r u \rangle_{m,\gamma} \langle v \rangle_{m_0} + \langle u \rangle_{m,\gamma} \langle \Lambda_1^r v \rangle_{m_0} + \langle \Lambda^r v \rangle_{m,\gamma} \langle u \rangle_{m_0} + \langle v \rangle_{m,\gamma} \langle \Lambda_1^r u \rangle_{m_0} \\ (c) \langle \Lambda^r(\partial^{m_1} u \cdot \partial^{m_2} v) \rangle_{0,\gamma} &\lesssim \langle \Lambda^r u \rangle_{m,\gamma} \langle v \rangle_{m_0} + \langle u \rangle_{m,\gamma} \langle \Lambda_1^r v \rangle_{m_0} + \langle \Lambda_1^r v \rangle_m \langle u \rangle_{m_0,\gamma} + \langle v \rangle_m \langle \Lambda^r u \rangle_{m_0,\gamma}. \end{aligned}$$

Proof. At least one of m_1, m_2 is $\leq m_0$. If $m_2 \leq m_0$, estimate the left side of (3.133)(a) by applying Proposition 3.31(a) with $\sigma = 0, s = m - m_1, t = m_0 - m_2$. If $m_1 \leq m_0$, take $s = m_0 - m_1, t = m - m_2$. Estimate (b) is proved by the same argument, writing

$$(3.134) \quad e^{-\gamma t}(\partial^{m_1} u \cdot \partial^{m_2} v) = (e^{-\gamma t} \partial^{m_1} u) \cdot \partial^{m_2} v$$

in the case $m_2 \leq m_0$, and switching the exponential to the other factor in the case $m_1 \leq m_0$. The proof of (c) is similar. □

Remark 3.39. 1) We also need the exact analogues of Corollary 3.36 and Propositions 3.35, 3.37, where the tangential norms are replaced by the norms involving powers of ∂_{x_2} given in Definition 3.32. The statements and proofs are essentially identical, except that now we must have $m_0 > \frac{d+2}{2}$.

2) There is some redundancy in the results of sections 3.4.1 and 3.4.2. One might think that the nontame estimates simply follow from corresponding tame ones by taking $m = m_0$, but that is not true in all cases. For example, Proposition 3.34(b) implies Proposition 3.22(b), but Proposition 3.35(b) does not imply Proposition 3.23(b). In section 3.5 we use only the simpler nontame estimates, while in section 3.6 we use both tame and nontame estimates.

3.5 Uniform higher derivative estimates and proof of Theorem 3.7

In this section we prove Proposition 3.20. First note that for $\varepsilon \in (0, \varepsilon_0]$ and $T \in (0, T_\varepsilon^*]$, $v^s = v_T^s$ satisfies

$$(3.135) \quad |v^s|_{\infty, m} \leq C\varepsilon M_0,$$

where C depends on the Seeley extension but is independent of ε and T . Thus if ε_0 is small enough, the operators on the left in (3.86)-(3.88) will be admissible, that is, they will satisfy the properties listed in section 3.1.1. Moreover, the bound (3.135) allows us to apply the singular calculus to (3.86)-(3.88) to obtain estimates involving constants that are uniform with respect to ε .

Remark 3.40. Throughout this section we make essential use of the fact that $v = \nabla_\varepsilon u$ on the support of v^s and that $v^s = \nabla_\varepsilon u^s = (\nabla_\varepsilon u)^s$. This allows us to replace $\nabla_\varepsilon u$ by v , or v by $\nabla_\varepsilon u$, in products where a factor of v^s is present. This is important because, for example, we are able to control the norm $|\nabla_\varepsilon u / \varepsilon|_{\infty, m, \gamma}$ but not the norm $|v / \varepsilon|_{\infty, m, \gamma}$. We are able to control the first norm by estimating solutions of $\frac{1}{\varepsilon}$ (3.88), but we are not able to control the second norm by estimating solutions of $\frac{1}{\varepsilon}$ (3.86), since that would require us to commute a negative power of Λ_D through the equation, and the singular calculus gives inadequate uniform control of such commutators.

3.5.1 Outline of estimates

When $j = 1$ ²⁹ the terms in the first line of (3.78) will be controlled by applying the estimates (3.37)-(3.39) to the problem (3.86) (or rather, to the problem $\partial_{x', \theta}^\alpha$ (3.86), $|\alpha| \leq m$). The first two terms in the second line are controlled by applying estimate (3.37) to the problem “ $\sqrt{\varepsilon}$ (3.86).”

When $j = 2$ the terms in the first line of (3.78) and the first two terms in the second line will be controlled by similarly applying the estimates (3.40)-(3.42) to the problem (3.87).

²⁹This brief outline is filled out in the remainder of this section.

The last two terms in the second line of (3.78) when $j = 1$ (resp. $j=2$) are estimated by applying (3.57) (resp. (3.58)) to the problem $\frac{1}{\sqrt{\varepsilon}}$ (3.86) (resp. $\frac{1}{\sqrt{\varepsilon}}$ (3.87)).

Note that u satisfies a Neumann-type problem of the same kind as (3.86) for v_1 . Thus we will control $|\nabla_\varepsilon u/\varepsilon|_{\infty,m,\gamma}$ and $|\nabla_\varepsilon u/\varepsilon|_{0,m+1,\varepsilon}$ by applying the estimates (3.38) and (3.39), with $(\Lambda_{D,\gamma}v_1, D_{x_2}v_1)$ replaced by $(\Lambda_{D,\gamma}u, D_{x_2}u)$, to the problem $\frac{1}{\varepsilon}$ (3.88). This will handle the first and second terms in the third line of (3.78).

The term $\left| \Lambda^{\frac{1}{2}} \nabla_\varepsilon u / \sqrt{\varepsilon} \right|_{0,m+1,\gamma}$ is controlled by applying the estimate (3.37) with $(\Lambda_{D,\gamma}v_1, D_{x_2}v_1)$ replaced by $(\Lambda_{D,\gamma}u, D_{x_2}u)$ to the problem $\frac{1}{\sqrt{\varepsilon}}$ (3.88). These estimates use the control of the first two terms in the second line of (3.78). The estimate of $\left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma}$ is a byproduct of this estimate.

The term $|\Lambda^{\frac{1}{2}} \nabla_\varepsilon u / \varepsilon|_{0,m,\gamma}$ is controlled by applying (3.37) to $\frac{1}{\varepsilon}$ (3.88).

Remark 3.41. When the estimates (3.37)-(3.39) are applied to control singular norms of $\nabla_\varepsilon u$, note that these estimates give the same control of $\partial_{t,\varepsilon} u$ that they do of $\partial_{x_1,\varepsilon} u$. Thus, the arguments below that give control of $E_{m,\gamma}(v)$ also give the same control of the norm that is defined by replacing every occurrence of $\nabla_\varepsilon u$ in the third line of (3.78) by $(\partial_{t,\varepsilon} u, \nabla_\varepsilon u)$.

3.5.2 Preliminaries

We denote the interior and boundary forcing terms in the v_1 -problem (3.86) by \mathcal{F}_1 and \mathcal{G}_1 respectively, in the v_2 -problem (3.87) by \mathcal{F}_2 and \mathcal{G}_2 , and in the u -problem (3.88) by 0 and \mathcal{G} .

Letting $\partial = (\partial_{x'}, \partial_\theta) = (\partial_t, \partial_{x_1}, \partial_\theta)$ and $\partial_\gamma = (\partial_t + \gamma, \partial_{x_1}, \partial_\theta)$, we have, for example, $|\Lambda^r v|_{0,m,\gamma} :=$

$$(3.136) \quad |\Lambda_D^r(e^{-\gamma t} v)|_{H_\gamma^m} \sim \sum_{|\alpha| \leq m} |\partial_\gamma^\alpha \Lambda_D^r(e^{-\gamma t} v)|_{L^2(x_2, L^2(x', \theta))} = \sum_{|\alpha| \leq m} |\Lambda_D^r(e^{-\gamma t} \partial^\alpha v)|_{L^2(x_2, L^2(x', \theta))},$$

so we will prove Proposition 3.20 by applying the estimates (3.37)-(3.39), (3.40)-(3.42), and (3.57)-(3.58) to the problems satisfied by $\partial^\alpha v$ or $\partial^\alpha u$, where $|\alpha| \leq m$ or $|\alpha| \leq m + 1$, in some cases multiplied by a power of ε .

The functions \mathcal{F}_1 and \mathcal{F}_2 are sums of terms of the form

$$(3.137) \quad a(v^s) \partial_{x_{i,\varepsilon}} v_k^s \partial_{x_{j,\varepsilon}} v_l^s,$$

where $a = d_v A_\alpha$ is an analytic function of its arguments, i, j, k, l take values in $\{1, 2\}$, and we set $\partial_{x_{2,\varepsilon}} := \partial_{x_2}$. In the estimates below it will be convenient to suppress some subscripts and write terms like (3.137) as

$$(3.138) \quad a(v^s) d_\varepsilon v^s d_\varepsilon v^s$$

The term \mathcal{G}_1 has the form

$$(3.139) \quad d_g H(v_1^s, g_\varepsilon) \partial_{x_{1,\varepsilon}} g_\varepsilon, \quad \text{where } g_\varepsilon := \varepsilon^2 G,$$

and, with $b(v_1^s, g_\varepsilon)$ denoting an analytic function, we write this as³⁰

$$(3.140) \quad \mathcal{G}_1 = b(v_1^s, g_\varepsilon) d_\varepsilon g_\varepsilon; \quad \text{also, } \mathcal{G}_2 = \chi_0(t) H(v_1, g_\varepsilon).$$

³⁰As explained in Remark 3.18, it is important that v_1 and not v_1^s occurs in \mathcal{G}_2 . In boundary terms d_ε represents only $\partial_{x_{1,\varepsilon}}$, never ∂_{x_2} of course.

Since $H(0,0) = 0$ we compute that

$$(3.141) \quad \mathcal{G} = H(v_1^s, g_\varepsilon) - d_{v_1} H(v_1^s, g_\varepsilon) v_1^s = d_g H(0,0) g_\varepsilon + b(v_1^s, g_\varepsilon) ((v_1^s, g_\varepsilon), (v_1^s, g_\varepsilon)),$$

and we write

$$(3.142) \quad \mathcal{G} = C g_\varepsilon + b(v_1^s, g_\varepsilon) (v_1^s, g_\varepsilon)^2.$$

Remark 3.42. *In the estimates of interior and boundary forcing as well as the commutator estimates of this section, which depend only on the results of section 3.4.1 and do not make use of the basic energy estimates of section 3.2.1, we take $m > \frac{d+1}{2}$, or sometimes slightly larger. It is only in Propositions 3.73 and 3.80 that we require $m > 3d + 4 + \frac{d+1}{2}$.*

3.5.3 Estimates of interior forcing

In this section we estimate the interior forcing terms needed to control the terms in (3.78).

Lemma 3.43. *Assume $m > \frac{d+1}{2}$ ($d = 2$ now). We have*

$$(3.143) \quad |\Lambda^{\frac{1}{2}}(d_\varepsilon v^s d_\varepsilon v^s)|_{0,m,\gamma} \lesssim |\Lambda^{\frac{3}{2}} v^s|_{0,m,\gamma} |\Lambda_1 v^s|_{\infty,m} + |\Lambda v^s|_{\infty,m,\gamma} |\Lambda_1^{\frac{3}{2}} v^s|_{0,m}$$

in the case where both $d_\varepsilon = \partial_{x_1,\varepsilon}$; otherwise, $|\Lambda^{\frac{1}{2}} \partial_{x_2} v^s|_{0,m,\gamma}$, $|\Lambda_1^{\frac{1}{2}} \partial_{x_2} v^s|_{0,m}$, etc. appear in the obvious places on the right.

Proof. Consider the case where both $d_\varepsilon = \partial_{x_1,\varepsilon}$. Treating x_2 as a parameter and applying Corollary 3.24 “in the (t, x_1, θ) variables” we have

$$(3.144) \quad \langle \Lambda^{\frac{1}{2}}(d_\varepsilon v^s d_\varepsilon v^s) \rangle_{m,\gamma} \lesssim \langle \Lambda^{\frac{3}{2}} v^s \rangle_{m,\gamma} \langle \Lambda_1 v^s \rangle_m + \langle \Lambda v^s \rangle_{m,\gamma} \langle \Lambda_1^{\frac{3}{2}} v^s \rangle_m,$$

and the lemma follows immediately. □

Notations 3.44. 1) *In the statements and proofs below we will (with slight abuse) denote analytic functions with nonnegative coefficients like $k(\langle D_3 u_1 \rangle_m, \dots, \langle D_3 u_K \rangle_m)$, where $u = (u_1, \dots, u_K)$, simply by $h(\langle u \rangle_m)$, and the function h may change from term to term. In addition we will write sums like*

$$(3.145) \quad \langle u_1 \rangle_{m,\gamma} \sum_{j=1}^K \langle \Lambda_1^r u_j \rangle_m k_j(\langle D_3 u_1 \rangle_m, \dots, \langle D_3 u_K \rangle_m)$$

as in Proposition 3.25 simply as $\langle u \rangle_{m,\gamma} \langle \Lambda_1^r u \rangle h(\langle u \rangle_m)$.

2) *The symbol Q or Q_j , $j \in \mathbb{N}$, will always denote an increasing, continuous function from \mathbb{R}_+ to itself such that $Q(z) \geq z$. The symbol Q^o or Q_j^o will denote a function of this type such that $Q^o(0) = 0$. The meaning of Q or Q^o may change from term to term.*

Lemma 3.45. *Let $r > 0$ and $m > \frac{d+1}{2}$.*

$$(3.146) \quad \langle \Lambda^r(a(v^s)w) \rangle_{m,\gamma} \lesssim \langle \Lambda^r w \rangle_{m,\gamma} h(\langle v^s \rangle_m) + \langle w \rangle_{m,\gamma} \langle \Lambda_1^r v^s \rangle_m h(\langle v^s \rangle_m).$$

Proof. Write $a(v^s) = a(0) + b(v^s)v^s$, note that by Corollary 3.24

$$(3.147) \quad \langle \Lambda^r(b(v^s)v^s w) \rangle_{m,\gamma} \lesssim \langle \Lambda^r w \rangle_{m,\gamma} \langle b(v^s)v^s \rangle_m + \langle w \rangle_{m,\gamma} \langle \Lambda_1^r(b(v^s)v^s) \rangle_m,$$

and finish by applying Proposition 3.25(a). □

The next proposition gives control of $|\Lambda^{\frac{1}{2}} \mathcal{F}_j|_{0,m,\gamma}$, $j = 1, 2$.

Proposition 3.46. *Let $m > \frac{d+1}{2}$. For small enough $M_0 > 0$ as in (3.84) we have*

$$(3.148) \quad |\Lambda^{\frac{1}{2}}(a(v^s)d_\varepsilon v^s d_\varepsilon v^s)|_{0,m,\gamma} \lesssim E_{m,\gamma}(v^s) Q^o(E_m(v^s)) \lesssim Q^o(E_{m,T}(v^s)).$$

Proof. Treating x_2 as a parameter, we have by Lemma 3.45

$$(3.149) \quad \langle \Lambda^{\frac{1}{2}}(a(v^s)d_\varepsilon v^s d_\varepsilon v^s) \rangle_{m,\gamma} \lesssim \langle \Lambda^{\frac{1}{2}}(d_\varepsilon v^s d_\varepsilon v^s) \rangle_{m,\gamma} h(\langle v^s \rangle_m) + \langle d_\varepsilon v^s d_\varepsilon v^s \rangle_{m,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_m h(\langle v^s \rangle_m)$$

Since $|d_\varepsilon v^s d_\varepsilon v^s|_{0,m,\gamma} \lesssim |\Lambda v^s|_{0,m,\gamma} |\Lambda_1 v^s|_{\infty,m}$ and for $M_0 > 0$ small enough (depending on the polydisc of convergence of h),

$$(3.150) \quad \sup_{x_2 \geq 0} h(\langle v^s \rangle_m) \leq h(E_m(v^s)) \leq h(CE_{m,T}(v^s)), \text{ where } E_m(v^s) \leq CE_{m,T}(v^s) \text{ (recall (3.82))},$$

the lemma now follows from Lemma 3.43. The presence of the factors involving Λ_1 allows us to use Q^o instead of Q in (3.148). For the last inequality in (3.148) we use (3.82). □

We need the following variant of Lemma 3.45.

Lemma 3.47. *Let $r \geq 0$ and $m > \frac{d+1}{2}$.*

$$(3.151) \quad \begin{aligned} \langle \Lambda^r(a(v^s)w) \rangle_{m+1,\gamma} &\lesssim \langle \Lambda^r w \rangle_{m+1,\gamma} h(\langle v^s \rangle_m) + \langle w \rangle_{m+1,\gamma} \langle \Lambda_1^r v^s \rangle_m h(\langle v^s \rangle_m) + \\ &\langle \Lambda^r w \rangle_{m,\gamma} h(\langle v^s \rangle_{m+1}) + \langle w \rangle_{m,\gamma} \langle \Lambda_1^r v^s \rangle_{m+1} h(\langle v^s \rangle_{m+1}). \end{aligned}$$

Proof. Write $a(v^s) = a(0) + b(v^s)v^s$ and apply Proposition 3.38(c) with “ u ” = w , “ m ” = $m+1$, “ m_0 ” = m , observing for example that

$$(3.152) \quad \langle \Lambda_1^r(v^s b(v^s)) \rangle_m \lesssim \langle \Lambda_1^r v^s \rangle_m h(\langle v^s \rangle_m).$$

□

The next proposition gives control of $|\mathcal{F}_j|_{0,m+1,\gamma}$, $j = 1, 2$.

Proposition 3.48. *Let $m > \frac{d+1}{2}$. For small enough $M_0 > 0$ as in (3.84) we have*

$$(3.153) \quad |(a(v^s)d_\varepsilon v^s d_\varepsilon v^s)|_{0,m+1,\gamma} \lesssim E_{m,\gamma}(v^s) Q^o(E_m(v^s)) \lesssim Q^o(E_{m,T}(v^s)).$$

Proof. Treating x_2 as a parameter, we have by Lemma 3.47 (with $r = 0$)

$$(3.154) \quad \langle a(v^s) d_\varepsilon v^s d_\varepsilon v^s \rangle_{m+1, \gamma} \lesssim \langle d_\varepsilon v^s d_\varepsilon v^s \rangle_{m+1, \gamma} h(\langle v^s \rangle_m) + \langle d_\varepsilon v^s d_\varepsilon v^s \rangle_{m, \gamma} h(\langle v^s \rangle_{m+1}).$$

A standard Moser estimate gives

$$(3.155) \quad \langle d_\varepsilon v^s d_\varepsilon v^s \rangle_{m+1, \gamma} \lesssim \langle d_\varepsilon v^s \rangle_{m+1, \gamma} |d_\varepsilon v^s|_{L^\infty} \lesssim \langle \Lambda v^s \rangle_{m+1, \gamma} |\Lambda_1 v^s|_{\infty, m}.$$

In addition,

$$(3.156) \quad |v^s|_{\infty, m+1} \leq \left| \left(\frac{\Lambda_1 v^s}{D_{x_2} v^s} \right) \right|_{0, m+1} \text{ implies } \sup_{x_2 \geq 0} h(\langle v^s \rangle_{m+1}) \leq h(E_m(v^s)),$$

so we can finish by taking the L^2 norm of (3.154) in x_2 . □

To control the terms in the second line of (3.78) we must estimate $\sqrt{\varepsilon} |\Lambda^{\frac{1}{2}} \mathcal{F}_j|_{0, m+1, \gamma}$, $j = 1, 2$.

Lemma 3.49. *Assume $m > \frac{d+1}{2} + 1$. Then $\sqrt{\varepsilon} |\Lambda^{\frac{1}{2}} (d_\varepsilon v^s d_\varepsilon v^s)|_{0, m+1, \gamma} \lesssim$*

$$(3.157) \quad \left(\sqrt{\varepsilon} |\Lambda^{\frac{3}{2}} v^s|_{0, m+1, \gamma} + |\Lambda v^s|_{0, m+1, \gamma} \right) |\Lambda_1 v^s|_{\infty, m} + \sqrt{\varepsilon} |\Lambda v^s|_{\infty, m, \gamma} |\Lambda_1^{\frac{3}{2}} v^s|_{0, m} \lesssim E_{m, \gamma}(v_s) E_m(v_s).$$

Proof. Treating x_2 as a parameter, we estimate a sum of terms of the form

$$(3.158) \quad \sqrt{\varepsilon} \langle \Lambda^{\frac{1}{2}} ((\partial^{m_1} d_\varepsilon v^s) \cdot (\partial^{m_2} d_\varepsilon v^s)) \rangle_{0, 0, \gamma} := A,$$

where $m_1 + m_2 = m + 1$. In the case where one of m_1 or m_2 is $m + 1$, say $m_1 = m + 1$, we apply Proposition 3.31(b) with “ u ” = $\partial^{m+1} d_\varepsilon v^s$ and “ v ” = $d_\varepsilon v^s$ to obtain

$$(3.159) \quad A \lesssim \sqrt{\varepsilon} \langle \Lambda^{\frac{3}{2}} v^s \rangle_{m+1, \gamma} \langle \Lambda_1 v^s \rangle_m + \langle \Lambda v^s \rangle_{m+1, \gamma} \langle \Lambda_1 v^s \rangle_m.$$

On the last term we have used $\sqrt{\varepsilon} \langle \Lambda_1^{\frac{3}{2}} v^s \rangle_{m-\frac{1}{2}} \lesssim \langle \Lambda_1 v^s \rangle_m$. In the other case where both m_1 and m_2 are $\leq m$ we apply Proposition 3.31 with $s = m - m_1$ and $t = m - m_2$ to obtain

$$(3.160) \quad A \lesssim \sqrt{\varepsilon} \langle \Lambda^{\frac{3}{2}} v^s \rangle_{m, \gamma} \langle \Lambda_1 v^s \rangle_m + \sqrt{\varepsilon} \langle \Lambda v^s \rangle_{m, \gamma} \langle \Lambda_1^{\frac{3}{2}} v^s \rangle_m.$$

To finish we take the $L^2(x_d)$ norm in these inequalities. □

Proposition 3.50 ($\sqrt{\varepsilon} |\Lambda^{\frac{1}{2}} \mathcal{F}_j|_{0, m+1, \gamma}$). *Assume $m > \frac{d+1}{2} + 1$. For small enough M_0 we have .*

$$(3.161) \quad \sqrt{\varepsilon} |\Lambda^{\frac{1}{2}} (a(v^s) d_\varepsilon v^s d_\varepsilon v^s)|_{0, m+1, \gamma} \lesssim E_{m, \gamma}(v^s) Q^o(E_m(v^s)) \lesssim Q^o(E_{m, T}(v^s))$$

Proof. Treating x_2 as a parameter, we apply Lemma 3.45 to get $\sqrt{\varepsilon} \langle \Lambda^{\frac{1}{2}} (a(v^s) d_\varepsilon v^s d_\varepsilon v^s) \rangle_{0, m+1, \gamma} \lesssim$

$$(3.162) \quad \sqrt{\varepsilon} \langle \Lambda^{\frac{1}{2}} (d_\varepsilon v^s d_\varepsilon v^s) \rangle_{m+1, \gamma} h(\langle v^s \rangle_{m+1}) + \sqrt{\varepsilon} \langle d_\varepsilon v^s d_\varepsilon v^s \rangle_{m+1, \gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} h(\langle v^s \rangle_{m+1}).$$

We finish by taking the $L^2(x_d)$ norm and by applying Lemma 3.49 and (3.155).³¹ □

³¹Control of $\sqrt{\varepsilon} |\Lambda^{\frac{1}{2}} v^s|_{\infty, m+1}$ comes from the first term in the second lines of (3.78).

To control the term

$$(3.163) \quad \left| \begin{pmatrix} \Lambda v_j / \sqrt{\varepsilon} \\ D_{x_2} v_j / \sqrt{\varepsilon} \end{pmatrix} \right|_{0,m,\gamma}^{(2)}$$

we must estimate $\frac{1}{\sqrt{\varepsilon}} |\mathcal{F}_j|_{0,m,\gamma}$.

Proposition 3.51. *Let $m > \frac{d+1}{2}$. For small enough $M_0 > 0$ we have*

$$(3.164) \quad \frac{1}{\sqrt{\varepsilon}} |a(v^s) d_\varepsilon v^s d_\varepsilon v^s|_{0,m,\gamma} \lesssim Q^o(E_{m,T}(v^s)).$$

Proof. Treating x_2 as a parameter, we have by Corollary 3.24

$$(3.165) \quad \frac{1}{\sqrt{\varepsilon}} \langle (a(v^s) d_\varepsilon v^s d_\varepsilon v^s) \rangle_{m,\gamma} \lesssim \frac{1}{\sqrt{\varepsilon}} \langle (d_\varepsilon v^s d_\varepsilon v^s) \rangle_{m,\gamma} h(\langle v^s \rangle_m).$$

Since $|d_\varepsilon v^s d_\varepsilon v^s|_{0,m,\gamma} \lesssim |\Lambda v^s|_{0,m,\gamma} |\Lambda_1 v^s|_{\infty,m}$, the result follows. \square

3.5.4 Boundary forcing I

Here we estimate the forcing terms given by \mathcal{G}_1 and \mathcal{G} for the problems (3.86) and (3.88) respectively. The next two Propositions, which are needed to estimate the terms in the first line of (3.78) when $j = 1$, give control of $\langle \Lambda \mathcal{G}_1 \rangle_{m,\gamma}$ and $\langle \Lambda^{\frac{1}{2}} \mathcal{G}_1 \rangle_{m+1,\gamma}$; recall $\mathcal{G}_1 = b(v_1^s, g_\varepsilon) d_\varepsilon g_\varepsilon$.

Proposition 3.52. *Let $m > \frac{d+1}{2}$. For small enough constants $\varepsilon_1 > 0$ and $M_0 > 0$ we have for $\varepsilon \in (0, \varepsilon_1]$ ³²*

$$(3.166) \quad \langle \Lambda \mathcal{G}_1 \rangle_{m,\gamma} \lesssim \langle \Lambda d_\varepsilon g_\varepsilon \rangle_{m,\gamma} h(\langle v_1^s \rangle_m) + (\langle \Lambda_1 v_1^s \rangle_m + \varepsilon) h(\langle v_1^s \rangle_m) \lesssim (M_G + \varepsilon) Q(E_{m,T}(v^s)).$$

Proof. Write $b(v_1^s, g_\varepsilon) = b(0, 0) + c(v_1^s, g_\varepsilon)$ where $c(0, 0) = 0$. By Corollary 3.24

$$(3.167) \quad \begin{aligned} \langle \Lambda(c(v_1^s, g_\varepsilon) d_\varepsilon g_\varepsilon) \rangle_{m,\gamma} &\lesssim \langle \Lambda d_\varepsilon g_\varepsilon \rangle_{m,\gamma} \langle c(v_1^s, g_\varepsilon) \rangle_m + \langle d_\varepsilon g_\varepsilon \rangle_{m,\gamma} \langle \Lambda_1 c(v_1^s, g_\varepsilon) \rangle_m \leq \\ &M_G h(\langle v_1^s \rangle_m) + \varepsilon (\langle \Lambda_1 v_1^s \rangle_m + \varepsilon) h(\langle v_1^s \rangle_m). \end{aligned}$$

Here we have used Proposition 3.25(a) and the fact that, since $g_\varepsilon = \varepsilon^2 G$ with $G \in H^{m+3}(b\Omega)$, we have for $\varepsilon_1 > 0$ small enough (depending on the polydisk of convergence of c)

$$(3.168) \quad \langle c(v_1^s, g_\varepsilon) \rangle_m \leq h(\langle v_1^s \rangle_m) \text{ and } \langle d_\varepsilon g_\varepsilon \rangle_{m,\gamma} \leq \varepsilon M_G \text{ for } \varepsilon \in (0, \varepsilon_1].$$

\square

Proposition 3.53. *Let $m > \frac{d+1}{2}$. For small enough $\varepsilon_1 > 0$ and $M_0 > 0$ as in (3.84) we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.169) \quad \langle \Lambda^{\frac{1}{2}} \mathcal{G}_1 \rangle_{m+1,\gamma} \lesssim \sqrt{\varepsilon} h(\langle v_1^s \rangle_{m+1}) + \varepsilon (\langle \Lambda_1^{\frac{1}{2}} v_1^s \rangle_{m+1} + \varepsilon^{\frac{3}{2}}) h(\langle v_1^s \rangle_{m+1}) \lesssim \sqrt{\varepsilon} Q(E_{m,T}(v^s)).$$

Proof. By Corollary 3.24 we have $\langle \Lambda^{\frac{1}{2}}(b(v_1^s, g_\varepsilon) d_\varepsilon g_\varepsilon) \rangle_{m+1,\gamma} \lesssim$

$$(3.170) \quad \langle \Lambda^{\frac{1}{2}} d_\varepsilon g_\varepsilon \rangle_{m+1,\gamma} h(\langle v_1^s, g_\varepsilon \rangle_{m+1}) + \langle d_\varepsilon g_\varepsilon \rangle_{m+1,\gamma} \langle \Lambda_1^{\frac{1}{2}}(v_1^s, g_\varepsilon) \rangle_{m+1} h(\langle v_1^s, g_\varepsilon \rangle_{m+1}) \leq \sqrt{\varepsilon} Q(E_m(v^s)).$$

\square

³²Here M_G is as in (3.90).

To control the terms in the second line of (3.78) when $j = 1$ we must estimate $\sqrt{\varepsilon}\langle\Lambda\mathcal{G}_1\rangle_{m+1,\gamma}$.

Proposition 3.54. *Let $m > \frac{d+1}{2}$. For small enough $\varepsilon_1 > 0$ and $M_0 > 0$ as in (3.84) we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.171) \quad \sqrt{\varepsilon}\langle\Lambda\mathcal{G}_1\rangle_{m+1,\gamma} \lesssim \sqrt{\varepsilon}h(\langle v_1^s \rangle_{m+1}) + \varepsilon^{\frac{3}{2}}(\langle\Lambda_1 v_1^s \rangle_{m+1} + \varepsilon)h(\langle v_1^s \rangle_{m+1}) \lesssim \sqrt{\varepsilon}Q(E_{m,T}(v^s)).$$

Proof. By Corollary 3.24 we have $\sqrt{\varepsilon}\langle\Lambda(b(v_1^s, g_\varepsilon)d_\varepsilon g_\varepsilon)\rangle_{m+1,\gamma} \lesssim$

$$(3.172) \quad \sqrt{\varepsilon}\langle\Lambda d_\varepsilon g_\varepsilon\rangle_{m+1,\gamma}h(\langle v_1^s, g_\varepsilon \rangle_{m+1}) + \sqrt{\varepsilon}\langle d_\varepsilon g_\varepsilon \rangle_{m+1,\gamma}\langle\Lambda_1(v_1^s, g_\varepsilon)\rangle_{m+1}h(\langle v_1^s, g_\varepsilon \rangle_{m+1}).$$

□

To control the terms (3.163) with $j = 1$ we must estimate $\frac{1}{\sqrt{\varepsilon}}\langle\Lambda^{\frac{1}{2}}\mathcal{G}_1\rangle_{m,\gamma}$.

Proposition 3.55. *Let $m > \frac{d+1}{2}$. For small enough constants $\varepsilon_1 > 0$ and $M_0 > 0$ we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.173) \quad \frac{1}{\sqrt{\varepsilon}}\langle\Lambda^{\frac{1}{2}}\mathcal{G}_1\rangle_{m,\gamma} \lesssim (M_G + \sqrt{\varepsilon})Q(E_{m,T}(v^s)).$$

Proof. Parallel to Proposition 3.52 we have

$$(3.174) \quad \frac{1}{\sqrt{\varepsilon}}\langle\Lambda^{\frac{1}{2}}(c(v_1^s, g_\varepsilon)d_\varepsilon g_\varepsilon)\rangle_{m,\gamma} \lesssim \frac{1}{\sqrt{\varepsilon}}\langle\Lambda^{\frac{1}{2}}d_\varepsilon g_\varepsilon\rangle_{m,\gamma}\langle c(v_1^s, g_\varepsilon) \rangle_m + \frac{1}{\sqrt{\varepsilon}}\langle d_\varepsilon g_\varepsilon \rangle_{m,\gamma}\langle\Lambda_1^{\frac{1}{2}}c(v_1^s, g_\varepsilon)\rangle_m \leq \\ M_G h(\langle v_1^s \rangle_m) + \sqrt{\varepsilon}(\langle\Lambda_1^{\frac{1}{2}}v_1^s \rangle_m + \varepsilon^{\frac{3}{2}})h(\langle v_1^s \rangle_m).$$

□

To control the terms

$$(3.175) \quad \left| \frac{\nabla_\varepsilon u}{\varepsilon} \right|_{\infty, m, \gamma}, \left| \frac{\nabla_\varepsilon u}{\varepsilon} \right|_{0, m+1, \gamma}, \left| \frac{\Lambda^{\frac{1}{2}} \nabla_\varepsilon u}{\varepsilon} \right|_{0, m, \gamma}, \left| \frac{\Lambda^{\frac{1}{2}} \nabla_\varepsilon u}{\sqrt{\varepsilon}} \right|_{0, m+1, \gamma}, \text{ and } \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1, \gamma},$$

in the third line of (3.78), we use that fact that $v = \nabla_\varepsilon u$ on the support of v^s and apply the estimates (3.37)-(3.39) and (3.57) to the problems $\frac{1}{\varepsilon}$ (3.88) and $\frac{1}{\sqrt{\varepsilon}}$ (3.88). For this we must estimate the forcing terms

$$(3.176) \quad \frac{1}{\varepsilon}\langle\Lambda\mathcal{G}\rangle_{m,\gamma}, \frac{1}{\varepsilon}\langle\Lambda^{\frac{1}{2}}\mathcal{G}\rangle_{m+1,\gamma}, \text{ and } \frac{1}{\sqrt{\varepsilon}}\langle\Lambda\mathcal{G}\rangle_{m+1,\gamma},$$

where $\mathcal{G} = Cg_\varepsilon + b(v_1^s, g_\varepsilon)(v_1^s, g_\varepsilon)^2$ in the notation of (3.142).

Proposition 3.56. *Let $m > \frac{d+1}{2}$. For small enough $\varepsilon_1 > 0$ and $M_0 > 0$ we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.177) \quad \frac{1}{\varepsilon}\langle\Lambda\mathcal{G}\rangle_{m,\gamma} \lesssim M_G + (E_m^2(v^s) + \varepsilon^2 M_G^2)h(\langle v^s \rangle_m) \lesssim M_G Q(E_{m,T}(v^s)) + Q^o(E_{m,T}(v^s))$$

Proof. Clearly, $\frac{1}{\varepsilon}\langle\Lambda g_\varepsilon\rangle_{m,\gamma} \lesssim M_G$. Writing v^s in place of v_1^s and $b(v^s, g_\varepsilon) = b(0, 0) + c(v^s, g_\varepsilon)$ with $c(0, 0) = 0$, we have by Corollary 3.24 that $\frac{1}{\varepsilon}\langle\Lambda[c(v^s, g_\varepsilon)(v^s, g_\varepsilon)^2]\rangle_{m,\gamma} \lesssim$

$$(3.178) \quad \frac{1}{\varepsilon}\langle\Lambda(v^s, g_\varepsilon)^2\rangle_{m,\gamma}\langle v^s, g_\varepsilon \rangle_m h(\langle v^s \rangle_m) + \frac{1}{\varepsilon}\langle(v^s, g_\varepsilon)^2\rangle_{m,\gamma}\langle\Lambda_1(v^s, g_\varepsilon)\rangle_m h(\langle v^s \rangle_m) := A + B.$$

The estimate

$$(3.179) \quad \frac{1}{\varepsilon} \langle \Lambda(v^s v^s) \rangle_{m,\gamma} \lesssim \langle \Lambda v^s \rangle_{m,\gamma} \langle v^s / \varepsilon \rangle_m + \langle v^s / \varepsilon \rangle_{m,\gamma} \langle \Lambda_1 v^s \rangle_m \lesssim E_m^2(v^s)$$

and other similar ones yields,

$$(3.180) \quad \frac{1}{\varepsilon} \langle \Lambda(v^s, g_\varepsilon)^2 \rangle_{m,\gamma} \lesssim E_m^2(v^s) + \varepsilon^2 M_G^2,$$

so $A \lesssim (E_m^2(v^s) + \varepsilon^2 M_G^2) h(\langle v_s \rangle_m)$. The term B is treated similarly. □

Proposition 3.57. *Let $m > \frac{d+1}{2}$. For small enough $\varepsilon_1 > 0$ and $M_0 > 0$ we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.181) \quad \begin{aligned} (a) \quad & \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}} \mathcal{G} \rangle_{m+1,\gamma} \lesssim \sqrt{\varepsilon} M_G + \left(E_m^2(v^s) + \varepsilon^{\frac{5}{2}} M_G^2 \right) h(\langle v_s \rangle_{m+1}) \lesssim Q^o(E_{m,T}(v^s)) + \sqrt{\varepsilon} Q(E_{m,T}(v^s)) \\ (b) \quad & \frac{1}{\sqrt{\varepsilon}} \langle \Lambda \mathcal{G} \rangle_{m+1,\gamma} \lesssim \sqrt{\varepsilon} M_G + \left(E_m^2(v^s) + \varepsilon^{\frac{5}{2}} M_G^2 \right) h(\langle v_s \rangle_{m+1}) \lesssim Q^o(E_{m,T}(v^s)) + \sqrt{\varepsilon} Q(E_{m,T}(v^s)). \end{aligned}$$

Proof. (a) Clearly, $\frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}} g_\varepsilon \rangle_{m+1,\gamma} \lesssim \sqrt{\varepsilon} M_G$. By Proposition 3.38(c) we have in place of (3.179)

$$(3.182) \quad \begin{aligned} \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}}(v^s v^s) \rangle_{m+1,\gamma} & \lesssim \langle \Lambda^{\frac{1}{2}} v^s \rangle_{m+1,\gamma} \langle v^s / \varepsilon \rangle_m + \langle v^s / \sqrt{\varepsilon} \rangle_{m+1,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s / \sqrt{\varepsilon} \rangle_m + \\ & \langle \Lambda^{\frac{1}{2}} v^s / \sqrt{\varepsilon} \rangle_{m,\gamma} \langle v^s / \sqrt{\varepsilon} \rangle_{m+1} + \langle v^s / \varepsilon \rangle_{m,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} \lesssim E_m^2(v^s). \end{aligned}$$

The rest of the proof is essentially the same as that of Proposition 3.56. The proof of (b) is identical, except that we use control of $\sqrt{\varepsilon} \langle \Lambda v^s \rangle_{m+1,\gamma}$ here. □

3.5.5 Interior commutators

To estimate terms in the first line of (3.78) we must estimate norms involving interior commutators of the form

$$(3.183) \quad |\Lambda^{\frac{1}{2}} [a(v^s) d_\varepsilon^2, \partial^k] v|_{0,0,\gamma} \text{ and } |[a(v^s) d_\varepsilon^2, \partial^{k+1}] v|_{0,0,\gamma},$$

where $k \leq m$, $a = A_\alpha$, and ∂^k denotes $(\partial_{x'}, \partial_\theta)^\alpha$ for some multi-index such that $|\alpha| = k$. We will use ∂ to denote not only the tangential derivatives like ∂_t , ∂_{x_1} , and ∂_θ but also (with some abuse) $\varepsilon \partial_{x_1} + \beta_1 \partial_\theta = \varepsilon \partial_{x_1, \varepsilon}$.

Proposition 3.58. *Suppose $m > \frac{d+1}{2} + 1$. (a) When $d_\varepsilon^2 = \partial_{x_1, \varepsilon}^2$ or $\partial_{x_2} \partial_{x_1, \varepsilon}$ we have for $0 \leq k \leq m$*

$$(3.184) \quad |\Lambda^{\frac{1}{2}} [a(v^s) d_\varepsilon^2, \partial^k] v|_{0,0,\gamma} \lesssim |\Lambda^{\frac{3}{2}} v|_{0,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}) + |\Lambda v|_{\infty,m,\gamma} \left| \frac{\Lambda_1^{\frac{1}{2}} v^s}{\varepsilon} \right|_{0,m} h(|v^s|_{\infty,m}).$$

(b) When $d_\varepsilon^2 = \partial_{x_2}^2$, we have the same estimate with $|\Lambda^{\frac{3}{2}} v|_{0,m,\gamma}$ and $|\Lambda v|_{\infty,m,\gamma}$ replaced respectively by

$$(3.185) \quad |\Lambda^{\frac{1}{2}} \varepsilon \partial_{x_2}^2 v|_{0,m-1,\gamma} \text{ and } |\varepsilon \partial_{x_2}^2 v|_{\infty,m-1,\gamma}.$$

Proof. (a) Writing $a(v^s) = a(0) + b(v^s)v^s$ and taking $k = m$ (the “main” case), we must estimate a sum of terms of the form

$$(3.186) \quad |\Lambda^{\frac{1}{2}}[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon^2v)]|_{0,0,\gamma} = \frac{1}{\varepsilon}|\Lambda^{\frac{1}{2}}[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon\partial v)]|_{0,0,\gamma},$$

where $m_1 + m_2 = m$, $m_1 \geq 1$. Treating x_2 as a parameter we apply Proposition 3.31 “in the (x', θ) variables” with $\sigma = 0$, $s = m - m_2 - 1$ and $t = m - m_1$. Finish by taking the $L^2(x_2)$ norm of the inequality thus obtained. The proof of (b) is essentially the same.³³ \square

Remark 3.59. Observe that the right side of (3.184) is $\lesssim E_{m,\gamma}(v)Q(E_{m,T}(v^s))$. As we show below and in the next section, the same statement applies to all interior and boundary commutators.

Proposition 3.60. Suppose $m > \frac{d+1}{2} + 1$. (a) When $d_\varepsilon^2 = \partial_{x_1,\varepsilon}^2$ or $\partial_{x_2}\partial_{x_1,\varepsilon}$ we have for $0 \leq k \leq m + 1$

$$(3.187) \quad |[a(v^s)d_\varepsilon^2, \partial^k]v|_{0,0,\gamma} \lesssim |\Lambda v|_{0,m+1,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}) + |\Lambda v|_{\infty,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{0,m+1} h(|v^s|_{\infty,m+1}).$$

(b) When $d_\varepsilon^2 = \partial_{x_2}^2$, we have the same estimate with $|\Lambda v|_{0,m+1,\gamma}$ and $|\Lambda v|_{\infty,m,\gamma}$ replaced respectively by

$$(3.188) \quad |\varepsilon\partial_{x_2}^2 v|_{0,m,\gamma} \text{ and } |\varepsilon\partial_{x_2}^2 v|_{\infty,m-1,\gamma}.$$

Proof. (a) Taking $k = m + 1$, we must estimate a sum of terms of the form

$$(3.189) \quad |[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon^2v)]|_{0,0,\gamma} = \frac{1}{\varepsilon}|[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon\partial v)]|_{0,0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. In the case $m_1 = m + 1$ we obtain easily

$$(3.190) \quad A \lesssim |v^s b(v^s)/\varepsilon|_{0,m+1} |e^{-\gamma t} d_\varepsilon \partial v|_{L^\infty} \lesssim |\Lambda v|_{\infty,m,\gamma} |v^s/\varepsilon|_{0,m+1} h(|v^s|_{\infty,m+1}).$$

In the remaining case, we have $m_1 \geq 1$, $m_2 \geq 1$. Treating x_2 as a parameter, we estimate A by applying Proposition 3.31 with $r = 0$, $\sigma = 0$, $s = (m + 1) - (m_2 + 1)$ and $t = m - m_1$, and taking the $L^2(x_2)$ norm of the inequality thus obtained. Again, the proof of (b) is the similar. \square

The next estimate is needed to control terms in the second line of (3.78).

Proposition 3.61. Suppose $m > \frac{d+1}{2} + 2$. (a) When $d_\varepsilon^2 = \partial_{x_1,\varepsilon}^2$ or $\partial_{x_2}\partial_{x_1,\varepsilon}$ we have for $0 \leq k \leq m + 1$

$$(3.191) \quad \begin{aligned} \sqrt{\varepsilon}|\Lambda^{\frac{1}{2}}[a(v^s)d_\varepsilon^2, \partial^k]v|_{0,0,\gamma} &\lesssim \left(\left| \frac{\Lambda^{\frac{1}{2}}v^s}{\sqrt{\varepsilon}} \right|_{0,m+1} + \left| \frac{v^s}{\varepsilon} \right|_{0,m+1} \right) |\Lambda v|_{\infty,m,\gamma} h(|v^s|_{\infty,m+1}) + \\ &\left(|\sqrt{\varepsilon}\Lambda^{\frac{3}{2}}v|_{0,m+1,\gamma} + |\Lambda v|_{0,m+1,\gamma} \right) \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}). \end{aligned}$$

(b) When $d_\varepsilon^2 = \partial_{x_2}^2$, we have the same estimate with $|\Lambda v|_{\infty,m,\gamma}$, $|\sqrt{\varepsilon}\Lambda^{\frac{3}{2}}v|_{0,m+1,\gamma}$, and $|\Lambda v|_{0,m+1,\gamma}$ replaced by

$$(3.192) \quad |\varepsilon\partial_{x_2}^2 v|_{\infty,m-1,\gamma}, \quad |\varepsilon^{\frac{3}{2}}\Lambda^{\frac{1}{2}}\partial_{x_2}^2 v|_{0,m,\gamma}, \quad \text{and } |\varepsilon\partial_{x_2}^2 v|_{0,m,\gamma}.$$

respectively.

³³The formulas in (a) and (b) agree if one regards $\varepsilon\partial_{x_2}$ (resp. ∂_{x_2}) as “equivalent” to a tangential derivative ∂ (resp. Λ_D).

Proof. Parallel to (3.189) we must estimate a sum of terms of the form

$$(3.193) \quad \sqrt{\varepsilon} |\Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon^2 v)]|_{0,0,\gamma} = \frac{1}{\sqrt{\varepsilon}} |\Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon \partial v)]|_{0,0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. In the case $m_1 = m + 1$ we apply Proposition 3.31(b) (in the (x', θ) variables) with $m - \frac{3}{2}$ playing the role of “ s ” and “ u ” = $d_\varepsilon \partial v$ to obtain

$$(3.194) \quad A \lesssim \left(\left| \frac{\Lambda^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right|_{0,m+1} + \left| \frac{v^s}{\varepsilon} \right|_{0,m+1} \right) |\Lambda v|_{\infty,m,\gamma} h(|v^s|_{\infty,m+1}).$$

Here we have used $m - \frac{3}{2} > \frac{d+1}{2}$ to estimate

$$(3.195) \quad |\sqrt{\varepsilon} \Lambda^{\frac{1}{2}} d_\varepsilon \partial v|_{\infty,m-\frac{3}{2},\gamma} \leq |\Lambda v|_{\infty,m,\gamma}.$$

In the case $m_1 = m$ we similarly apply Proposition 3.31(b) with “ s ” = $m - \frac{5}{2}$ and “ t ” = 1 to obtain (3.194) again.

In the case $m_1 \leq m - 1$ we apply Proposition 3.31 with $\sigma = 0$, $t = m - 1 - m_1$ and $s = (m + 1) - (m_2 + 1)$ to obtain

$$(3.196) \quad A \lesssim \left(|\sqrt{\varepsilon} \Lambda^{\frac{3}{2}} v|_{0,m+1,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m-1} + |\Lambda v|_{0,m+1,\gamma} \left| \frac{\Lambda^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right|_{\infty,m-1} \right) h(|v^s|_{\infty,m}).$$

To finish we observe $|\frac{\Lambda^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}}|_{\infty,m-1} \lesssim |\frac{v^s}{\varepsilon}|_{\infty,m}$.

□

To control the terms (3.175) we must estimate the interior commutators

$$(3.197) \quad \frac{1}{\varepsilon} |\Lambda^{\frac{1}{2}} [a(v^s) d_\varepsilon^2, \partial^m] u|_{0,0,\gamma}, \frac{1}{\varepsilon} |[a(v^s) d_\varepsilon^2, \partial^{m+1}] u|_{0,0,\gamma}, \text{ and } \frac{1}{\varepsilon^{1/2}} |\Lambda^{1/2} [a(v^s) d_\varepsilon^2, \partial^{m+1}] u|_{0,0,\gamma}.$$

In these estimates we sometimes write v in place of $d_\varepsilon u$ when a factor of v^s is present.

Proposition 3.62. *Suppose $m > \frac{d+1}{2}$. For $0 \leq k \leq m$ we have*

$$(3.198) \quad \frac{1}{\varepsilon} |\Lambda^{\frac{1}{2}} [a(v^s) d_\varepsilon^2, \partial^k] u|_{0,0,\gamma} \lesssim |\Lambda^{\frac{3}{2}} v|_{0,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}) + |\Lambda v|_{\infty,m,\gamma} \left| \frac{\Lambda^{\frac{1}{2}} v^s}{\varepsilon} \right|_{0,m} h(|v^s|_{\infty,m})$$

Proof. Parallel to (3.186), to prove (a) we must estimate a sum of terms of the form

$$(3.199) \quad \frac{1}{\varepsilon} |\Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon^2 u)]|_{0,0,\gamma} = \frac{1}{\varepsilon} |\Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon v)]|_{0,0,\gamma},$$

where $m = m_1 + m_2$, $m_1 \geq 1$. To finish apply Proposition 3.31 with $s = m - m_2$, $t = m - m_1$. □

Proposition 3.63. *Suppose $m > \frac{d+1}{2}$. For $0 \leq k \leq m + 1$ we have*

$$(3.200) \quad \frac{1}{\varepsilon} |[a(v^s) d_\varepsilon^2, \partial^k] u|_{0,0,\gamma} \lesssim |\Lambda v|_{0,m+1,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}) + |\Lambda v|_{\infty,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{0,m+1} h(|v^s|_{\infty,m+1}).$$

Proof. (a) Parallel to (3.189) we must estimate a sum of terms of the form

$$(3.201) \quad \frac{1}{\varepsilon} |[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon^2 u)]|_{0,0,\gamma} = \frac{1}{\varepsilon} |[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon v)]|_{0,0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. In the case $m_1 = m + 1$ we obtain

$$(3.202) \quad A \lesssim |v^s b(v^s)/\varepsilon|_{0,m+1} |e^{-\gamma t} d_\varepsilon v|_{L^\infty} \lesssim |\Lambda v|_{\infty,m,\gamma} |v^s/\varepsilon|_{0,m+1} h(|v^s|_{\infty,m+1}).$$

In the remaining case, we have $m_1 \geq 1$, $m_2 \geq 1$, and we estimate A by applying Proposition 3.31 with $r = 0$, $\sigma = 0$, $s = m + 1 - m_2$ and $t = m - m_1$, and taking the $L^2(x_2)$ norm of the inequality thus obtained. \square

Proposition 3.64. *Suppose $m > \frac{d+1}{2} + 1$. For $0 \leq k \leq m + 1$ we have*

$$(3.203) \quad \frac{1}{\sqrt{\varepsilon}} |\Lambda^{\frac{1}{2}} [a(v^s) d_\varepsilon^2, \partial^k] u|_{0,0,\gamma} \lesssim \left(\left| \frac{\Lambda^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right|_{0,m+1} + \left| \frac{v^s}{\varepsilon} \right|_{0,m+1} \right) |\Lambda v|_{\infty,m,\gamma} h(|v^s|_{\infty,m+1}) + |\sqrt{\varepsilon} \Lambda^{\frac{3}{2}} v|_{0,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}).$$

Proof. Parallel to (3.193) we must estimate a sum of terms of the form

$$(3.204) \quad \frac{1}{\sqrt{\varepsilon}} |\Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon^2 u)]|_{0,0,\gamma} = \frac{1}{\sqrt{\varepsilon}} |\Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon v)]|_{0,0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$ and $m_1 \geq 1$. We can now repeat the proof of Proposition 3.61 in the case $m_1 = m + 1$ with “ u ” = $d_\varepsilon v$ and using “ s ” = $m - \frac{1}{2}$ in place of $m - \frac{3}{2}$. In the case $m_1 \leq m$ we apply Proposition 3.31 with $t = m - m_1$ and $s = m - m_2$ to obtain

$$(3.205) \quad A \lesssim \left(|\sqrt{\varepsilon} \Lambda^{\frac{3}{2}} v|_{0,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} + |\Lambda v|_{\infty,m,\gamma} \left| \frac{\Lambda^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right|_{0,m} \right) h(|v^s|_{\infty,m}).$$

\square

To control the terms (3.163) we must estimate the commutator $\frac{1}{\sqrt{\varepsilon}} |[a(v^s) d_\varepsilon^2, \partial^k] v|_{0,0,\gamma}$, $k \leq m$.

Proposition 3.65. *Suppose $m > \frac{d+1}{2} + 1$. (a) When $d_\varepsilon^2 = \partial_{x_1,\varepsilon}^2$ or $\partial_{x_2} \partial_{x_1,\varepsilon}$ we have for $0 \leq k \leq m$*

$$(3.206) \quad \frac{1}{\sqrt{\varepsilon}} |[a(v^s) d_\varepsilon^2, \partial^k] v|_{0,0,\gamma} \lesssim \left| \frac{\Lambda v}{\sqrt{\varepsilon}} \right|_{0,m,\gamma} \left| \frac{v^s}{\varepsilon} \right|_{\infty,m} h(|v^s|_{\infty,m}).$$

(b) When $d_\varepsilon^2 = \partial_{x_2}^2$, we have the same estimate with $\left| \frac{\Lambda v}{\sqrt{\varepsilon}} \right|_{0,m,\gamma}$ replaced by $\left| \frac{\varepsilon \partial_{x_2}^2 v}{\sqrt{\varepsilon}} \right|_{0,m-1,\gamma}$.

Proof. (a) Taking $k = m$, we must estimate a sum of terms of the form

$$(3.207) \quad \frac{1}{\sqrt{\varepsilon}} |[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon^2 v)]|_{0,0,\gamma} = \frac{1}{\varepsilon^{3/2}} |[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon \partial v)]|_{0,0,\gamma} := A,$$

where $m_1 + m_2 = m$, $m_1 \geq 1$. Treating x_2 as a parameter, we estimate A by applying Proposition 3.31 with $r = 0$, $\sigma = 0$, $s = m - (m_2 + 1)$ and $t = m - m_1$, and taking the $L^2(x_2)$ norm of the inequality thus obtained. Again, the proof of (b) is similar. \square

3.5.6 Boundary commutators

The boundary commutators vanish for the Dirichlet problem (3.87), but not for the Neumann-type problem (3.86). To estimate terms in the first lines of (3.78), we must estimate norms involving boundary commutators of the form

$$(3.208) \quad \langle \Lambda[a(v^s)d_\varepsilon, \partial^k]v \rangle_{0,\gamma} \text{ and } \langle \Lambda^{\frac{1}{2}}[a(v^s)d_\varepsilon, \partial^{k+1}]v \rangle_{0,\gamma},$$

where $k \leq m$, $a(v^s) = d_{v_1}H(v_1^s, h(v^s))$, and $v = v_1$.

Proposition 3.66. *Suppose $m > \frac{d+1}{2} + 1$. We have for $0 \leq k \leq m$*

$$(3.209) \quad \langle \Lambda[a(v^s)d_\varepsilon, \partial^k]v \rangle_{0,\gamma} \lesssim \langle \Lambda v \rangle_{m,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m h(\langle v^s \rangle_m) + \left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \Lambda_1 v^s \rangle_m h(\langle v^s \rangle_m).$$

Proof. Writing $a(v^s) = a(0) + b(v^s)v^s$ and taking $k = m$, we must estimate a sum of terms of the form

$$(3.210) \quad \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon v)] \rangle_{0,\gamma} = \frac{1}{\varepsilon} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}\partial v)] \rangle_{0,\gamma},$$

where $m_1 + m_2 = m$, $m_1 \geq 1$. To finish apply Proposition 3.31 with $r = 1$, $\sigma = 0$, $s = m - (m_2 + 1)$, and $t = m - m_1$. \square

Proposition 3.67. *Suppose $m > \frac{d+1}{2} + 1$. We have for $0 \leq k \leq m + 1$*

$$(3.211) \quad \langle \Lambda^{\frac{1}{2}}[a(v^s)d_\varepsilon, \partial^k]v \rangle_{0,\gamma} \lesssim \left(\left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} + \left\langle \frac{\Lambda^{\frac{1}{2}} v}{\sqrt{\varepsilon}} \right\rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}) + \left(\langle \Lambda^{\frac{1}{2}} v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \left\langle \frac{\Lambda_1^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right\rangle_m \right) h(\langle v^s \rangle_m).$$

Proof. Taking $k = m + 1$, we must estimate a sum of terms of the form

$$(3.212) \quad \langle \Lambda^{\frac{1}{2}}[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon v)] \rangle_{0,\gamma} = \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}}[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}\partial v)] \rangle_{0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. In the case $m_1 = m + 1$ we apply Proposition 3.31 with $t = 0$, $s = m - 1$ to obtain

$$(3.213) \quad A \lesssim \left(\left\langle \frac{\Lambda^{\frac{1}{2}} v}{\sqrt{\varepsilon}} \right\rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} + \left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}).$$

In the case $m_1 \leq m$ we apply Proposition 3.31 with $r = \frac{1}{2}$, $\sigma = 0$, $s = m + 1 - (m_2 + 1)$, and $t = m - m_1$ to obtain

$$(3.214) \quad A \lesssim \left(\langle \Lambda^{\frac{1}{2}} v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \left\langle \frac{\Lambda_1^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right\rangle_m \right) h(\langle v^s \rangle_m).$$

\square

The next estimate is needed to control terms in the second line of (3.78).

Proposition 3.68. *Suppose $m > \frac{d+1}{2} + 1$. We have for $0 \leq k \leq m + 1$*

$$(3.215) \quad \begin{aligned} \sqrt{\varepsilon} \langle \Lambda[a(v^s)d_\varepsilon, \partial^k]v \rangle_{0,\gamma} &\lesssim \left(\left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \sqrt{\varepsilon} \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} + \langle \Lambda v \rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}) + \\ &\left(\langle \sqrt{\varepsilon} \Lambda v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \langle \Lambda_1 v^s \rangle_m \right) h(\langle v^s \rangle_m) \end{aligned}$$

Proof. Taking $k = m + 1$, we must estimate a sum of terms of the form

$$(3.216) \quad \sqrt{\varepsilon} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon v)] \rangle_{0,\gamma} = \frac{1}{\sqrt{\varepsilon}} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}\partial v)] \rangle_{0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. In the case $m_1 = m + 1$ we apply Proposition 3.31 with $t = 0$, $s = m - 1$ to obtain

$$(3.217) \quad A \lesssim \left(\left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \sqrt{\varepsilon} \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} + \langle \Lambda v \rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}).$$

In the case $m_1 \leq m$ we apply Proposition 3.31 with $s = m + 1 - (m_2 + 1)$, and $t = m - m_1$ to obtain

$$(3.218) \quad A \lesssim \left(\langle \sqrt{\varepsilon} \Lambda v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \langle \Lambda_1 v^s \rangle_m \right) h(\langle v^s \rangle_m).$$

□

To control the terms (3.175) we must estimate for u in (3.88) the boundary commutators

$$(3.219) \quad \frac{1}{\varepsilon} \langle \Lambda[a(v^s)d_\varepsilon, \partial^m]u \rangle_{0,\gamma}, \quad \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}}[a(v^s)d_\varepsilon, \partial^{m+1}]u \rangle_{0,\gamma}, \quad \text{and} \quad \frac{1}{\sqrt{\varepsilon}} \langle \Lambda[a(v^s)d_\varepsilon, \partial^{m+1}]u \rangle_{0,\gamma}.$$

Proposition 3.69. *Suppose $m > \frac{d+1}{2}$. We have for $0 \leq k \leq m$*

$$(3.220) \quad \frac{1}{\varepsilon} \langle \Lambda[a(v^s)d_\varepsilon, \partial^k]u \rangle_{0,\gamma} \lesssim \langle \Lambda v \rangle_{m,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m h(\langle v^s \rangle_m) + \left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \Lambda_1 v^s \rangle_m h(\langle v^s \rangle_m)$$

Proof. Parallel to (3.210), to prove (a) we must estimate a sum of terms of the form

$$(3.221) \quad \frac{1}{\varepsilon} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}d_\varepsilon u)] \rangle_{0,\gamma} = \frac{1}{\varepsilon} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2}v)] \rangle_{0,\gamma},$$

where $m_1 + m_2 = m$, $m_1 \geq 1$. To finish apply Proposition 3.31 with $r = 1$, $\sigma = 0$, $s = m - m_2$, and $t = m - m_1$.

□

Proposition 3.70. *Suppose $m > \frac{d+1}{2}$. We have for $0 \leq k \leq m + 1$*

$$(3.222) \quad \begin{aligned} \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}}[a(v^s)d_\varepsilon, \partial^k]u \rangle_{0,\gamma} &\lesssim \left(\left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} + \left\langle \frac{\Lambda^{\frac{1}{2}} v}{\sqrt{\varepsilon}} \right\rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}) + \\ &\left(\langle \Lambda^{\frac{1}{2}} v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \left\langle \frac{\Lambda_1^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right\rangle_m \right) h(\langle v^s \rangle_m). \end{aligned}$$

Proof. Parallel to (3.212) we must estimate a sum of terms of the form

$$(3.223) \quad \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon u)] \rangle_{0,\gamma} = \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} v)] \rangle_{0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. In the case $m_1 = m + 1$ we apply Proposition 3.31 to obtain

$$(3.224) \quad A \lesssim \left(\left\langle \frac{\Lambda^{\frac{1}{2}} v}{\sqrt{\varepsilon}} \right\rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} + \left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}).$$

In the case $m_1 \leq m$ we apply Proposition 3.31 with $r = \frac{1}{2}$, $\sigma = 0$, $s = m + 1 - m_2$, and $t = m - m_1$ to obtain

$$(3.225) \quad A \lesssim \left(\langle \Lambda^{\frac{1}{2}} v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \left\langle \frac{\Lambda_1^{\frac{1}{2}} v^s}{\sqrt{\varepsilon}} \right\rangle_m \right) h(\langle v^s \rangle_m).$$

□

Proposition 3.71. *Suppose $m > \frac{d+1}{2}$. We have for $0 \leq k \leq m + 1$*

$$(3.226) \quad \frac{1}{\sqrt{\varepsilon}} \langle \Lambda[a(v^s)d_\varepsilon, \partial^k]u \rangle_{0,\gamma} \lesssim \left(\left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \langle \sqrt{\varepsilon} \Lambda_1^{\frac{1}{2}} v^s \rangle_{m+1} + \langle \Lambda v \rangle_{m,\gamma} \left\langle \frac{v^s}{\sqrt{\varepsilon}} \right\rangle_{m+1} \right) h(\langle v^s \rangle_{m+1}) + \left(\langle \sqrt{\varepsilon} \Lambda v \rangle_{m+1,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m + \left\langle \frac{\nabla_\varepsilon u}{\sqrt{\varepsilon}} \right\rangle_{m+1,\gamma} \langle \Lambda_1 v^s \rangle_m \right) h(\langle v^s \rangle_m)$$

Proof. Taking $k = m + 1$, we must estimate a sum of terms of the form

$$(3.227) \quad \frac{1}{\sqrt{\varepsilon}} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon u)] \rangle_{0,\gamma} = \frac{1}{\sqrt{\varepsilon}} \langle \Lambda[(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} v)] \rangle_{0,\gamma} := A,$$

where $m_1 + m_2 = m + 1$, $m_1 \geq 1$. Comparison with (3.216) shows that we can repeat the proof of Proposition 3.68, now applying Proposition 3.31 with $t = 0$, $s = m$ in the case $m_1 = m + 1$, and with $t = m - m_1$, $s = m + 1 - m_2$ in the case $m_1 \leq m$.

□

To control the terms (3.163) with $j = 1$ we must estimate $\frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} [a(v^s)d_\varepsilon, \partial^k]v \rangle_{0,\gamma}$, $k \leq m$ and $v = v_1$.

Proposition 3.72. *Suppose $m > \frac{d+1}{2} + 1$. We have for $0 \leq k \leq m$ and $v = v_1$*

$$(3.228) \quad \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} [a(v^s)d_\varepsilon, \partial^k]v \rangle_{0,\gamma} \lesssim \left\langle \frac{\Lambda^{\frac{1}{2}} v}{\sqrt{\varepsilon}} \right\rangle_{m,\gamma} \left\langle \frac{v^s}{\varepsilon} \right\rangle_m h(\langle v^s \rangle_m) + \left\langle \frac{\nabla_\varepsilon u}{\varepsilon} \right\rangle_{m,\gamma} \left\langle \frac{\Lambda_1^{\frac{1}{2}} v}{\sqrt{\varepsilon}} \right\rangle_m h(\langle v^s \rangle_m).$$

Proof. Parallel to (3.210) we must estimate a sum of terms of the form

$$(3.229) \quad \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} d_\varepsilon v)] \rangle_{0,\gamma} = \frac{1}{\varepsilon} \langle \Lambda^{\frac{1}{2}} [(\partial^{m_1}(b(v^s)v^s)) \cdot (\partial^{m_2} \partial v)] \rangle_{0,\gamma},$$

where $m_1 + m_2 = m$, $m_1 \geq 1$. To finish apply Proposition 3.31 with $r = \frac{1}{2}$, $\sigma = 0$, $s = m - (m_2 + 1)$, and $t = m - m_1$.

□

3.5.7 Boundary forcing II

Here we estimate the forcing given by \mathcal{G}_2 in the problem (3.87). For this we require

Proposition 3.73. *Suppose $m > 3d + 4 + \frac{d+1}{2}$. There exist positive constants ε_0, γ_0 such that for $j \in J_h \cup J_e$, $\varepsilon \in (0, \varepsilon_0]$, and each T with $0 < T \leq T_\varepsilon^*$,*

$$(3.230) \quad E_{m,\gamma}(v_1) + \langle \phi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m,\gamma} + \langle \phi_j \Lambda v_1 \rangle_{m+1,\gamma} + \sqrt{\varepsilon} \langle \phi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m+1,\gamma} + \frac{1}{\sqrt{\varepsilon}} \langle \phi_j \Lambda v_1 \rangle_{m,\gamma} \lesssim \frac{1}{\sqrt{\gamma}} \{Q_1(E_{m,T}(v^s)) \cdot E_{m,\gamma}(v_1) + Q_2^o(E_{m,T}(v^s)) + (M_G + \sqrt{\varepsilon})Q_3(E_{m,T}(v^s))\} \text{ for } \gamma \geq \gamma_0.$$

Proof. For all terms in the definition of $E_{m,\gamma}(v_1)$, domination by the right side of (3.230) follows directly from the linearized estimates of section 3.2.1 and the propositions of sections 3.5.3, 3.5.4, 3.5.5, and 3.5.6. The ‘‘right side’’ of every commutator proposition needed to estimate these terms is dominated by $Q_1(E_{m,T}(v^s)) \cdot E_{m,\gamma}(v_1)$, and the right side of every forcing proposition is dominated by $Q_2^o(E_{m,T}(v^s)) + (M_G + \sqrt{\varepsilon})Q_3(E_{m,T}(v^s))$. The estimates of section 3.2.1 show that $1/\sqrt{\gamma}$ is the correct factor in (3.230).

Consider for example the first term in the first line of (3.78) when $j = 1$. By (3.37) we have

$$(3.231) \quad \left| \begin{pmatrix} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{pmatrix} \right|_{0,m,\gamma} \lesssim \gamma^{-1} \left(|\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{0,m,\gamma} + \langle \Lambda \mathcal{G}_1 \rangle_{m,\gamma} + |\Lambda^{\frac{1}{2}} \mathcal{F}_{1,com}|_{0,m,\gamma} + \langle \Lambda \mathcal{G}_{1,com} \rangle_{m,\gamma} + \mathcal{E}_{m,\gamma} \right),$$

where $\mathcal{F}_{1,com}$ and $\mathcal{G}_{1,com}$ denote the interior and boundary commutators estimated in Propositions 3.58 and 3.66, and $\mathcal{E}_{m,\gamma}$ denotes the error term

$$(3.232) \quad \mathcal{E}_{m,\gamma} := \left| \begin{pmatrix} \Lambda v_1 / \sqrt{\varepsilon} \\ D_{x_2} v_1 / \sqrt{\varepsilon} \end{pmatrix} \right|_{0,m,\gamma}.$$

These propositions, along with the interior and boundary forcing estimates in Propositions 3.46 and 3.52, show that the right side of (3.231) is dominated by the right side of (3.230). By the estimate (3.37), a term involving ϕ_j like $\langle \phi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m,\gamma}$ is controlled as soon as the left side of (3.231) is. However, for this ϕ_j term we require the factor $\frac{1}{\sqrt{\gamma}}$ in place of γ^{-1} on the right side of (3.231). \square

The next Corollary is immediate.

Corollary 3.74. *Let $m > 3d + 4 + \frac{d+1}{2}$. There exist positive constants $\varepsilon_1 = \varepsilon_1(M_0)$, $\gamma_m = \gamma_m(M_0)$ such that for $\varepsilon \in (0, \varepsilon_1]$ and each T with $0 < T \leq T_\varepsilon^*$,*

$$(3.233) \quad E_{m,\gamma}(v_1) \leq \frac{1}{\sqrt{\gamma}} [Q_2^o(E_{m,T}(v^s)) + (M_G + \sqrt{\varepsilon})Q_3(E_{m,T}(v^s))] \text{ for } \gamma \geq \gamma_m.$$

The function $\gamma(M_0)$ increases with M_0 .

Remark 3.75. *1) When we apply Proposition 3.29 to estimate $\langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m,\gamma}$ for $j \in J_h$, we must control norms like $\langle \Lambda_1(\chi^*(t)v_1) \rangle_m$ (also $\langle \chi^* v_1 \rangle_m$), where v_1 and not v_1^s occurs, and $\text{supp } \chi^* \subset (-1, 2)$. Using Proposition 3.15 we have*

$$(3.234) \quad E_m(\chi^*(t)v_1) \lesssim E_{m,T=2}(v_1) \lesssim e^{\gamma_m(M_0)^2} E_{m,\gamma_m(M_0)}(v_1)$$

Applying Corollary 3.74 with $\gamma = \gamma_m(M_0)$ we obtain

$$(3.235) \quad \begin{aligned} E_m(\chi^*(t)v_1) &\lesssim e^{\gamma_m(M_0)^2} [Q_2^o(E_m, T(v^s)) + (M_G + \sqrt{\varepsilon})Q_3(E_m(v^s))] \lesssim \\ &e^{\gamma_m(M_0)^2} [Q_2^o(M_0) + (M_G + \sqrt{\varepsilon})Q_3(M_0)]. \end{aligned}$$

2) The right side of (3.235) is small when M_G , M_0 , and ε_1 are small. The constant M_G is small when T_0 as in (3.84) is small, since $G(x', \theta) = 0$ in $t < 0$.

To control terms in the first line of (3.78) when $j = 2$, we must estimate $\langle \Lambda \mathcal{G}_2 \rangle_{m, \gamma}$ and $\langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m+1, \gamma}$; recall $\mathcal{G}_2 = \chi_0(t)H(v_1, g_\varepsilon)$.

Proposition 3.76. *Let $m > \frac{d+1}{2}$. For small enough positive constants ε_1 , M_G (3.90), and M_0 (3.84) we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.236) \quad \begin{aligned} (a) \quad \langle \Lambda \mathcal{G}_2 \rangle_{m, \gamma} &\lesssim (\langle \Lambda v_1 \rangle_{m, \gamma} + \varepsilon) [Q_1(E_m(v^s)) + Q_2(M_G)] \\ (b) \quad \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m+1, \gamma} &\lesssim (\langle \Lambda^{\frac{1}{2}} v_1 \rangle_{m+1, \gamma} + \varepsilon^{\frac{3}{2}}) [Q_1(E_m(v^s)) + Q_2(M_G)]. \end{aligned}$$

Proof. Choose $\chi^*(t)$ supported in $(-1, 2)$ such that $\chi_0 \chi^* = \chi_0$ and write $\chi^*(t)v_1 = v_1^*$. By Proposition 3.25 we have

$$(3.237) \quad \langle \Lambda \mathcal{G}_2 \rangle_{m, \gamma} \leq \langle \Lambda(v_1^*, g_\varepsilon) \rangle_{m, \gamma} h(\langle v_1^*, g_\varepsilon \rangle_m) + \langle v_1^*, g_\varepsilon \rangle_{m, \gamma} \langle \Lambda_1(v_1^*, g_\varepsilon) \rangle_m h(\langle v_1^*, g_\varepsilon \rangle_m).$$

Then (3.236)(a) follows since

$$(3.238) \quad \langle \Lambda(v_1^*, g_\varepsilon) \rangle_{m, \gamma} \lesssim \langle \Lambda v_1 \rangle_{m, \gamma} + \varepsilon M_G,$$

and the factors involving $\langle \cdot \rangle_m$ norms on the right in (3.237) are controlled using (3.235). In particular, ε_1 , M_0 , and M_G must be taken small enough so that $\langle v_1^*, g_\varepsilon \rangle_m$ lies in the region of convergence of h . The commutator $\langle [\Lambda, \chi^*]v_1 \rangle_{m, \gamma}$ is estimated using Proposition A.14. The inequality (3.236)(b) is proved similarly. \square

Proposition 3.77. *Let $m > \frac{d+1}{2}$ and let ϕ_j and ψ_j , $j \in J_h$, be singular symbols related as in Notation 3.27. For small enough positive constants ε_1 , M_G , and M_0 we have for $\varepsilon \in (0, \varepsilon_1]$*

$$(3.239) \quad \begin{aligned} (a) \quad \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m, \gamma} &\lesssim \left(\langle \psi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m, \gamma} + \langle \Lambda v_1 \rangle_{m, \gamma} + \sqrt{\varepsilon} \right) [Q_1(E_m(v^s)) + Q_2(M_G)] \\ (b) \quad \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m+1, \gamma} &\lesssim \left(\langle \psi_j \Lambda v_1 \rangle_{m+1, \gamma} + \langle \Lambda^{\frac{1}{2}} v_1 \rangle_{m+1, \gamma} + \varepsilon \right) [Q_1(E_m(v^s)) + Q_2(M_G)] \end{aligned}$$

Proof. The argument is parallel to the proof of Proposition 3.76, except that Proposition 3.29 is used in place of Proposition 3.25. Commutators like $\langle [\psi_j \Lambda^{\frac{3}{2}}, \chi^*(t)]v_1 \rangle_{m, \gamma}$ are estimated using Proposition A.14. \square

To control the terms in the second line of (3.78) when $j = 2$, we must estimate

$$(3.240) \quad \sqrt{\varepsilon} \langle \Lambda \mathcal{G}_2 \rangle_{m+1, \gamma}, \quad \sqrt{\varepsilon} \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m+1, \gamma}, \quad \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m, \gamma}, \quad \text{and} \quad \frac{1}{\sqrt{\varepsilon}} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m, \gamma}, \quad \text{where } j \in J_h.$$

The proofs of the next two propositions are essentially repetitions of the proofs of Propositions 3.76 and 3.77.

Proposition 3.78. Let $m > \frac{d+1}{2}$. For small enough positive constants ε_1 , M_G , and M_0 we have for $\varepsilon \in (0, \varepsilon_1]$

$$(3.241) \quad \begin{aligned} (a) \quad & \sqrt{\varepsilon} \langle \Lambda \mathcal{G}_2 \rangle_{m+1, \gamma} \lesssim (\sqrt{\varepsilon} \langle \Lambda v_1 \rangle_{m+1, \gamma} + \varepsilon^{\frac{3}{2}}) [Q_1(E_m(v^s)) + Q_2(M_G)] \\ (b) \quad & \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m, \gamma} \lesssim \left(\frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} v_1 \rangle_{m, \gamma} + \varepsilon \right) [Q_1(E_m(v^s)) + Q_2(M_G)]. \end{aligned}$$

Proposition 3.79. Let $m > \frac{d+1}{2}$ and let ϕ_j and ψ_j , $j \in J_h$, be singular symbols related as in Notation 3.27. For small enough positive constants ε_1 , M_G , and M_0 we have for $\varepsilon \in (0, \varepsilon_1]$

$$(3.242) \quad \begin{aligned} (a) \quad & \sqrt{\varepsilon} \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m+1, \gamma} \lesssim \left(\sqrt{\varepsilon} \langle \psi_j \Lambda^{\frac{3}{2}} v_1 \rangle_{m+1, \gamma} + \sqrt{\varepsilon} \langle \Lambda v_1 \rangle_{m+1, \gamma} + \varepsilon \right) [Q_1(E_m(v^s)) + Q_2(M_G)] \\ (b) \quad & \frac{1}{\sqrt{\varepsilon}} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m, \gamma} \lesssim \left(\frac{1}{\sqrt{\varepsilon}} \langle \psi_j \Lambda v_1 \rangle_{m, \gamma} + \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} v_1 \rangle_{m, \gamma} + \sqrt{\varepsilon} \right) [Q_1(E_m(v^s)) + Q_2(M_G)] \end{aligned}$$

The next proposition is the last step in the proof of Proposition 3.20.

Proposition 3.80. Suppose $m > 3d + 4 + \frac{d+1}{2}$. There exist positive constants ε_0 , γ_0 such that for $\varepsilon \in (0, \varepsilon_0]$ and each T with $0 < T \leq T_\varepsilon^*$,

$$(3.243) \quad E_{m, \gamma}(v_2) \lesssim \gamma^{-1} E_{m, \gamma}(v) Q(E_{m, T}(v^s)) + (\gamma^{-\frac{1}{2}} + \sqrt{\varepsilon}) Q(E_{m, T}(v^s)) \text{ for } \gamma \geq \gamma_0.$$

Proof. 1. For all terms in the definition of $E_{m, \gamma}(v_2)$, domination by the right side of (3.243) follows directly from the linearized estimates of section 3.2.1 and the propositions of sections 3.5.3, 3.5.4, 3.5.5, and 3.5.7. The ‘‘right side’’ of every interior or boundary commutator proposition needed to estimate these terms is dominated by $Q(E_{m, T}(v^s)) \cdot E_{m, \gamma}(v)$,³⁴ the right side of every interior forcing proposition is dominated by $Q^o(E_{m, T}(v^s))$, and the boundary forcing terms involving \mathcal{G} (3.176) are dominated by

$$(3.244) \quad Q^o(E_{m, T}(v^s)) + (M_G + \sqrt{\varepsilon}) Q(E_{m, T}(v^s)) \leq Q(E_{m, T}(v^s)) \text{ since } M_G \leq 1.$$

The boundary forcing terms involving \mathcal{G}_2 have been estimated in this section. Those that do not involve $\phi_{j, D}$, $j \in J_h$ are dominated by

$$(3.245) \quad (E_{m, \gamma}(v_1) + \sqrt{\varepsilon}) [Q(E_{m, T}(v^s)) + Q(M_G)] \leq (E_{m, \gamma}(v_1) + \sqrt{\varepsilon}) Q(E_{m, T}(v^s)).$$

Using Proposition 3.73, we see that the boundary forcing terms involving $\phi_{j, D}$ on the left in Propositions 3.77 and 3.79 are dominated by

$$(3.246) \quad \begin{aligned} & \frac{1}{\sqrt{\gamma}} \{ Q_1(E_{m, T}(v^s)) \cdot E_{m, \gamma}(v_1) + Q_2^o(E_{m, T}(v^s)) + (M_G + \sqrt{\varepsilon}) Q_3(E_{m, T}(v^s)) \} [Q(E_{m, T}(v^s)) + Q(M_G)] \leq \\ & \frac{1}{\sqrt{\gamma}} [E_{m, \gamma}(v_1) Q(E_{m, T}(v^s)) + Q(E_{m, T}(v^s))]. \end{aligned}$$

The coefficients on the right in (3.243) are explained below.

³⁴Although the boundary commutators for the v_2 problem (3.87) are zero, the boundary commutators for (3.88) are not; such commutators are needed to control terms in the third row of (3.78) when $j = 1$ or $j = 2$.

2. Consider for example the second term in the first line of (3.78) when $j = 2$. By (3.41) we have

$$(3.247) \quad \left| \begin{pmatrix} \Lambda v_2 \\ D_{x_2} v_2 \end{pmatrix} \right|_{\infty, m, \gamma} \lesssim \gamma^{-1} |\mathcal{F}_2|_{0, m+1, \gamma} + \gamma^{-1} |\mathcal{F}_{2, com}|_{0, m+1, \gamma} + \\ (\langle \Lambda \mathcal{G}_2 \rangle_{m, \gamma} + \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m+1, \gamma}) + \gamma^{-\frac{1}{2}} \sum_{j \in J_h} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m+1, \gamma},$$

The discussion in step 1 shows that the right side of (3.247) is dominated by

$$(3.248) \quad \gamma^{-1} Q(E_{m, T}(v^s)) + \gamma^{-1} E_{m, \gamma}(v) Q(E_{m, T}(v^s)) + (E_{m, \gamma}(v_1) + \sqrt{\varepsilon}) Q(E_{m, T}(v^s)) + \\ \gamma^{-1} [E_{m, \gamma}(v_1) Q(E_{m, T}(v^s)) + Q(E_{m, T}(v^s))].$$

By Corollary 3.74 $E_{m, \gamma}(v_1) \leq \gamma^{-\frac{1}{2}} Q(E_{m, T}(v^s))$, so we see that the right side of (3.248) is dominated by the right side of (3.243). □

Proof of Proposition 3.20. The estimate of Proposition 3.20 follows immediately from Corollary 3.74 and Proposition 3.80. □

3.6 Local existence and continuation for the singular problems with ε fixed.

For fixed $\varepsilon \in (0, 1]$ we prove here existence and continuation theorems for solutions $v_1^\varepsilon, v_2^\varepsilon, u^\varepsilon$ to the triple of coupled systems (3.11), (3.12), (3.13). We also establish the relation $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_{T_ε} . These results are used in the proof of Proposition 3.16, which establishes a uniform time of existence with respect to ε for solutions of the singular nonlinear problems.

Recall that we describe these systems as singular both because of the factors of $\frac{1}{\varepsilon}$ that appear and because of the fact that $\partial_{x'}$ and ∂_θ derivatives occur in the combination $\partial_{x'} + \beta \frac{\partial_\theta}{\varepsilon}$. Even when we fix ε , the second singular feature is still present. Thus, we still need to use the singular calculus to prove estimates.

Definition 3.81. For $m \geq 0$ and $T \in \mathbb{R}$ set $\mathcal{N}^m(\Omega_T) = \{w : |w|_{m, T}^* < \infty\}$, where

$$(3.249) \quad |w|_{m, T}^* = \left| \begin{pmatrix} \Lambda_1^{\frac{3}{2}} w \\ D_{x_2} \Lambda_1^{\frac{1}{2}} w \end{pmatrix} \right|_{m, T} + \left\langle \begin{pmatrix} \Lambda_1 w \\ D_{x_2} w \end{pmatrix} \right\rangle_{m, T}.$$

The norms $|w|_{m, \gamma}^*$, $|w|_{m, \gamma, T}^*$, and $|w|_m^*$ are defined by substituting (m, γ) , etc. for (m, T) in (3.249); recall Definition 3.32.

Remark 3.82. 1. Let $m > \frac{d+1}{2} + 1$. Observe that if $(v^\varepsilon, u^\varepsilon)$ is a solution of (3.11), (3.12), (3.13) such that $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_T and $E_{m, T}(v^\varepsilon) < \infty$, then $v^\varepsilon \in \mathcal{N}^{m+1}(\Omega_T)$ and we have

$$(3.250) \quad |v^\varepsilon|_{m+1, T}^* \leq C_\varepsilon E_{m, T}(v^\varepsilon).$$

Indeed, the control of tangential derivatives is immediate, and since the boundary is noncharacteristic, the equation can be used in the standard way for ε fixed to control ∂_{x_2} derivatives. (Note: We need estimates as

in section 3.4 involving higher ∂_{x_2} derivatives for this.) On the other hand for any $(v^\varepsilon, u^\varepsilon) \in \mathcal{N}^{m+1}(\Omega_T)$ with $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ we have

$$(3.251) \quad E_{m,T}(v^\varepsilon) \leq C_\varepsilon |v^\varepsilon|_{m+1,T}^*.$$

2. Let $\delta > 0$ and $\varepsilon > 0$ be fixed. Suppose $(v^\varepsilon, u^\varepsilon) \in \mathcal{N}^{m+1}(\Omega_T)$ for some $T > 0$ with $\nabla_\varepsilon u^\varepsilon = v^\varepsilon$ and vanishes in $t < 0$. For sufficiently small T_ε with $0 < T_\varepsilon < T$ we have $E_{m,T_\varepsilon}(v) < \delta$. This is easily seen by first approximating v in the $\mathcal{N}^{m+1}(\Omega_T)$ norm by a C^∞ function with compact support in $\overline{\Omega_T}$.

3.6.1 Local existence.

Proposition 3.83 (Local existence for fixed ε). *Let $m > 3d + 4 + \frac{d+1}{2}$ and consider the problems (3.11), (3.12), (3.13) for ε fixed, where $G(x', \theta) \in H^{m+3}(b\Omega)$ (see section 4) and vanishes in $t < 0$. There exist $T > 0$ and unique solutions $v = (v_1, v_2) \in \mathcal{N}^{m+1}(\Omega_T)$ to (3.11), (3.12) and $u \in \mathcal{N}^{m+1}(\Omega_T)$ to (3.13) vanishing in $t < 0$. Moreover, we have $v = \nabla_\varepsilon u$ on Ω_T .*

Proof. **1. A priori estimates and existence for the linearized problems.** Consider the linearized problems for ε fixed

$$(3.252) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_1 + \sum_{|\alpha|=2} A_\alpha(w) \partial_{x,\varepsilon}^\alpha v_1 &= \mathcal{F}_1 \text{ on } \Omega \\ \partial_{x_2} v_1 - d_{v_1} H(w_1, h(w)) \partial_{x_1,\varepsilon} v_1 &= \mathcal{G}_1 \text{ on } x_2 = 0 \end{aligned}$$

and

$$(3.253) \quad \begin{aligned} \partial_{t,\varepsilon}^2 v_2 + \sum_{|\alpha|=2} A_\alpha(w) \partial_{x,\varepsilon}^\alpha v_2 &= \mathcal{F}_2 \text{ on } \Omega \\ v_2 &= \mathcal{G}_2 \text{ on } x_2 = 0. \end{aligned}$$

Here the functions $v_i, w, \mathcal{F}_i, \mathcal{G}_i$ all vanish in $t < 0$.

Let $k \in \{0, 1, \dots, m+1\}$. For the problem 3.252 we have: for $|w|_{m+1}^*$ sufficiently small, there exist constants $\gamma_{m+1}(k), C = C_{m+1}(k)$ (depending on $|w|_{m+1}^*, k$, and ε) such that for $\gamma \geq \gamma_{m+1}(k)$

$$(3.254) \quad \begin{aligned} \gamma \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right|_{k,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right\rangle_{k,\gamma}^2 + \sum_{J_h \cup J_e} \left\langle \phi_j \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right\rangle_{k,\gamma}^2 \leq \\ \frac{1}{\gamma} C \left(|\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{k,\gamma}^2 + \langle \Lambda \mathcal{G}_1 \rangle_{k,\gamma}^2 \right). \end{aligned}$$

For the problem 3.253 we have: for $|w|_{m+1}^*$ sufficiently small, there exist constants $\gamma_{m+1}(k), C = C_{m+1}(k)$ (depending on $|w|_{m+1}^*, k$, and ε) such that for $\gamma \geq \gamma_{m+1}(k)$

$$(3.255) \quad \begin{aligned} \gamma \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_2 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_2 \end{array} \right) \right|_{k,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda v_2 \\ D_{x_2} v_2 \end{array} \right) \right\rangle_{k,\gamma}^2 \leq \\ C \left(\frac{1}{\gamma} |\Lambda^{\frac{1}{2}} \mathcal{F}_2|_{k,\gamma}^2 + \gamma \langle \Lambda \mathcal{G}_2 \rangle_{k,\gamma}^2 + \sum_{j \in J_h} \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{k,\gamma}^2 \right). \end{aligned}$$

The estimates (3.254) and (3.255) are derived under the assumption of having sufficiently regular v_i . When $k = 0$ they follow directly from (3.37) and (3.40). Estimates of $|\cdot|_{0,k,\gamma}$ and $\langle \cdot \rangle_{k,\gamma}$ norms follow by applying the $k = 0$ estimate to the tangentially differentiated problems $\partial^k(3.252)$, $\partial^k(3.253)$. As in section 3.5 the singular norm estimates of section 3.4.1 are needed here; in particular, Proposition 3.31(b)

is again used to estimate commutators.³⁵ Estimates of $\left| \begin{pmatrix} D_{x_2}^j \left(\Lambda^{\frac{3}{2}} v_i \right) \\ D_{x_2}^j \left(D_{x_2} \Lambda^{\frac{1}{2}} v_i \right) \end{pmatrix} \right|_{0,k-j,\gamma}$ for $j > 0$ are proved as usual by induction on j using the equation and the noncharacteristic boundary assumption. Proposition 3.33 is needed here.

We have the following existence theorem for the singular linearized problems with ε fixed:

Theorem 3.84. *Consider the singular linear problems (3.252), (3.253) for $\varepsilon > 0$ fixed and assume $d = 2$. Let $m > 3d + 4 + \frac{d+1}{2}$ and suppose $|w|_{m+1}^* < \infty$. Suppose the data $\mathcal{F}_i, \mathcal{G}_i, i = 1, 2$ vanish in $t < 0$ and are such that for $k \in \{0, 1, \dots, m+1\}$ the right sides of (3.254), (3.255) are finite. For $|w|_{m+1}^*$ sufficiently small, there exist constants $\gamma_{m+1}(k), C = C_{m+1}(k)$ (depending on $|w|_{m+1}^*, k$, and ε) such that for $\gamma \geq \gamma_{m+1}(k)$ the problems (3.252), (3.253) admit unique solutions $v_i, i = 1, 2$ vanishing in $t < 0$ satisfying the estimates (3.254), (3.255).*

The theorem follows from the a priori estimates by a classical duality argument using the properties (P6) and (P7) described in section 3.1.1. We refer to [ST88], p. 279 or to [CP82], Chapter 7 for this kind of argument.

Remark 3.85. *An important consequence of Theorem 3.84 that we use in the proof of Proposition 3.20 and later in the error analysis is that the linearized singular problems exhibit causality [CP82]; solutions in $t < T$ are unaffected by changing the forcing terms and coefficients in $t > T$.*

2. Iteration schemes. For $T_0 > 0$ to be chosen, as before we let $\chi_0(t) \geq 0$ be a C^∞ function that is equal to 1 on a neighborhood of $[0, 1]$ and compactly supported in $(-1, 2)$. We use iteration schemes similar to those in [ST88], initialized by $v^0 = 0$. In $t < 0$ we have $G = 0$ and $v_k = 0, u_k = 0$ for all k .

(3.256)

$$\begin{aligned} \partial_{t,\varepsilon}^2 v_1^{k+1} + \sum_{|\alpha|=2} A_\alpha(v^k) \partial_{x,\varepsilon}^\alpha v_1^{k+1} &= - \left[\sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_1,\varepsilon} (A_\alpha(v^k)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_1^k - \partial_{x_1,\varepsilon} (A_{(0,2)}(v^k)) \partial_{x_2} v_2^k \right] \\ \partial_{x_2} v_1^{k+1} - d_{v_1} H(v_1^k, h(v^k)) \partial_{x_1,\varepsilon} v_1^{k+1} &= d_g H(v_1^k, \varepsilon^2 G) \partial_{x_1,\varepsilon} (\varepsilon^2 G) \text{ on } x_2 = 0. \end{aligned}$$

(3.257)

$$\begin{aligned} \partial_{t,\varepsilon}^2 v_2^{k+1} + \sum_{|\alpha|=2} A_\alpha(v^k) \partial_{x,\varepsilon}^\alpha v_2^{k+1} &= - \left[\sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_2} (A_\alpha(v^k)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_1^k - \partial_{x_2} (A_{(0,2)}(v^k)) \partial_{x_2} v_2^k \right] \\ v_2^{k+1} &= \chi_0(t) H(v_1^{k+1}, \varepsilon^2 G) \text{ on } x_2 = 0. \end{aligned}$$

(3.258)

$$\begin{aligned} \partial_{t,\varepsilon}^2 u^{k+1} + \sum_{|\alpha|=2} A_\alpha(v^k) \partial_{x,\varepsilon}^\alpha u^{k+1} &= 0 \\ \partial_{x_2} u^{k+1} - d_{v_1} H(v_1^k, h(v^k)) \partial_{x_1,\varepsilon} u^{k+1} &= \left[H(v_1^k, \varepsilon^2 G(x', \theta)) - d_{v_1} H(v_1^k, \varepsilon^2 G) v_1^k \right] \text{ on } x_2 = 0. \end{aligned}$$

³⁵As in section 3.5, interior norms are estimated by first doing a tangential estimate in (x', θ) for fixed x_2 , and then taking an $L^2(x_2)$ norm.

This is actually a family of problems parametrized by T , where $0 < T \leq T_0$. Each problem is solved on Ω . The function v^k appearing in the coefficients of (3.256)–(3.258) is really v_T^k (with the T suppressed), where v_T^k is a Seeley extension of $v_k|_{\Omega_T}$ to Ω chosen as in Proposition 3.15. Similarly, G here denotes a Seeley extension of $G|_{b\Omega_{T_0}}$ to $H^{m+3}(b\Omega)$. We can suppose T_0 is small enough so that $\langle G \rangle_{m+3, T_0} \leq 1$.

The function v_1^{k+1} appearing in the boundary condition of (3.257) is taken to be the *same* as that in (3.256), not a Seeley extension of $v_1^{k+1}|_{\Omega_T}$, for the reasons explained in Remark 3.18.

3. High norm boundedness. We make the following induction assumption:

- there exists $T > 0$ independent of k such that for $j = 0, \dots, k$ we have $|v_j|_{m+1, T}^* < \delta$ for δ to be chosen.³⁶

Using the estimates (3.254), (3.255) together with Proposition 3.15 (which relates $|\cdot|_{m, \gamma}^*$ and $|\cdot|_{m, T}^*$ norms) and the estimates of nonlinear functions in section 3.4.1, we can complete the induction step by an argument modeled on [ST88], pages 286–289.³⁷

We denote the interior and boundary forcing terms in (3.256), (3.257) by $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{G}_1, \mathcal{G}_2$, respectively. By Proposition 3.33 and Corollary 3.24 we have

$$(3.259) \quad |\Lambda^{\frac{1}{2}} \mathcal{F}_i|_{m+1, \gamma} \lesssim C(\delta) \text{ and } \langle \Lambda \mathcal{G}_1 \rangle_{m+1, \gamma} \leq C(\delta, \langle \Lambda_1^2 G \rangle_{m+1, T_0}).$$

Thus, estimate (3.254) applied to the problem (3.256), yields for $\gamma \geq \gamma_{m+1}(m+1) := \gamma_{m+1}^*$ and $j \in J_h \cup J_e$:

$$(3.260) \quad \gamma |v_1^{k+1}|_{m+1, \gamma}^* + \langle \phi_j \Lambda^{\frac{3}{2}} v_1^{k+1} \rangle_{m+1, \gamma} \leq K_1(\delta, \langle \Lambda_1^2 G \rangle_{m+1, T_0}), \text{ where } \langle \Lambda_1^2 G \rangle_{m+1, T_0} \lesssim \langle G \rangle_{m+3, T_0}.$$

The estimates giving (3.259) show that K_1 can be taken to be a continuous function such that $K_1(0, 0) = 0$.

To estimate v_2^{k+1} with (3.255) we write

$$(3.261) \quad \chi_0(t) H(v_1^{k+1}, \varepsilon^2 G) = h(\chi^* v_1^{k+1}, \varepsilon^2 \chi^* G)(\chi^* v_1^{k+1}, \varepsilon^2 \chi^* G),$$

where $\chi^*(t)$ is a smooth cutoff, compactly supported in $(-1, 2)$, and equal to one on the support of χ_0 . Using (3.260) and

$$(3.262) \quad \langle \Lambda_1 v_1^{k+1} \rangle_{m+1, T=2} \lesssim e^{\gamma_{m+1}^*} \langle \Lambda v_1^{k+1} \rangle_{m+1, \gamma_{m+1}^*},$$

we obtain

$$(3.263) \quad \langle \Lambda_1(\chi^* v_1^{k+1}) \rangle_{m+1} \lesssim e^{\gamma_{m+1}^*} K_1 \text{ and } \langle \Lambda(\chi^* v_1^{k+1}) \rangle_{m+1, \gamma} \lesssim \frac{1}{\gamma} K_1.$$

Applying Corollary 3.24 to (3.261), this gives

$$(3.264) \quad \langle \Lambda(\chi_0 H(v_1^{k+1}, \varepsilon^2 G)) \rangle_{m+1, \gamma} \lesssim C(K_1, \gamma_{m+1}^*)(\gamma^{-1} + \langle \Lambda G \rangle_{m+1, T_0}).$$

Similarly, applying Proposition 3.29 and using the simplified notation explained in Notation 3.44, we obtain with $z = (\chi^* v_1^{k+1}, \varepsilon^2 \chi^* G)$:

$$(3.265) \quad \begin{aligned} & \langle \phi_j \Lambda^{\frac{3}{2}}(\chi_0 H(v_1^{k+1}, \varepsilon^2 G)) \rangle_{m+1, \gamma} \lesssim \\ & \left(\langle \Lambda z \rangle_{m+1, \gamma} \langle \Lambda_1 z \rangle_{m+1} h(\langle z \rangle_{m+1}) + \langle \psi \Lambda z \rangle_{m+1, \gamma} \langle \Lambda_1^{\frac{1}{2}} z \rangle_{m+1} h(\langle z \rangle_{m+1}) \right) + \langle \psi \Lambda^{\frac{3}{2}} z \rangle_{m+1, \gamma} h(\langle z \rangle_{m+1}) \lesssim \\ & (\gamma^{-1} + \langle \Lambda G \rangle_{m+1, T_0}) C(K_1, \gamma_{m+1}^*) + \left(\gamma^{-1/2} K_1 + \langle \Lambda^{\frac{3}{2}} G \rangle_{m+1, T_0} \right). \end{aligned}$$

³⁶This choice is made precise later in this step; for now we claim that the estimates in this step are valid for δ and T_0 small enough.

³⁷We mainly use our Corollary 3.24, the singular Rauch-type lemma Proposition 3.29, and Proposition 3.33. The use of Proposition 3.29 permits a simplification of the argument in [ST88]; for example, dyadic decompositions are not needed here.

In view of (3.259), (3.264), and (3.265) the estimate (3.255) applied to the problem (3.257) gives

$$(3.266) \quad |v_2^{k+1}|_{m+1,\gamma}^* \leq \gamma^{-1}K_2 + K_3\langle \Lambda G \rangle_{m+1,T_0} + \gamma^{-\frac{1}{2}}K_4.$$

Choice of δ and T_0 . We first fix δ sufficiently small so that the estimates (3.254), (3.255) apply to the problems (3.256), (3.257) when $|v^k|_{m+1,T}^* < \delta$. We also need δ and T_0 small enough so that $|v^k|_{m+1}$, $\langle \chi^* v_1^{k+1}, \chi^* \varepsilon^2 G \rangle_{m+1}$ and $\langle v_1^k, \varepsilon^2 G \rangle_{m+1}$ lie in the domain of convergence of the analytic functions (like h in (3.265), for example) that appear on the right in the estimates of section 3.4 that we use. To see that $\langle \chi^* v_1^{k+1} \rangle_{m+1}$ can be made small by taking δ and T_0 small, one uses the estimate (3.263) and the fact that $K_1(0,0) = 0$. A further possible reduction of T_0 occurs at the end of this step.

Taking $\gamma = \frac{1}{T}$ in (3.266) and using Proposition 3.15, we find

$$(3.267) \quad |v_2^{k+1}|_{m+1,T}^* \lesssim TK_2 + K_3\langle \Lambda G \rangle_{m+1,T_0} + T^{\frac{1}{2}}K_4.$$

After reducing T_0 if necessary, the right side of (3.267) will be $< \delta$ for small enough $T > 0$.

Uniform boundedness of the iterates $|u^k|_{m+1,T}^*$ with respect to k now follows by applying the estimate (3.254) to the problem (3.258).

4. Low norm contraction. Having the uniform boundedness of the iterates v^k and u^k in the norm $|\cdot|_{m+1,T}^*$ allows us to repeat the low norm contraction argument of [ST88], p. 290, to show that the sequence v^k is Cauchy in the $\left| \begin{array}{c} \Lambda v^k \\ D_{x_2} v^k \end{array} \right|_{0,T}$ norm for a possibly smaller $T > 0$. The same applies to the u^k .

This argument uses the estimates of Proposition 3.13 and, of course, some of the nonlinear estimates of section 3.4. A standard interpolation argument implies $v^k \rightarrow v$ and $u^k \rightarrow u$ in $\mathcal{N}^m(\Omega_T)$, where v and u both lie in $\mathcal{N}^{m+1}(\Omega_T)$ and satisfy (3.11), (3.12), (3.13) on Ω_T .

5. Show $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_T . By a computation similar to the one on p. 291 of [ST88], we see that the differences $v_1 - \partial_{x_1,\varepsilon} u$ and $v_2 - \partial_{x_2} u$ satisfy a coupled system of singular, linear equations on Ω_T with boundary and interior forcing terms that are identically zero. One can therefore apply the estimates of Proposition 3.13 to see that these differences are both zero on Ω_T . \square

3.6.2 Continuation.

We will prove the continuation theorem using estimates of the following form that are tame with respect to the norm $|\cdot|_{k,\gamma}^*$.

Proposition 3.86. *Let $m_0 > 3d + 4 + \frac{d+1}{2}$ and suppose $k \in \{0, 1, \dots, m_0 + \lceil \frac{m_0+1}{2} \rceil\}$, where $\lceil r \rceil$ denotes the largest integer $\leq r$. Suppose that the function w occurring in the coefficients of the problem 3.252 has compact t -support contained in $[0, 2)$ and let $\chi^*(t)$ be a C^∞ function equal to one on $\text{supp } w$ with support in $(-1, 2)$.*

For the problem 3.252 we have: there exist $\gamma_0 > 0$ and an increasing function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for $\gamma \geq \gamma_0$,

$$\begin{aligned}
(3.268) \quad & \gamma \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right|_{k,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right\rangle_{k,\gamma}^2 + \sum_{J_h \cup J_e} \left\langle \phi_j \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right\rangle_{k,\gamma}^2 \leq \\
& \frac{1}{\gamma} K^2(|w|_{m_0}^*) \left(\left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_1 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_1 \end{array} \right) \right|_{k,\gamma}^2 + (|\chi^* v_1|_{m_0}^*)^2 \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} w \\ D_{x_2} \Lambda^{\frac{1}{2}} w \end{array} \right) \right|_{k,\gamma}^2 + |\Lambda^{\frac{1}{2}} \mathcal{F}_1|_{k,\gamma}^2 \right) + \\
& \frac{1}{\gamma} K^2(|w|_{m_0}^*) \left(\left\langle \left(\begin{array}{c} \Lambda v_1 \\ D_{x_2} v_1 \end{array} \right) \right\rangle_{k,\gamma}^2 + (|\chi^* v_1|_{m_0}^*)^2 \left\langle \left(\begin{array}{c} \Lambda w \\ D_{x_2} w \end{array} \right) \right\rangle_{k,\gamma}^2 + \langle \Lambda \mathcal{G}_1 \rangle_{k,\gamma}^2 \right).
\end{aligned}$$

For the problem 3.253 we have: there exist $\gamma_0 > 0$ and an increasing function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for $\gamma \geq \gamma_0$,

$$\begin{aligned}
(3.269) \quad & \gamma \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_2 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_2 \end{array} \right) \right|_{k,\gamma}^2 + \gamma \left\langle \left(\begin{array}{c} \Lambda v_2 \\ D_{x_2} v_2 \end{array} \right) \right\rangle_{k,\gamma}^2 \leq \\
& \frac{1}{\gamma} K^2(|w|_{m_0}^*) \left(\left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} v_2 \\ D_{x_2} \Lambda^{\frac{1}{2}} v_2 \end{array} \right) \right|_{k,\gamma}^2 + (|\chi^* v_2|_{m_0}^*)^2 \left| \left(\begin{array}{c} \Lambda^{\frac{3}{2}} w \\ D_{x_2} \Lambda^{\frac{1}{2}} w \end{array} \right) \right|_{k,\gamma}^2 + |\Lambda^{\frac{1}{2}} \mathcal{F}_2|_{k,\gamma}^2 \right) + \\
& K^2(|w|_{m_0}^*) \left(\gamma \langle \Lambda \mathcal{G}_2 \rangle_{k,\gamma}^2 + \sum_{j \in J_h} \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{k,\gamma}^2 \right).
\end{aligned}$$

In these estimates $|w|_{L^\infty(\Omega)}$ must be small enough, γ_0 depends on

$$(3.270) \quad \left| \frac{w}{\varepsilon} \right|_{C^{0,n}} + \left| \frac{w}{\varepsilon} \right|_{CH^{s_0}} + |\partial_{x_2} w|_{L^\infty(\Omega)}, \text{ where } n \geq 3d + 4, \ s_0 > \frac{d+1}{2} + 2,$$

and $K(|w|_{m_0}^*)$ depends also on ε and k .

Proof. **1.** One first proves the same estimates, but where the cutoffs χ^* are absent in the factors $|\chi^* v_j|_{m_0}^*$.

2. The proof is similar to that of estimates 3.254 and 3.255. Again, the starting point is the $k = 0$ estimate given by (3.37) and (3.40). Estimates of $|\cdot|_{0,k,\gamma}$ and $\langle \cdot \rangle_{k,\gamma}$ norms follow by applying the $k = 0$ estimate to the tangentially differentiated problems $\partial^k(3.252)$, $\partial^k(3.253)$. Estimates of commutators are now done using the tame estimate of Proposition 3.38(b).

3. As before, estimates of $\left| \left(\begin{array}{c} D_{x_2}^j \left(\Lambda^{\frac{3}{2}} v_i \right) \\ D_{x_2}^j \left(D_{x_2} \Lambda^{\frac{1}{2}} v_i \right) \end{array} \right) \right|_{0,k-j,\gamma}$ for $j > 0$ are proved by induction on j using

the equation and the noncharacteristic boundary assumption. We set $U = \left(\begin{array}{c} e^{\gamma t} \Lambda_D e^{-\gamma t} v_i \\ D_{x_2} v_i \end{array} \right)$ and estimate $|\Lambda^{\frac{1}{2}} D_{x_2}^j U|_{0,k-j,\gamma}$ using the first order singular equation (3.45). For j such that $j \leq k$, $1 \leq j \leq \lfloor \frac{m_0+1}{2} \rfloor$ (range I), we apply the tame estimate of Proposition 3.38(b), while for j such that $j \leq k$, $\lfloor \frac{m_0+1}{2} \rfloor \leq j \leq m_0 + \lfloor \frac{m_0+1}{2} \rfloor$ (range II), we apply Proposition 3.31(b).

For example, in the estimate of $|\Lambda^{\frac{1}{2}} D_{x_2}^j U|_{0,k-j,\gamma}$ we must estimate terms of the form

$$(3.271) \quad |\Lambda^{\frac{1}{2}} (a(w) d_\varepsilon D_{x_2}^{j-1} U)|_{0,k-j,\gamma}.$$

Since ε is fixed, we treat d_ε like a nonsingular tangential derivative. Thus, we are led to consider a sum of terms of form

$$(3.272) \quad |\Lambda^{\frac{1}{2}} \left((\partial^{k_1} z) (\partial^{k_2} D_{x_2}^{j-1} U) \right) |_{0,0,\gamma},$$

where, with $a(w) = a(0) + b(w)w$,

$$(3.273) \quad z := b(w)w, \quad k_1 + k_2 = k - (j - 1), \quad k_2 \geq 1.$$

We first do a tangential estimate for x_2 fixed. For j in range I we apply Proposition 3.38(b) with “ m ” = $k - (j - 1)$ and “ m_0 ” = $m_0 - (j - 1)$; the factors involving z are estimated using Remark 3.39. For j in range II we apply Proposition 3.31(b) with “ u ” = $D_{x_2}^{j-1} U$, “ v ” = z , “ s ” = $k - k_2 - (j - 1)$, “ t ” = $m_0 - k_1$; factors involving z are estimated using Proposition 3.33. In both cases the remaining $L^2(x_2)$ norm is easily controlled by arguments like those in section 3.5

4. The cutoff functions χ^* appearing in the estimates (3.268), (3.269) can be inserted because the factors $|v_j|_{m_0}^*$ only arise from terms like (3.272) in which “free” factors of $\partial^l w$ appear for some l . The support assumption on w thus permits the insertion of χ^* . \square

Proposition 3.87 (Continuation of solutions for fixed ε). (a) Let $m > 3d + 4 + \frac{d+1}{2}$. For ε fixed, suppose we have a solution $(v^\varepsilon, u^\varepsilon) \in \mathcal{N}^{m+1}(\Omega_{T_1})$ of (3.11), (3.12), (3.13) with $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ for some $0 < T_1 < T_0$, where $v^\varepsilon = 0$ in $t < 0$ and is such that $E_{m,T_1}(v^\varepsilon) \leq \frac{M_0}{N}$ for M_0 as in (3.84). For N large enough there exists $T_2 > T_1$ and an extension of $(v^\varepsilon, u^\varepsilon)$ to a solution on Ω_{T_2} with $v^\varepsilon \in \mathcal{N}^{m+1}(\Omega_{T_2})$ and $v^\varepsilon = \nabla_\varepsilon u^\varepsilon$ on Ω_{T_2} , where T_2 depends on ε and M_0 .³⁸

(b) There exists T_3 with $T_1 < T_3 \leq T_2$ such that $E_{m,T_3}(v^\varepsilon) < M_0$.

Proof. As usual, we often suppress superscripts ε below. We note that part (b) follows easily from part (a) by taking T_3 close enough to T_1 .

1. Translation. By downward translation in time we reduce to the case where $T_1 = 0$ and where the given solution $v = (v_1, v_2) \in \mathcal{N}^{m+1}(\Omega_0)$ vanishes in $t < -T_1$ and satisfies $E_{m,0}(v) \leq \frac{M_0}{N}$.

2. Iteration scheme. Setting $g = \varepsilon^2 G$ (which now vanishes in $t < -T_1$), let us write the translates of (3.256), (3.257), and (3.258) as

$$(3.274) \quad \begin{aligned} \mathcal{L}(v^k)v_1^{k+1} &= \mathcal{F}_1(v^k, \nabla_\varepsilon v^k) \text{ on } \Omega \\ \mathcal{B}_1(v^k)v_1^{k+1} &= \mathcal{G}_1(v_1^k, g) \text{ on } x_2 = 0, \\ v^{k+1} &= v_1 \text{ in } t < 0. \end{aligned}$$

$$(3.275) \quad \begin{aligned} \mathcal{L}(v^k)v_2^{k+1} &= \mathcal{F}_2(v^k, \nabla_\varepsilon v^k) \text{ on } \Omega \\ v_2^{k+1} &= \mathcal{G}_2(v_1^{k+1}, g) \text{ on } x_2 = 0, \\ v_2^{k+1} &= v_2 \text{ in } t < 0. \end{aligned}$$

$$(3.276) \quad \begin{aligned} \mathcal{L}(v^k)u^{k+1} &= 0 \text{ on } \Omega \\ \mathcal{B}_1(v^k)u^{k+1} &= \mathcal{G}(v_1^k, g) \text{ on } x_2 = 0, \\ u^{k+1} &= u \text{ in } t < 0. \end{aligned}$$

³⁸The size of N depends on the norm of the Seeley extension operator used in step 2 of the proof.

We focus now on the continuation of v . The treatment of u is similar.

Let $(v^e, u^e) \in \mathcal{N}^{m+1}(\Omega)$ be an extension of $(v, u)|_{\Omega_0}$ to Ω supported in $[-T_1, 2 - T_1]$ and such that $v^e = \nabla_\varepsilon u^e$ and $E_m(v^e) < \frac{M_0}{2}$. If we initialize the schemes (3.274), (3.275) with $v^0 = v^e$, the iterates may be written $v^k = v^e + z^k$, where the z^k satisfy

$$(3.277) \quad \begin{aligned} \mathcal{L}(v^e + z^k)z_1^{k+1} &= \mathcal{F}_1(v^e + z^k, \nabla_\varepsilon(v^e + z^k)) - \mathcal{L}(v^e + z^k)v_1^e := \mathbb{F}_1, \\ \mathcal{B}_1(v^e + z^k)z_1^{k+1} &= \mathcal{G}_1(v_1^e + z_1^k, g) - \mathcal{B}_1(v^e + z^k)v_1^e := \mathbb{G}_1 \text{ on } x_2 = 0, \\ z_1^{k+1} &= 0 \text{ in } t < 0. \end{aligned}$$

$$(3.278) \quad \begin{aligned} \mathcal{L}(v^e + z^k)z_2^{k+1} &= \mathcal{F}_2(v^e + z^k, \nabla_\varepsilon(v^e + z^k)) - \mathcal{L}(v^e + z^k)v_2^e := \mathbb{F}_2 \\ z_2^{k+1} &= \mathcal{G}_2(v_1^e + z_1^{k+1}, g) - v_2^e := \mathbb{G}_2 \text{ on } x_2 = 0 \\ z_2^{k+1} &= 0 \text{ in } t < 0. \end{aligned}$$

3. Lower regularity continuation. In the forcing terms on the right sides of (3.277) and (3.278) derivatives of v^e occur, so we first obtain a lower regularity continuation. Just as in the proof of Proposition 3.83, we introduce Seeley extensions z_T^k with support in $[0, 2 - T_1]$ and regard (3.277), (3.278) as a family of problems on Ω parametrized by T , where now $0 < T \leq T_0 - T_1$; thus, every z^k (as opposed to z^{k+1}) inside a coefficient or forcing term is really z_T^k . We make the induction assumption:

- there exists $T > 0$ independent of k such that for $j = 0, \dots, k$ we have (dropping the subscript on z^j) $|z^j|_{m,T}^* < \delta$ and thus $|z_T^j|_m^* < C_1\delta$ for a small enough δ .³⁹

From this assumption and the regularity of v^e we obtain (arguing as in the earlier proof)

$$(3.279) \quad |\Lambda^{\frac{1}{2}}\mathbb{F}_i|_{m,\gamma} < \infty \text{ and } \langle \Lambda\mathbb{G}_1 \rangle_{m,\gamma} < \infty;$$

moreover, \mathbb{F}_i and \mathbb{G}_i vanish in $t < 0$.

Applying the argument used to prove Proposition 3.83 to the scheme (3.277), (3.278), we obtain a solution $v = v^e + z \in \mathcal{N}^m(\Omega_{T_2'})$ to (3.11), (3.12) for some $T_2' > 0$. Henceforth, let v^e, z^j, z_T^j , and z denote the upward translations by T_1 of the similarly denoted functions we have just defined. We now have a continuation $v := v^e + z \in \mathcal{N}^m(\Omega_{T_2})$, where $T_2 = T_1 + T_2' > T_1$ and $v = 0$ in $t < 0$, and the functions $v^j = v^e + z^j$ satisfy

$$(3.280) \quad \begin{aligned} \mathcal{L}(v^k)v_1^{k+1} &= \mathcal{F}_1(v^k, \nabla_\varepsilon v^k) \text{ on } \Omega \\ \mathcal{B}_1(v^k)v_1^{k+1} &= \mathcal{G}_1(v_1^k, g) \text{ on } x_2 = 0, \\ v_1^{k+1} &= v_1 \text{ in } t < T_1. \end{aligned}$$

$$(3.281) \quad \begin{aligned} \mathcal{L}(v^k)v_2^{k+1} &= \mathcal{F}_2(v^k, \nabla_\varepsilon v^k) \text{ on } \Omega \\ v_2^{k+1} &= \mathcal{G}_2(v_1^{k+1}, g) \text{ on } x_2 = 0, \\ v_2^{k+1} &= v_2 \text{ in } t < T_1, \end{aligned}$$

where as before every v^k (as opposed to v^{k+1}) appearing inside a coefficient or forcing term is really v_T^k . Recall that v_1 and v_2 vanish in $t < 0$.

In the rest of the continuation argument the norm $|\cdot|_{m,T_2}^*$ will have the role usually played by a Lipschitz-type norm in such arguments.

³⁹Here, as in the proof of Proposition 3.83, we require δ and M_0 to be small enough so that the estimates (3.254), (3.255) apply.

Remark 3.88. 1) From our construction there exists a constant $R > 0$ such that $v_{T_2}^k = v^e + z_{T_2}^k$ has support in a fixed compact subset of $[0, 2)$ and satisfies

$$(3.282) \quad |v_{T_2}^k|_{m, T_2}^* \leq R \text{ and } |v_{T_2}^k|_m^* \leq CR \text{ for all } k.$$

2) Since $E_m(v^e) \leq \frac{M_0}{2}$, and $|z_{T_2}^k|_m^* < C\delta$ for all k , we have in particular that

$$(3.283) \quad |v_{T_2}^k|_{\infty, m} \leq \frac{M_0}{2} + C\delta \text{ for all } k,$$

a fact that will be used in the next step.

3) Let $\chi^*(t)$ as usual denote a cutoff support in $(-1, 2)$ and equal to one on the compact subset of $[0, 2)$ in which the $v_{T_2}^k$ have support, and let v_1^{k+1} be the solution of (3.280) (not a Seeley extension). By the argument that gave (3.263), but applied in the construction of z_1^{k+1} satisfying (3.277), we obtain that there exists a constant \mathcal{K} independent of k such that

$$(3.284) \quad |\chi^* v_1^{k+1}|_m^* \leq \mathcal{K}.$$

4. Higher regularity. To show $v \in \mathcal{N}^{m+1}(\Omega_{T_2})$, it is enough to show that the iterates v^j in (3.280), (3.281) are uniformly bounded in $\mathcal{N}^{m+1}(\Omega_{T_2})$, since we already have the iterates converging to v in a lower norm. We make the following induction assumption:

- There exist positive constants $P > 0$ and γ_c independent of k such that for $j = 1, \dots, k$ we have

$$(3.285) \quad |v^k|_{m+1, \gamma}^* \leq P \text{ for } \gamma \geq \gamma_c.$$

Once this is shown to hold for all k , it then follows from Proposition 3.15 that

$$(3.286) \quad |v^k|_{m+1, T_2}^* \leq Ce^{\gamma_c T_2} |v^k|_{m+1, \gamma_c}^* \leq Ce^{\gamma_c T_2} P \text{ for all } k.$$

Part (b) of the Proposition follows as in part (2) of Remark 3.82.

5. Induction step. Let $a_{j, \gamma}^{k+1} = |v_j^{k+1}|_{m+1, \gamma}^*$ for $j = 1, 2$. We will complete the induction step by applying the estimates (3.268), (3.269) with “ k ” = $m + 1$ and “ m_0 ” = m , together with the estimates of sections 3.4.1 and 3.4.2, to the problems (3.280), (3.281). In these estimates K, K_j denote nonnegative continuous functions of one or more arguments which increase as any one argument increases. Also, $v^k = v_{T_2}^k$, but v^{k+1} is not a Seeley extension.

We first apply Corollary 3.36 (with Remark 3.39) to estimate \mathcal{F}_i in (3.274), (3.275), obtaining

$$(3.287) \quad |\Lambda^{\frac{1}{2}} \mathcal{F}_i(v^k, \nabla_\varepsilon v^k)|_{m+1, \gamma} \leq K_1(R)P \text{ for } j = 1, 2.$$

Here and below we use (3.283) to bound the analytic functions like $h(\langle v^k, \varepsilon^2 G \rangle_m)$ that appear by constants C ; we suppose that M_0, δ , and M_G have been chosen small enough so that the arguments of these functions lie in their domains of convergence. Since,

$$(3.288) \quad \langle \Lambda(v_1^k, \varepsilon^2 G) \rangle_{m+1, \gamma} \lesssim P + M_G,$$

Corollary 3.36 yields

$$(3.289) \quad \langle \Lambda \mathcal{G}_1(v_1^k, \varepsilon^2 G) \rangle_{m+1, \gamma} \leq C_1(P + PR + M_G).$$

Applying the estimate 3.268 to (3.280) and using (3.284), we obtain for $j \in J_h$ and $\gamma \geq \gamma_0$

$$(3.290) \quad \begin{aligned} a_{1,\gamma}^{k+1} + \gamma^{-\frac{1}{2}} \langle \phi_j \Lambda^{\frac{3}{2}} v_1^{k+1} \rangle_{m+1,\gamma} &\leq \frac{1}{\gamma} K(R) \left(a_{1,\gamma}^{k+1} + \mathcal{K}P + K_1(R)P \right) + \\ &\frac{1}{\gamma} K(R) \left(a_{1,\gamma}^{k+1} + \mathcal{K}P + C_1(P + PR + M_G) \right) \leq \frac{1}{\gamma} (2K(R)a_{1,\gamma}^{k+1} + K_2(R)P + K_3(R)M_G). \end{aligned}$$

Thus,

$$(3.291) \quad \begin{aligned} a_{1,\gamma}^{k+1} + \gamma^{-\frac{1}{2}} \langle \phi_j \Lambda^{\frac{3}{2}} v_1^{k+1} \rangle_{m+1,\gamma} &\leq \gamma^{-1} K_4(R)(P + M_G) := \gamma^{-1} K_5(R, P, M_G) \\ &\text{for } \gamma \geq \max(4K(R), \gamma_0) := \gamma_a, \text{ where } K_5(R, 0, 0) = 0. \end{aligned}$$

This implies

$$(3.292) \quad |\Lambda(\chi^* v_1^{k+1})|_{m+1,\gamma} \lesssim \frac{1}{\gamma} K_5(R, P, M_G) \text{ for } \gamma \geq \gamma_a,$$

and since $\text{supp } \chi^* \subset (-1, 2)$, this gives

$$(3.293) \quad |\Lambda_1(\chi^* v_1^{k+1})|_{m+1} \lesssim e^{\gamma a^2} K_5(R, P, M_G) := K_6.$$

Application of the (nontame) estimate of Proposition 3.25(c) yields with $z = (\chi^* v_1^{k+1}, \varepsilon^2 \chi^* G)$:

$$(3.294) \quad \begin{aligned} \langle \Lambda \mathcal{G}_2 \rangle_{m+1,\gamma} = \langle \Lambda(\chi_0 H(v_1^{k+1}, \varepsilon^2 G)) \rangle_{m+1,\gamma} &\lesssim (\langle \Lambda z \rangle_{m+1,\gamma} + \langle z \rangle_{m+1,\gamma} \langle \Lambda_1 z \rangle_{m+1}) h(\langle z \rangle_{m+1}) \lesssim \\ &\left(\frac{K_5}{\gamma} + M_G \right) + \left(\frac{K_5}{\gamma} + M_G \right) (K_6 + M_G) \lesssim \frac{K_7}{\gamma} + M_G. \end{aligned}$$

In the above estimate we have used (3.293), noting that since $K_5(R, 0, 0) = 0$ we will have $\langle z \rangle_{m+1}$ in the domain of convergence of h provided P and M_G are small enough.

Similarly, applying Proposition 3.29 and using (3.291) to estimate $\langle \psi \Lambda^{\frac{3}{2}} v_1^{k+1} \rangle_{m+1,\gamma}$, we obtain

$$(3.295) \quad \begin{aligned} \langle \phi_j \Lambda^{\frac{3}{2}}(\chi_0 H(v_1^{k+1}, \varepsilon^2 G)) \rangle_{m+1,\gamma} &\lesssim \\ \langle \Lambda z \rangle_{m+1,\gamma} \langle \Lambda_1 z \rangle_{m+1} h(\langle z \rangle_{m+1}) &+ \langle \psi \Lambda z \rangle_{m+1,\gamma} \langle \Lambda_1^{\frac{1}{2}} z \rangle_{m+1} h(\langle z \rangle_{m+1}) + \langle \psi \Lambda^{\frac{3}{2}} z \rangle_{m+1,\gamma} h(\langle z \rangle_{m+1}) \lesssim \\ \left(\frac{K_5}{\gamma} + M_G \right) (K_6 + M_G) &+ \left(\frac{K_5}{\gamma} + M_G \right) (K_6 + M_G) + \left(\frac{K_5}{\gamma^{\frac{1}{2}}} + M_G \right) \lesssim \frac{K_8}{\gamma^{\frac{1}{2}}} + 1. \end{aligned}$$

With (3.284), (3.294) and (3.295) we can now apply the estimate (3.269) to (3.281) to obtain

$$(3.296) \quad \begin{aligned} a_{2,\gamma}^{k+1} &\lesssim \frac{1}{\gamma} K(R) \left(a_{2,\gamma}^{k+1} + \mathcal{K}P + K_1(R)P \right) + \\ &K(R) \left[\left(\frac{K_7}{\gamma} + M_G \right) + \frac{1}{\gamma^{\frac{1}{2}}} \left(\frac{K_8}{\gamma^{\frac{1}{2}}} + 1 \right) \right]. \end{aligned}$$

Thus, for large enough γ

$$(3.297) \quad a_{2,\gamma}^{k+1} \lesssim \frac{1}{\gamma} K_9 P + K_{10} M_G + \frac{1}{\gamma^{\frac{1}{2}}} K_{11} < \frac{P}{2},$$

if γ is large enough and M_G small enough. Since (3.291) implies $a_{1,\gamma}^{k+1} \leq \frac{P}{2}$ for γ large enough, so this completes the induction step. \square

Chapter 4

Approximate solutions

This chapter is devoted to the construction of an approximate solution (leading term and corrector) to the coupled, singular nonlinear problems (3.11)-(3.13) on a short time interval. We will use the result of Proposition 2.3, but otherwise this chapter can be read independently of chapter 2 and all but the first few pages of chapter 3. We introduce some notations here that differ from those used in chapter 2, but which we have found more suitable for estimating the nonlinear interactions that appear in the first corrector.

4.1 Introduction

We seek initially an approximate solution to the original nonlinear, nonsingular problem (3.4) of the form

$$(4.1) \quad U_a^\varepsilon(t, x) = u_a(t, x, \theta, z) \Big|_{\theta = \frac{\beta \cdot (t, x_1)}{\varepsilon}, z = \frac{x_2}{\varepsilon}} \text{ on } \Omega_{T_1}$$

for some $T_1 > 0$, where¹

$$(4.2) \quad \begin{aligned} u_a(t, x, \theta, z) &= \varepsilon^2 u_\sigma(t, x, \theta, z) + \varepsilon^3 u_\tau(t, x, \theta, z) \text{ with} \\ u_\sigma(t, x, \theta, z) &= \sum_{j=1}^4 \sigma_j(t, x, \theta + \omega_j z) r_j \text{ and } u_\tau(t, x, \theta, z) = \sum_{j=1}^4 (\chi_\varepsilon(D_\theta) \tau_j(t, x, \theta, z)) r_j. \end{aligned}$$

Here the σ_j and τ_j are scalar profiles constructed in section 4.2, and the ω_j, r_j are characteristic roots and vectors satisfying

$$(4.3) \quad \det L(\beta, \omega_j) = 0 \text{ and } L(\beta, \omega_j) r_j = 0,$$

where $L(\xi', \xi_2)$ is the 2×2 matrix symbol defined below in (4.22). The operator $\chi_\varepsilon(D_\theta)$ is a low-frequency cutoff operator that is discussed below. The functions

$$(4.4) \quad u_a^\varepsilon(t, x, \theta) := u_a(t, x, \theta, \frac{x_2}{\varepsilon}) \text{ and } v_a^\varepsilon = \nabla_\varepsilon u_a^\varepsilon$$

will then turn out to be approximate solutions of the singular problems (3.11)-(3.13) on the same time interval.

¹In chapter 2 u_σ and u_τ were denoted $u^{(1)}$ and $u^{(2)}$ respectively.

When $U_a^\varepsilon(t, x)$ is plugged into the system (3.4), we obtain

$$(4.5) \quad \begin{aligned} \partial_t^2 U_a^\varepsilon + \sum_{|\alpha|=2} A_\alpha(\nabla U_a^\varepsilon) \partial_x^\alpha U_a^\varepsilon &= F_a(t, x, \theta, z)_{\theta=\frac{\beta \cdot (t, x_1)}{\varepsilon}, z=\frac{x_2}{\varepsilon}} \text{ on } \Omega_{T_1} \\ h(\nabla U_a^\varepsilon) &= g_a(t, x_1, \theta)|_{\theta=\frac{\beta \cdot (t, x_1)}{\varepsilon}} \text{ on } x_2 = 0, \end{aligned}$$

for interior and boundary profiles F_a, g_a described below. All functions here are zero in $t < 0$.

The coefficients $A_\alpha(v)$ are polynomials in v of order two, so we write

$$(4.6) \quad A_\alpha(v) = A_\alpha(0) + L_\alpha(v) + Q_\alpha(v),$$

a sum of constant, linear, and quadratic parts. Let us set

$$(4.7) \quad D_\varepsilon = (\partial_{x_1} + \frac{\beta_1}{\varepsilon} \partial_\theta, \partial_{x_2} + \frac{1}{\varepsilon} \partial_z)$$

and define²

$$(4.8) \quad \partial_{ss}^\alpha = \partial_x^\alpha, \partial_{fs}^\alpha = \begin{cases} 2\beta_1 \partial_{x_1 \theta}, & \alpha = (2, 0) \\ \beta_1 \partial_{x_2 \theta} + \partial_{x_1 z}, & \alpha = (1, 1) \\ 2\partial_{x_2 z}, & \alpha = (0, 2) \end{cases}, \partial_{ff}^\alpha = \begin{cases} \beta_1^2 \partial_{\theta \theta}, & \alpha = (2, 0) \\ \beta_1 \partial_{\theta z}, & \alpha = (1, 1) \\ \partial_{zz}, & \alpha = (0, 2) \end{cases}.$$

Next set

$$(4.9) \quad \begin{aligned} L_{ss} &= \partial_t^2 + \sum_{|\alpha|=2} A_\alpha(0) \partial_{ss}^\alpha, \quad L_{fs} = 2\beta_0 \partial_{t\theta} + \sum_{|\alpha|=2} A_\alpha(0) \partial_{fs}^\alpha, \quad L_{ff} = \beta_0^2 \partial_{\theta\theta} + \sum_{|\alpha|=2} A_\alpha(0) \partial_{ff}^\alpha \\ N(u_\sigma) &= \sum_{|\alpha|=2} L_\alpha(\beta_1 \partial_\theta u_\sigma, \partial_z u_\sigma) \partial_{ff}^\alpha u_\sigma. \end{aligned}$$

The function $h(v)$ in the boundary condition of (4.5) is a cubic polynomial in v satisfying $h(0) = 0$, so we write

$$(4.10) \quad h(v) = \ell(v) + q(v) + c(v),$$

a sum of linear, quadratic, and cubic parts, and define

$$(4.11) \quad \ell_s(u_\sigma) = \ell(\partial_{x_1} u_\sigma, \partial_{x_2} u_\sigma), \quad \ell_f(u_\tau) = \ell(\beta_1 \partial_\theta u_\tau, \partial_z u_\tau), \quad n(u_\sigma) = q(\beta_1 \partial_\theta u_\sigma, \partial_z u_\sigma).$$

One attempts to construct u_σ and u_τ so that the terms in F_a of orders $O(\varepsilon^0)$ and $O(\varepsilon)$ vanish, and so that the terms in $g_a - g$, where $g = \varepsilon^2 G$, of orders $O(\varepsilon)$ and $O(\varepsilon^2)$ vanish. This leads directly to interior and boundary equations that one *would like* u_σ and u_τ to satisfy:

$$(4.12) \quad \begin{aligned} (a) \quad &L_{ff} u_\sigma = 0 \\ (b) \quad &L_{ff} u_\tau + L_{fs} u_\sigma + N(u_\sigma) = 0 \text{ in } x_2 \geq 0, z \geq 0 \end{aligned}$$

and

$$(4.13) \quad \begin{aligned} (a) \quad &\ell_f(u_\sigma) = 0 \\ (b) \quad &\ell_f u_\tau + \ell_s u_\sigma + n(u_\sigma) = G(t, x_1, \theta) \text{ on } x_2 = 0, z = 0. \end{aligned}$$

²Here, for example, we write $\partial_{x_1 \theta}$ for $\partial_{x_1} \partial_\theta$.

We now introduce a new unknown $\mathcal{U}_\tau(t, x, \theta, z)$ that will be related to u_τ by $u_\tau = \chi_\varepsilon(D_\theta)\mathcal{U}_\tau$. The amplitude equation of Proposition 2.2 is a solvability condition for the following problem (which is the same as (2.14)):

$$(4.14) \quad \begin{aligned} (a) \quad & L_{ff}\mathcal{U}_\tau = -(L_{fs}u_\sigma + N(u_\sigma)) \text{ on } x_2 = 0, z \geq 0 \\ (b) \quad & \ell_f\mathcal{U}_\tau + \ell_s u_\sigma + n(u_\sigma) = G(t, x_1, \theta) \text{ on } x_2 = 0, z = 0. \end{aligned}$$

More precisely, it is a solvability condition for the Fourier transform of this problem with respect to θ . The traces $\hat{\sigma}_j(t, x_1, 0, k)$ turn out to be constant multiples of $\hat{w}(t, x_1, k)$, where $w(t, x_1, \theta)$ is the solution of the amplitude equation 2.64.

For each k one is able to construct $\hat{\mathcal{U}}_\tau(t, x, k, z)$ so that the transform of (4.14)(b) holds and the transform of (4.14)(a) holds in $\{x_2 \geq 0, z \geq 0\}$. A difficulty is that $\hat{\mathcal{U}}_\tau(t, x, k, z)$ is so singular at $k = 0$ that one cannot take the inverse transform to define \mathcal{U}_τ ; in fact, (4.54) shows that $\hat{\mathcal{U}}_\tau(t, x, k, z) = O(\frac{1}{k^2})$. The division by k^2 in (4.54) reflects the two integrations needed in this second order problem. Since $\mathcal{U}_\tau(t, x, \theta, z)$ is not well-defined, we can not actually solve (4.14). Moreover, $\hat{\mathcal{U}}_\tau(t, x, k, z)$ is too large to be of any direct use in the error analysis.

This is the reason for $\chi_\varepsilon(D_\theta)$ in the definition of u_τ (4.2). For some $b > 0$ to be chosen later and a C^∞ cutoff $\chi(s)$ vanishing near $s = 0$ with $\chi = 1$ on $|s| \geq 1$, we define

$$(4.15) \quad \hat{u}_\tau(t, x, k, z) = \chi\left(\frac{k}{\varepsilon^b}\right) \hat{\mathcal{U}}_\tau(t, x, k, z) := \chi_\varepsilon(k) \hat{\mathcal{U}}_\tau(t, x, k, z),$$

and observe that u_τ , which is well-defined, satisfies

$$(4.16) \quad \begin{aligned} (a) \quad & L_{ff}u_\tau + \chi_\varepsilon(D_\theta)(L_{fs}u_\sigma + N(u_\sigma)) = 0 \text{ on } x_2 \geq 0, z \geq 0 \\ (b) \quad & \ell_f u_\tau + \chi_\varepsilon(D_\theta)(\ell_s u_\sigma + n(u_\sigma)) = \chi_\varepsilon(D_\theta)G(t, x_1, \theta) \text{ on } x_2 = 0, z = 0. \end{aligned}$$

The use of $\chi_\varepsilon(D_\theta)$ introduces new errors of course, but we show in chapter 5 that if the exponent b is chosen correctly, the errors will converge to zero at a computable rate as $\varepsilon \rightarrow 0$, because of the presence of the factor of ε^3 on u_τ . This kind of low-frequency cutoff was first used in the rigorous study of pulses propagating in the interior in work of Alterman-Rauch [AR03].

Using (4.12)(a), (4.13)(a), and (4.16), we can now write the interior error profile in (4.5) as

$$(4.17) \quad \begin{aligned} F_a(t, x, \theta, z) := & L_{ss}u_a + \varepsilon^2 L_{fs}u_\tau + \sum_{|\alpha|=2} Q_\alpha(D_\varepsilon u_a) D_\varepsilon^\alpha u_a + \left(\sum_{|\alpha|=2} L_\alpha(D_\varepsilon u_a) D_\varepsilon^\alpha u_a - \varepsilon N(u_\sigma) \right) + \\ & \varepsilon(1 - \chi_\varepsilon(D_\theta))(L_{fs}u_\sigma + N(u_\sigma)). \end{aligned}$$

Similarly, we can write the boundary profile in (4.5) as $g_a = g + h_a$, where the boundary error profile is

$$(4.18) \quad \begin{aligned} h_a(t, x_1, \theta) := & \varepsilon^3 \ell_s(u_\tau) + c(D_\varepsilon u_a) + (q(D_\varepsilon u_a) - \varepsilon^2 n(u_\sigma)) + \\ & \varepsilon^2 [(\chi_\varepsilon(D_\theta) - 1)G + (1 - \chi_\varepsilon(D_\theta))(\ell_s(u_\sigma) + n(u_\sigma))] \text{ at } x_2 = 0, z = 0. \end{aligned}$$

Remark 4.1. 1. The terms in the first line of (4.17) would have been the only terms to appear if there were no need to introduce $\chi_\varepsilon(D_\theta)$. These terms are all (formally) $O(\varepsilon^2)$ or smaller.

2. The second line represents the ‘‘low frequency cutoff error’’ incurred by introducing $\chi_\varepsilon(D_\theta)$.

3. Remarks analogous to (1) and (2) apply to the first and second lines of (4.18).

4. The approximate solution exhibits amplification, as often happens in weakly stable problems, in the following sense. The approximate solution is $O(\varepsilon^2)$, and its gradient is of size $O(\varepsilon)$, given boundary data of size $O(\varepsilon^2)$. If the uniform Lopatinskii condition were satisfied, we would expect the solution to be of size $O(\varepsilon^3)$ and its gradient of size $O(\varepsilon^2)$ in this second-order problem.

5. Even if one assumes $G(t, x_1, \theta)$ decays exponentially as $|\theta| \rightarrow \infty$, the profiles $\sigma_j(t, x, \theta)$ defining u_σ generally exhibit no better than $H^s(t, x, \theta)$ “decay” in θ . This reflects that fact that $\hat{\sigma}_j(t, x, k)$ may be discontinuous at $k = 0$. This loss of θ -decay from data to solution is typical of evanescent pulses, and occurs even in problems where the uniform Lopatinskii condition is satisfied [Willig].

4.2 Construction of the leading term and corrector

1. Some notation. As in Lardner [Lar83], the operator L_{ss} can be written

$$(4.19) \quad L_{ss} = \partial_{tt} - \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \partial_{x_1 x_1} - \begin{pmatrix} 0 & r-1 \\ r-1 & 0 \end{pmatrix} \partial_{x_1 x_2} - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{x_2 x_2},$$

where $r > 1$ is the ratio of the squares of pressure c_d and shear c_s velocities.³ The boundary frequency β has the form $(-c, 1)$ for a c to be chosen, so the operators L_{fs} and L_{ff} are

$$(4.20) \quad \begin{aligned} L_{fs} &= -2c\partial_{t\theta} - \begin{pmatrix} 2r & 0 \\ 0 & 2 \end{pmatrix} \partial_{x_1\theta} - \begin{pmatrix} 0 & r-1 \\ r-1 & 0 \end{pmatrix} (\partial_{x_1 z} + \partial_{x_2\theta}) - \begin{pmatrix} 2 & 0 \\ 0 & 2r \end{pmatrix} \partial_{x_2 z}, \\ L_{ff} &= \begin{pmatrix} c^2 - r & 0 \\ 0 & c^2 - 1 \end{pmatrix} \partial_{\theta\theta} - \begin{pmatrix} 0 & r-1 \\ r-1 & 0 \end{pmatrix} \partial_{\theta z} - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{zz}, \end{aligned}$$

and

$$(4.21) \quad \begin{aligned} \ell_s &= \begin{pmatrix} 0 & 1 \\ r-2 & 0 \end{pmatrix} \partial_{x_1} + \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{x_2} \\ \ell_f &= \begin{pmatrix} 0 & 1 \\ r-2 & 0 \end{pmatrix} \partial_\theta + \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_z. \end{aligned}$$

With $\xi' = (\sigma, \xi_1) = (\xi_0, \xi_1)$ the symbol of L_{ss} is

$$(4.22) \quad L(\xi', \xi_2) = \begin{pmatrix} \sigma^2 - (r-1)\xi_1^2 - |\xi|^2 & -(r-1)\xi_1\xi_2 \\ -(r-1)\xi_1\xi_2 & \sigma^2 - (r-1)\xi_2^2 - |\xi|^2 \end{pmatrix}$$

with characteristic roots ω_j and vectors r_j satisfying

$$(4.23) \quad \det L(\beta, \omega_j) = 0 \text{ and } L(\beta, \omega_j)r_j = 0, j = 1, \dots, 4.$$

The numbers ω_1 and ω_2 are pure imaginary with positive imaginary part. We have $\omega_3 = \overline{\omega_1}$, $\omega_4 = \overline{\omega_2}$ and

$$(4.24) \quad \omega_1^2 = c^2 - 1, \quad \omega_2^2 = \frac{c^2}{r} - 1,$$

³We have $c_s^2 = \mu$ and $c_d^2 = (\lambda + 2\mu)$, where λ, μ are the Lamé constants. The form (4.19) is obtained by taking units of time so that $c_s = 1$. Observe $r = c_d^2/c_s^2 > 1$ since $\lambda + \mu > 0$.

the numbers on the right in (4.24) being negative since β lies in the elliptic region. If we define $q = q(c) > 0$ by

$$(4.25) \quad q^2 = -\omega_1\omega_2,$$

then the condition for $\beta = (-c, 1)$ to be a Rayleigh frequency is that⁴

$$(4.26) \quad (2 - c^2)^2 = 4q^2(c), \text{ or equivalently } 2 - c^2 = 2q.$$

For the existence of $\beta = (-c, 1)$ in the elliptic region satisfying (4.26) we refer, for example, to [Tay77]. We take

$$(4.27) \quad r_1 = \begin{pmatrix} -\omega_1 \\ 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 \\ \omega_2 \end{pmatrix}, r_3 = \bar{r}_1, r_4 = \bar{r}_2.$$

2. First-order system for the σ_j . Consider a problem of the form

$$(4.28) \quad L_{ff}u = f(t, x, \theta, z) \text{ in } \{x_2 \geq 0, z \geq 0\}, \quad \ell_{ff}u = g(t, x_1, \theta) \text{ on } x_2 = z = 0.$$

Taking the Fourier transform with respect to θ and setting $\hat{U} = \begin{pmatrix} \hat{u} \\ \partial_z \hat{u} \end{pmatrix}$, we obtain the 4×4 first order system

$$(4.29) \quad \partial_z \hat{U} - G(\beta, k)\hat{U} = \hat{\mathcal{F}}, \quad C(\beta, k)\hat{U} = \hat{g},$$

where $G(\beta, k) = \begin{pmatrix} 0 & 1 \\ D & B \end{pmatrix}$ with

$$(4.30) \quad D = k^2 \begin{pmatrix} r - c^2 & 0 \\ 0 & \frac{1-c^2}{r} \end{pmatrix}, \quad B = ik \begin{pmatrix} 0 & 1-r \\ \frac{1}{r} - 1 & 0 \end{pmatrix} \text{ and } \hat{\mathcal{F}} = \begin{pmatrix} 0 \\ - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}^{-1} f \end{pmatrix}$$

and

$$(4.31) \quad C(\beta, k) = \left(ik \begin{pmatrix} 0 & 1 \\ r-2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \right).$$

The matrix $G(\beta, k)$ has eigenvalues $ik\omega_j$, $j = 1, \dots, 4$ corresponding respectively to the right eigenvectors

$$(4.32) \quad R_1 = \begin{pmatrix} -\omega_1 \\ 1 \\ -ik\omega_1^2 \\ ik\omega_1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 \\ \omega_2 \\ ik\omega_2 \\ ik\omega_2^2 \end{pmatrix}, R_3 = \begin{pmatrix} \omega_1 \\ 1 \\ -ik\omega_1^2 \\ -ik\omega_1 \end{pmatrix}, R_4 = \begin{pmatrix} 1 \\ -\omega_2 \\ -ik\omega_2 \\ ik\omega_2^2 \end{pmatrix}.$$

Using the R_k to diagonalize $G(\beta, k)$ we see that solutions of $\partial_z \hat{U} - G(\beta, k)\hat{U} = 0$ that decay as $z \rightarrow +\infty$ must have the form

$$(4.33) \quad \hat{U}_j(t, x, k, z) = e^{ik\omega_j z} \hat{\sigma}_j(t, x, k) R_j,$$

⁴This is the condition for the 2×2 matrix \mathcal{B}_{Lop} defined (4.35) to have vanishing determinant. Equation 4.26 is equivalent to equation (2.42). The ω_j appearing here are obtained from the ω_j in (2.44) by multiplying the latter by i .

where the $\hat{\sigma}_j$ are scalar functions to be determined such that

$$(4.34) \quad k\text{Im } \omega_j \geq 0 \text{ on } \text{supp } \hat{\sigma}_j.$$

This explains the form of u_σ in (4.2), where r_j is the vector given by the first two components of R_j .

A short calculation using the relations (4.24)-(4.26) shows that

$$(4.35) \quad [C(\beta, k)R_1 \ C(\beta, k)R_2] = ik \begin{pmatrix} 2 - c^2 & 2\omega_2 \\ 2\omega_1 & c^2 - 2 \end{pmatrix} := ik\mathcal{B}_{Lop},$$

and that

$$(4.36) \quad \ker \mathcal{B}_{Lop} = \text{span} \begin{pmatrix} \omega_2 \\ -q \end{pmatrix}, \quad \text{coker } \mathcal{B}_{Lop} = \text{span} (q \ \omega_2).$$

In order for (the Fourier transform in θ of) (4.13)(a) to hold we must have, when $k > 0$,

$$(4.37) \quad C(\beta, k)(\hat{U}_1 + \hat{U}_2) = 0 \text{ on } x_2 = 0, z = 0 \text{ for } \hat{U}_i \text{ as in (4.33),}$$

and thus in view of (4.36)

$$(4.38) \quad \begin{pmatrix} \hat{\sigma}_1(t, x_1, 0, k) \\ \hat{\sigma}_2(t, x_1, 0, k) \end{pmatrix} = \alpha(t, x_1, k) \begin{pmatrix} \omega_2 \\ -q \end{pmatrix},$$

for some scalar amplitude α . We take α to be $-2i\hat{w}(t, x_1, k)$, where w is the function constructed using Proposition 2.3 to satisfy the solvability condition (2.20).⁵ With the traces of the $\hat{\sigma}_j$ thereby determined, we extend the $\hat{\sigma}_j$ into $x_2 \geq 0$ by setting

$$(4.39) \quad \hat{\sigma}_j(t, x_1, x_2, k) := \psi(x_2)\hat{\sigma}_j(t, x_1, 0, k),$$

where $\psi \in C^\infty$ is compactly supported and equal to 1 near $x_2 = 0$.

3. First-order system for the τ_j . To proceed further we must now consider the inhomogenous problem (4.14) for \mathcal{U}_τ . We look for the solution as a sum of a homogeneous solution \mathcal{U}_τ^h and a ‘‘particular’’ solution \mathcal{U}_τ^p as

$$(4.40) \quad \mathcal{U}_\tau = \mathcal{U}_\tau^h + \mathcal{U}_\tau^p, \text{ where}$$

$$\mathcal{U}_\tau = \sum_{j=1}^4 \tau_j(t, x, \theta, z)r_j, \quad \mathcal{U}_\tau^h = \sum_{j=1}^4 \tau_j^h(t, x, \theta, z)r_j, \quad \mathcal{U}_\tau^p = \sum_{j=1}^4 \tau_j^p(t, x, \theta, z)r_j.$$

The Fourier transform of (4.14) can be written as the first order system (4.29)-(4.31), where now

$$(4.41) \quad \begin{aligned} (a) \ f &= f_\sigma := -(L_{fs}u_\sigma + N(u_\sigma)) \\ (b) \ g &= -[\ell_s u_\sigma + n(u_\sigma)] + G(t, x_1, \theta) \text{ on } x_2 = 0, z = 0. \end{aligned}$$

The interior forcing term $\hat{\mathcal{F}}$ can be written

$$(4.42) \quad \hat{\mathcal{F}} = \sum_{j=1}^4 \hat{F}_j R_j, \text{ with } \hat{F}_j = L_j \hat{\mathcal{F}},$$

⁵The factor of $-2i$ could be replaced by one if we multiplied r_1 and r_2 in (4.27) by $-2i$.

where the L_j are left eigenvectors of $G(\beta, k)$ associated to ω_j chosen so that $L_m R_n = \delta_{mn}$. The L_j are given by

$$(4.43) \quad \begin{aligned} L_1(k) &= (-ik(r - c^2), -ik\omega_1, \omega_1, -r)/(-2i\omega_1 c^2 k), \quad L_2(k) = (ik\omega_2 r, ik(c^2 - 1), 1, r\omega_2)/2i\omega_2 c^2 k, \\ L_3(k) &= \bar{L}_1(-k), \quad L_4(k) = \bar{L}_2(-k). \end{aligned}$$

The decoupled interior system for the $\hat{\tau}_j^p$ is then

$$(4.44) \quad (\partial_z - ik\omega_j)\hat{\tau}_j^p = \hat{F}_j, \quad j = 1, \dots, 4.$$

From (4.43) and the form of \mathcal{F} in (4.30), we see that

$$(4.45) \quad \begin{aligned} \hat{F}_j &= \ell_j(k)\hat{f}_\sigma \text{ for } f_\sigma \text{ as in (4.41) and row vectors } \ell_j(k) \text{ given by:} \\ \ell_1(k) &= (\omega_1, -1)/(-2i\omega_1 c^2 k), \quad \ell_2(k) = (1, \omega_2)/(2i\omega_2 c^2 k), \quad \ell_3(k) = \bar{\ell}_1(-k), \quad \ell_4(k) = \bar{\ell}_2(-k). \end{aligned}$$

4. Formulas for the $\hat{\tau}_j$. Decaying solutions of (4.44) are given by

$$(4.46) \quad \hat{\tau}_j^p(t, x, k, z) = \begin{cases} \int_0^z e^{ik\omega_j(z-s)} \hat{F}_j(t, x, k, s) ds, & k > 0 \\ \int_{+\infty}^z e^{ik\omega_j(z-s)} \hat{F}_j(t, x, k, s) ds, & k < 0 \end{cases} \quad \text{for } j = 1, 2,$$

with the same formulas for $j = 3, 4$, except that $k > 0$ (resp. $k < 0$) is now associated with $\int_{+\infty}^z$ (resp. \int_0^z). For the homogeneous parts we have

$$(4.47) \quad \hat{\tau}_j^h(t, x, k, z) = e^{ik\omega_j z} \hat{\tau}_j^*(t, x, k), \quad \text{where } \text{supp } \hat{\tau}_j^* \subset \begin{cases} \{k \geq 0\}, & j = 1, 2 \\ \{k \leq 0\}, & j = 3, 4 \end{cases}$$

for functions $\hat{\tau}_j^*$ to be determined.

To complete the determination of the τ_j we now examine the Fourier transform in θ of the boundary equation in (4.28). As in (4.29) this can be written

$$(4.48) \quad C(\beta, k)\hat{U}_\tau = \hat{g}, \quad \text{where } g = -[\ell_s u_\sigma + n(u_\sigma)] + G(t, x_1, \theta) \text{ and } \hat{U}_\tau := \begin{pmatrix} \hat{u}_\tau \\ \partial_z \hat{u}_\tau \end{pmatrix}$$

on $x_2 = 0, z = 0$. We have

$$(4.49) \quad \hat{U}_\tau = \hat{U}_\tau^h + \hat{U}_\tau^p, \quad \text{where } \hat{U}_\tau^h = \sum_{j=1}^4 \hat{\tau}_j^* R_j, \quad \hat{U}_\tau^p = \sum_{j=1}^4 \hat{\tau}_j^p R_j \text{ on } x_2 = 0, z = 0,$$

so (4.48) becomes for $k > 0$

$$(4.50) \quad C(\beta, k)\hat{U}_\tau^h = ik\mathcal{B}_{Lop} \begin{pmatrix} \hat{\tau}_1^* \\ \hat{\tau}_2^* \end{pmatrix} = \hat{g} - C(\beta, k)\hat{U}_\tau^p \text{ on } x_2 = 0, z = 0.$$

The matrix \mathcal{B}_{Lop} is singular, but the function $\hat{w}(t, x_1, k)$, which determines the traces of the $\hat{\sigma}_j$ on $x_2 = 0$ (4.38), was chosen precisely so that the right side of (4.50) lies in the range of \mathcal{B}_{Lop} . Thus, we can solve for the $\hat{\tau}_j^*(t, x_1, 0, k)$. We extend the $\hat{\tau}_j^*$ to $x_2 \geq 0$ by setting

$$(4.51) \quad \hat{\tau}_j^*(t, x_1, x_2, k) = \psi(x_2)\hat{\tau}_j^*(t, x_1, 0, k)$$

for ψ as in (4.39).

This completes the construction of the $\hat{\sigma}_j, \hat{\tau}_j$. It remains to examine the regularity of these functions.

5. Regularity of the $\hat{\sigma}_j$. We introduce the notation

$$(4.52) \quad \hat{H}^s(t, x_1, k) = \{\hat{u}(t, x_1, k), \text{ where } u \in H^s(t, x_1, \theta)\}, \quad |\hat{u}|_{\hat{H}^s(t, x_1, k)} := |\langle k \rangle^s \hat{u}(t, x_1, k)|_{L^2(k, H^s(t, x_1))}.$$

The space $\hat{H}^s(t, x, k)$ is defined similarly.

It follows from Proposition 2.3 that $\hat{w}(t, x_1, k) \in \hat{H}^s$ when $G \in H^s(t, x_1, \theta)$, provided $s > d + 3$. Thus, we have from (4.38) and (4.39),

Proposition 4.2. *Assume $s > d + 3$ and $G(t, x_1, \theta) \in H^s(t, x_1, \theta)$. Then*

$$(4.53) \quad \hat{\sigma}_j(t, x_1, 0, k) \in \hat{H}^s(t, x_1, k) \text{ and } \hat{\sigma}_j(t, x_1, x_2, k) \in \hat{H}^s(t, x, k).$$

6. Regularity of the $\hat{\tau}_j$.

Proposition 4.3. *Assume $s > d + 3$ and $G(t, x_1, \theta) \in H^s(t, x_1, \theta)$. We have*

$$(4.54) \quad \hat{\tau}_j(t, x, k, z) = \frac{1}{k^2} T_j(t, x, k, z)$$

where

$$(4.55) \quad T_j(t, x, k, z) \in C_c(x_2; L^\infty(z, \hat{H}^{s-2}(t, x_1, k))) \cap C(x_2, z; \hat{H}^{s-2}(t, x_1, k)).$$

Proof. **1.** We take $j = 1$ and using (4.46) consider first

$$(4.56) \quad \hat{\tau}_1^p(t, x, k, z) = \int_{+\infty}^z e^{ik\omega_1(z-s)} \hat{F}_1(t, x, k, s) ds, \quad k < 0$$

for \hat{F}_1 as in (4.45). From (4.45) we see that \hat{F}_1 is a linear combination of terms of the form

$$(4.57) \quad \begin{aligned} & (a) \frac{1}{k} \widehat{\partial \partial_\theta \sigma_j}, \\ & (b) \frac{1}{k} \partial_\theta \widehat{\sigma_m} \partial_{\theta\theta} \widehat{\sigma_n}, \end{aligned}$$

where $\sigma_j = \sigma_j(t, x, \theta + \omega_j z)$.

Let us consider first the case (b) when $m \neq n$. We then obtain for the corresponding term in $\hat{\tau}_1^p$ a scalar multiple of:

$$(4.58) \quad \begin{aligned} \hat{\tau}_{mn}^p(t, x, k, z) & := \frac{1}{k^2} T_{mn}^p(t, x, k, z) \text{ where} \\ T_{mn}^p(t, x, k, z) & = k e^{ik\omega_1 z} \int_{+\infty}^z \int e^{-ik\omega_1 s + i(k-k')\omega_m s + ik'\omega_n s} \widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k') dk' ds. \end{aligned}$$

For a given (x_2, z) the norm $|T_{mn}^p(t, x, k, z)|_{\hat{H}^{s-2}(t, x_1, k)}$ can be estimated by considering just the $L^2(k, H^{s-2}(t, x_1))$ and $L^2(t, x_1, \hat{H}^{s-2}(k))$ norms. Since $\text{Im}(k-k')\omega_m \geq 0$ (resp. $\text{Im} k'\omega_n \geq 0$) on $\text{supp } \hat{\sigma}_m$ (resp. $\hat{\sigma}_n$), we obtain for the $H^{s-2}(t, x_1)$ norm:

$$(4.59) \quad \begin{aligned} |T_{mn}^p(t, x, k, z)|_{H^{s-2}(t, x_1)} & \leq |k| \int_z^{+\infty} \int e^{ik\omega_1(z-s)} |\widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k')|_{H^{s-2}(t, x_1)} dk' ds = \\ & \frac{|k|}{ik\omega_1} \int |\widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k')|_{H^{s-2}(t, x_1)} dk', \end{aligned}$$

which yields

$$(4.60) \quad |T_{mn}^p(t, x, k, z)|_{L^2(k, H^{s-2}(t, x_1))} \lesssim |\hat{\sigma}_m|_{\hat{H}^s(t, x_1, k)} |\hat{\sigma}_n|_{\hat{H}^s(t, x_1, k)}.$$

Here we have used a Moser estimate in the (t, x_1) variables, and observed that, for example,

$$(4.61) \quad \left| \widehat{|\partial_{\theta\theta}\sigma_n|}(t, x_1, x_2, k) \right|_{L^\infty(t, x_1)} \Big|_{L^1(k)} \lesssim \left| \langle k \rangle^{\frac{1}{2} + \delta} \widehat{\partial_{\theta\theta}\sigma_n}(t, x_1, x_2, k) \right|_{H^{\frac{d}{2} + \delta}(t, x_1)} \Big|_{L^2(k)} \lesssim |\hat{\sigma}_n|_{\hat{H}^{\frac{d}{2} + 2 + \frac{1}{2} + 2\delta}(t, x_1, k)}.$$

The L^2 norm of the convolution in k' can then be estimated by Young's inequality. The $L^2(t, x_1, \hat{H}^{s-2}(k))$ norm is estimated similarly, using

$$(4.62) \quad \langle k \rangle^{s-2} \lesssim \langle k - k' \rangle^{s-2} + \langle k' \rangle^{s-2}$$

in place of the Moser estimate.

Continuity of T_{mn}^p in x_2 is evident from the special x_2 dependence of the $\hat{\sigma}_j$ (4.39), while continuity in z follows from the dominated convergence theorem.

2. In the case $k > 0$ we define T_{mn}^p with $\int_{+\infty}^z \dots ds$ replaced by $\int_0^z \dots ds$, and find that $|T_{mn}^p(t, x, k, z)|_{H^{s-2}(t, x_1)}$ is again dominated by the right side of (4.59). The rest of the estimate goes as before. The estimates of the contributions to $\hat{\tau}_1^p$ in the case (b) when $m = n$ or case (a) are similar (or easier).

We observe that in the case (4.57)(b) when $m = n = 1$, the integral (4.56) is zero since $\hat{\sigma}_1(t, x, k)$ is supported in $k \geq 0$, so one just needs to estimate the $\int_0^z \dots ds$ integral that defines $\hat{\tau}_1^p(t, x, k, z)$ when $k > 0$. In this case the ds integral produces a factor of z ("secular growth"). Using the fact that

$$(4.63) \quad \sup_{z \geq 0} z e^{-|k\omega_j|z} \approx \frac{1}{|k|},$$

we see that this factor of z has the same effect (introducing an extra factor of $1/|k|$) as the ds integral in (4.59). Hence one obtains the same estimate for the contribution of (4.57)(b) to $\hat{\tau}_1^p$ when $m = n = 1$ as when $m \neq n$ or when $m = n \neq 1$.

3. From (4.50) and the expression for g (4.48), we see that the regularity of $\hat{\tau}_j^*(t, x_1, 0, k)$, $j = 1, 2$ for $k > 0$ is determined by the regularity of

$$(4.64) \quad \frac{\widehat{\ell_s u_\sigma}}{k}, \quad \frac{\widehat{n(u_\sigma)}}{k}, \quad \frac{\widehat{G}}{k}, \quad \text{and } \widehat{U}_\tau^p \text{ on } x_2 = 0, z = 0.$$

The first three terms have the form $\frac{1}{k} h(t, x_1, k)$, where h lies respectively in \hat{H}^{s-1} , \hat{H}^{s-1} , and H^{s+1} , while (4.54) for $\hat{\tau}_j^p$ implies $k^2 \widehat{U}_\tau^p \in \hat{H}^{s-2}$ at $x_2 = 0, z = 0$. With (4.51) we see that

$$(4.65) \quad \hat{\tau}_j^h(t, x, k, z) = \hat{\tau}_j^*(t, x, k) e^{ik\omega_j z}$$

also satisfies (4.54). □

Chapter 5

Error Analysis and proof of Theorem 3.8

In this chapter we show that the approximate solution is close in a precise sense to the exact solution constructed in Theorem 3.7. We will use the notation and estimates of chapter 3, especially sections 3.2.1, 3.3, 3.4, and 3.5.

5.1 Introduction

The error analysis is performed on functions of (t, x, θ) . We write

$$\begin{aligned}
 u_a^\varepsilon(t, x, \theta) &:= \varepsilon^2 u_\sigma^\varepsilon(t, x, \theta) + \varepsilon^3 u_\tau^\varepsilon(t, x, \theta), \text{ where} \\
 u_\sigma^\varepsilon(t, x, \theta) &:= u_\sigma(t, x, \theta, z)|_{z=\frac{x_2}{\varepsilon}} = \sum_{j=1}^4 \left(\sigma_j(t, x, \theta + \omega_j z)|_{z=\frac{x_2}{\varepsilon}} \right) r_j \text{ and} \\
 u_\tau^\varepsilon(t, x, \theta) &:= u_\tau(t, x, \theta, z)|_{z=\frac{x_2}{\varepsilon}} = \sum_{j=1}^4 \left(\chi_\varepsilon(D\theta)\tau_j(t, x, \theta, z)|_{z=\frac{x_2}{\varepsilon}} \right) r_j.
 \end{aligned}
 \tag{5.1}$$

When u_a^ε is plugged into the system (3.8) we obtain

$$\begin{aligned}
 \partial_{t,\varepsilon}^2 u_a^\varepsilon + \sum_{|\alpha|=2} A_\alpha(\nabla_\varepsilon u_a^\varepsilon) \partial_{x,\varepsilon}^\alpha u_a^\varepsilon &= F_a^\varepsilon := F_a(t, x, \theta, \frac{x_2}{\varepsilon}) \text{ on } \Omega_{T_1} \\
 h(\nabla_\varepsilon u_a^\varepsilon) = g_a \text{ or } \partial_{x_2} u_a^\varepsilon &= H(\partial_{x_1,\varepsilon} u_a^\varepsilon, g_a) \text{ on } x_2 = 0
 \end{aligned}
 \tag{5.2}$$

for some $T_1 > 0$ and F_a, g_a as in (4.17), (4.18). Here u_a^ε is zero in $t < 0$.

Remark 5.1. *The functions u_σ^ε and u_τ^ε must be constructed to be real-valued. The analysis of Chapter 2 shows that the amplitude $w(t, x_1, \theta)$ is real-valued. The fact that the ω_j and r_j occur in complex conjugate pairs permits one to construct u_σ^ε as a real-valued function. Since the function $\chi(s)$ giving the low frequency cutoff in (5.1) can be chosen to be an even function, one can similarly construct u_τ^ε to be real-valued for the same reason. This remark is used in solving the systems (5.7)-(5.9).*

5.1.1 Extension of approximate solutions to the whole space

Recall that the estimates of $(v^\varepsilon, \nabla u^\varepsilon)$ leading to the proof of Theorem 3.7 had to be performed initially on the whole half-space Ω . We estimated functions $(v^\varepsilon, \nabla u^\varepsilon)$ on Ω that were solutions to the modified,

singular *linear* systems (3.86)-(3.88), whose restrictions to Ω_T for $0 < T < T_\varepsilon$ coincided (by causality, Remark (3.85)) with solutions provided by Theorem 3.83 to the nonlinear singular problems (3.11)-(3.13).

Similarly, the estimates in the error analysis must be done on the whole space. However, the approximate solutions u_a^ε and $v_a^\varepsilon := \nabla_\varepsilon u_a^\varepsilon$ constructed in Chapter 4 are defined just on $(-\infty, T_1]$ for some $T_1 > 0$. Thus, we must extend these functions to the whole half-space in order to compare them to the functions $(v^\varepsilon, u^\varepsilon)$. The form of the modified systems (3.86)-(3.88) suggests that we will need two kinds of extensions. First, we will need Seeley extensions v_a^s to Ω of $v_a^\varepsilon|_{\Omega_T}$ for $0 < T \leq T_1$, that we can hope to prove are close in the $E_{m,T}$ norm to the Seeley extensions v^s appearing in the coefficients of the systems (3.86)-(3.88). These Seeley extensions are good approximate solutions only for $t \leq T$; for later times they are useless as approximate solutions to the modified systems (3.86)-(3.88). Thus, we also need extensions to Ω of $v_a^\varepsilon|_{\Omega_T}$ $u_a^\varepsilon|_{\Omega_T}$, that we can hope to prove are close in the $E_{m,\gamma}$ norm to the solutions on Ω of the modified systems (3.86)-(3.88). These extensions will be constructed as solutions on Ω of the systems (5.7)-(5.9) below, which should be viewed as approximations to the systems (3.86)-(3.88). We now explain how to obtain these extensions in more detail.

The first step is to extend the error terms. With $g(t, x_1, \theta) = \varepsilon^2 G$ as before, we have

$$(5.3) \quad \begin{aligned} F_a(t, x, \theta) &= \mathcal{F}(u_\sigma(t, x, \theta, z), u_\tau(t, x, \theta, z))|_{z=\frac{x_2}{\varepsilon}} \\ g_a(t, x_1, \theta) &= g(t, x_1, \theta) + h_a(t, x_1, \theta), \text{ where } h_a(t, x_1, \theta) = \mathcal{H}(u_\sigma(t, x_1, 0, \theta, 0), u_\tau(t, x_1, 0, \theta, 0)), \end{aligned}$$

where \mathcal{F} and \mathcal{H} are functions that may be read off from the formulas (4.17), (4.18).¹

The functions $u_\sigma(t, x, \theta, z)$, $u_\tau(t, x, \theta, z)$, and $u_a(t, x, \theta, z)$ are built out of the component functions $\sigma_j(t, x, \theta)$, $j = 1, \dots, 4$. For any T satisfying $0 < T \leq T_1$ we let $\sigma_{j,T}^s$ denote a Seeley extension of $\sigma_j|_{\Omega_T}$ to Ω defined as in (3.83), and we denote by u_σ^s , the function of (t, x, θ, z) built out of the extended σ_j . This is the same as the Seeley extension to Ω of $u_\sigma|_{\Omega_T}$. Next we define $u_\tau^s(t, x, \theta, z)$ to be the Seeley extension of $u_\tau|_{\Omega_T}$. This is *not* the same as the extension obtained by replacing σ_j by $\sigma_{j,T}^s$ in the definition of u_τ .² We set

$$(5.4) \quad u_a^s = \varepsilon^2 u_\sigma^s + \varepsilon^3 u_\tau^s \text{ and } v_a^s(t, x, \theta, z) = D_\varepsilon u_a^s(t, x, \theta, z)$$

for D_ε as in (4.7) and observe that

$$(5.5) \quad \nabla_\varepsilon u_a^{s,\varepsilon}(t, x, \theta) = v_a^{s,\varepsilon}(t, x, \theta).$$

Recalling that g has already been extended, we now define (using superscript e for “extension”)

$$(5.6) \quad \begin{aligned} F_a^e(t, x, \theta) &= \mathcal{F}(u_\sigma^s(t, x, \theta, z), u_\tau^s(t, x, \theta, z))|_{z=\frac{x_2}{\varepsilon}} \\ g_a^e(t, x_1, \theta) &= g(t, x_1, \theta) + h_a^e(t, x_1, \theta), \text{ where } h_a^e(t, x_1, \theta) = \mathcal{H}(u_\sigma^s(t, x_1, 0, \theta, 0), u_\tau^s(t, x_1, 0, \theta, 0)). \end{aligned}$$

We need additional extensions of $v_a^\varepsilon|_{\Omega_T}$ and $u_a^\varepsilon|_{\Omega_T}$ to Ω that are obtained by solving on Ω the following three linear systems for the respective unknowns v_{1a}^ε , v_{2a}^ε , and u_a^ε . In these systems all functions have arguments (t, x, θ) and we suppress superscripts ε ; as usual, subscripts T are suppressed on Seeley

¹The arguments of \mathcal{F} and \mathcal{G} should also involve derivatives of (u_σ, u_τ) , but we have suppressed these in the notation.

²The Seeley extension of a product is not the same as the product of the Seeley extensions.

extensions.

$$\begin{aligned}
(5.7) \quad & (a) \partial_{t,\varepsilon}^2 v_{1a} + \sum_{|\alpha|=2} A_\alpha(v_a^s) \partial_{x,\varepsilon}^\alpha v_{1a} = \\
& - \left[\sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_1,\varepsilon} (A_\alpha(v_a^s)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_{1a}^s - \partial_{x_1,\varepsilon} (A_{(0,2)}(v_a^s)) \partial_{x_2} v_{2a}^s \right] + \partial_{x_1,\varepsilon} F_a^e, \\
& (b) \partial_{x_2} v_{1a} - d_{v_1} H(v_{1a}^s, h(v_a^s)) \partial_{x_1,\varepsilon} v_{1a} = d_g H(v_{1a}^s, g_a^e) \partial_{x_1,\varepsilon} (g_a^e) \text{ on } x_2 = 0.
\end{aligned}$$

$$\begin{aligned}
(5.8) \quad & (a) \partial_{t,\varepsilon}^2 v_{2a} + \sum_{|\alpha|=2} A_\alpha(v_a^s) \partial_{x,\varepsilon}^\alpha v_{2a} = \\
& - \left[\sum_{|\alpha|=2, \alpha_1 \geq 1} \partial_{x_2} (A_\alpha(v_a^s)) \partial_{x_1,\varepsilon}^{\alpha_1-1} \partial_{x_2}^{\alpha_2} v_{1a}^s - \partial_{x_2} (A_{(0,2)}(v_a^s)) \partial_{x_2} v_{2a}^s \right] + \partial_{x_2} F_a^e, \\
& (b) v_{2a} = \chi_0(t) H(v_{1a}, g_a^e) \text{ on } x_2 = 0.
\end{aligned}$$

$$\begin{aligned}
(5.9) \quad & (a) \partial_{t,\varepsilon}^2 u_a + \sum_{|\alpha|=2} A_\alpha(v_a^s) \partial_{x,\varepsilon}^\alpha u_a = F_a^e \\
& (b) \partial_{x_2} u_a - d_{v_1} H(v_{1a}^s, h(v_a^s)) \partial_{x_1,\varepsilon} u_a = H(v_{1a}^s, g_a^e) - d_{v_1} H(v_{1a}^s, g_a^e) v_{1a}^s \text{ on } x_2 = 0.
\end{aligned}$$

In (5.8) $\chi_0(t) \geq 0$ is the same C^∞ function (that is equal to 1 on a neighborhood of $[0, 1]$ and supported in $(-1, 2)$) as in (3.87).

The systems (5.7)-(5.9) for (v_a, u_a) should be compared to the systems (3.86)-(3.88) for (v, u) . In view of the smallness of F_a^e and $g_a^e - g$ and the expected, but still unproved, smallness of $v^s - v_a^s$, we can hope to show that (v_a, u_a) is close to (v, u) in the $E_{m,\gamma}$ norm on Ω . We carry out this strategy in the remainder of this chapter by studying the error equations computed below.

Remark 5.2. 1. The above three systems are solved on the full domain Ω . Each system depends on the parameters ε and T , which are usually suppressed in our notation.

2. Since the underlying linearized problem corresponding to both (5.7) and (5.9) is just weakly stable, it is important that the functions v_a^s appearing in the coefficients here be real-valued. That is so as a consequence of remark 5.1.

3. The definitions of $v_a^s = v_{a,T}^s$, F_a^e , g_a^e and causality (see remark (3.85)) imply that the solutions v_{1a} , v_{2a} , and u_a of (5.7), (5.8), and (5.9) are equal to the already constructed, similarly denoted functions on Ω_T . The various Seeley extensions depend on T , so the solutions v_a and u_a change as T changes.

5.1.2 Error equations

By subtracting the equations (5.7), (5.8), (5.9) from the equations (3.86), (3.87), (3.88) we obtain the following equations on Ω for the error functions

$$(5.10) \quad w = (w_1, w_2) = v - v_a \text{ and } z = u - u_a.$$

With slight abuse let us write the terms in brackets on the right sides of (5.7)(a) and (5.8)(a) respectively as $b_j(v_a^s) d_\varepsilon v_a^s d_\varepsilon v_a^s$, $j = 1, 2$ and similarly write the corresponding terms in (3.86) and (3.87) as $b_j(v^s) d_\varepsilon v^s d_\varepsilon v^s$, $j = 1, 2$.

$$\begin{aligned}
(5.11) \quad & (a) \partial_{t,\varepsilon}^2 w_1 + \sum_{|\alpha|=2} A_\alpha(v^s) \partial_{x,\varepsilon}^\alpha w_1 = \\
& - \sum_{|\alpha|=2} (A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x,\varepsilon}^\alpha v_{1a} + b_1(v^s) d_\varepsilon v^s d_\varepsilon v^s - b_1(v_a^s) d_\varepsilon v_a^s d_\varepsilon v_a^s - \partial_{x_1,\varepsilon} F_a^e, \\
& (b) \partial_{x_2} w_1 - d_{v_1} H(v_1^s, h(v^s)) \partial_{x_1,\varepsilon} w_1 = -[d_{v_1} H(v_1^s, h(v^s)) - d_{v_1} H(v_{1a}^s, h(v_a^s))] \partial_{x_1,\varepsilon} v_{1a} + \\
& \quad d_g H(v_1^s, g) \partial_{x_1,\varepsilon} g - d_g H(v_{1a}^s, g_a^e) \partial_{x_1,\varepsilon} g_a^e \text{ on } x_2 = 0.
\end{aligned}$$

$$\begin{aligned}
(5.12) \quad & (a) \partial_{t,\varepsilon}^2 w_2 + \sum_{|\alpha|=2} A_\alpha(v^s) \partial_{x,\varepsilon}^\alpha w_2 = \\
& - \sum_{|\alpha|=2} (A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x,\varepsilon}^\alpha v_{2a} + b_2(v^s) d_\varepsilon v^s d_\varepsilon v^s - b_2(v_a^s) d_\varepsilon v_a^s d_\varepsilon v_a^s - \partial_{x_2} F_a^e, \\
& (b) w_2 = \chi_0(t) [H(v_1, g) - H(v_{1a}, g_a^e)] \text{ on } x_2 = 0.
\end{aligned}$$

$$\begin{aligned}
(5.13) \quad & (a) \partial_{t,\varepsilon}^2 z + \sum_{|\alpha|=2} A_\alpha(v^s) \partial_{x,\varepsilon}^\alpha z = - \sum_{|\alpha|=2} (A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x,\varepsilon}^\alpha u_a - F_a^e, \\
& (b) \partial_{x_2} z - d_{v_1} H(v_1^s, h(v^s)) \partial_{x_1,\varepsilon} z = H(v_1^s, g) - d_{v_1} H(v_1^s, g) v_1^s \\
& \quad - [H(v_{1a}^s, g_a^e) - d_{v_1} H(v_{1a}^s, g_a^e) v_{1a}^s] + [d_{v_1} H(v_1^s, h(v^s)) - d_{v_1} H(v_{1a}^s, h(v_a^s))] v_{1a} \text{ on } x_2 = 0.
\end{aligned}$$

Remark 5.3. 1. We note again that these equations depend on ε and T as parameters, and so the same is true of the solutions. Just like the trios (3.86)-(3.88) and (5.7)-(5.9), the equations (5.11)-(5.13) must be estimated simultaneously.

2. Smallness of F_a^e and h_a^e in appropriate norms on Ω will follow, provided one has smallness of F_a and h_a in the corresponding time-localized norms on Ω_T , from continuity properties of Seeley extensions, and the rules of section 3.4 for computing norms of nonlinear functions of u in terms of norms of u .

3. The smallness of F_a^e and h_a^e will imply the smallness of $(w, \nabla_\varepsilon z)$ on Ω_{T_2} for T_2 small enough, but independent of ε .

The error analysis will be accomplished by estimating the error systems (5.11)-(5.13) on Ω in the $E_{m,\gamma}$ norm. Since the right sides of these systems include terms that contain factors given by derivatives of (v_a, u_a) , we shall also need to estimate the approximate solution systems (5.7)-(5.9) in the $E_{m,\gamma}$ norm.

The next two sections are devoted to estimating the functions F_a^e and $h_a^e = g_a^e - g$ that determine the size of the forcing in the approximate solution and error systems. Each of these functions is constructed from the building blocks u_σ^s and u_τ^s , so the first step is to estimate these components in a variety of singular norms; the estimates of F_a^e and h_a^e then follow by applying the results of section 3.4. Having estimates of F_a^e and h_a^e , we estimate (v_a, u_a) in section 5.4, and finally estimate the error (w, z) in section 5.5. The estimates of section 5.5 have much in common with the estimates of section 3.5, the main difference being that new kinds of forcing terms such as, for example,

$$- \sum_{|\alpha|=2} (A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x,\varepsilon}^\alpha v_{1a} \text{ in (5.11)}$$

are encountered.

5.2 Building block estimates

In these estimates the functions being estimated are evaluated at $(t, x_1, x_2, \theta, \frac{x_2}{\varepsilon})$ after the indicated derivatives are taken. Recall that

$$(5.14) \quad u_\sigma(t, x, \theta, z) = \sum_{j=1}^4 \sigma_j(t, x, \theta + \omega_j z) r_j \text{ and } u_\tau(t, x, \theta, z) = \sum_{j=1}^4 (\chi_\varepsilon(D_\theta) \tau_j(t, x, \theta, z)) r_j.$$

The functions u_σ (resp., u_τ) are built out of the constituent functions $\sigma_j(t, x, \theta)$, $j = 1, \dots, 4$ (resp., the σ_j and $G(t, x_1, \theta)$). The notation $C(m+r)$ indicates a constant that depends on norms

$$\sup_{x_2 \geq 0} |\sigma_j(t, x_1, x_2, \theta)|_{H^{m+r}(t, x_1, \theta)} \text{ or } |G(t, x_1, \theta)|_{H^{m+r}(t, x_1, \theta)}$$

of the constituent functions.³ In view of the special way x_2 dependence enters into the definitions of u_σ and u_τ (recall (4.39)), the estimates take exactly the same form if the $\langle \cdot \rangle_{m, \gamma}$ norms on the left are replaced by $|\cdot|_{0, m, \gamma}$ or $|\cdot|_{\infty, m, \gamma}$ norms.

The constant $p > 0$ is the one that appears in the low-frequency cutoff $\chi_\varepsilon(k) = \chi(\frac{k}{p})|_{p=\varepsilon^b}$, for $b > 0$ to be chosen. We recall that $\chi \geq 0$ and $\text{supp } \chi(k/p) \subset \{k : |k| \geq p\}$.

We will sometimes use ∂_f to represent ∂_θ or ∂_z and ∂_s to represent ∂_t , ∂_{x_1} , or ∂_{x_2} .

Remark 5.4. In the estimates of this section we write u_σ , u_τ to indicate the Seeley extensions $u_{\sigma, T}^s$, $u_{\tau, T}^s$. Recall that the Seeley extension of $u_{\tau, T}$ is not the same as the extension of $u_{\tau, T}$ obtained by replacing the constituent functions $\sigma_{j, T}$ by their Seeley extensions. Nevertheless, we need to estimate $u_{\tau, T}^s$ in terms of the $\sigma_{j, T}^s$. This is justified by noting that, for example,

$$(5.15) \quad \langle \Lambda^{\frac{1}{2}} \partial_f u_{\tau, T}^s \rangle_{m, \gamma} \leq C \langle \Lambda^{\frac{1}{2}} \partial_f u_{\tau, T}^s \rangle_{m, T, \gamma}$$

and the time-localized norm on the right can be realized as an infimum over norms of extensions of $u_{\tau, T}$ to the full-half space.

We begin with estimates of u_σ .

Proposition 5.5. Let $m \geq 0$.

$$(5.16) \quad \begin{aligned} \langle \Lambda^{\frac{k}{2}} \partial_f u_\sigma \rangle_{m, \gamma} &\lesssim \frac{C(m+1+\frac{k}{2})}{\varepsilon^{k/2}}, \quad k = 0, 1, 2, 3, 4 \\ \langle \Lambda^{\frac{k}{2}} \partial_{ff} u_\sigma \rangle_{m, \gamma} &\lesssim \frac{C(m+2+\frac{k}{2})}{\varepsilon^{k/2}}, \quad k = 0, 1, 2, 3 \\ \langle \Lambda^{\frac{k}{2}} \partial_s u_\sigma \rangle_{m, \gamma} &\lesssim \frac{C(m+1+\frac{k}{2})}{\varepsilon^{k/2}}, \quad k = 0, 1, 2, 3, 4. \end{aligned}$$

Proof. Since $|X, \gamma| \lesssim |\xi', \gamma| + |\frac{k}{\varepsilon}|$, we can for $r \geq 0$ estimate norms of $\Lambda_D^r w$ by estimating pieces corresponding to $|\xi', \gamma|^r$ and $|\frac{k}{\varepsilon}|^r$. Let us write the operators associated to these pieces as $\Lambda_{x', D}^r$ and $\Lambda_{\theta, D}^r$ respectively.

As in the proof of Proposition 4.3 we will use

$$(5.17) \quad |u(t, x_1, k)|_{\hat{H}^m(t, x_1, k)} \lesssim |u(t, x_1, k)|_{L^2(k, H^m(t, x_1))} + |u(t, x_1, k)|_{L^2(t, x_1, \hat{H}^m(k))}.$$

³The index $m+r$ can often be reduced in the case of G norms, but we wish to lighten the notation by not indicating this.

Given the form (5.1) of u_σ and recalling that

$$(5.18) \quad \text{supp } \hat{\sigma}_j(t, x, k) \subset \{k : k \text{Im } \omega_j \geq 0\},$$

the estimates are immediate. □

The next proposition gives estimates of $\partial_f u_\tau$.

Proposition 5.6. *Let $m > \frac{d+1}{2}$.*

$$(5.19) \quad \begin{aligned} \langle \partial_f u_\tau \rangle_{m,\gamma} &\lesssim C(m+2)/p \\ \langle \Lambda^{\frac{1}{2}} \partial_f u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+2)}{\varepsilon^{1/2} p^{1/2}} + \frac{C(m+2+\frac{1}{2})}{p} \\ \langle \Lambda \partial_f u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+2)}{\varepsilon} + \frac{C(m+3)}{p} \\ \langle \Lambda^{\frac{3}{2}} \partial_f u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+2+\frac{1}{2})}{\varepsilon^{3/2}} + \frac{C(m+3+\frac{1}{2})}{p} \\ \langle \Lambda^2 \partial_f u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+3)}{\varepsilon^2} + \frac{C(m+4)}{p}. \end{aligned}$$

Proof. **1.** We give the proof of the second estimate, which is typical of the rest. Consider, for example, $\langle \Lambda^{1/2} \partial_\theta u_\tau \rangle_{m,\gamma}$ and in particular $\langle \Lambda^{1/2} \partial_\theta \chi_\varepsilon(D_\theta) \tau_1^p \rangle_{m,\gamma}$. We estimate first the part of τ_1^p considered in (4.58), where $k < 0$ and $m \neq n$: Recall

$$(5.20) \quad \begin{aligned} \hat{\tau}_{mn}^p(t, x, k, z) &= e^{ik\omega_1 z} \int_{+\infty}^z \int e^{-ik\omega_1 s + i(k-k')\omega_m s + ik'\omega_n s} \frac{1}{k} \widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k') dk' ds := \\ &\int_{+\infty}^z e^{ik\omega_1(z-s)} \hat{F}_{mn}(t, x, k, s) ds. \end{aligned}$$

Here we will suppress some expressions involving γ such as factors of $e^{-\gamma t}$. Using (5.17) and (5.18) we estimate $\langle \Lambda_\theta^{1/2} \partial_\theta \chi_\varepsilon(D_\theta) \tau_{mn}^p \rangle_{m,\gamma}$ as follows:

$$(5.21) \quad \begin{aligned} &\left| \frac{|k|^{3/2}}{\varepsilon^{1/2}} \chi(k/p) \int_{+\infty}^z \int e^{ik\omega_1(z-s)} \left| \frac{1}{k} \widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k') \right|_{H^m(t, x_1)} dk' ds \right|_{L^2(k)} \lesssim \\ &\left| \frac{1}{\varepsilon^{1/2} p^{1/2}} \int \left| \widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k') \right|_{H^m(t, x_1)} dk' \right|_{L^2(k)} \lesssim \frac{C(m+2)}{\varepsilon^{1/2} p^{1/2}}. \end{aligned}$$

Here we have taken account of the factor of $1/k$ introduced by the ds integral, and have estimated the convolution in k using the observation (4.61) and Young's inequality. When $\Lambda_{\theta,D}^{1/2}$ is replaced by $\Lambda_{x',D}^{1/2}$ a parallel argument shows that the right side of (5.21) is replaced by $\frac{C(m+2+\frac{1}{2})}{p}$. The $L^2(t, x_1, \hat{H}^m(k))$ norm is estimated similarly after writing $\langle k \rangle^{1/2} \lesssim \langle k-k' \rangle^{1/2} + \langle k' \rangle^{1/2}$.

When $k > 0$ the integral $\int_{+\infty}^z \dots ds$ in (5.21) is replaced by $\int_0^z \dots ds$, but the resulting estimate is the same. The cases $m = n$ also yield the same estimate⁴; this is true even in the case $m = n = 1$ leading to secular growth for the reason given in step 2 of the proof of Proposition 4.3.

⁴One actually gets a better estimate in this case, since $\partial_\theta \sigma_m \partial_{\theta\theta} \sigma_m = \frac{1}{2} \partial_\theta (\partial_\theta \sigma_m)^2$.

As in step 3 of the proof of Proposition 4.3, we find that τ_1^h satisfies the same estimate.

2. To estimate $\langle \Lambda^{1/2} \partial_z \chi_\varepsilon(D_\theta) \tau_1^p \rangle_{m,\gamma}$ we first use (5.20) to write

$$(5.22) \quad \partial_z \hat{\tau}_{mn}^p(t, x, k, z) = \hat{F}_{mn}(t, x, k, z) + \int_{+\infty}^z ik\omega_1 e^{ik\omega_1(z-s)} \hat{F}_{mn}(t, x, k, z) ds.$$

The contribution of the second term can be estimated exactly as the corresponding term in step 1. Moreover, since the ds integral introduces a factor of $1/k$ it is clear that the contribution of the first term satisfies the same estimate. The remaining details are straightforward. \square

Next we consider estimates of $\partial_{ff} u_\tau$.

Proposition 5.7. *Let $m > \frac{d+1}{2}$.*

$$(5.23) \quad \begin{aligned} \langle \Lambda^{k/2} \partial_{\theta\theta} u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+2+\frac{k}{2})}{\varepsilon^{k/2}}, k = 0, 1, \dots, 3 \\ \langle \partial_{zz} u_\tau \rangle_{m,\gamma} &\lesssim C(m+3)/p \\ \langle \Lambda^{\frac{1}{2}} \partial_{zz} u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+3)}{\varepsilon^{1/2} p^{1/2}} + \frac{C(m+3+\frac{1}{2})}{p} \\ \langle \Lambda^{\frac{3}{2}} \partial_{zz} u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+3+\frac{1}{2})}{\varepsilon^{3/2}} + \frac{C(m+4+\frac{1}{2})}{p} \\ \langle \partial_{\theta z} u_\tau \rangle_{m,\gamma} &\lesssim C(m+2) \\ \langle \Lambda^{\frac{1}{2}} \partial_{\theta z} u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+2+\frac{1}{2})}{\varepsilon^{1/2}} \\ \langle \Lambda^{\frac{3}{2}} \partial_{\theta z} u_\tau \rangle_{m,\gamma} &\lesssim \frac{C(m+3+\frac{1}{2})}{\varepsilon^{3/2}}. \end{aligned}$$

Proof. The $\partial_{\theta\theta}$ estimates are obvious. For the estimates involving $\partial_{zz} \hat{\tau}_{mn}^p$ for $\hat{\tau}_{mn}^p$ as in (5.20), we use (5.22) to obtain

$$(5.24) \quad \begin{aligned} \partial_{zz} \hat{\tau}_{mn}^p &= \int [i(k-k')\omega_m + ik'\omega_n] e^{i(k-k')\omega_m z + ik'\omega_n z} \frac{1}{k} \widehat{\partial_\theta \sigma_m}(t, x, k-k') \widehat{\partial_{\theta\theta} \sigma_n}(t, x, k') dk + \\ &ik\omega_1 \hat{F}_{mn}(t, x, k, z) + \int_{+\infty}^z (ik\omega_1)^2 e^{ik\omega_1(z-s)} \hat{F}_{mn}(t, x, k, s) ds = A + B + C. \end{aligned}$$

The term C is estimated exactly like $\partial_{\theta\theta} \hat{\tau}_{mn}^p$, and B satisfies the same estimates (since the ds integral in C introduces a factor of $1/k$). One shows by arguments already used in this section that in each ∂_{zz} estimate the contribution of A is dominated by the terms that appear on the right in Proposition 5.7. The $\partial_{\theta z}$ estimates are similar but simpler. \square

Finally, we have estimates of $\partial_s u_\tau$.

Proposition 5.8. *Let $m > \frac{d+1}{2}$.*

$$\begin{aligned}
(5.25) \quad & \langle \partial_s u_\tau \rangle_{m,\gamma} \lesssim C(m+3)/p^2 \\
& \langle \Lambda^{\frac{1}{2}} \partial_s u_\tau \rangle_{m,\gamma} \lesssim \frac{C(m+3)}{\varepsilon^{1/2} p^{3/2}} + \frac{C(m+3+\frac{1}{2})}{p^2} \\
& \langle \Lambda \partial_s u_\tau \rangle_{m,\gamma} \lesssim \frac{C(m+3)}{\varepsilon p} + \frac{C(m+4)}{p^2} \\
& \langle \Lambda^{\frac{3}{2}} \partial_s u_\tau \rangle_{m,\gamma} \lesssim \frac{C(m+3)}{\varepsilon^{3/2} p^{1/2}} + \frac{C(m+4+\frac{1}{2})}{p^2} \\
& \langle \Lambda^2 \partial_s u_\tau \rangle_{m,\gamma} \lesssim \frac{C(m+3)}{\varepsilon^2} + \frac{C(m+5)}{p^2}
\end{aligned}$$

Proof. We omit the details of the proof, since it uses only the arguments given earlier in this section. \square

Remark 5.9. *Observe that ∂_{sf} (resp. ∂_{ss}) estimates of u_σ or u_τ can be obtained immediately from ∂_f (resp. ∂_s) estimates by increasing the argument of $C(\cdot)$ by 1 for each $C(\cdot)$ that appears. For example,*

$$(5.26) \quad \langle \Lambda^{\frac{1}{2}} \partial_f u_\tau \rangle_{m,\gamma} \lesssim \frac{C(m+2)}{\varepsilon^{1/2} p^{1/2}} + \frac{C(m+2+\frac{1}{2})}{p} \text{ implies } \langle \Lambda^{\frac{1}{2}} \partial_{sf} u_\tau \rangle_{m,\gamma} \lesssim \frac{C(m+3)}{\varepsilon^{1/2} p^{1/2}} + \frac{C(m+3+\frac{1}{2})}{p}.$$

Similarly, estimates of u_σ or u_τ (without the ∂_s) can be obtained by decreasing the argument of $C(\cdot)$ in ∂_s estimates.

5.3 Forcing estimates

Interior forcing. Here we give estimates of the terms appearing in $F_a^e(t, x, \theta)$ (5.6). The estimates starting in Proposition 5.11 are based on the assumption that $1 \geq p \geq \varepsilon^{\frac{1}{2}-\delta}$ for some $\delta > 0$. In that case the building block estimates show that any given norm of u_σ is always bigger than (i.e., satisfies an estimate with a bigger right hand side) than the same norm of εu_τ .⁵ For example,

$$\begin{aligned}
(5.27) \quad & \langle \Lambda^{3/2} \partial_{zz} u_\sigma \rangle_{m,\gamma} \lesssim \frac{1}{\varepsilon^{3/2}}, \text{ while } \langle \Lambda^{3/2} \varepsilon \partial_{zz} u_\tau \rangle_{m,\gamma} \lesssim \varepsilon \left(\frac{1}{\varepsilon^{3/2}} + \frac{1}{p} \right) \\
& \langle \partial_s u_\sigma \rangle_{m,\gamma} \lesssim 1, \text{ while } \langle \partial_s \varepsilon u_\tau \rangle_{m,\gamma} \lesssim \frac{\varepsilon}{p^2}.
\end{aligned}$$

This observation allows us to greatly reduce the number of estimates that have to be done in estimating F_a^e or g_a^e .

We have the following estimates for the “low frequency cutoff” error in (4.17).

Proposition 5.10. *Let $m > \frac{d+1}{2}$.*

$$\begin{aligned}
(5.28) \quad & (a) \left| \frac{1}{\varepsilon} \Lambda^{1/2} [\varepsilon(1 - \chi_\varepsilon(D_\theta))(L_{fs} u_\sigma + N(u_\sigma))] \right|_{0,m,\gamma} \leq C(m+1) \frac{p^3}{\varepsilon^{1/2}} + C(m+2+\frac{1}{2}) p^{1/2} \\
& (b) \left| \Lambda^{3/2} [\varepsilon(1 - \chi_\varepsilon(D_\theta))(L_{fs} u_\sigma + N(u_\sigma))] \right|_{0,m,\gamma} \leq C(m+2) \frac{p^2}{\varepsilon^{1/2}} + C(m+3+\frac{1}{2}) \varepsilon.
\end{aligned}$$

⁵When constructing profiles it is not uncommon, but more importantly not circular, to assume at one stage that a profile has a given form, and then later to construct a profile with that form.

Proof. 1. Preliminaries. Since $|X, \gamma| \lesssim |\xi', \gamma| + |\frac{k}{\varepsilon}|$, we can for $r \geq 0$ estimate norms of $\Lambda_D^r w$ by estimating pieces corresponding to $|\xi', \gamma|^r$ and $|\frac{k}{\varepsilon}|^r$, thereby taking advantage of the helpful factor $|k|^r$ which is $\lesssim p^r$ on $\text{supp}(1 - \chi(k/p)) \subset [-p, p]$.

We will again use (5.17) and we recall from (4.39) that

$$(5.29) \quad \hat{\sigma}_j(t, x_1, x_2, k) = \psi(x_2) \hat{\sigma}_j(t, x_1, 0, k), \text{ where } \psi \in C_c^\infty \text{ and } \psi = 1 \text{ near } x_2 = 0.$$

Below it will be convenient to suppress some arguments (t, x_1, x_2) as well as some expressions involving γ such as factors of $e^{-\gamma t}$.

2. Part (a). To estimate the $L^2(x_2, L^2(k, H^m(t, x_1)))$ norm of the term involving $N(u_\sigma)$ we consider first

$$(5.30) \quad \left| \frac{\sqrt{|k|}}{\sqrt{\varepsilon}} (1 - \chi(k/p)) \int |\widehat{\partial_{\theta\theta}\sigma_m}(k - k')|_{H^m(t, x_1)} e^{i(k-k')\omega_m \frac{x_2}{\varepsilon}} |\widehat{\partial_{\theta\theta}\sigma_n}(k')|_{H^m(t, x_1)} e^{ik'\omega_n \frac{x_2}{\varepsilon}} dk' \right|_{L^2(k, x_2)} := A,$$

and recall that the support properties of the $\hat{\sigma}_j$ imply both $i(k - k')\omega_m \frac{x_2}{\varepsilon} \leq 0$ and $ik'\omega_n \frac{x_2}{\varepsilon} \leq 0$ on the support of the integrand in (5.30). We break up the k' integral into pieces on $|k'| \leq p$, $|k'| \geq p$ that we call A_1, A_2 respectively. To estimate A_1 we ignore the exponentials (which are ≤ 1), write

$$(5.31) \quad |\widehat{\partial_{\theta\theta}\sigma_n}(k')| = |k'^2 \hat{\sigma}_n(k')| \leq p^2 |\hat{\sigma}_n(k')|,$$

and use Cauchy-Schwarz to estimate the dk' integral to obtain

$$(5.32) \quad A_1 \lesssim \frac{\sqrt{p}}{\sqrt{\varepsilon}} \cdot p^2 \cdot C(m+1) \cdot \sqrt{p}.$$

Here the last \sqrt{p} is $|1|_{L^2(\{|k| \leq p\})}$.

To estimate A_2 we ignore the first exponential and integrate the (square of the) second one to obtain

$$(5.33) \quad |e^{ik'\omega_n \frac{x_2}{\varepsilon}}|_{L^2(x_2)} \lesssim \frac{\sqrt{\varepsilon}}{\sqrt{p}},$$

after using (5.29) to neglect the x_2 dependence in the $\hat{\sigma}_j$. Thus we find

$$(5.34) \quad A_2 \lesssim \frac{\sqrt{p}}{\sqrt{\varepsilon}} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{p}} \cdot C(m+2) \cdot \sqrt{p},$$

where the final \sqrt{p} arises as before.

For the analogue of A obtained when $\Lambda_{\theta, D}^{1/2}$ is replaced by $\Lambda_{x', D}^{1/2}$, we clearly obtain

$$A \leq C(m+2 + \frac{1}{2})\sqrt{p},$$

so this completes the $L^2(x_2, L^2(k, H^m(t, x_1)))$ estimate in part (a) for the term that involves $N(u_\sigma)$. Writing

$$(5.35) \quad \langle k \rangle^m \lesssim \langle k - k' \rangle^m + \langle k' \rangle^m,$$

we obtain the same estimate for the $L^2(x_2, L^2(t, x_1, \hat{H}^m(k)))$ norm of the term involving $N(u_\sigma)$ in part (a) by very similar arguments.

3. To estimate the $L^2(x_2, L^2(k, H^m(t, x_1)))$ norm of the term involving $L_{fs}u_\sigma$ in (5.28)(a), instead of (5.30) we first consider

$$(5.36) \quad \left| \frac{\sqrt{|k|}}{\sqrt{\varepsilon}} (1 - \chi(k/p)) k |\widehat{\partial_s \sigma_m}(k)|_{H^m(t, x_1)} e^{ik\omega_m \frac{x_2}{\varepsilon}} \right|_{L^2(k, x_2)} := A,$$

whence

$$(5.37) \quad A \lesssim C(m+1)p.$$

Here again we have used (5.29) and computed $|e^{ik\omega_m \frac{x_2}{\varepsilon}}|_{L^2(x_d)} \lesssim \frac{\sqrt{\varepsilon}}{\sqrt{|k|}}$.

For the analogue of A obtained when $\Lambda_{\theta, D}^{1/2}$ is replaced by $\Lambda_{x', D}^{1/2}$, we have

$$A \leq C(m+1 + \frac{1}{2})p,$$

so this completes the $L^2(x_2, L^2(k, H^m(t, x_1)))$ estimate in part (a) for the term that involves $L_{fs}u_\sigma$. Again, we obtain the same estimate for the $L^2(x_2, L^2(t, x_1, \hat{H}^m(k)))$ norm.

4. Part (b). This estimate is simpler than the one in part (a), since one can take advantage of the helpful factor $|k|^{3/2}$ in $\frac{|k|^{3/2}}{\varepsilon^{3/2}}$; in particular, there is no need to break up the dk' integral as in step 2 or compute $L^2(x_d)$ norms of exponentials. Thus, we omit the details. \square

We now omit some of the constants $C(m+r)$. Eventually we will take r to be the largest such index appearing in the building block estimates.

Proposition 5.11. *Let $m > \frac{d+1}{2}$. For $k = 1, 3$ we have*

$$(5.38) \quad \begin{aligned} (a) \quad & \langle \Lambda^{k/2} L_{ss} u_a \rangle_{m, \gamma} \lesssim \varepsilon^{2-\frac{k}{2}} \\ (b) \quad & \langle \Lambda^{k/2} (\varepsilon^2 L_{fs} u_\tau) \rangle_{m, \gamma} \lesssim \varepsilon^{2-\frac{k}{2}} \\ (c) \quad & \langle \Lambda^{k/2} [\sum_{|\alpha|=2} Q_\alpha (D_\varepsilon u_a) D_\varepsilon^\alpha u_a] \rangle_{m, \gamma} \lesssim \varepsilon^{2-\frac{k}{2}} \\ (d) \quad & \langle \Lambda^{k/2} [\sum_{|\alpha|=2} L_\alpha (D_\varepsilon u_a) D_\varepsilon^\alpha u_a - \varepsilon N(u_\sigma)] \rangle_{m, \gamma} \lesssim \varepsilon^{2-\frac{k}{2}} \end{aligned}$$

Proof. **1. Preliminaries.** Recall

$$(5.39) \quad \begin{aligned} u_a &= \varepsilon^2 u_\sigma + \varepsilon^3 u_\tau \\ D_\varepsilon &= (\partial_{x_1} + \frac{\beta_1}{\varepsilon} \partial_\theta, \partial_{x_2} + \frac{1}{\varepsilon} \partial_z). \end{aligned}$$

With some abuse we will write $D_\varepsilon = \partial_s + \frac{\partial_f}{\varepsilon}$ and will, for example, write products of derivatives of possibly different components of u_σ as “ $\partial_f u_\sigma \partial_f u_\sigma$ ”.

We will use the observation explained at the beginning of this section that u_σ is always bigger than εu_τ in any given building block norm as long as $0 < \varepsilon^{\frac{1}{2}-\delta} \leq p \leq 1$. Below we take $k = 3$ to illustrate the proofs.

2. (a). The biggest contribution is by Proposition 5.5 and Remark 5.9:

$$(5.40) \quad \varepsilon^2 \langle \Lambda^{3/2} \partial_{ss} u_\sigma \rangle_{m,\gamma} \lesssim \varepsilon^2 \cdot \frac{1}{\varepsilon^{3/2}} = \varepsilon^{1/2}.$$

3. (c). For a given α the largest contribution comes from products $\varepsilon^6 \left(\frac{\partial_f u_\sigma}{\varepsilon} \frac{\partial_f u_\sigma}{\varepsilon} \frac{\partial_{ff} u_\sigma}{\varepsilon^2} \right)$. Using Corollary 3.24(a) and Proposition 5.5 we obtain

$$(5.41) \quad \varepsilon^2 \langle \Lambda^{3/2} (\partial_f u_\sigma \partial_f u_\sigma \partial_{ff} u_\sigma) \rangle_{m,\gamma} \lesssim \varepsilon^2 \cdot \frac{1}{\varepsilon^{3/2}} = \varepsilon^{1/2}.$$

4. (d). After $\varepsilon N(u_\sigma)$ is subtracted away and after examining the size of terms involving products $\partial_f u_\tau \partial_{ff} u_\tau$, one sees that the worst terms for a given α are $\varepsilon^4 \left(\partial_s u_\sigma \frac{\partial_{ff} u_\sigma}{\varepsilon^2} \right)$ and $\varepsilon^4 \left(\frac{\partial_f u_\sigma}{\varepsilon} \frac{\partial_{sf} u_\sigma}{\varepsilon} \right)$. For these we obtain the desired estimate as above. \square

The next corollary is an immediate consequence of the previous two Propositions.

Corollary 5.12. *Let $m > \frac{d+1}{2}$.*

$$(5.42) \quad \begin{aligned} (a) \quad & \Lambda^{1/2} \partial_{x_1, \varepsilon} F_a^e|_{0,m,\gamma} \lesssim \sqrt{\varepsilon} + \left(\frac{p^2}{\sqrt{\varepsilon}} + \varepsilon \right) + \sqrt{\varepsilon} \lesssim \sqrt{\varepsilon} + \frac{p^2}{\sqrt{\varepsilon}} \\ (b) \quad & \frac{1}{\varepsilon} |\Lambda^{1/2} F_a^e|_{0,m,\gamma} \leq \frac{p^3}{\varepsilon^{1/2}} + p^{1/2} + \sqrt{\varepsilon}. \end{aligned}$$

Proposition 5.13. *There exists a constant $T_2 > 0$ such that $v_a^s = \nabla_\varepsilon u_a^s$ satisfies*

$$(5.43) \quad E_{m,T}(v_a^s) \lesssim 1$$

uniformly for $\varepsilon \in (0, 1]$ and $0 \leq T \leq T_2$. ⁶

Proof. To prove this one must show that the quantities $|\Lambda^{3/2} v_a^s|_{0,m,\gamma}$, $|\Lambda v_a^s|_{\infty,m,\gamma}$, $|\frac{\nabla_\varepsilon u_a^s}{\varepsilon}|_{\infty,m,\gamma}$, etc., appearing in the definition of $E_{m,T}(v_a^s)$, are all $\lesssim 1$.

Consider for example $|\Lambda^{3/2} v_a^s|_{0,m,\gamma}$. The worst terms are $\varepsilon^2 |\Lambda_\theta^{3/2} D_\varepsilon u_\sigma|_{0,m,\gamma}$, for which we perform the $L^2(x_2, k, H^m(t, x_1))$ estimate as follows. Writing $D_\varepsilon = \partial_s + \frac{\partial_f}{\varepsilon}$ we obtain for the worst (∂_f/ε) part:

$$(5.44) \quad \begin{aligned} \varepsilon \left| \frac{|k|^{3/2}}{\varepsilon^{3/2}} k \psi(x_2) |\hat{\sigma}_m(t, x_1, 0, k)|_{H^m(t, x_1)} e^{ik\omega_m \frac{x_2}{\varepsilon}} \right|_{L^2(x_2, k)} &\lesssim \\ |k^2 |\hat{\sigma}_m(t, x_1, 0, k)|_{H^m(t, x_1)}|_{L^2(k)} &\lesssim C(m+2), \end{aligned}$$

since $|e^{ik\omega_m \frac{x_2}{\varepsilon}}|_{L^2(x_2)} \lesssim \frac{\varepsilon^{1/2}}{|k|^{1/2}}$. The remaining estimates are similar or simpler. \square

Boundary forcing estimates. Here we carry out the estimation of g_a^e (5.6). The proof of the next proposition is similar to that of propositions 5.10 and 5.11 but simpler.

⁶Recall that v_a^s depends on T as a parameter.

Proposition 5.14. *Let $m > \frac{d+1}{2}$. For $k = 0, \dots, 4$ we have*

$$\begin{aligned}
(5.45) \quad & \langle \Lambda^{k/2} \varepsilon^2 G \rangle_{m,\gamma} \lesssim \varepsilon^{2-\frac{k}{2}} \\
& \langle \Lambda^{k/2} \varepsilon^3 \ell_s(u_\tau) \rangle_{m,\gamma} \lesssim \frac{\varepsilon^{3-\frac{k}{2}}}{p^{2-\frac{k}{2}}} \\
& \langle \Lambda^{k/2} c(D_\varepsilon u_a) \rangle_{m,\gamma} \lesssim \varepsilon^{3-\frac{k}{2}} \\
& \langle \Lambda^{k/2} [q(D_\varepsilon u_a) - \varepsilon^2 n(u_\sigma)] \rangle_{m,\gamma} \lesssim \varepsilon^{3-\frac{k}{2}} \\
& \langle \Lambda^{k/2} [\varepsilon^2 (\chi_\varepsilon(D_\theta) - 1) G] \rangle_{m,\gamma} \leq p^{k/2} \varepsilon^{2-\frac{k}{2}} C(m) + \varepsilon^2 C(m + \frac{k}{2}) \\
& \langle \Lambda^{k/2} [\varepsilon^2 (1 - \chi_\varepsilon(D_\theta)) (\ell_s(u_\sigma) + n(u_\sigma))] \rangle_{m,\gamma} \leq p^{k/2} \varepsilon^{2-\frac{k}{2}} C(m+1) + \varepsilon^2 C(m+1 + \frac{k}{2}).
\end{aligned}$$

This yields the immediate corollary:

Corollary 5.15. *Let $m > \frac{d+1}{2}$.*

$$\begin{aligned}
(5.46) \quad & (a) \langle g - g_a^e \rangle_{m,\gamma} \lesssim \frac{\varepsilon^3}{p^2} + \varepsilon^3 + \varepsilon^2 \lesssim \frac{\varepsilon^3}{p^2} + \varepsilon^3 + \varepsilon^2 \lesssim \varepsilon^2 \\
& (b) \langle \Lambda^{1/2} (g - g_a^e) \rangle_{m,\gamma} \lesssim \frac{\varepsilon^{5/2}}{p^{3/2}} + \varepsilon^{5/2} + p^{1/2} \varepsilon^{3/2} + \varepsilon^2 \lesssim \frac{\varepsilon^{5/2}}{p^{3/2}} + p^{1/2} \varepsilon^{3/2} \\
& (c) \langle \Lambda (g - g_a^e) \rangle_{m,\gamma} \lesssim \frac{\varepsilon^2}{p} + \varepsilon^2 + p\varepsilon \lesssim p\varepsilon \\
& (d) \langle \Lambda^{3/2} (g - g_a^e) \rangle_{m,\gamma} \lesssim \frac{\varepsilon^{3/2}}{p^{1/2}} + \varepsilon^{3/2} + p^{3/2} \varepsilon^{1/2} + \varepsilon^2 \lesssim \frac{\varepsilon^{3/2}}{p^{1/2}} + p^{3/2} \varepsilon^{1/2} \\
& (e) \langle \Lambda^2 (g - g_a^e) \rangle_{m,\gamma} \lesssim \varepsilon + p^2 + \varepsilon^2 \lesssim p^2.
\end{aligned}$$

5.4 Estimates of the extended approximate solution

In order to estimate solutions of the error equations (5.11)-(5.13) we first need estimates uniform in ε for solutions of (5.7)-(5.9) in the $E_{m,\gamma}(v_a)$ norm. Except for the boundary term in (5.8) the coefficients of (5.7)-(5.9) depend on v_a^s and the forcing is given by F_a^e and g_a^e .

The next proposition and its proof are very similar to Proposition 3.20. Given the forcing estimates of section 5.3, the proof is somewhat simpler, as we explain below.

Proposition 5.16. *Let $m > 3d + 4 + \frac{d+1}{2}$ and $G \in H^{m+3}(b\Omega)$. There exist positive constants M_G as in (3.90) and T_2 such that the following is true. There exist positive constants ε_0, γ_0 , and there exist increasing functions $Q_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2$, with $Q_i(z) \geq z$ such that for $\varepsilon \in (0, \varepsilon_0]$ and each T with $0 < T \leq T_2$, the solution to (5.7)-(5.9) satisfies*

$$(5.47) \quad E_{m,\gamma}(v_a) \leq \gamma^{-1} E_{m,\gamma}(v_a) Q_1(E_{m,T}(v_a^s)) + (\gamma^{-\frac{1}{2}} + \sqrt{\varepsilon}) Q_2(E_{m,T}(v_a^s)) \text{ for } \gamma \geq \gamma_0.$$

Proof. 1. Contrary to the situation in Proposition 3.20, we already know (Prop. 5.13) that there exist positive constants T_2, M_0 such that

$$(5.48) \quad E_{m,T}(v_a^s) \leq M_0 \text{ for } T \leq T_2, \varepsilon \in (0, 1].$$

Moreover, since the constants $C(m+r)$ appearing in the estimates of section 5.2 can be made as small as desired by taking T_2 small, the same applies to M_0 .

2. The two terms on the right side of (5.47) correspond to the terms on the right in (3.91); the first comes from commutators, and the second from forcing. The commutator analysis here is identical to that given for Prop. 3.20, since it depends only on the control of the coefficients given by (5.48). Moreover, the forcing estimates of section 5.3 show that

$$(5.49) \quad F_a^e = o(1) \text{ and } g_a^e = \varepsilon^2 G + o(1) \text{ as } \varepsilon \rightarrow 0$$

in every one of the norms that is used in the forcing estimate. Thus, the second term of (5.47) has the same form as the second term of (3.91). □

Corollary 5.17. *Under the assumptions of Prop. 5.16 we have, after enlarging γ_0 if necessary,*

$$(5.50) \quad E_{m,\gamma}(v_a) \leq (\gamma^{-\frac{1}{2}} + \sqrt{\varepsilon})Q_2(E_{m,T}(v_a^s)) \text{ for } \gamma \geq \gamma_0.$$

5.5 Endgame

The next Proposition is the analogue of Proposition 3.73 for the trio of error equations (5.11)-(5.13). The main difference in the proof is that there are now new kinds of forcing terms to deal with.

Henceforth we will denote functions $Q_i(z)$ like those in Proposition 5.16 simply by $Q(z)$, and we will allow Q to change from term to term, or even from factor to factor within a given term. In the next proposition T_1 and T_2 are the positive constants appearing in Theorem 3.7 and Proposition 5.16 respectively.

Proposition 5.18. *Suppose $m > 3d + 4 + \frac{d+1}{2}$, $G \in H^{m+3}(b\Omega)$, and $0 < \varepsilon^{\frac{1}{2}-\delta} \leq p \leq 1$ for some $\delta > 0$. There exist positive constants $T_3 \leq \min\{T_1, T_2\}$, ε_0, γ_0 such that for $j \in J_h \cup J_e$, $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geq \gamma_0$, and $T \leq T_3$ the solution to the error system (5.11)-(5.13) satisfies*

$$(5.51) \quad \begin{aligned} & E_{m,\gamma}(w_1) + \langle \phi_j \Lambda^{\frac{3}{2}} w_1 \rangle_{m,\gamma} + \langle \phi_j \Lambda w_1 \rangle_{m+1,\gamma} + \sqrt{\varepsilon} \langle \phi_j \Lambda^{\frac{3}{2}} w_1 \rangle_{m+1,\gamma} + \frac{1}{\sqrt{\varepsilon}} \langle \phi_j \Lambda w_1 \rangle_{m,\gamma} \lesssim \\ & \frac{1}{\sqrt{\gamma}} \{ Q_1(E_{m,T}(v^s)) \cdot E_{m,\gamma}(w_1) + E_{m,T}(w^s) E_{m+1,\gamma}(v_a) Q_2(E_{m,T}(v_a^s, v^s)) + \\ & E_{m,T}(w^s) Q_3(E_{m,T}(v_a^s, v^s)) + (\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p}) \}. \end{aligned}$$

Proof. To prove (5.51) we use (3.37)-(3.39) to estimate the solutions of (5.11) and (5.13). The arguments have much in common with those of section 3.5, so here we will emphasize what is different.

1. The first term on the right in (5.51) arises by exactly the same commutator analysis as given in section 3.5. See Proposition 3.73.

Let us denote the right sides of the interior equations (5.11)(a)-(5.13)(a) by $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}$ respectively, and the right sides of the corresponding boundary equations by $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}$ respectively. These terms differ from the forcing terms estimated in section 5, and the error analysis depends on their careful estimation. In particular, to estimate the terms which define $E_{m,\gamma}(w_1)$ in the first, second, and third lines of (3.78)

respectively, one must respectively estimate

$$\begin{aligned}
(5.52) \quad & (a) |\Lambda^{1/2} \mathcal{F}_1|_{0,m,\gamma}, |\mathcal{F}_1|_{0,m+1,\gamma}, \langle \Lambda \mathcal{G}_1 \rangle_{m,\gamma}, \langle \Lambda^{1/2} \mathcal{G}_1 \rangle_{m+1,\gamma} \\
& (b) \sqrt{\varepsilon} |\Lambda^{1/2} \mathcal{F}_1|_{0,m+1,\gamma}, \frac{1}{\sqrt{\varepsilon}} |\mathcal{F}_1|_{0,m,\gamma}, \sqrt{\varepsilon} \langle \Lambda \mathcal{G}_1 \rangle_{m+1,\gamma}, \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{1/2} \mathcal{G}_1 \rangle_{m,\gamma} \\
& (c) \frac{1}{\varepsilon} |\Lambda^{1/2} \mathcal{F}|_{0,m,\gamma}, \frac{1}{\varepsilon} |\mathcal{F}|_{0,m+1,\gamma}, \langle \Lambda \mathcal{G} \rangle_{m,\gamma}, \frac{1}{\varepsilon} \langle \Lambda^{1/2} \mathcal{G} \rangle_{m+1,\gamma}.
\end{aligned}$$

Below we will focus on estimating the terms in (5.52)(a) and (c). We will omit the (similar) details for (b).

2. Interior forcing in (5.11). We show that for $|\alpha| = 2$

$$\begin{aligned}
(5.53) \quad & (a) |\Lambda^{1/2} [(A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x,\varepsilon}^\alpha v_{1a}]|_{0,m,\gamma} \leq E_{m,T}(w^s) E_{m+1,\gamma}(v_{1a}) Q(E_{m,T}(v_a^s, v^s)) \\
& (b) |[(A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x,\varepsilon}^\alpha v_{1a}]|_{0,m+1,\gamma} \leq E_{m,T}(w^s) E_{m+1,\gamma}(v_{1a}) Q(E_{m,T}(v_a^s, v^s)).
\end{aligned}$$

We write $A_\alpha(v^s) - A_\alpha(v_a^s) = f(v_a^s, v^s) w^s$, and reduce to considering the case $\alpha = (2, 0)$ by using the equation and the noncharacteristic boundary assumption to treat $\alpha = (1, 1)$ or $(0, 2)$. Applying Corollary 3.24(a) with x_2 fixed, we have

$$\begin{aligned}
(5.54) \quad & \langle \Lambda^{1/2} [f(v_a^s, v^s) w^s \partial_{x,\varepsilon}^\alpha v_{1a}] \rangle_{m,\gamma} \leq \\
& \langle \Lambda^{1/2} \varepsilon \partial_{x,\varepsilon}^\alpha v_{1a} \rangle_{m,\gamma} \langle f(v_a^s, v^s) \frac{w^s}{\varepsilon} \rangle_m + \langle \varepsilon \partial_{x,\varepsilon}^\alpha v_{1a} \rangle_{m,\gamma} \langle \Lambda_1^{1/2} [f(v_a^s, v^s) \frac{w^s}{\varepsilon}] \rangle_m \leq \\
& \langle \Lambda^{3/2} v_{1a} \rangle_{m+1,\gamma} h(\langle v_a^s, v^s \rangle_m) \langle \frac{w^s}{\varepsilon} \rangle_m + \langle \Lambda v_{1a} \rangle_{m+1,\gamma} \langle \Lambda_1^{1/2} (v_a^s, v^s) \rangle_m h(\langle v_a^s, v^s \rangle_m) \langle \frac{w^s}{\varepsilon} \rangle_m + \\
& \langle \Lambda v_{1a} \rangle_{m+1,\gamma} h(\langle v_a^s, v^s \rangle_m) \langle \Lambda^{1/2} \frac{w^s}{\varepsilon} \rangle_m = A + B + C.
\end{aligned}$$

Next take the $L^2(x_d)$ norm of both sides. In the final estimate the factors $|\Lambda^{3/2} v_{1a}|_{0,m+1,\gamma}$, $|\Lambda v_{1a}|_{0,m+1,\gamma}$, and $|\Lambda^{1/2} \frac{w^s}{\varepsilon}|_{0,m}$ should respectively appear in each of the three terms on the right; the remaining factors in those terms all involve $L^\infty(x_d)$ norms. This gives (5.53)(a).

After again writing $w^s \partial_{x,\varepsilon}^\alpha v_{1a} = \frac{w^s}{\varepsilon} \varepsilon \partial_{x,\varepsilon}^\alpha v_{1a}$, one can prove (5.53)(b) by an argument similar to the proof of Proposition 3.48.

3. An argument similar to that in step 2 but more straightforward yields

$$\begin{aligned}
(5.55) \quad & |\Lambda^{1/2} [b(v^s) d_\varepsilon v^s d_\varepsilon v^s - b(v_a^s) d_\varepsilon v_a^s d_\varepsilon v_a^s]|_{0,m,\gamma} \leq E_{m,T}(w^s) Q(E_{m,T}(v_a^s, v^s)) \\
& |b(v^s) d_\varepsilon v^s d_\varepsilon v^s - b(v_a^s) d_\varepsilon v_a^s d_\varepsilon v_a^s]|_{0,m+1,\gamma} \leq E_{m,T}(w^s) Q(E_{m,T}(v_a^s, v^s))
\end{aligned}$$

4. Corollary 5.12(a) gives

$$(5.56) \quad |\Lambda^{1/2} \partial_{x_1,\varepsilon} F_a^e|_{0,m,\gamma} \lesssim \sqrt{\varepsilon} + \left(\frac{p^2}{\sqrt{\varepsilon}} + \varepsilon\right) + \sqrt{\varepsilon} \lesssim \sqrt{\varepsilon} + \frac{p^2}{\sqrt{\varepsilon}}.$$

The same estimate clearly holds for $|\partial_{x_1,\varepsilon} F_a^e|_{0,m+1,\gamma}$.

5. Boundary forcing in (5.11). An argument like that in step 2 yields

$$\begin{aligned}
(5.57) \quad & \langle \Lambda [d_{v_1} H(v_1^s, h(v^s)) - d_{v_1} H(v_{1a}^s, h(v_a^s))] \partial_{x_1,\varepsilon} v_{1a} \rangle_{m,\gamma} \leq E_{m,T}(w^s) E_{m+1,\gamma}(v_{1a}) Q_2(E_{m,T}(v_a^s, v^s)) \\
& \langle \Lambda^{1/2} [d_{v_1} H(v_1^s, h(v^s)) - d_{v_1} H(v_{1a}^s, h(v_a^s))] \partial_{x_1,\varepsilon} v_{1a} \rangle_{m+1,\gamma} \leq E_{m,T}(w^s) E_{m+1,\gamma}(v_{1a}) Q_2(E_{m,T}(v_a^s, v^s)).
\end{aligned}$$

We claim

$$(5.58) \quad \langle \Lambda [d_g H(v_1^s, g) \partial_{x_1, \varepsilon} g - d_g H(v_{1a}^s, g_a^e) \partial_{x_1, \varepsilon} g_a^e] \rangle_{m, \gamma} \lesssim E_{m, T}(w^s) Q(E_{m, T}(v_1^s, v_{1a}^s)) + \varepsilon + p^2,$$

and the same estimate holds for $\langle \Lambda^{1/2} [d_g H(v_1^s, g) \partial_{x_1, \varepsilon} g - d_g H(v_{1a}^s, g_a^e) \partial_{x_1, \varepsilon} g_a^e] \rangle_{m+1, \gamma}$.

Writing $d_g H(v_1^s, g) \partial_{x_1, \varepsilon} g - d_g H(v_{1a}^s, g_a^e) \partial_{x_1, \varepsilon} g_a^e$ in the obvious way as a sum of three differences, we see that Corollary 5.15(a),(c),(e) and Corollary 3.24(a) imply (5.58).

6. Interior forcing in 5.13. The above estimates show that the terms arising from the application of (3.37)-(3.39) to the system (5.11) are dominated by the right side of (5.51), so we now turn to the z -system, (5.13), which is needed to estimate terms like $|\nabla_\varepsilon z / \varepsilon|_{\infty, m, \gamma}$.⁷

Corollary 5.12(b) gives

$$(5.59) \quad \frac{1}{\varepsilon} |\Lambda^{1/2} F_a^e|_{0, m, \gamma} \leq \frac{p^3}{\varepsilon^{1/2}} + p^{1/2} + \sqrt{\varepsilon}.$$

Next we show that for $|\alpha| = 2$:

$$(5.60) \quad \begin{aligned} (a) \quad & \frac{1}{\varepsilon} |\Lambda^{1/2} [(A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x, \varepsilon}^\alpha u_a]|_{0, m, \gamma} \leq E_{m, T}(w^s) E_{m, \gamma}(v_a) Q(E_{m, T}(v_a^s, v^s)) \\ (b) \quad & \frac{1}{\varepsilon} |[(A_\alpha(v^s) - A_\alpha(v_a^s)) \partial_{x, \varepsilon}^\alpha u_a]|_{0, m+1, \gamma} \leq E_{m, T}(w^s) E_{m, \gamma}(v_a) Q(E_{m, T}(v_a^s, v^s)) \end{aligned}$$

For fixed x_2 we estimate for $|\alpha| = 1$

$$(5.61) \quad \begin{aligned} & \frac{1}{\varepsilon} \langle \Lambda^{1/2} [f(v_a^s, v^s) w^s \partial_{x, \varepsilon}^\alpha v_a] \rangle_{m, \gamma} \leq \\ & \langle \Lambda^{1/2} \partial_{x, \varepsilon}^\alpha v_a \rangle_{m, \gamma} \langle f(v_a^s, v^s) \frac{w^s}{\varepsilon} \rangle_m + \langle \partial_{x, \varepsilon}^\alpha v_a \rangle_{m, \gamma} \langle \Lambda_1^{1/2} [f(v_a^s, v^s) \frac{w^s}{\varepsilon}] \rangle_m, \end{aligned}$$

so applying rules of section 3.4 and taking the $L^2(x_2)$ norm of both sides gives (5.60)(a). The proof of (5.60)(b) is parallel to that of (5.53)(b).

7. Boundary forcing in 5.13. As in (3.141), (3.142) let us write

$$(5.62) \quad H(v_1^s, g) - d_{v_1} H(v_1^s, g) v_1^s = Cg + b(v_1^s, g)(v_1^s, g)^2 := F(v_1^s, g).$$

We express the boundary forcing term in (5.13) as

$$(5.63) \quad \mathcal{G} = [F(v_1^s, g) - F(v_{1a}^s, g_a^e)] + [d_{v_1} H(v_1^s, h(v^s)) - d_{v_1} H(v_{1a}^s, h(v_a^s))] v_{1a} = A + B$$

By arguments similar to step 2 we obtain

$$(5.64) \quad \frac{1}{\varepsilon} \langle \Lambda B \rangle_{m, \gamma} \leq E_{m, \gamma}(v_{1a}) E_{m, T}(w^s) Q(E_{m, T}(v^s, v_a^s)).$$

Writing A as a sum of two differences, using the special form of F , and using (5.46)(a),(c), we obtain

$$(5.65) \quad \begin{aligned} \frac{1}{\varepsilon} \langle \Lambda A \rangle_{m, \gamma} & \lesssim E_{m, T}(w^s) Q(E_{m, T}(v^s, v_a^s)) + \left(\frac{\varepsilon^2}{p^2} + \varepsilon^2 + \varepsilon \right) + \left(\frac{\varepsilon}{p} + \varepsilon + p \right) \lesssim \\ & E_{m, T}(w^s) Q(E_{m, T}(v^s, v_a^s)) + \frac{\varepsilon}{p} + p. \end{aligned}$$

⁷We will use the fact that a remark analogous to Remark 3.40 applies to w and $\nabla_\varepsilon z$.

This gives

$$\frac{1}{\varepsilon} \langle \Lambda(A+B) \rangle_{m,\gamma} \lesssim E_{m,\gamma}(v_{1a}) E_{m,T}(w^s) Q(E_{m,T}(v^s, v_a^s)) + E_{m,T}(w^s) Q(E_{m,T}(v^s, v_a^s)) + \frac{\varepsilon}{p} + p$$

and similarly

$$\frac{1}{\varepsilon} \langle \Lambda^{1/2}(A+B) \rangle_{m+1,\gamma} \lesssim E_{m,\gamma}(v_{1a}) E_{m,T}(w^s) Q(E_{m,T}(v^s, v_a^s)) + E_{m,T}(w^s) Q(E_{m,T}(v^s, v_a^s)) + \frac{\varepsilon}{p} + p.$$

8. The same interior and boundary forcing estimates are satisfied by the terms in (5.52)(b). Combining these results we obtain (5.51). \square

By enlarging γ_0 if necessary and using Corollary 5.17, we obtain

Corollary 5.19. *Suppose $m > 3d + 4 + \frac{d+1}{2}$ and $0 < \varepsilon^{\frac{1}{2}-\delta} \leq p \leq 1$ for some $\delta > 0$. There exist positive constants T_3, ε_0 , and γ_0 such that for $j \in J_h \cup J_e$, $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geq \gamma_0$, and $T \leq T_3$:*

$$(5.66) \quad E_{m,\gamma}(w_1) + \langle \phi_j \Lambda^{\frac{3}{2}} w_1 \rangle_{m,\gamma} + \langle \phi_j \Lambda w_1 \rangle_{m+1,\gamma} + \sqrt{\varepsilon} \langle \phi_j \Lambda^{\frac{3}{2}} w_1 \rangle_{m+1,\gamma} + \frac{1}{\sqrt{\varepsilon}} \langle \phi_j \Lambda w_1 \rangle_{m,\gamma} \lesssim \frac{1}{\sqrt{\gamma}} \{ E_{m,T}(w^s) Q(E_{m+1,T}(v_a^s)) Q(E_{m,T}(v_a^s, v^s)) + (\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p}) \}.$$

To complete the error analysis we will apply (3.40)-(3.42) to the Dirichlet problem (5.12) in order to estimate $E_{m,\gamma}(w_2)$. This requires us to estimate the terms in (5.52)(a),(b) where $\mathcal{F}_1, \mathcal{G}_1$ are replaced by $\mathcal{F}_2, \mathcal{G}_2$. In addition we must estimate norms of \mathcal{G}_2 involving the singular pseudodifferential operators ϕ_{jD}, ψ_{jD} for $j \in J_h$ (recall Prop. 3.77, for example). The estimates of \mathcal{F}_2 are identical to those already given of \mathcal{F}_1 . In the next two Propositions we give the required estimates of \mathcal{G}_2 . These estimates are quite similar to those of Propositions 3.76, 3.77, 3.78, and 3.79 of section 3.5, and are proved in the same way.

Proposition 5.20. *Suppose $m > 3d + 4 + \frac{d+1}{2}$ and $0 < \varepsilon^{\frac{1}{2}-\delta} \leq p \leq 1$ for some $\delta > 0$. Let ϕ_j and ψ_j , $j \in J_h$ be singular symbols related as in Notation 3.27. There exist positive constants $\varepsilon_0, \gamma_0, T_3$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geq \gamma_0$, $T \leq T_3$ we have*

$$(5.67) \quad \begin{aligned} (a) \quad & \langle \Lambda \mathcal{G}_2 \rangle_{m,\gamma} \lesssim \langle \Lambda w_1 \rangle_{m,\gamma} Q(E_{m,T}(v^s, v_a^s)) + \varepsilon \\ (b) \quad & \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m+1,\gamma} \lesssim \langle \Lambda^{1/2} w_1 \rangle_{m+1,\gamma} Q(E_{m,T}(v^s, v_a^s)) + \varepsilon \\ (c) \quad & \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m,\gamma} \lesssim \left(\langle \psi_j \Lambda^{\frac{3}{2}} w_1 \rangle_{m,\gamma} + \langle \Lambda w_1 \rangle_{m,\gamma} \right) Q(E_{m,T}(v^s, v_a^s)) + \sqrt{\varepsilon} \\ (d) \quad & \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m+1,\gamma} \lesssim \left(\langle \psi_j \Lambda w_1 \rangle_{m+1,\gamma} + \langle \Lambda^{\frac{1}{2}} w_1 \rangle_{m+1,\gamma} \right) Q(E_{m,T}(v^s, v_a^s)) + \varepsilon. \end{aligned}$$

Proposition 5.21. *With notation as in the previous proposition we have for $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geq \gamma_0$, $T \leq T_3$:*

$$(5.68) \quad \begin{aligned} (a) \quad & \sqrt{\varepsilon} \langle \Lambda \mathcal{G}_2 \rangle_{m+1,\gamma} \lesssim \sqrt{\varepsilon} \langle \Lambda w_1 \rangle_{m+1,\gamma} Q(E_{m,T}(v^s, v_a^s)) + \varepsilon^{3/2} \\ (b) \quad & \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m,\gamma} \lesssim \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} w_1 \rangle_{m,\gamma} Q(E_{m,T}(v^s, v_a^s)) + \sqrt{\varepsilon} \\ (c) \quad & \sqrt{\varepsilon} \langle \phi_j \Lambda^{\frac{3}{2}} \mathcal{G}_2 \rangle_{m+1,\gamma} \lesssim \left(\sqrt{\varepsilon} \langle \psi_j \Lambda^{\frac{3}{2}} w_1 \rangle_{m+1,\gamma} + \sqrt{\varepsilon} \langle \Lambda w_1 \rangle_{m+1,\gamma} \right) Q(E_{m,T}(v^s, v_a^s)) + \varepsilon \\ (d) \quad & \frac{1}{\sqrt{\varepsilon}} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m,\gamma} \lesssim \left(\frac{1}{\sqrt{\varepsilon}} \langle \psi_j \Lambda w_1 \rangle_{m,\gamma} + \frac{1}{\sqrt{\varepsilon}} \langle \Lambda^{\frac{1}{2}} w_1 \rangle_{m,\gamma} \right) Q(E_{m,T}(v^s, v_a^s)) + \sqrt{\varepsilon}. \end{aligned}$$

The following Proposition is the analogue for the error system of Proposition 3.80.

Proposition 5.22. *Suppose $m > 3d + 4 + \frac{d+1}{2}$, $G \in H^{m+3}(b\Omega)$, and $0 < \varepsilon^{\frac{1}{2}-\delta} \leq p \leq 1$ for some $\delta > 0$. There exist positive constants $\varepsilon_0, \gamma_0, T_3$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geq \gamma_0$, and $T \leq T_3$ the solution to the error system (5.11)-(5.13) satisfies*

$$(5.69) \quad E_{m,\gamma}(w_2) \lesssim \frac{1}{\gamma} E_{m,\gamma}(w_2) Q(E_{m,T}(v^s)) + \frac{1}{\sqrt{\gamma}} \left[E_{m,T}(w^s) Q(E_{m+1,T}(v_a^s)) Q(E_{m,T}(v_a^s, v^s)) + \left(\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p} \right) \right] + \sqrt{\varepsilon}.$$

Proof. Given the earlier results of this section, the proof is quite similar to that of Proposition 3.80. Parallel to step 2 of the earlier proof, consider for example the term $\left| \begin{pmatrix} \Lambda w_2 \\ D_{x_2} w_2 \end{pmatrix} \right|_{\infty, m, \gamma}$. By (3.41) applied to (5.12) we have

$$(5.70) \quad \left| \begin{pmatrix} \Lambda w_2 \\ D_{x_2} w_2 \end{pmatrix} \right|_{\infty, m, \gamma} \lesssim \gamma^{-1} |\mathcal{F}_{2,com}|_{0, m+1, \gamma} + \gamma^{-1} |\mathcal{F}_2|_{0, m+1, \gamma} + (\langle \Lambda \mathcal{G}_2 \rangle_{m, \gamma} + \langle \Lambda^{\frac{1}{2}} \mathcal{G}_2 \rangle_{m+1, \gamma}) + \gamma^{-\frac{1}{2}} \sum_{j \in J_h} \langle \phi_j \Lambda \mathcal{G}_2 \rangle_{m+1, \gamma} := A + B + C + D,$$

where $\mathcal{F}_{2,com}$ is the interior commutator. By the interior commutator arguments of section 3.5 we have

$$(5.71) \quad A \lesssim \frac{1}{\gamma} E_{m,\gamma}(w_2) Q(E_{m,T}(v^s)).$$

Corollary 5.17 and the estimates in steps 2, 3, and 4 of the proof of Proposition 5.18 apply to \mathcal{F}_2 to show

$$(5.72) \quad B \lesssim \frac{1}{\gamma} \left[E_{m,T}(w^s) Q(E_{m+1,T}(v_a^s)) Q(E_{m,T}(v_a^s, v^s)) + \sqrt{p} + \frac{p^2}{\sqrt{\varepsilon}} \right].$$

From Proposition 5.20(a),(b) we have

$$(5.73) \quad C \lesssim E_{m,\gamma}(w_1) Q(E_{m,T}(v_a^s, v^s)) + \sqrt{\varepsilon},$$

while Proposition 5.20(d) gives

$$(5.74) \quad D \lesssim \frac{1}{\sqrt{\gamma}} \left[\sum_{j \in J_h} \left(\langle \psi_j \Lambda w_1 \rangle_{m+1, \gamma} + \langle \Lambda^{\frac{1}{2}} w_1 \rangle_{m+1, \gamma} \right) Q(E_{m,T}(v^s, v_a^s)) + \sqrt{\varepsilon} \right].$$

Next apply Corollary 5.19 to obtain

$$(5.75) \quad C + D \lesssim \frac{1}{\sqrt{\gamma}} \left[E_{m,T}(w^s) Q(E_{m+1,T}(v_a^s)) Q(E_{m,T}(v_a^s, v^s)) + \left(\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p} \right) \right] + \sqrt{\varepsilon}.$$

Thus, we see that $A + B + C + D$ is dominated by the right side of (5.69). The other terms making up $E_{m,\gamma}(w_2)$ are estimated in the same way using the earlier results of this section. \square

Combining Corollary 5.19 with Proposition 5.22 we immediately obtain the following corollary by enlarging γ_0 if necessary:

Corollary 5.23. *Suppose $m > 3d + 4 + \frac{d+1}{2}$ and $0 < \varepsilon^{\frac{1}{2}-\delta} \leq p \leq 1$ for some $\delta > 0$. There exist positive constants $\varepsilon_0, \gamma_0, T_3$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geq \gamma_0$, and $T \leq T_3$ we have*

$$(5.76) \quad E_{m,\gamma}(w) \lesssim \frac{1}{\sqrt{\gamma}} \left[E_{m,T}(w^s)Q(E_{m+1,T}(v_a^s))Q(E_{m,T}(v_a^s, v^s)) + \left(\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p} \right) \right] + \sqrt{\varepsilon}.$$

Corollary 5.24. *Suppose $m > 3d + 4 + \frac{d+1}{2}$ and $\delta > 0$. There exist positive constants $T_4 \leq T_3, \varepsilon_0$, and C_δ such that for $\varepsilon \in (0, \varepsilon_0]$, $T \leq T_4$ the solution to the error system (5.11)-(5.13) satisfies*

$$(5.77) \quad E_{m,T}(w) \leq C_\delta \varepsilon^{\frac{1}{4}-\delta}.$$

Proof. Since $E_{m,\gamma}(w) \gtrsim e^{-\gamma T} E_{m,T}(w)$, we obtain from (5.76)

$$(5.78) \quad E_{m,T}(w) \lesssim e^{\gamma T} \left(\frac{1}{\sqrt{\gamma}} \left[E_{m,T}(w^s)Q(E_{m+1,T}(v_a^s))Q(E_{m,T}(v_a^s, v^s)) + \left(\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p} \right) \right] + \sqrt{\varepsilon} \right)$$

for $\gamma \geq \gamma_0$ as in Corollary 5.23. Thus, for $T \leq 1/\gamma_0$ we have

$$(5.79) \quad E_{m,T}(w) \lesssim \sqrt{T} \left[E_{m,T}(w^s)Q(E_{m+1,T}(v_a^s))Q(E_{m,T}(v_a^s, v^s)) + \left(\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p} \right) \right] + \sqrt{\varepsilon}$$

Since $w = w_T^s$ on Ω_T for $T \leq T_3$, by choosing $0 < T_4 \leq 1/\gamma_0$ small enough and setting $p = \varepsilon^{\frac{1}{2}-2\delta}$, we deduce (5.77) for $T \leq T_4$ from (5.79). □

Chapter 6

Some extensions

6.1 Extension to general isotropic hyperelastic materials.

We now describe the minor changes in the analysis that are needed when assumption (A1) in chapter 3 is replaced by assumption (A1g), that is, when we pass from a Saint Venant-Kirchhoff system to general isotropic hyperelastic materials. Theorems 3.7 and 3.8 both remain true as stated under assumption (A1g).

The amplitude equation in Chapter 2 was already treated in this more general context; in particular, Proposition 2.3 applies to these more general materials.

The construction and estimation of the approximate solution in chapter 4 goes through almost without change. In the definition of the coefficients A_α as in (4.6),

$$(6.1) \quad A_\alpha(v) = A_\alpha(0) + L_\alpha(v) + Q_\alpha(v),$$

only the term $Q_\alpha(v)$ changes; it must now be defined as the sum of all terms in the expansion of $A_\alpha(v)$ that are quadratic or of higher order. Similarly, in the definition of the boundary function $h(v)$ as in (4.10),

$$(6.2) \quad h(v) = \ell(v) + q(v) + c(v),$$

only the term $c(v)$ changes; it must now be defined as the sum of all terms in the expansion of $h(v)$ that are cubic or of higher order. We observe that these changes have no effect on the interior and boundary profile equations (4.12) and (4.13). Thus, the construction of the σ_j and τ_j and the building block estimates are unaffected.

However, these changes do affect the expressions for the interior and boundary error profiles $F_a(t, x, \theta, z)$ (4.17) and $h_a(t, x_1, \theta)$ (4.18), which respectively contain the terms

$$\sum_{|\alpha|=2} Q_\alpha(D_\varepsilon u_a) D_\varepsilon^\alpha u_a \text{ and } c(D_\varepsilon u_a).$$

Under assumption (A1) these terms were estimated in Proposition 5.11(c) and in the third estimate of Proposition 5.14. The estimates remain true as stated under assumption (A1g), but a small change is needed in the proofs. For example, in the proof of Proposition 5.11(c), instead of using Corollary 3.24(a) for products, one should use Proposition 3.25(c) for analytic functions $f(u)$.¹ The cubic and higher order

¹Here we use the analyticity assumption on $W(E)$ in (A1g).

terms in $Q_\alpha(v)$ are readily seen to make contributions to the estimate that are negligible compared to the quadratic terms.

It remains to check that properties (P1)-(P7) of section 3.1.1 continue to hold under the assumption (A1g). This is clear for (P1)-(P4), since the problem (P^0, B^0) is unchanged.² The verification of (P5), given in [ST88], p. 283 for the SVK system, is based on showing that the Lopatinskii determinant $\det b^+(v, z)$ as in (3.22) is real for $\gamma = 0$. The argument is based on symmetry properties of the coefficient matrices $A_\alpha(v)$ defining P^v (3.16), which arise as consequences of the fact that these matrices are derived from a stored energy function $W(\nabla u)$. For example, the argument uses the fact that the (α, β) component of $A_{(1,1)}(v)$ is given by

$$(6.3) \quad c_{\alpha 1 \beta 2}(v) + c_{\alpha 2 \beta 1}(v), \text{ where } c_{\alpha j \beta \ell}(v) = \frac{\partial^2 W(v)}{\partial u_{\alpha, j} \partial u_{\beta, \ell}},$$

and thus we have the matrix equalities

$$(6.4) \quad (c_{\alpha 2 \beta 1}(v)) = (c_{\beta 1 \alpha 2}(v)) = (c_{\alpha 1 \beta 2}(v))^t.$$

This property is clearly unaffected by the passage from (A1) to (A1g). Similarly, the verification of (P6) and (P7) on p. 286 of [ST88] for the SVK system continues to hold under assumption (A1g) because the matrices $A_\alpha(v)$ are constructed from derivatives of $W(v)$.

6.2 Extension to wavetrains.

Next we summarize how the method used in this paper to treat surface pulses extends to treat surface wavetrains. We explain in Remark 6.1 that another method is available for wavetrains that should yield more detailed qualitative information about the solutions.

We restrict attention to the 2D case. The boundary data (3.6) is now given by $\varepsilon^2 G(t, x_1, \theta)$, where $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Writing $G(t, x_1, \theta) = \sum_{n \in \mathbb{Z}} G^n(t, x_1) e^{in\theta}$, we assume $G^0 = 0$.

We work with exactly the same Sobolev spaces and singular operators as before, except now $\Omega = \{(t, x_1, x_2, \theta) \in \mathbb{R}^3 \times \mathbb{T} : x_2 > 0\}$, $k \in \mathbb{Z}$, where k is the variable dual to θ , and integrals $\int \dots dk$ are replaced by sums over $k \in \mathbb{Z}$. A version of the singular calculus for wavetrains parallel to the calculus for pulses used here was developed in [CGW14b]; in fact the wavetrain calculus is “better behaved”, since θ now lies in a compact set.

Except for the changes just mentioned, the results of chapter 3 on the existence of exact solutions hold with identical statements and proofs. The main changes in the treatment of wavetrains occur in the construction of approximate solutions. Unlike a pulse a wavetrain has a well-defined mean, and in the construction of approximate solutions one has to separate out the initial boundary value problems satisfied by the means of profiles. As in (4.2) we look for the profile of the approximate solution in the form $u_a = \varepsilon^2 u_\sigma + \varepsilon^3 u_\tau$, where now

$$(6.5) \quad \begin{aligned} u_\sigma(t, x, \theta, z) &= \sum_{n \in \mathbb{Z}} u_\sigma^n(t, x, z) e^{in\theta} \\ u_\sigma^0(t, x, z) &= \underline{u}_\sigma^0(t, x) + u_\sigma^{0,*}(t, x, z) \text{ with } \lim_{z \rightarrow \infty} u_\sigma^{0,*} = 0, \end{aligned}$$

²Properties (P1)-(P7) were verified for the SVK system in [ST88] under the assumption $\mu > 0, \lambda > 0$. The discussion in chapter 2 and an inspection of [ST88] shows that her arguments apply just as well under the assumption $\mu > 0, \lambda + \mu > 0$.

and a similar expansion holds for u_τ . We define $\underline{u}_\sigma^0(t, x)$ to be the *mean* of u_σ .

We seek u_σ and u_τ as solutions of the interior and boundary profile equations (4.12), (4.13) of the form

$$(6.6) \quad u_\sigma(t, x, \theta, z) = \sum_{j=1}^4 \sigma_j(t, x, \theta + \omega_j z) r_j \quad \text{and} \quad u_\tau(t, x, \theta, z) = \sum_{j=1}^4 \tau_j(t, x, \theta, z) r_j,$$

where the low frequency cutoff $\chi_\varepsilon(D_\theta)$ acting on the τ_j in (4.2) is now absent.

The determination of $u_\sigma - u_\sigma^0$ and of $u_\tau - u_\tau^0$ follows the procedure used in chapter 4 to determine the $\hat{\sigma}_j(t, x, k)$ and the $\hat{\tau}_j(t, x, k, z)$ for $k \neq 0$. In particular, the traces $\hat{\sigma}_j(t, x_1, 0, k)$, $k \neq 0$ are constant multiples of $\hat{w}(t, x_1, k)$, where $\hat{w}(t, x_1, k)$ is the solution of the amplitude equation (2.2). The building block estimates for $u_\sigma - u_\sigma^0$ and $u_\tau - u_\tau^0$ are readily seen to be the same as the estimates for u_σ and u_τ given in section 5.2, except for the change that every occurrence of p is replaced by the number 1. For example, parallel to the third estimate in Proposition 5.7 we now have instead

$$(6.7) \quad \langle \Lambda^{\frac{1}{2}} \partial_{zz} (u_\tau - u_\tau^0) \rangle_{m, \gamma} \lesssim \frac{C(m+3)}{\varepsilon^{1/2}} + C(m+3 + \frac{1}{2}).$$

Additional work beyond what is given in chapter 4 is needed to determine u_σ^0 and u_τ^0 , and this work has essentially been carried out in Chapter 2 of the thesis of Marcou [Mar10]. In that chapter Marcou considers a simplified version of the elasticity equations in which a number of the nonlinear terms, including quadratic terms, have been thrown away. Nevertheless, her method does apply directly to the full equations and shows that $u_\sigma^0 = 0$ and that $u_\tau^{0,*}$ is given by a simple integral formula.³ Since we are only concerned with solving the profile equations (4.12), (4.13), we have no need to construct the mean $\underline{u}_\tau^0(t, x)$; we can set it equal to 0. To complete the building block estimates one needs to combine estimates like (6.7) with corresponding estimates of $u_\tau^0 = u_\tau^{0,*}$. Such estimates follow readily from the integral formulas given in [Mar10]. One finds in each case that the estimate already obtained (or a better one) continues to hold for u_τ^0 . Thus, for example, (6.7) continues to hold when $u_\tau - u_\tau^0$ is replaced by u_τ .

The forcing estimates of section 5.3 are adapted to the wavetrain case by using the building block estimates, modified in the way we have just described, as before, and observing that terms involving positive powers of p on the right in the estimates of section 5.3 always arise from forcing terms in which the low frequency cutoff $\chi_\varepsilon(D_\theta)$ appears. Such p -terms are therefore absent now. Thus, for example, in the estimates of Corollary 5.12, only the term $\sqrt{\varepsilon}$ should appear on the right. In estimates (c) and (d) of Corollary 5.15 the right sides are now ε^2 and $\varepsilon^{3/2}$, respectively. Similarly, the last term on the right in the estimate of Proposition 5.18 is now ε , and on the right side of Proposition 5.22,

$$(6.8) \quad \frac{1}{\sqrt{\gamma}} \left(\frac{p^2}{\sqrt{\varepsilon}} + \sqrt{p} + \frac{\varepsilon}{p} \right) + \sqrt{\varepsilon}$$

should be replaced by $\sqrt{\varepsilon}$. The same substitution should be made in the estimate of Corollary 5.23, and this yields the rate of convergence

$$(6.9) \quad E_{m,T}(w) \lesssim \sqrt{\varepsilon}$$

in Corollary 5.24.

³This formula is provided in equations (2.6.5), (2.6.6) of [Mar10], where $k = 3$ and H_{k-1}^0, K_{k-1}^0 should be replaced by the analogous (quadratic) terms which appear in the full elasticity equations.

Remark 6.1. *We believe it should be possible to construct arbitrarily high order approximate solutions to the elasticity equations in the wavetrain case, and then to show that these approximate solutions are close to exact solutions on an ε -independent time interval by some variant of the method introduced by [Guè93]. With high order approximate solutions it should not be necessary to consider singular problems; one attempts to solve directly the error equation satisfied by the difference between exact and approximate solutions.*

Such a result would yield more precise information than the result for wavetrains described above. For example, the second chapter of [Mar10] indicates that one should expect “internal rectification” to occur in the leading corrector; in other words one expects u_τ to have in general a nonzero mean $\underline{u}_\tau^0(t, x)$. That property cannot be detected by our result for wavetrains, but could be detectable using this alternative method. As we have already mentioned, there is no hope of constructing high order approximate solutions in the case of pulses.

6.3 The case of dimensions $d \geq 3$.

In this section we restrict attention to the Saint Venant-Kirchoff model (3.1). For $d \geq 3$ the solution of the amplitude equation and the construction of the approximate solution goes through as in $d = 2$. However, there is a serious difficulty in $d \geq 3$ in constructing Kreiss symmetrizers for the linearized problem. As we explain below, the linearized problem has characteristics of variable multiplicity; moreover, these include characteristics which fail to be *algebraically regular* in the sense of [MZ05], and which are at the same time *glancing*.

We now let $\Omega = \{(t, x) : x = (x_1, \dots, x_d), x_d > 0\}$ and denote dual variables by $(\sigma, \xi) = (\sigma, \xi_1, \dots, \xi_d)$. We regard ξ as a column vector, so $\xi \xi^t$ is the $d \times d$ matrix $(a_{ij}) = (\xi_i \xi_j)$. Writing $\phi(t, x) = x + U(t, x)$ as in (3.4) and setting $\theta(t, x) = \nabla U$, one obtains for the $d \times d$ principal matrix symbol of the interior equation in (3.1) linearized at $\nabla \phi = I + \theta$:⁴

$$(6.10) \quad \begin{aligned} P(\theta, \sigma, \xi) = & -\sigma^2 I + (\lambda + \mu)(I + \theta)\xi \xi^t (I + \theta)^t + \mu |\xi|^2 (I + \theta)(I + \theta^t) + \\ & [(\lambda \operatorname{tr} E(I + \theta) - \mu) |\xi|^2 + \mu \xi^t (I + \theta)^t (I + \theta) \xi] I. \end{aligned}$$

When $\theta = 0$, we have $E(I) = 0$ and this reduces to $-\sigma^2 I + (\lambda + \mu)\xi \xi^t + \mu |\xi|^2 I$, a system for which the characteristic equation is

$$(6.11) \quad (\sigma^2 - \mu |\xi|^2)^{d-1} (\sigma^2 - (\lambda + 2\mu) |\xi|^2) = 0.$$

The roots $\sigma = \pm \sqrt{\mu} |\xi|$, $\sigma = \pm \sqrt{\lambda + 2\mu} |\xi|$ have multiplicities $d - 1$ and 1 respectively. Here the factors $\sigma - c |\xi|$ occurring in (6.11) have double roots in ξ_d when $\sigma = c |(\xi_1, \dots, \xi_{d-1}, 0)|$. We then refer to $(\theta = 0, \sigma, \xi_1, \dots, \xi_{d-1}, 0)$ as a *glancing mode*.

Proposition 6.2. *Assume $d = 3$. Let $\underline{\xi} = (1, 0, 0)$ and consider the glancing mode $(\theta = 0, \sqrt{\mu}, \underline{\xi})$, where $\sigma = \sqrt{\mu}$ is a root of multiplicity two of $P(0, \sigma, \underline{\xi}) = 0$. For θ near 0 the double root splits into roots $\sigma_j(\theta)$ satisfying $P(\theta, \sigma_j(\theta), \underline{\xi}) = 0$, where*

$$(6.12) \quad \sigma_j(\theta) = \sqrt{\mu} + n_j(\theta), \quad j = 1, 2$$

and the $n_j(\theta)$ are distinct, continuous functions of θ which fail to be C^1 at $\theta = 0$.

⁴Recall that $E(I + \theta) = \frac{1}{2}(\theta^t \theta + \theta + \theta^t)$.

Proof. The only things to check are that the n_j are distinct and fail to be C^1 at $\theta = 0$. We choose θ so that $\theta_{ij} = 0$, except for the θ_{23} and θ_{33} components which vary near 0. Write

$$(6.13) \quad P(\theta, \sigma, \xi) + \sigma^2 I = M(\theta, \xi) + S(\theta, \xi)I,$$

where $S(\theta, \xi)I$ is given by the second line of (6.10). For this choice of θ , the matrix $M(\theta, \underline{\xi}) + S(\theta, \underline{\xi})I$ has the form

$$(6.14) \quad \begin{pmatrix} \lambda + 2\mu + S & 0 & 0 \\ 0 & \mu + S + \mu\theta_{23}^2 & \mu\theta_{23} \\ 0 & \mu\theta_{23} & \mu + S + \mu(\theta_{33}^2 + \theta_{33}) \end{pmatrix}.$$

The eigenvalues of the lower 2×2 block are computed to be

$$(6.15) \quad \beta_{\pm} = (\mu + S(\theta, \underline{\xi})) + \frac{\mu}{2}(\theta_{23}^2 + \theta_{33}^2 + \theta_{33}) \mp \frac{\mu}{2}\sqrt{[\theta_{23}^2 - (\theta_{33}^2 + \theta_{33})]^2 + 4\theta_{23}^2}.$$

The argument of the square root is $\theta_{33}^2 + 4\theta_{23}^2 + O(|\theta|^3)$, so the square root term fails to be C^1 in θ near $\theta = 0$. □

Remark 6.3. 1. We refer to [MZ05] for the precise definition of algebraically regular multiple characteristics. Roughly, a characteristic mode of multiplicity $m \geq 2$ is algebraically regular when it splits smoothly with respect to small changes of parameters. Thus, the glancing mode in Proposition 6.2 fails to be algebraically regular.

2. The state of the art in Kreiss symmetrizers is represented by the papers [Mét00] and [MZ05]. The first paper constructs smooth Kreiss symmetrizers for symmetric hyperbolic systems, including high order systems, but only for systems with characteristics of constant multiplicity. Since we are dealing here with characteristics of variable multiplicity, the first paper does not apply.

The second paper treats only first order hyperbolic systems, and constructs smooth Kreiss symmetrizers in certain situations where characteristics of variable multiplicity are present. The extension of the results of [MZ05] to higher order systems appears to be nontrivial, and has not yet been done as far as we know. Moreover, the results of [MZ05] do not appear to cover first order problems with characteristic modes of multiplicity $m \geq 2$ that fail to be algebraically regular and are at the same time glancing. Thus, Proposition 6.2 indicates that even if we had an extension of [MZ05] to higher order systems, it might not apply to the kinds of variable multiplicity characteristics that we encounter in this problem. If Theorems 3.7 and 3.8 do actually extend to $d \geq 3$, it appears that further development of the theory of smooth Kreiss symmetrizers will be necessary to prove such an extension by our methods.

Appendix A

Singular pseudodifferential calculus for pulses

In this Appendix, we summarize the parts of the singular pulse calculus constructed in [CGW14b] that are needed in Chapter 3. The calculus of [CGW14b] was constructed with applications to first-order hyperbolic systems in mind. The systems of elasticity equations considered in this work are second-order, so some extensions of the calculus are needed; these are given in section A.5 below.

First we define the singular Sobolev spaces used to describe mapping properties. The variable in \mathbb{R}^{d+1} is denoted (x, θ) , $x \in \mathbb{R}^d$, $\theta \in \mathbb{R}$, and the associated frequency is denoted (ξ, k) . We consider a fixed vector $\beta \in \mathbb{R}^d \setminus \{0\}$. Then for $s \in \mathbb{R}$ and $\varepsilon \in (0, 1]$, the anisotropic Sobolev space $H^{s,\varepsilon}(\mathbb{R}^{d+1})$ is defined by

$$H^{s,\varepsilon}(\mathbb{R}^{d+1}) := \left\{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) / \widehat{u} \in L^2_{\text{loc}}(\mathbb{R}^{d+1}) \text{ and } \int_{\mathbb{R}^{d+1}} \left(1 + \left| \xi + \frac{k\beta}{\varepsilon} \right|^2 \right)^s |\widehat{u}(\xi, k)|^2 d\xi dk < +\infty \right\}.$$

Here \widehat{u} denotes the Fourier transform of u on \mathbb{R}^{d+1} . The space $H^{s,\varepsilon}(\mathbb{R}^{d+1})$ is equipped with the family of norms

$$\forall \gamma \geq 1, \quad \forall u \in H^{s,\varepsilon}(\mathbb{R}^{d+1}), \quad \|u\|_{H^{s,\varepsilon,\gamma}}^2 := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \left(\gamma^2 + \left| \xi + \frac{k\beta}{\varepsilon} \right|^2 \right)^s |\widehat{u}(\xi, k)|^2 d\xi dk.$$

When m is an integer, the space $H^{m,\varepsilon}(\mathbb{R}^{d+1})$ coincides with the space of functions $u \in L^2(\mathbb{R}^{d+1})$ such that the derivatives, in the sense of distributions,

$$\left(\partial_{x_1} + \frac{\beta_1}{\varepsilon} \partial_\theta \right)^{\alpha_1} \dots \left(\partial_{x_d} + \frac{\beta_d}{\varepsilon} \partial_\theta \right)^{\alpha_d} u, \quad \alpha_1 + \dots + \alpha_d \leq m,$$

belong to $L^2(\mathbb{R}^{d+1})$. In the definition of the norm $\|\cdot\|_{H^{m,\varepsilon,\gamma}}$, one power of γ counts as much as one derivative.

A.1 Symbols

Our singular symbols are built from the following sets of classical symbols.

Definition A.1. Let $\mathcal{O} \subset \mathbb{R}^N$ be an open subset that contains the origin. For $m \in \mathbb{R}$ we let $\mathbf{S}^m(\mathcal{O})$ denote the class of all functions $\sigma : \mathcal{O} \times \mathbb{R}^d \times [1, \infty) \rightarrow \mathbb{C}^{N \times N}$, $N \geq 1$, such that σ is \mathcal{C}^∞ on $\mathcal{O} \times \mathbb{R}^d$ and for all compact sets $K \subset \mathcal{O}$:

$$\sup_{v \in K} \sup_{\xi' \in \mathbb{R}^d} \sup_{\gamma \geq 1} (\gamma^2 + |\xi|^2)^{-(m-|\nu|)/2} |\partial_v^\alpha \partial_{\xi'}^\nu \sigma(v, \xi, \gamma)| \leq C_{\alpha, \nu, K}.$$

Let $\mathcal{C}_b^k(\mathbb{R}^{d+1})$, $k \in \mathbb{N}$, denote the space of continuous and bounded functions on \mathbb{R}^{d+1} , whose derivatives up to order k are continuous and bounded. Let us first define the singular symbols.

Definition A.2 (Singular symbols). Fix $\beta \in \mathbb{R}^d \setminus 0$, let $m \in \mathbb{R}$, and let $n \in \mathbb{N}$. Then we let S_n^m denote the set of families of functions $(a^{\varepsilon, \gamma})_{\varepsilon \in (0, 1], \gamma \geq 1}$ that are constructed as follows:

$$(A.1) \quad \forall (x, \theta, \xi, k) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}, \quad a^{\varepsilon, \gamma}(x, \theta, \xi, k) = \sigma \left(\varepsilon V(x, \theta), \xi + \frac{k\beta}{\varepsilon}, \gamma \right),$$

where $\sigma \in \mathbf{S}^m(\mathcal{O})$, V belongs to the space $\mathcal{C}_b^n(\mathbb{R}^{d+1})$ and where furthermore V takes its values in a convex compact subset K of \mathcal{O} that contains the origin (for instance K can be a closed ball centered round the origin).

All results below extend to the case where in place of a function V that is independent of ε , the representation (A.1) is considered with a function V_ε that is indexed by ε , provided that we assume that all functions $\varepsilon V_\varepsilon$ take values in a fixed convex compact subset K of \mathcal{O} that contains the origin, and $(V_\varepsilon)_{\varepsilon \in (0, 1]}$ is a bounded family of $\mathcal{C}_b^n(\mathbb{R}^{d+1})$.

A.2 Definition of operators and action on Sobolev spaces

To each symbol $a = (a^{\varepsilon, \gamma})_{\varepsilon \in (0, 1], \gamma \geq 1} \in S_n^m$ given by the formula (A.1), we associate a singular pseudodifferential operator $a_D^{\varepsilon, \gamma}$, with $\varepsilon \in (0, 1]$ and $\gamma \geq 1$, whose action on a function $u \in \mathcal{S}(\mathbb{R}^{d+1}; \mathbb{C}^N)$ is defined by

$$(A.2) \quad a_D^{\varepsilon, \gamma} u(x, \theta) := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} e^{i(\xi \cdot x + k\theta)} \sigma \left(\varepsilon V(x, \theta), \xi + \frac{k\beta}{\varepsilon}, \gamma \right) \widehat{u}(\xi, k) \, d\xi \, dk.$$

Let us briefly note that for the Fourier multiplier $\sigma(v, \xi, \gamma) = i\xi_1$, the corresponding singular operator is $\partial_{x_1} + (\beta_1/\varepsilon)\partial_\theta$. We now describe the action of singular pseudodifferential operators on Sobolev spaces. Detailed proofs of all results stated below can be found in [CGW14b].

Remark A.3. We will usually write $a_D u$ instead of $a_D^{\varepsilon, \gamma} u$ for the function defined in (A.2). Also we often write X for $\xi + k\beta/\varepsilon$.

Proposition A.4. Let $n \geq d+1$, and let $a \in S_n^m$ with $m \leq 0$. Then a_D in (A.2) defines a bounded operator on $L^2(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation (A.1), such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|a_D u\|_0 \leq \frac{C}{\gamma^{|m|}} \|u\|_0.$$

The constant C in Proposition A.4 depends uniformly on the compact set in which V takes its values and on the norm of V in \mathcal{C}_b^{d+1} . For operators defined by symbols of order $m > 0$, we have:

Proposition A.5. *Let $n \geq d + 1$, and let $a \in S_n^m$ with $m > 0$. Then a_D in (A.2) defines a bounded operator from $H^{m,\varepsilon}(\mathbb{R}^{d+1})$ to $L^2(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation (A.1), such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds*

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|a_D u\|_0 \leq C \|u\|_{H^{m,\varepsilon,\gamma}}.$$

The next proposition describes the smoothing effect of operators of order -1 .

Proposition A.6. *Let $n \geq d + 2$, and let $a \in S_n^{-1}$. Then a_D in (A.2) defines a bounded operator from $L^2(\mathbb{R}^{d+1})$ to $H^{1,\varepsilon}(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation (A.1), such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds*

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|a_D u\|_{H^{1,\varepsilon,\gamma}} \leq C \|u\|_0.$$

Remark A.7. In applications of the pulse calculus, we verify the hypothesis that for V as in (A.1), $V \in \mathcal{C}_b^n(\mathbb{R}^{d+1})$, by showing $V \in H^s(\mathbb{R}^{d+1})$ for some $s > \frac{d+1}{2} + n$.

A.3 Adjoints and products

For proofs of the following results we refer again to [CGW14b]. The two first results deal with adjoints of singular pseudodifferential operators while the last two deal with products.

Proposition A.8. *Let $a = \sigma(\varepsilon V, X, \gamma) \in S_n^0$, $n \geq 2(d+1)$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let a^* denote the conjugate transpose of the symbol a . Then a_D and $(a^*)_D$ act boundedly on L^2 and there exists a constant $C \geq 0$ such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds*

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|(a_D)^* u - (a^*)_D u\|_0 \leq \frac{C}{\gamma} \|u\|_0.$$

If $n \geq 3d + 3$, then for another constant C , there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|(a_D)^* u - (a^*)_D u\|_{H^{1,\varepsilon,\gamma}} \leq C \|u\|_0,$$

uniformly in ε and γ .

Proposition A.9. *Let $a = \sigma(\varepsilon V, X, \gamma) \in S_n^1$, $n \geq 3d + 4$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let a^* denote the conjugate transpose of the symbol a . Then a_D and $(a^*)_D$ map $H^{1,\varepsilon}$ into L^2 and there exists a family of operators $R^{\varepsilon,\gamma}$ that satisfies*

- there exists a constant $C \geq 0$ such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|R^{\varepsilon,\gamma} u\|_0 \leq C \|u\|_0,$$

- the following duality property holds

$$\forall u, v \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \langle a_D u, v \rangle_{L^2} - \langle u, (a^*)_D v \rangle_{L^2} = \langle R^{\varepsilon,\gamma} u, v \rangle_{L^2}.$$

In particular, the adjoint $(a_D)^*$ for the L^2 scalar product maps $H^{1,\varepsilon}$ into L^2 .

Proposition A.10. (a) Let $a, b \in S_n^0$, $n \geq 2(d+1)$, and suppose $b = \sigma(\varepsilon V, X, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then there exists a constant $C \geq 0$ such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|a_D b_D u - (ab)_D u\|_0 \leq \frac{C}{\gamma} \|u\|_0.$$

If $n \geq 3d+3$, then for another constant C , there holds

$$(A.3) \quad \begin{aligned} \forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad & \|a_D b_D u - (ab)_D u\|_{H^{1,\varepsilon,\gamma}} \leq C \|u\|_0, \\ & \|a_D b_D u - (ab)_D u\|_0 \leq C \|u\|_{H^{-1,\varepsilon,\gamma}}, \end{aligned}$$

uniformly in ε and γ .

(b) Let $a \in S_n^1, b \in S_n^0$ or $a \in S_n^0, b \in S_n^1$, $n \geq 3d+4$, and in each case suppose $b = \sigma(\varepsilon V, X, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then there exists a constant $C \geq 0$ such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|a_D b_D u - (ab)_D u\|_0 \leq C \|u\|_0.$$

The proof of the first estimate of (A.3) in [CGW14b] gives an explicit amplitude for the remainder $a_D b_D u - (ab)_D u$, and from this it is clear that the adjoint of $a_D b_D u - (ab)_D u$ has the same mapping property. Duality therefore implies the second estimate in (A.3).

Proposition A.11. Let $a \in S_n^{-1}, b \in S_n^1$, $n \geq 3d+4$, and suppose $b = \sigma(\varepsilon V, X, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then $a_D b_D$ defines a bounded operator on $H^{1,\varepsilon}$ and there exists a constant $C \geq 0$ such that for all $\varepsilon \in (0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|a_D b_D u - (ab)_D u\|_{H^{1,\varepsilon,\gamma}} \leq C \|u\|_0.$$

Our next result is Gårding's inequality, either for symbols of degree 0 or 1.

Theorem A.12. (a) Let $\sigma \in \mathbf{S}^0$ satisfy $\operatorname{Re} \sigma(v, \xi, \gamma) \geq C_K > 0$ for all v in a compact subset K of \mathcal{O} . Let now $a \in S_n^0$, $n \geq 2d+2$ be given by $a = \sigma(\varepsilon V, X, \gamma)$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$ and is valued in a convex compact subset K . Then for all $\delta > 0$, there exists γ_0 which depends uniformly on V , the constant C_K and δ , such that for all $\gamma \geq \gamma_0$ and all $u \in \mathcal{S}(\mathbb{R}^{d+1})$, there holds

$$\operatorname{Re} \langle a_D u; u \rangle_{L^2} \geq (C_K - \delta) \|u\|_0^2.$$

(b) Let $\sigma \in \mathbf{S}^1$ satisfy $\operatorname{Re} \sigma(v, \xi, \gamma) \geq C_K \langle \xi, \gamma \rangle$ for all v in a compact subset K of \mathcal{O} . Let now $a \in S_n^1$, $n \geq 3d+4$ be given by $a = \sigma(\varepsilon V, X, \gamma)$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$ and is valued in a convex compact subset K . Then for all $\delta > 0$, there exists γ_0 which depends uniformly on V , the constant C_K and δ , such that for all $\gamma \geq \gamma_0$ and all $u \in \mathcal{S}(\mathbb{R}^{d+1})$, there holds

$$\operatorname{Re} \langle a_D u; u \rangle_{L^2} \geq (C_K - \delta) \|u\|_{H^{1/2,\varepsilon,\gamma}}^2.$$

A.4 Extended calculus

At times we use a slight extension of the singular calculus. For given parameters $0 < \delta_1 < \delta_2 < 1$, we choose a cutoff $\chi^e(\xi, k\beta/\varepsilon, \gamma)$ such that

$$(A.4) \quad \begin{aligned} 0 &\leq \chi^e \leq 1, \\ \chi^e\left(\xi, \frac{k\beta}{\varepsilon}, \gamma\right) &= 1 \text{ on } \left\{ (\gamma^2 + |\xi|^2)^{1/2} \leq \delta_1 \left| \frac{k\beta}{\varepsilon} \right| \right\}, \\ \text{supp}\chi^e &\subset \left\{ (\gamma^2 + |\xi|^2)^{1/2} \leq \delta_2 \left| \frac{k\beta}{\varepsilon} \right| \right\}, \end{aligned}$$

and define a corresponding Fourier multiplier χ_D in the extended calculus by the formula (A.2) with $\chi^e(\xi, k\beta/\varepsilon, \gamma)$ in place of $\sigma(\varepsilon V, X, \gamma)$. Composition laws involving such operators are proved in [CGW14b], but here we need only the fact that part (a) of Proposition A.10 (composition of two zero order singular operators) holds when either a or b is replaced by an extended cutoff χ^e .

A.5 Commutator estimates

In the proofs of this section we ignore some constant factors such as powers of 2π . For a given amplitude $c(x, \theta, y, \omega, X, \gamma)$, $\widetilde{Op}(c)$ denotes the operator defined by

$$(A.5) \quad \widetilde{Op}(c)u(x, \theta) = \int e^{i(x-y)\xi + i(\theta-\omega)k} c(x, \theta, y, \omega, X, \gamma) u(y, \omega) dy d\omega d\xi dk.$$

Lemma A.13. *Let $r \geq 0$ and let $M_{e^{iy\eta + i\omega l}}$ denote the operator that multiplies $u(y, \omega)$ by $e^{iy\eta + i\omega l}$. Then*

$$(A.6) \quad \|\Lambda_D^r M_{e^{iy\eta + i\omega l}} u\|_0 \lesssim \|\Lambda_D^r u\|_0 + \langle \eta \rangle^r \|u\|_0 + \left\langle \frac{l}{\varepsilon} \right\rangle^r \|u\|_0.$$

Proof. This follows directly from the definition of the norm on the left and

$$(A.7) \quad \left\langle \xi + \eta + \frac{(k+l)\beta}{\varepsilon}, \gamma \right\rangle^r \lesssim \langle X, \gamma \rangle^r + \langle \eta \rangle^r + \left\langle \frac{l}{\varepsilon} \right\rangle^r.$$

□

Proposition A.14. *i.) Set $x = (x_0, x'') = (t, x'') \in \mathbb{R}^{d+1}$ and $y = (y_0, y'') \in \mathbb{R}^{d+1}$, let $a(X, \gamma)$ be a singular symbol of order $r \geq 1$, and let $\chi(x_0) \in H^\infty(\mathbb{R})$. Then for $m \in \{0, 1, 2, \dots\}$ we have*

$$(A.8) \quad \begin{aligned} (a) \quad &|[a_D, \chi]u|_{H^\eta} \lesssim |\Lambda_D^{r-1} u|_{H^\eta} \\ (b) \quad &|[a_D, \chi]u|_{m, \gamma} \lesssim |\Lambda_D^{r-1} u|_{m, \gamma}. \end{aligned}$$

ii.) The same estimates hold when $\chi(x) \in H^\infty(\mathbb{R}^{d+1})$.

iii.) When $\chi(x, \theta) \in H^\infty(\mathbb{R}^{d+1} \times \mathbb{R})$, we have¹

$$(A.9) \quad |[a_D, \chi]u|_{m, \gamma} \lesssim |\Lambda_D^{r-1} u|_{m, \gamma} + \frac{|u|_{m, \gamma}}{\varepsilon^{r-1}}.$$

¹It is easy to give a version of this proposition for $\chi \in H^s$ for s large enough.

Proof. 1. It suffices to prove (a) for $m = 0$, since derivatives with respect to (x'', θ) commute with the commutator and $\partial_t^k([a_D, \chi]u)$ is a linear combination of terms of the form

$$(A.10) \quad [a_D, \chi^{(k_1)}] \partial_t^{k_2} u, \text{ where } k_1 + k_2 = k.$$

Replacing u by $e^{-\gamma t}u$ then yields (b).

2. The function $[a_D, \chi]u$ may be written as (some constant factors are ignored here and below)

$$(A.11) \quad \int e^{i(x-y)\xi + i(\theta-\omega)k} a(X, \gamma) [\chi(x_0) - \chi(y_0)] u(y, \omega) dy d\omega d\xi dk = \\ i \int e^{i(x-y)\xi + i(\theta-\omega)k} \partial_{\xi_0} a(X, \gamma) \left(\int_0^1 \chi'(y_0 + s(x_0 - y_0)) ds \right) u(y, \omega) dy d\omega d\xi dk.$$

Write $\chi'(y_0 + s(x_0 - y_0)) = \int \widehat{\chi}'(\eta_0) e^{i\eta_0(y_0 + s(x_0 - y_0))} d\eta_0$, let $\widetilde{Op}(A_{s, \eta_0})$ be the operator associated to the amplitude of order $r - 1$ given by

$$(A.12) \quad e^{i\eta_0 s x_0} \cdot \partial_{\xi_0} a(X, \gamma) \cdot e^{i\eta_0(1-s)y_0},$$

and observe that

$$(A.13) \quad [a_D, \chi] = \int_0^1 \int \widetilde{Op}(A_{s, \eta_0}) \widehat{\chi}'(\eta_0) d\eta_0 ds.$$

We have

$$(A.14) \quad \widetilde{Op}(A_{s, \eta_0}) = M_{e^{i\eta_0 s x_0}} \circ (\partial_{\xi_0} a)_D \circ M_{e^{i\eta_0(1-s)y_0}},$$

where M denotes multiplication. Thus, by Lemma A.13

$$(A.15) \quad |\widetilde{Op}(A_{s, \eta_0})u|_{L^2} \lesssim |\Lambda_D^{r-1}(e^{i\eta_0(1-s)y_0}u)|_{L^2} \lesssim |\Lambda_D^{r-1}u|_{L^2} + |\eta_0|^{(r-1)}|u|_{L^2}.$$

Together with (A.13) this gives the estimate (a) when $m = 0$.

3. The proofs of (ii) and (iii) involve only minor changes. □

Proposition A.15. For $j = 1, 0$ let $a^j = \sigma^j(\varepsilon V, X, \gamma) \in S_n^j$, $n \geq 3d + 4$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let $\phi(X, \gamma)$ be a singular symbol of order 0. We have

$$(A.16) \quad (a) \quad \|[a_D^1, \phi_D]u\|_0 \lesssim \|u\|_0 \\ (b) \quad \|[a_D^0, \phi_D]u\|_0 \lesssim \|\Lambda_D^{-1}u\|_0.$$

Proof. Part (a) follows directly from Proposition A.10(b) and part (b) follows from Proposition A.10(a). □

Proposition A.16. Let the singular symbols a^j have the product form $a^j = \sigma(\varepsilon V)b^j(X, \gamma) \in S_n^j$, $n \geq 3d + 4$, $j = 0, 1$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 2$.

$$(A.17) \quad (a) \quad \|[a_D^1, \Lambda_D^{\frac{1}{2}}]u\|_0 \lesssim \|\Lambda_D^{\frac{1}{2}}u\|_0 + \|u/\sqrt{\varepsilon}\|_0 \\ (b) \quad \|\Lambda_D^{\frac{1}{2}}[a_D^1, \Lambda_D^{\frac{1}{2}}]u\|_0 \lesssim \|\Lambda_D u\|_0 + \|\Lambda_D^{\frac{1}{2}}u/\sqrt{\varepsilon}\|_0 + \|u/\varepsilon\|_0 \\ (c) \quad \|[a_D^1, \Lambda_D^{-\frac{1}{2}}]u\|_0 \lesssim \gamma^{-\frac{1}{2}}\|u\|_0 \\ (d) \quad \|[a_D^0, \Lambda_D^{\frac{1}{2}}]u\|_0 \lesssim \gamma^{-\frac{1}{2}}\|u\|_0.$$

Proof. **1. Proof of (a).** We write²

$$(A.18) \quad \Lambda_D^{\frac{1}{2}} a_D = \Lambda_D^{\frac{1}{2}} [(a^*)_D^* + R],$$

where the operator R is given by the amplitude

$$(A.19) \quad \begin{aligned} & (\sigma(\varepsilon V(x, \theta) - \sigma(\varepsilon V(y, \omega))) b(X, \gamma) = r_1 + r_2, \text{ with} \\ r_1 &= \left(\int_0^1 d_v \sigma(\varepsilon V(y + s(x - y), \theta)) \varepsilon \partial_y V(y + s(x - y), \theta) ds \right) (x - y) b(X, \gamma) \\ r_2 &= \left(\int_0^1 d_v \sigma(\varepsilon V(y, \omega + s(\theta - \omega))) \varepsilon \partial_\omega V(y, \omega + s(\theta - \omega)) ds \right) (\theta - \omega) b(X, \gamma). \end{aligned}$$

The operator $a_D \Lambda_D^{\frac{1}{2}} - \Lambda_D^{\frac{1}{2}} (a^*)_D^*$ is given by the amplitude

$$(A.20) \quad [(\sigma(\varepsilon V(x, \theta)) - \sigma(\varepsilon V(y, \omega))) b(X, \gamma)] \Lambda_D^{\frac{1}{2}} = (r_1 + r_2) \Lambda_D^{\frac{1}{2}}.$$

Thus, we have

$$(A.21) \quad [a_D, \Lambda_D^{\frac{1}{2}}] = \widetilde{Op} \left((r_1 + r_2) \Lambda_D^{\frac{1}{2}} \right) - \Lambda_D^{\frac{1}{2}} \circ \widetilde{Op}(r_1 + r_2).$$

2. We focus on the “worst” terms in (A.21), those involving r_2 . Integration by parts with respect to k using $\partial_k (e^{i(x-y)\xi + i(\theta-\omega)k}) = i(\theta - \omega) e^{i(x-y)\xi + i(\theta-\omega)k}$ shows that

$$(A.22) \quad \widetilde{Op} \left(r_2 \Lambda_D^{\frac{1}{2}} \right) = \widetilde{Op}(f), \text{ where } f(x, \theta, y, \omega, X, \gamma) = \int_0^1 F(y, \omega + s(\theta - \omega)) ds \cdot c(X, \gamma),$$

with c of order $\frac{1}{2}$ and $F(y, \omega)$ satisfying

$$(A.23) \quad \int \langle \eta, l \rangle |\hat{F}(\eta, l)| d\eta dl \lesssim |F(y, \omega)|_{H^{s_0-1}(y, \omega)}.$$

We have

$$(A.24) \quad F(y, \omega + s(\theta - \omega)) = \int e^{ils\theta} \hat{F}(\eta, l) e^{i\eta y + il\omega(1-s)} d\eta dl.$$

Letting $\widetilde{Op}(A_{s, \eta, l})$ be the operator associated to the amplitude $e^{ils\theta} \hat{F}(\eta, l) c(X, \gamma) e^{i\eta y + il\omega(1-s)}$, we have with obvious notation

$$(A.25) \quad \widetilde{Op}(A_{s, \eta, l}) = M_{e^{ils\theta}} \hat{F}(\eta, l) c_D M_{e^{i\eta y + il\omega(1-s)}}.$$

We obtain

$$(A.26) \quad \begin{aligned} \|\widetilde{Op}(A_{s, \eta, l}) u\|_0 &= \|\hat{F}(\eta, l) c_D \Lambda_D^{-\frac{1}{2}} \Lambda_D^{\frac{1}{2}} M_{e^{i\eta y + il\omega(1-s)}} u\|_0 \lesssim \\ &|\hat{F}(\eta, l)| \|\Lambda_D^{\frac{1}{2}} M_{e^{i\eta y + il\omega(1-s)}} u\|_0 \lesssim |\hat{F}(\eta, l)| \left(\|\Lambda_D^{\frac{1}{2}} u\|_0 + \langle \eta \rangle^{\frac{1}{2}} \|u\|_0 + \langle l \rangle^{\frac{1}{2}} \frac{\|u\|_0}{\sqrt{\varepsilon}} \right), \end{aligned}$$

²Here $(a^*)_D^*$ denotes the adjoint of $(a^*)_D$.

where the last inequality follows from Lemma A.13. With (A.22) and (A.23) this implies

$$(A.27) \quad \|\widetilde{Op}(r_2\Lambda^{\frac{1}{2}})u\|_0 \lesssim \left(\|\Lambda_D^{\frac{1}{2}}u\|_0 + \frac{\|u\|_0}{\sqrt{\varepsilon}} \right).$$

To see that the operator $\Lambda_D^{\frac{1}{2}}\widetilde{Op}(r_2)$ in (A.21) satisfies

$$(A.28) \quad \|\Lambda_D^{\frac{1}{2}}\widetilde{Op}(r_2)u\|_0 \lesssim \left(\|\Lambda_D^{\frac{1}{2}}u\|_0 + \frac{\|u\|_0}{\sqrt{\varepsilon}} \right),$$

we first decompose $\widetilde{Op}(r_2)$ into operators like those in (A.25), where now c_D in (A.25) has order zero. The estimate (A.28) then follows readily.

The terms involving r_1 in (A.21) are treated similarly, but satisfy slightly better estimates since, for example, one integrates by parts with respect to ξ instead of k to define the analogue of f in (A.22).

3. Proof of (b). With $\widetilde{Op}(A_{s,\eta,l})$ as in (A.25), two applications of Lemma A.13 yield

$$(A.29) \quad \|\Lambda^{\frac{1}{2}}\widetilde{Op}(A_{s,\eta,l})u\|_0 \lesssim |\hat{F}(\eta,l)| \left(\|\Lambda_D u\|_0 + \langle l \rangle^{\frac{1}{2}} \frac{\|\Lambda_D^{\frac{1}{2}}u\|_0}{\sqrt{\varepsilon}} + \langle \eta \rangle \|u\|_0 + \langle \eta \rangle^{\frac{1}{2}} \langle l \rangle^{\frac{1}{2}} \frac{\|u\|_0}{\sqrt{\varepsilon}} + \left\langle \frac{l}{\varepsilon} \right\rangle \|u\|_0 \right),$$

so (b) follows from (A.23).

4. The proof of (c) is similar to the proof of (b) but simpler. For example, the operators c_D that appear as in (A.25) but now in the decomposition of $\widetilde{Op}(r_2\Lambda^{-\frac{1}{2}})$ have order $-\frac{1}{2}$, so the right side of (A.27) is replaced by $\gamma^{-\frac{1}{2}}\|u\|_0$. The proof of (d) is almost the same as that of (c). \square

Proposition A.17. *Let the singular symbols a^j have the product form $a^j = \sigma^j(\varepsilon V)b^j(X, \gamma) \in S_n^j$, $n \geq 3d + 4$, $j = 0, 1$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 2$.*

$$(A.30) \quad \|\Lambda_D^{\frac{1}{2}}[a_D^1, a_D^0]u\|_0 \lesssim \|\Lambda^{\frac{1}{2}}u\|_0 + \|u/\sqrt{\varepsilon}\|_0.$$

Proof. Using a classical “**“-argument” to analyze compositions³, we obtain

$$(A.31) \quad \begin{aligned} [a_D^1, a_D^0] &= \widetilde{Op} [a^1(\varepsilon V(x, \theta), X, \gamma)(a^0(\varepsilon V(x, \theta), X, \gamma) - a^0(\varepsilon V(y, \omega), X, \gamma))] - \\ &\widetilde{Op} [a^0(\varepsilon V(x, \theta), X, \gamma)(a^1(\varepsilon V(x, \theta), X, \gamma) - a^1(\varepsilon V(y, \omega), X, \gamma))] + \\ &a_D^1 \widetilde{Op} [a^0(\varepsilon V(x, \theta), X, \gamma) - a^0(\varepsilon V(y, \omega), X, \gamma)] - a_D^0 \widetilde{Op} [a^1(\varepsilon V(x, \theta), X, \gamma) - a^1(\varepsilon V(y, \omega), X, \gamma)]. \end{aligned}$$

The product form of the symbols allows us to analyze these operators using Fourier decompositions as in step 2 of the proof of Proposition A.16. The result then follows by application of Lemma A.13. \square

We also need an estimate like (A.30) for symbols that are *not* of product form.

³See the proof of Proposition 11 of [CGW14b], for example.

Corollary A.18. *Let the singular symbols a^j have the form $a^j = \sigma^j(\varepsilon V, X, \gamma) \in S_n^j$, $n \geq 3d + 4$, $j = 0, 1$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 2$, and where $\sigma^j(w, \xi, \gamma)$ is homogeneous of degree j in (ξ, γ) . Then*

$$(A.32) \quad \|\Lambda_D^{\frac{1}{2}}[a_D^1, a_D^0]u\|_0 \lesssim \|\Lambda^{\frac{1}{2}}u\|_0 + \|u/\sqrt{\varepsilon}\|_0.$$

Proof. One can reduce to the product case considered above by expanding the symbols $\sigma^j(w, \xi, \gamma)$ in terms of spherical harmonics. \square

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