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► **To cite this version:**

Julien Grivaux, Pascal Hubert. Loci in strata of meromorphic differentials with fully degenerate Lyapunov spectrum. *Journal of modern dynamics*, 2014, 8 (1), pp.61-73. 10.3934/jmd.2014.8.61 . hal-01299437

HAL Id: hal-01299437

<https://hal.science/hal-01299437>

Submitted on 18 Nov 2019

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LOCI IN STRATA OF MEROMORPHIC QUADRATIC DIFFERENTIALS WITH FULLY DEGENERATE LYAPUNOV SPECTRUM

J. GRIVAUX AND P. HUBERT

ABSTRACT. We construct explicit closed $GL(2; \mathbb{R})$ -invariant loci in strata of meromorphic quadratic differentials of arbitrary large dimension with fully degenerate Lyapunov spectrum. This answers a question of Forni-Matheus-Zorich.

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1. INTRODUCTION

Lyapunov exponents of the Teichmüller flow have been studied a lot since the work of Zorich ([Zor97], [Zor99]) and Forni [For02]. Their understanding is important for applications to the dynamics of interval exchange transformations and polygonal billiards. A big breakthrough is the Eskin-Kontsevich-Zorich formula for the sum of positive Lyapunov exponents [EKZ11b]. Given a $SL(2; \mathbb{R})$ -invariant suborbifold of a stratum of quadratic differentials, they relate the sum $\lambda_1 + \dots + \lambda_g$ to the Siegel-Veech constant of the invariant locus¹.

By a theorem of Kontsevich and Forni, the sum $\lambda_1 + \dots + \lambda_g$ is also the integral over the invariant locus of the curvature of the Hodge bundle along Teichmüller

2010 *Mathematics Subject Classification*. Primary: 30F60, 32G15, 32G20; Secondary: 37H15.

¹For quadratic differentials, two bundles can be considered. In this article, we will only be interested in the bundle with fiber $H^1(X, \mathbb{R})$ over a Riemann surface X . The Lyapunov exponents of this bundle are often denoted by $\lambda_1^+, \dots, \lambda_g^+$.

discs ([For02], [EKZ11b]). Using this interpretation, every Lyapunov exponent is computed for cyclic covers of the sphere branched over 4 points ([EKZ11a], [FMZ11], see also [BM10], and [Wri12] for abelian covers). For some cyclic covers, Forni-Matheus-Zorich remarked that the sum $\lambda_1 + \dots + \lambda_g$ is equal to zero [FMZ11, Thm. 35]. This surprising fact means that the complex structure of the underlying Riemann surface is constant along the Teichmüller disc. Forni-Matheus-Zorich ask whether it is possible to construct other invariant loci with this property (see [FMZ11, p. 312]). The content of this article is to give a simple explanation of the phenomenon discovered by Forni-Matheus-Zorich.

Theorem 1. *There exist closed $GL(2; \mathbb{R})$ -invariant loci of quadratic differentials of arbitrarily large dimension with zero Lyapunov exponents.*

This result can be interpreted in the following way: the projection of such a locus to the moduli space of *compact* Riemann surfaces is a point. Remark that the situation for strata of *abelian* differentials is completely different: there are finitely many invariant suborbifolds with fully degenerate Lyapunov spectrum (meaning in this setting that all exponents are zero except λ_1 which is 1), and they are arithmetic Teichmüller curves (see [Möl11], [For06], [FMZ11] and [Aul13]).

Acknowledgements We thank John Hubbard, Howard Masur and Christopher Leininger for helpful discussions. We also thank heartily Dmitri Zvonkine for sharing a very valuable idea.

2. BACKGROUND MATERIAL

2.1. The Teichmüller flow for translation surfaces.

A translation surface is a pair (X, ω) where X is a compact Riemann surface and ω is a holomorphic one-form on X . If $S(\omega)$ is the set of the zeroes of ω , there exists an open covering of $\tilde{X} = X \setminus S(\omega)$ and holomorphic charts $\varphi_i: U_i \rightarrow X$ such that $\varphi_i^* \omega = dz$ for all i . For such an atlas, the transition functions are translations. The form ω induces a flat metric $|\omega|^2$ on the open surface \tilde{X} , whose area is the integral $\frac{i}{2} \int_{\tilde{X}} \omega \wedge \bar{\omega}$. We could have taken meromorphic 1-forms instead of holomorphic ones, but in that case the area of the surface is never finite.

There is a natural action of $GL(2, \mathbb{R})$ on translation surfaces given as follows: first we pick an atlas of charts of \tilde{X} where all transitions functions are translations by some complex vectors v_{ij} which we will consider as vectors in \mathbb{R}^2 . Then, for any M in $GL(2, \mathbb{R})$, we get an open surface \tilde{X}_M defined by an atlas whose transition functions are translations by $M.v_{ij}$. This surface is diffeomorphic to \tilde{X}_M . Therefore, we can fill the holes and extend the complex structure in a unique way: the result is a compact Riemann surface X_M diffeomorphic to X endowed with a meromorphic differential ω_M of finite volume, hence holomorphic. The action of $GL(2; \mathbb{R})$ is defined by the formula $M.(X, \omega) = (X_M, \omega_M)$. The action of $SL(2; \mathbb{R})$ preserves the volume of translation surfaces.

The subgroup of $\mathrm{SL}(2; \mathbb{R})$ of matrices M such that $M.(X, \omega) = (X, \omega)$ up to diffeomorphism is called the Veech group of (X, ω) . If $M_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then the curve $(X_t, \omega_t) := M_t.(X, \omega)$ is called the orbit under the Teichmüller flow of (X, ω) . If M_θ is a rotation in $\mathrm{SO}(2, \mathbb{R})$, then $M_\theta.(X, \omega) = (X, e^{i\theta}\omega)$.

If we fix multiplicities (m_1, \dots, m_r) such that $\sum_{i=1}^r m_i = 2g - 2$, the stratum of translation surfaces $\mathcal{H}(m_1, \dots, m_r)$ is the set of translations surfaces (X, ω) where ω has r distinct zeroes of multiplicities m_1, \dots, m_r modulo diffeomorphism. The normalized stratum $\mathcal{H}_1(m_1, \dots, m_r)$ is the locus of flat surfaces with unit area in $\mathcal{H}(m_1, \dots, m_r)$, and the projective stratum $\mathrm{P}\mathcal{H}(m_1, \dots, m_r)$ is obtained by taking the quotient of $\mathcal{H}(m_1, \dots, m_r)$ under the natural \mathbb{C}^\times -action on forms. Strata and projective strata are complex orbifolds of respective dimensions $2g + r - 1$ and $2g + r - 2$ if $g \geq 2$.

There are standard coordinates on the stratum $\mathcal{H}(m_1, \dots, m_r)$, called period coordinates. Fix (X, ω) in this stratum, and let $A_1, \dots, A_g, B_1, \dots, B_g$ be a symplectic basis of $H_1(X, \mathbb{Z})$ and C_1, \dots, C_{r-1} be $r - 1$ paths joining a zero of ω to all the $r - 1$ other zeroes. The map

$$(X, \omega) \rightarrow \left(\int_{A_1} \omega, \dots, \int_{A_g} \omega, \int_{B_1} \omega, \dots, \int_{B_g} \omega, \int_{C_1} \omega, \dots, \int_{C_{r-1}} \omega \right)$$

yields an orbifold chart on $\mathcal{H}(m_1, \dots, m_r)$. These charts allow to define a canonical volume element on $\mathcal{H}(m_1, \dots, m_r)$, $\mathcal{H}_1(m_1, \dots, m_r)$, and $\mathrm{P}\mathcal{H}(m_1, \dots, m_r)$. By classical results of Masur and Veech, connected components of projective strata have finite volume.

Let $\mathbb{H} = \mathrm{SL}(2; \mathbb{R})/\mathrm{SO}(2)$ denote the Poincaré upper-half plane. For any (X, ω) in a projective stratum, the $\mathrm{SL}(2; \mathbb{R})$ -action factorizes to a holomorphic map

$$\mathbb{H} \rightarrow \mathrm{P}\mathcal{H}(m_1, \dots, m_r)$$

which is an immersion. The image of this map is called a Teichmüller disc, it is stable under the Teichmüller flow. Besides, Teichmüller discs induce a smooth foliation with holomorphic leaves on $\mathrm{P}\mathcal{H}(m_1, \dots, m_r)$.

Assume that the Veech group Γ of (X, ω) is a lattice in $\mathrm{SL}(2; \mathbb{R})$. Then the image \mathbb{H}/Γ of the corresponding Teichmüller disc in the projective stratum is called a Teichmüller curve.

All these considerations generalize to the so-called half-translation surfaces, which are pairs (X, q) where q is a quadratic holomorphic (for the time being) differential on X . The transition functions of a well-chosen atlas of charts on the open surface are half translations, that is either translations or flips. The area of the flat metric on \tilde{X} is $\frac{1}{2} \int_X |q|$, and we still have an action of $\mathrm{GL}(2, \mathbb{R})$ as well as a Teichmüller flow. The period coordinates on strata of quadratic differentials are obtained as follows: for any (X, q) in a stratum, we take the twofold branched covering $\mathfrak{p}: \widehat{X} \rightarrow X$ given by the holonomy representation of q , which is given by a morphism from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$. Let j be the corresponding involution acting on

\widehat{X} . The quadratic differential p^*q is the square of an abelian differential ω . The period coordinates of (X, q) are obtained by taking j -anti-invariant absolute and relative periods of (X, q) .

However, a major difference happens for quadratic differentials: it is possible to take meromorphic quadratic differentials and still get finite volume for the corresponding flat surface. More precisely, (X, q) has finite volume if and only if q has poles of order at most one. Therefore we have strata, normalized strata and projective strata $\mathcal{Q}(m_1, \dots, m_r)$, $\mathcal{Q}_1(m_1, \dots, m_r)$ and $\text{PQ}(m_1, \dots, m_r)$, where $\sum_{i=1}^r m_i = 4g - 4$ and each m_i is either positive or equal to -1 .

Let S be a finite subset of X of cardinal n , so that (X, S) gives a point in the marked Teichmüller space $\mathcal{T}_{g,n}$ (genus g with n marked points). The cotangent space of $\mathcal{T}_{g,n}$ at X is exactly the space $\mathcal{Q}_S(X)$ of holomorphic quadratic differentials on $X \setminus S$ with poles of order one at some points of S . There is a norm on $\mathcal{Q}_S(X)$ given by $\|q\| = \int_X |q|$, as well as a dual norm on $\mathcal{Q}_S(X)^*$. The corresponding distance on $\mathcal{T}_{g,n}$ is the Teichmüller metric.

Let us fix (X, S) as well as an element q in $\mathcal{Q}_S(X)$. There is a complex linear form μ_q on $\mathcal{Q}_S(X)$ given by scalar product with the L^∞ Beltrami differential $\frac{|q|}{q}$:

$$\mu_q(\tilde{q}) = \int_X \tilde{q} \frac{|q|}{q}.$$

Note that $\mu_q(q) = \int_X |q| > 0$ so that μ_q is nonzero. Besides, we have $\|\mu_q\| = 1$. The map $q \rightarrow \mu_q$ gives a non-linear isomorphism between the unit spheres of $\mathcal{Q}_S(X)$ and $\mathcal{Q}_S(X)^*$, hence between the unitary cotangent space $U^*\mathcal{T}_{g,n}$ and the unitary tangent space $U\mathcal{T}_{g,n}$.

If (X, q) is given and S is the set of poles of q , the Teichmüller flow of (X, q) introduced formerly is the geodesic flow (for the Teichmüller metric) on $\mathcal{T}_{g,n}$ starting from X in the direction μ_q .

2.2. The period mapping.

For any compact Riemann surface X , $H^1(X, \mathbb{C})$ is the orthogonal sum (for the intersection form) of $\Omega(X)$ and $\overline{\Omega(X)}$. Besides, the composition

$$\psi : H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathbb{C}) \xrightarrow{\text{pr}_1} \Omega(X)$$

is an isomorphism. The Hodge norm $\|\cdot\|_{\text{Hodge}}$ is the unique norm on $H^1(X, \mathbb{R})$ making ψ an isometry.

Let us now consider a local holomorphic family of curves, that is a proper holomorphic submersion $\pi : \mathfrak{X} \rightarrow \mathcal{B}$ whose fibers $(X_b)_{b \in \mathcal{B}}$ are compact Riemann surfaces of some genus g , where \mathcal{B} is a small ball in \mathbb{C}^n . The Hodge bundle \mathcal{H} is a holomorphic vector bundle on \mathcal{B} of rank g whose fiber at each point b is the vector space $\mathcal{H}_b := \Omega(X_b)$. The local system $R^1\pi_* \mathbb{R}_{\mathfrak{X}}$ is trivial, which means

that we can canonically identify all the vector spaces $H^1(X_b, \mathbb{R})$ to some fixed real vector space \mathbb{V} of dimension $2g$. The local period map

$$\xi: \mathcal{B} \rightarrow \text{Gr}(g, \mathbb{V}^{\mathbb{C}})$$

associates to any b the subspace \mathcal{H}_b in the Grassmannian of g -dimensional complex subspaces of $\mathbb{V}^{\mathbb{C}}$. The derivative of ξ at a point b in \mathcal{B} is a linear map from $T_b^{1,0}\mathcal{B}$ to $\text{End}(\mathcal{H}_b, \mathbb{V}^{\mathbb{C}}/\mathcal{H}_b)$, which is isomorphic to $\text{End}(\mathcal{H}_b, \overline{\mathcal{H}}_b)$.

The differential of ξ can be explicitly computed: ξ induces a classifying map $\xi_{\text{Teich}}: \mathcal{B} \rightarrow \mathcal{T}_g$. Then we have the following formula due to Ahlfors: for any vector v in $T_b\mathcal{B}$ and any elements α and β in $\Omega(X_b)$,

$$(\beta, \overline{\xi'_v(\alpha)}) = \int_X \alpha \otimes \beta \cdot \xi'_{\text{Teich}}(v).$$

In this formula, $\xi'_{\text{Teich}}(v)$ is a tangent vector to \mathcal{T}_g , hence represented by a Beltrami differential which is a tensor field on X of type $(-1, 1)$. Thus, the integrand in the above formula is of type $(2, 0) + (-1, 1) = (1, 1)$. We can also think of $\xi'_{\text{Teich}}(v)$ as a linear form on $Q(X)$; in this case the above formula reads

$$(\beta, \overline{\xi'_v(\alpha)}) = \xi'_{\text{Teich}}(v) \{ \alpha \otimes \beta \}.$$

It is also possible to give another interpretation on ξ' . For this we consider the exact sequence of holomorphic vector bundles

$$0 \rightarrow \mathcal{H} \rightarrow \mathbb{V} \otimes \mathcal{O}_{\mathcal{B}} \rightarrow \overline{\mathcal{H}} \rightarrow 0.$$

The bundle $\mathbb{V} \otimes \mathcal{O}_{\mathcal{B}}$ carries a natural flat connection (the Gauß-Manin connection), but \mathcal{H} is not in general a flat sub-bundle of $\mathbb{V} \otimes \mathcal{O}_{\mathcal{B}}$. A precise way to measure this (see formula (2) below) is the second fundamental form σ associated with this exact sequence and the Gauß-Manin connection; it is a $(1, 0)$ -form with values in $\text{Hom}(\mathcal{H}, \overline{\mathcal{H}})$. A simple calculation shows that

$$(1) \quad \sigma = \xi'.$$

The Hodge bundle \mathcal{H} carries a natural metric given by the intersection form, its curvature form is given by the formula

$$(2) \quad \Theta_{\mathcal{H}} = \sigma^* \wedge \sigma.$$

By " \wedge " we mean composition on the fiber and wedge-product on the base. In particular, $i \text{Tr} \Theta_{\mathcal{H}}$ is a positive $(1, 1)$ -form on \mathcal{B} .

For any compact half-translation surface (X, q) , Forni's B-form is a bilinear form on $\Omega(X)$ defined by

$$B_q(\alpha, \beta) = \int_X \alpha \otimes \beta \frac{|q|}{q}.$$

If $\xi'_{\text{Teich}}(v)$ has unit norm, we can write it as μ_q for some holomorphic quadratic differential on X . Then we have $(\beta, \overline{\xi'_v(\alpha)}) = B_q(\alpha, \beta)$. In the case of a Teichmüller orbit (X_t, q_t) , if we differentiate along the vector field $\frac{\partial}{\partial t}$, we get the formula

$$(3) \quad (\beta, \overline{\xi'_{\partial_t}(\alpha)}) = B_{q_t}(\alpha, \beta).$$

Applying Cauchy-Schwarz inequality, $|B_q(\alpha, \beta)| \leq \|\alpha\| \times \|\beta\|$ with equality if and only if there exists a holomorphic one-form ω and two complex constants c and c' such that $q = \omega^2$, $\alpha = c\omega$ and $\beta = c'\omega$. In particular, if q is meromorphic with simple poles, $\|B_q\| < 1$ where $\|\cdot\|$ denotes the operator norm.

We recall now Forni's inequality: let (X, q) be a half-translation surface, (X_t, q_t) be its orbit under the Teichmüller flow, v be in $H^1(X, \mathbb{R})$ and $t \rightarrow v_t$ be its parallel transport under the Teichmüller flow for the Gauß-Manin connection. We write $v_t = \chi_t + \overline{\chi_t}$ where χ_t is in $\Omega(X_t)$. Then a simple calculation gives

$$\partial_t \|v_t\|_{\text{Hodge}} = B_{q_t}(\chi_t, \chi_t).$$

Combined with the inequality $\|B\| \leq 1$, this gives Forni's inequality

$$(4) \quad \left| \partial_t \{\log \|v_t\|_{\text{Hodge}}\} \right| \leq 1.$$

2.3. Lyapunov exponents of the KZ cocycle.

The parallel transport for the Gauß-Manin connection of vectors of $H^1(X, \mathbb{R})$ under the Teichmüller flow is called the Kontsevich-Zorich cocycle. Recall that the Teichmüller flow is ergodic on every connected component \mathfrak{D}_1 of the normalized stratum $\mathcal{Q}_1(m_1, \dots, m_r)$. By Osceledets' theorem, it is possible to associate $2g$ Lyapunov exponents to this cocycle.

Forni's inequality (4) implies that the KZ cocycle is log-integrable, so that the Lyapunov exponents are well-defined. Since the cocycle is symplectic, the Lyapunov spectrum is of the form $\{-\lambda_1, -\lambda_2, \dots, -\lambda_g, \lambda_g, \lambda_{g-1}, \dots, \lambda_1\}$ where $\lambda_1 \geq \dots \geq \lambda_g$.

Note that the exponents λ_i are called λ_i^+ in numerous papers (e.g. in [EKZ11b], [FMZ11]). The exponents λ_i^- will never be considered in the article.

By (4), all λ_i 's are at most one. If the component \mathfrak{D}_1 is orientable, which means that every quadratic differential occurring in the stratum is the square of an abelian differential, then the top Lyapunov exponent λ_1 equals one. If not, the norm of Forni's B form is strictly smaller than one so that $\lambda_1 < 1$.

For any (X, q) in a stratum $\text{PQ}(m_1, \dots, m_r)$, the Poincaré metric on \mathbb{H} induces a metric on the Teichmüller disc passing through (X, q) . The corresponding volume element defines a relative $(1, 1)$ form dV_{Teich} , where by "relative" we mean relative with respect to the foliation by Teichmüller discs. If Θ is the curvature of the Hodge bundle on $\text{PQ}(m_1, \dots, m_r)$, its trace is also a relative $(1, 1)$ form on the projective stratum. Let $\Lambda: \text{PQ}(m_1, \dots, m_r) \rightarrow \mathbb{R}$ be defined by the formula

$$\Lambda = \frac{\text{Tr } \Theta}{dV_{\text{Teich}}}.$$

Then Kontsevich-Forni's main formula for the Lyapunov exponents is

$$\lambda_1 + \dots + \lambda_g = \int_{\mathfrak{D}} \Lambda(X, q) dV$$

where \mathfrak{D} is the projection of \mathfrak{D}_1 in the projective stratum and dV is the normalized volume element on \mathfrak{D} of total mass one. For any (X, q) in \mathfrak{D} , let $\theta_1, \dots, \theta_g$ be the eigenvalues of Forni's B-form in the direction of the Teichmüller flow when diagonalized in an orthonormal basis for the intersection form. Using formulæ (2), (1) and (3), we see that

$$(5) \quad \lambda_1 + \dots + \lambda_g = \int_{\mathfrak{D}} \left\{ |\theta_1(X, q)|^2 + \dots + |\theta_g(X, q)|^2 \right\} dV$$

Forni's inequality implies that $|\theta_i(X, q)| \leq 1$ for all i so that $\lambda_1 + \dots + \lambda_g \leq g$.

Thanks to the main result of [EM13], any closed $\mathrm{SL}(2; \mathbb{R})$ -invariant locus in the projective stratum $\mathrm{PQ}(m_1, \dots, m_r)$ is affine in period coordinates, hence carries a natural $\mathrm{SL}(2; \mathbb{R})$ -invariant probability measure. It is also possible to define Lyapunov exponents for this measure, and formula (5) holds.

If (X, q) is any half-translation surface, the closure of its $\mathrm{SL}(2; \mathbb{R})$ -orbit in the normalized stratum is affine in period coordinates. It follows from [CE13] that almost every direction θ , the real Teichmüller flow of $(X, e^{i\theta}q)$ is Osceledets-generic for the corresponding natural probability measure. Therefore it makes sense to consider Lyapunov exponents of (X, q) , and formula (5) is still valid if we integrate on the closure of the $\mathrm{PGL}(2; \mathbb{R})$ -orbit.

3. THE FORNI LOCUS

3.1. General properties. Let \mathfrak{D} be a connected component of the projective stratum $\mathrm{PQ}(m_1, \dots, m_r)$.

Definition 1. *The Forni locus of \mathfrak{D} is the set of elements (X, q) in \mathfrak{D} such that for all holomorphic 1-forms α and β on X , $B_q(\alpha, \beta) = 0$.*

Remark 1. *In the terminology used in [For06], the Forni locus is called the rank 0 locus.*

Let us now recall Noether's theorem (see [FK92, p. 104 & 159]):

Proposition 1. *Let X be a compact Riemann surface of genus g and*

$$\tau: \mathrm{Sym}^2 \Omega^1(X) \rightarrow \mathcal{Q}(X)$$

be the multiplication map.

- (i) *If X is not hyperelliptic or if $g \leq 2$, τ is surjective.*
- (ii) *If X is hyperelliptic, $\mathrm{Im}(\tau)$ has codimension $g - 2$ in $\mathcal{Q}(X)$ and consists of the quadratic differentials invariant by the hyperelliptic involution.*

Since τ is the transpose of the derivative of the period map, Noether's result has the following geometric interpretation:

Proposition 2 (Infinitesimal Torelli's theorem).

Let $\xi: \mathcal{T}_g \rightarrow \mathbb{H}_g$ be the period map. Then ξ is an immersion outside the hyperelliptic locus or everywhere if $g \leq 2$, and the restriction of ξ to the hyperelliptic locus is also an immersion.

Remark that Forni's B -form factors through $\text{Im } \tau$, and can be extended naturally to $\mathcal{Q}(X)$ by the formula $B_q(\tilde{q}) = \int_X \tilde{q} \frac{|q|}{q}$.

The key proposition of this section is:

Proposition 3. *Let (X, q) be a half-translation surface, n the number of poles of q , and \mathbb{D} be its Teichmüller disc. Then the following are equivalent:*

- (i) \mathbb{D} lies in the Forni locus.
- (ii) The forgetful map $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_g$ maps \mathbb{D} to a point.
- (iii) For any (X_t, q_t) in \mathbb{D} , the extension of B_{q_t} to $\mathcal{Q}(X_t)$ vanishes.
- (iv) All Lyapunov exponents of (X, q) are zero.

Proof.

(i) \Rightarrow (ii) Using (3), the composite map $\mathbb{D} \hookrightarrow \mathcal{T}_{g,n} \rightarrow \mathcal{T}_g \xrightarrow{\xi} \mathbb{H}_g$ has zero derivative. Assume that \mathbb{D} is not contained in the hyperelliptic locus. Thanks to the infinitesimal Torelli theorem, \mathbb{D} is mapped to a point via the forgetful map $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_g$. Assume now that \mathbb{D} is contained in the hyperelliptic locus. Then the restriction of ξ to this locus is again an immersion, and we can apply the same argument.

(ii) \Rightarrow (iii) If (X_t, q_t) is a point in \mathbb{D} , the derivative of projection of the Teichmüller flow of (X_t, q_t) on \mathcal{T}_g is the linear form $\tilde{q} \rightarrow B_{q_t}(\tilde{q})$ on $\mathcal{Q}(X_t)$.

(iii) \Rightarrow (i) Obvious.

(i) \Leftrightarrow (iv) Let \mathcal{V} be the closure of the $\text{PSL}(2; \mathbb{R})$ -orbit of X and ν the corresponding $\text{PSL}(2; \mathbb{R})$ -invariant probability measure. If $\lambda_1, \dots, \lambda_g$ are the Lyapunov exponents of (X, q) , then

$$\lambda_1 + \dots + \lambda_g = \int_{\mathcal{V}} \{\theta_1(X, q) + \dots + \theta_g(X, q)\} d\nu.$$

Since all θ_i 's are nonnegative and continuous functions, $\lambda_1 = \dots = \lambda_g = 0$ if and only if all θ_i 's vanish on \mathbb{D} . □

Corollary 1. *If q is a holomorphic quadratic differential on X , the Teichmüller disc of (X, q) is not included in the Forni locus. In particular some Lyapunov exponents are non-zero.*

Proof. If q is holomorphic, $B_q(q) > 0$ and we apply Proposition 3. □

By Proposition 3 the vanishing along a Teichmüller disc of the B -form and of its extension to the quadratic differentials are equivalent. This is no longer the case if we consider these properties for a single half-translation surface (X, q) , as shown by the example below.

Let X be a hyperelliptic surface of genus at least 3, let j be the hyperelliptic involution, and let q be an anti-invariant holomorphic quadratic differential (if X is the Riemann surface of a polynomial $w^2 - P(z)$, we can take $q = w^{-1}dz^{\otimes 2}$). Since any holomorphic 1-form on X is anti-invariant under j^* , $B_q = 0$. Hence we see that the following properties are satisfied:

- q is holomorphic.
- (X, q) is in the Forni locus.
- The extension of B_q to $Q(X)$ is nonzero since $B_q(q) > 0$.
- The Teichmüller disc of (X, q) is not contained in the Forni locus (see Corollary 1).

This also gives a concrete example showing that the Forni locus is not $SL(2; \mathbb{R})$ -invariant.

3.2. Galois coverings and pillow-tiled surfaces. Let (X, q) be a half-translation surface and (Y, π) be an arbitrary finite covering of X with branching locus S . Assume that for any point y in Y above S , the ramification index of π at y is at least 2. Then π^*q is holomorphic, so that B_{π^*q} is non zero on $Q(Y)$. Thanks to Corollary 1, the Teichmüller disc of (Y, π^*q) doesn't lie in the Forni locus. Using this observation, we have the following result:

Proposition 4. *Let (X, q) be a half-translation surface and (Y, π) be a finite Galois covering of X with branch locus S . If the Teichmüller disc of (Y, π^*q) lies in the Forni locus, then at least one pole of q does not belong to S .*

As a particular by-product, we get:

Corollary 2. *Let (X, q, π) be a pillow-tiled surface such that π is Galois. Then the Teichmüller disc of (X, q) lies in the Forni locus if and only if the branching locus of π contains at most three points.*

Proof. Let q_{st} be the standard meromorphic differential on \mathbb{P}^1 with four simple poles such that $q = \pi^*q_{st}$. By definition the branching locus of π lies in the set of poles of q_{st} . If X is in the Forni locus, according to Proposition 4, one of the poles of q_{st} is not a branching point of π .

Conversely, assume that the branching locus of π has less than four points. If $\{z_1, z_2, z_3, z_4\}$ are the four poles of q_{st} , let us assume that z_4 is not a branch point of π . The complex Teichmüller flow of (\mathbb{P}^1, q_{st}) is of the form (\mathbb{P}^1, q_t) where q_t has poles at z_1, z_2, z_3 and another point $z_4(t)$ such that $[z_1, z_2, z_3, z_4(t)] = t$. Let \mathring{X} be the open Riemann surface obtained by removing $\pi^{-1}\{z_1, z_2, z_3\}$. Then \mathring{X} is an unramified covering of $\mathbb{P}^1 \setminus \{z_1, z_2, z_3\}$. It follows that $(\mathring{X} \setminus \pi^{-1}(z_4(t)), \pi^*q_t)$ parametrizes the Teichmüller disc of (X, q) in $T^*\mathcal{T}_{g,n}$ (where n is the number of poles of q). This disc maps to $\{X\}$ via the forgetful map $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_g$. Thanks to Proposition 3, the Teichmüller disc of (X, q) lies in the Forni locus. \square

Let us now consider pillow-tiled surfaces arising as *cyclic* coverings of the projective line. They are given by a combinatorial datum (N, a_1, a_2, a_3, a_4) where

$0 < a_i \leq N$, $\gcd(a_1, a_2, a_3, a_4, N) = 1$ and $\sum_{i=1}^4 a_i \equiv 0 \pmod{N}$: the associated cyclic covering is the Riemann surface of the polynomial

$$w^N - (z - z_1)^{a_1} (z - z_2)^{a_2} (z - z_3)^{a_3} (z - z_4)^{a_4}.$$

In topological terms, if $(\gamma_i)_{1 \leq i \leq 4}$ are small loops around the z_i 's for $1 \leq i \leq 4$, then the kernel of the group morphism

$$\pi_1(\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}) \rightarrow \mathbb{Z}/N\mathbb{Z}$$

given by $\gamma_i \rightarrow a_i$ defines a true cyclic covering of $\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$ of degree N , which extends to a branched cyclic covering of the projective line.

In [FMZ11, Thm. 35], the authors prove that all Lyapunov exponents of the Teichmüller curve corresponding to a cyclic covering are 0 if one of the integers a_i equals N .

Proposition 5. *If (X, q) is a pillow-tiled surface obtained by a cyclic covering of \mathbb{P}^1 with combinatorial datum (N, a_1, a_2, a_3, a_4) , then the Teichmüller disc of (X, q) lies in the Forni locus if and only if one of the a_i 's equals N .*

Proof. Thanks to Corollary 2, it suffices to prove that the projection π of the covering is branched at three points or less if and only if one of the a_i 's equals N . If $\{z_1, z_2, z_3, z_4\}$ are the four points defining the cyclic cover, the ramification index of π at any point of $\pi^{-1}(z_i)$ is $\frac{N}{\gcd(N, a_i)}$. \square

3.3. Construction of Forni subspaces.

In this section, we provide the precise statement underlying Theorem 1 as well as its proof.

Let m_1, \dots, m_r and k be positive integers such that $(\sum_{i=1}^r m_i) - k = -4$, and let \mathcal{S} be the set of couples (q, x_1, \dots, x_{k-3}) such that q is a meromorphic differential on \mathbb{P}^1 with simple poles at $0, 1, \infty$ and the x_i 's, and q has r zeroes of order m_1, \dots, m_r . It is a smooth $\mathrm{GL}(2; \mathbb{R})$ -invariant submanifold of $T^* \mathcal{M}_{0, [k]}$ (where the bracket means that the points are ordered).

Let us fix a covering (Y, π) of \mathbb{P}^1 ramified over $0, 1$ and ∞ , and let g be the genus of Y . Put

$$n = \#\{y \in \pi^{-1}\{0, 1, \infty\} \text{ such that } \pi \text{ is unramified at } y\} + \deg(\pi) \times (k - 3)$$

We have a natural map

$$\chi: \mathcal{S} \rightarrow T_{\mathrm{orb}}^* \mathcal{M}_{g, n}$$

given by $\chi(q) = (Y, \pi^* q)$, where T_{orb}^* denotes the orbifold cotangent bundle.

Theorem 2. *Let \mathcal{W} be the image of χ .*

- (1) *The map $\chi: \mathcal{S} \rightarrow \mathcal{W}$ is a holomorphic orbifold map, which is a local immersion. Besides, \mathcal{W} is a suborbifold² of the orbifold cotangent bundle of $\mathcal{M}_{g, n}$ of dimension $r + k - 2$.*

²By suborbifold, we mean as usually done in this theory "locally finite union of suborbifolds".

- (2) \mathcal{W} is $\mathrm{GL}(2; \mathbb{R})$ -invariant and lies in the Forni locus, and the projection of \mathcal{W} by the map $T_{\mathrm{orb}}^* \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n} \rightarrow \mathcal{M}_g$ is $\{Y\}$.
- (3) The Lyapunov spectrum of \mathcal{W} is fully degenerate.

Proof. Let q be a point in \mathcal{S} , and U be a small neighborhood of q in \mathcal{S} . It is possible to lift locally χ to a smooth map $\widehat{\chi}$ from U to $T^* \mathcal{T}_{g,n}$, so that χ is a smooth orbifold map.

If q_1, q_2 are two elements in U such that $\chi(q_1) = \chi(q_2)$, then there exists φ in $\mathrm{Aut}(Y)$ such that $\varphi^*(Y, \pi^* q_1) = (Y, \pi^* q_2)$. Thus the fibers of $\widehat{\chi}|_U$ are finite. But $\widehat{\chi}$ is affine in period coordinates, so that it is an immersion on U .

Point (2) is proved using the same argument as in Corollary 2, which corresponds to the particular case $r = 0$.

Lastly, the fact that the Lyapunov spectrum of \mathcal{W} is totally degenerate results from the implication (ii) \Rightarrow (iv) in Proposition 3. \square

Remark 2. *In view of the existing terminology in the abelian case, it is natural to call \mathcal{W} a Forni subspace of the stratum.*

4. NUMERICAL CONSTRAINTS

In this section, given a half-translation surface (X, q) whose Teichmüller disc lies in the Forni locus, we give some numerical constraints relating the genus g of X , the number of n of simple poles of q and the degree d when (X, q) is a pillow-tiled surface.

Proposition 6. *Let (X, q) be a half-translation surface of genus $g \geq 1$ whose Teichmüller disc lies in the Forni locus, and let n be the number of poles of q . Then $n \geq \max(2g - 2, 2)$.*

Proof. The fact that the number n of poles of q must be at least two follows from [Kra81, Thm 4']. To get the lower bound $2g - 2$ in the proposition, we use [EKZ11b, Thm 2] for the closure of the $\mathrm{SL}(2; \mathbb{R})$ -orbit \mathcal{O} of (X, q) , which is contained in a stratum $\mathrm{PQ}((-1)^n, m_1, \dots, m_r)$: we get

$$\lambda_1 + \dots + \lambda_g = \frac{1}{24} \sum_{j=1}^r \frac{m_j(m_j + 4)}{m_j + 2} - \frac{n}{8} + \frac{\pi^2}{3} C_{\mathrm{area}}(\overline{\mathcal{O}})$$

where $C_{\mathrm{area}}(\overline{\mathcal{O}})$ is a Siegel-Veech constant of the locus $\overline{\mathcal{O}}$ which is nonnegative. Thus, if $\lambda_1 + \dots + \lambda_g = 0$,

$$\sum_{j=1}^r \frac{m_j(m_j + 4)}{m_j + 2} \leq 3n.$$

Since $(\sum m_j) - n = 4g - 4$,

$$2g - 2 \leq \sum_{j=1}^r \frac{m_j}{m_j + 2} + 2g - 2 \leq n.$$

\square

Remark 3. *We will see that this bound is asymptotically sharp at the end of this section.*

We now focus on the case of pillow-tiled surfaces. Let us start with a technical result:

Proposition 7. *Let X be a Riemann surface of genus g , $B(t_0, \varepsilon)$ a small ball in $\mathbb{C} \setminus \{0, 1, \infty\}$, and $\varphi: B(t_0, \varepsilon) \times X \rightarrow \mathbb{P}^1$ be a holomorphic map satisfying the following conditions:*

- (1) *For any t in $B(t_0, \varepsilon)$, φ_t is non-constant and the branching locus of φ_t is $\{0, 1, \infty, t\}$.*
- (2) *The configuration of the ramification points of φ_t remains constant with t .*

If d is the degree of the branched covers φ_t , then $d \geq 3(g - 1)$.

Proof. For any x in X , let $s(x) = \frac{\partial}{\partial t}|_{t=t_0} \varphi_t(x) \in T_{\varphi_{t_0}(x)} \mathbb{P}^1$. Then s is a holomorphic section of the holomorphic line bundle $\varphi_{t_0}^* \mathbb{P}^1$. Let x_0 be a ramification point of φ_{t_0} such that $\varphi_{t_0}(x_0) = 0$. Let us assume that $s(x_0) \neq 0$. By the implicit function theorem, the equation $\varphi_t(x) = 0$ has a unique solution $(t(x), x)$ depending holomorphically on x for (t, x) near (t_0, x_0) . Since $\varphi_{t(x)}(x) = 0$, we get

$$\partial_t \varphi(t(x), x) t'(x) + (\varphi_{t(x)})'(x) = 0.$$

By hypothesis, x is a ramification point of $\varphi_{t(x)}$, i.e. $(\varphi_{t(x)})'(x) = 0$. Besides, since $\partial_t \varphi(t(x), x) \rightarrow s(x_0)$ as $x \rightarrow x_0$, t' vanishes. Hence $\varphi_{t_0}(x)$ vanishes for x near x_0 , so that φ_{t_0} is constant and we get a contradiction. It follows that s vanishes at x_0 . The same result also holds over any ramification point of φ_{t_0} lying over 1 and ∞ . Lastly, if $\psi_t(x) = \varphi_t(x) - t$, the argument we used proves that for any ramification point x of ψ_{t_0} lying over 0, $\frac{\partial}{\partial t}|_{t=t_0} \psi_t(x) = 0$, which means that $s(x) = 1$. In particular s is nonzero.

We can now decompose the ramification divisor \mathcal{R} of the branched covering φ_{t_0} (which is the divisor of the section $d\varphi_{t_0}$ of $\varphi_{t_0}^* \mathbb{P}^1$) as the sum $\mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_\infty + \mathcal{R}_t$. Besides, we can assume that $\deg \mathcal{R}_t$ is smaller than $\deg \mathcal{R}_0$, $\deg \mathcal{R}_1$ and $\deg \mathcal{R}_\infty$, otherwise we move the points 0, 1, ∞ and t by a suitable homographic transformation. Besides, thanks to the Riemann-Hurwitz formula, we have

$$\deg \mathcal{R} = 2(g + d - 1)$$

Now s is a nonzero section of the line bundle $\mathcal{L} = \varphi_{t_0}^* \mathbb{P}^1(-\mathcal{R}_0 - \mathcal{R}_1 - \mathcal{R}_\infty)$; hence

$$0 \leq \deg \mathcal{L} = 2d - \deg \mathcal{R} + \deg \mathcal{R}_t \leq 2d - \frac{3}{4} \deg \mathcal{R} = \frac{d}{2} - \frac{3g}{2} + \frac{3}{2}.$$

The result follows. \square

Corollary 3. *Let (X, q, π) be a pillow-tiled surface of genus g whose Teichmüller disc lies in the Forni locus, and let d be the degree of π . If the ramification locus of π consists of 4 points, then $d \geq 3(g - 1)$.*

We end this section by proving that the estimate $n \geq 2g - 2$ of Proposition 6 is asymptotically sharp and that the condition on the ramification locus of π in Corollary 3 is necessary.

Let p be a prime number, and let Y be a cyclic covering of degree p of the projective line fully ramified at three distinct points z_1, z_2 and z_3 obtained by taking the Riemann surface of the polynomial

$$w^p - (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}$$

where $1 \leq a_1, a_2, a_3 \leq p - 1$ and $a_1 + a_2 + a_3 = p$. Then the genus of Y is $\frac{p-1}{2}$. If q is the pull-back of a meromorphic differential on the sphere with four simple poles at z_1, z_2, z_3 and another point z_4 , then π^*q has exactly p poles, and three zeros of order $p - 2$. Therefore, we have

$$\underbrace{n}_p = \underbrace{2g - 2}_{p-3} + \underbrace{\sum_j \frac{m_j}{m_j + 2}}_{3 - \frac{6}{p}} + \underbrace{\frac{\pi^2}{3} C_{\text{area}}}_{\frac{6}{p}}$$

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